IV. Short rate models: Pricing bonds in short rate models

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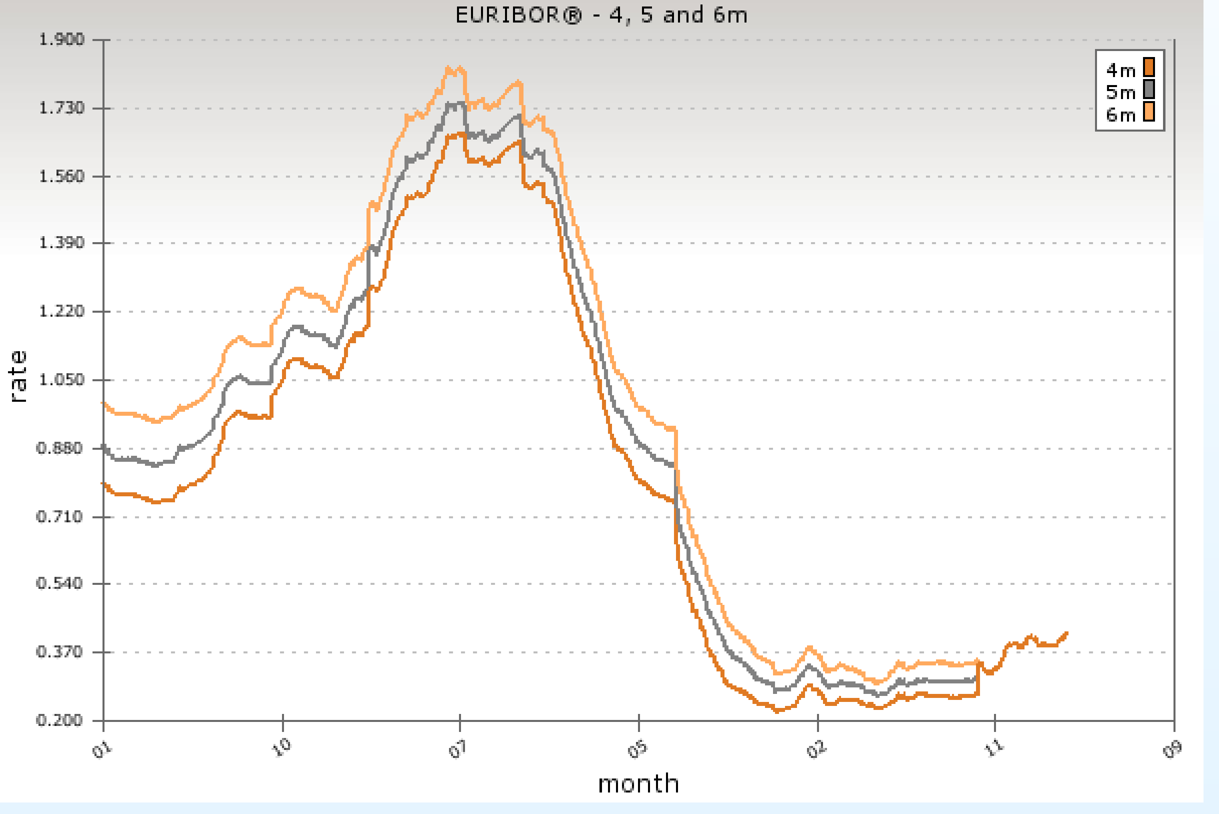
Interest rates

- Sa far we modelled the instantaneous interest rate
- Now: interest rates with different maturities



http://www.euribor-ebf.eu/

Bonds

- Bond
  - a security that in the predetermined time (called maturity of the bond) pays a predetermined amount of money - WLOG we take 1 USD
  - $P(t, T) =$ bond price at time t, if its maturity is at time T
  - $R(t, T) =$ interest rate with maturity T at time t
  - Relation between them:

$$P(t, T) = e^{-R(t,T)(T-t)} \Rightarrow R(t, T) = -\frac{\log P(t, T)}{T - t}$$

- In short rate models: bond price P is a solution to a PDE, $P = P(r, t, T)$

PDE for the bond price: derivation

- SDE for the short rate:

$$dr = \mu(t,r)dt + \sigma(t,r)dw$$

- Consider a bond with maturity T , then from Itō:

$$dP = \underbrace{\left(\frac{\partial P}{\partial t} + \mu\frac{\partial P}{\partial r} + \frac{\sigma^2}{2}\frac{\partial^2 P}{\partial r^2}\right)}_{\mu_B(t,r)} dt + \underbrace{\sigma\frac{\partial P}{\partial r}}_{\sigma_B(t,r)} dw$$

- Portfolio: 1 bond with maturity T_1 and Δ bonds with maturity T_2; its value:

$$\Pi = P(r,t,T_1) + \Delta P(r,t,T_2)$$

PDE for the bond price: derivation

- Change in the portfolio value:

$$\begin{aligned} d\Pi &= dP(r,t,T_1) + \Delta dP(r,t,T_2) \\ &= \left(\mu_B(r,t,T_1) + \Delta\mu_B(r,t,T_2)\right) dt \\ &\quad +\left(\sigma_B(r,t,T_1) + \Delta\sigma_B(r,t,T_2)\right) dw \end{aligned}$$

- We eliminate the randomness - by taking

$$\Delta = -\frac{\sigma_B(t,r,T_1)}{\sigma_B(t,r,T_2)},$$

then

$$d\Pi = \left(\mu_B(t,r,T_1) - \frac{\sigma_B(t,r,T_1)}{\sigma_B(t,r,T_2)}\mu_B(t,r,T_2)\right) dt$$

PDE for the bond price: derivation

- Yield of a riskless portfolio has to be r (instantaneous interest rate), i.e. $d\Pi = r\Pi dt$:

$$\mu_B(t,r,T_1) - \frac{\sigma_B(t,r,T_1)}{\sigma_B(t,r,T_2)}\mu_B(t,r,T_2) = r\Pi$$

- Substituting:

$$\mu_B(t,r,T_1) - \frac{\sigma_B(t,r,T_1)}{\sigma_B(t,r,T_2)}\mu_B(t,r,T_2)$$

$$= r\left(P(t,r,T_1) - \frac{\sigma_B(t,r,T_1)}{\sigma_B(t,r,T_2)}P(t,r,T_2)\right)$$

PDE for the bond price: derivation

- Maturities T_1, T_2 were arbitrary, hence there must be $\lambda = \lambda(r,t)$ such that for all t:

$$\lambda(r,t) = \frac{\mu_B(r,t,T) - rP(r,t,T)}{\sigma_B(r,t,T)}$$

- Function $\lambda = \lambda(r,t)$ does not depend on the maturity T; it is called market price of risk
- CONCLUSION: PDE for the bond price $P = P(r,t)$ is

$$\frac{\partial P}{\partial t} + (\mu(r,t) - \lambda(r,t)\sigma(r,t))\frac{\partial P}{\partial r} + \frac{\sigma^2(r,t)}{2}\frac{\partial^2 P}{\partial r^2} - rP = 0.$$

for $r \in (0,\infty), t \in (0,T)$ with terminal condition $P(r,T) = 1$ for $r \in (0,\infty)$

Closed form solutions

- Closed form solutions for the bond price
  - Vasicek model with market price of risk $\lambda(r,t) = \lambda$
  - CIR model with market price of risk $\lambda(r,t) = \lambda\sqrt{r}$
- We are looking for a solution in the form

$$P(r,\tau) = A(\tau)e^{-B(\tau)r},$$

  where $\tau = T - t$
- Substituting into the PDE $\Rightarrow$ we obtain a system of ordinary differential equations for the functions $A(\tau), B(\tau)$ $\Rightarrow$ this system can be solved explicitly

Vasicek model

- System of ODEs:

$$
\begin{aligned}
-\dot{A} + \frac{\sigma^2}{2}AB^2 - (\kappa\theta - \lambda\sigma)AB &= 0 \\
\dot{B} + \kappa B - 1 &= 0
\end{aligned}
$$

  with initial conditions $A(0) = 1, B(0) = 0$

- Functions A, B:

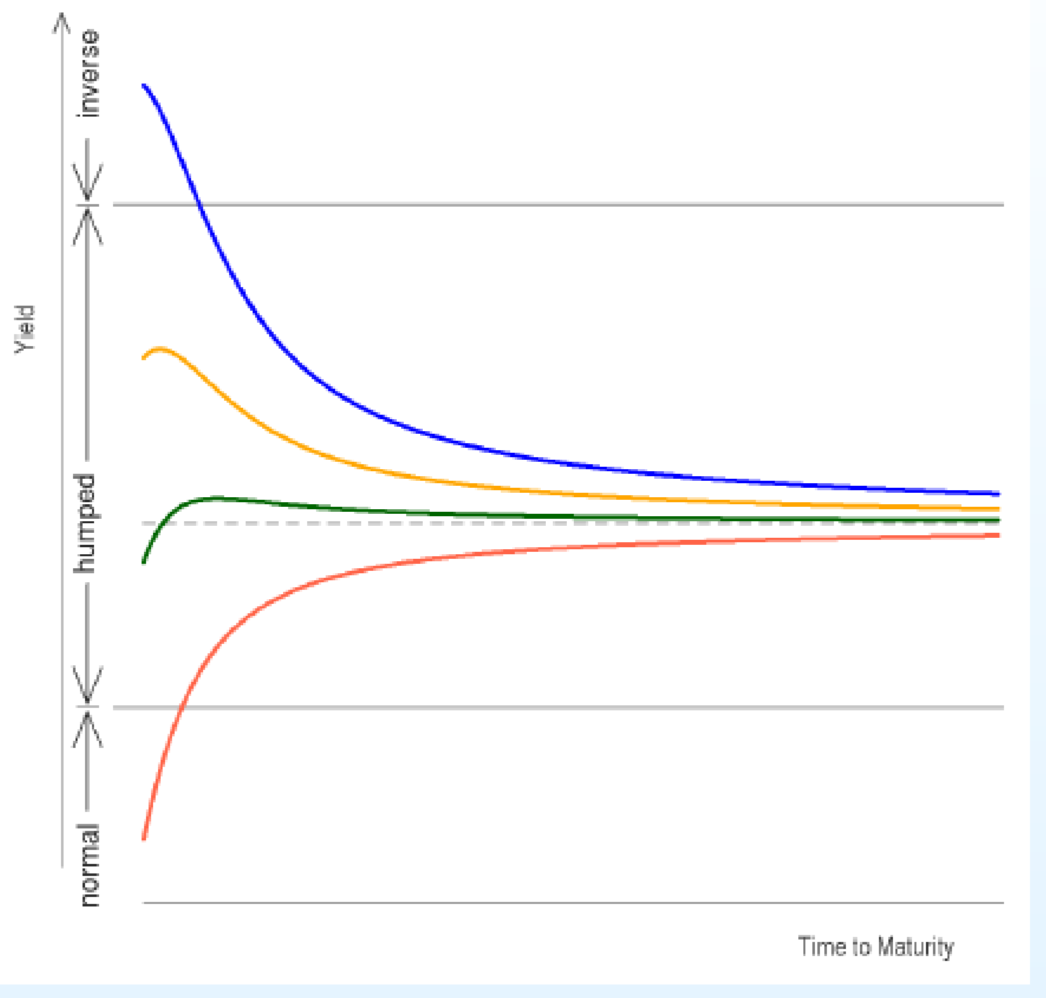
$$B(\tau) = \frac{1 - e^{-\kappa\tau}}{\kappa},$$

$$\log A(\tau) = \left[\frac{1}{\kappa}(1 - e^{-\kappa\tau}) - \tau\right] R_\infty - \frac{\sigma^2}{4\kappa^3}(1 - e^{-\kappa\tau})^2,$$

  where $R_\infty = \theta - \frac{\lambda\sigma}{\kappa} - \frac{\sigma^2}{2\kappa^2}$

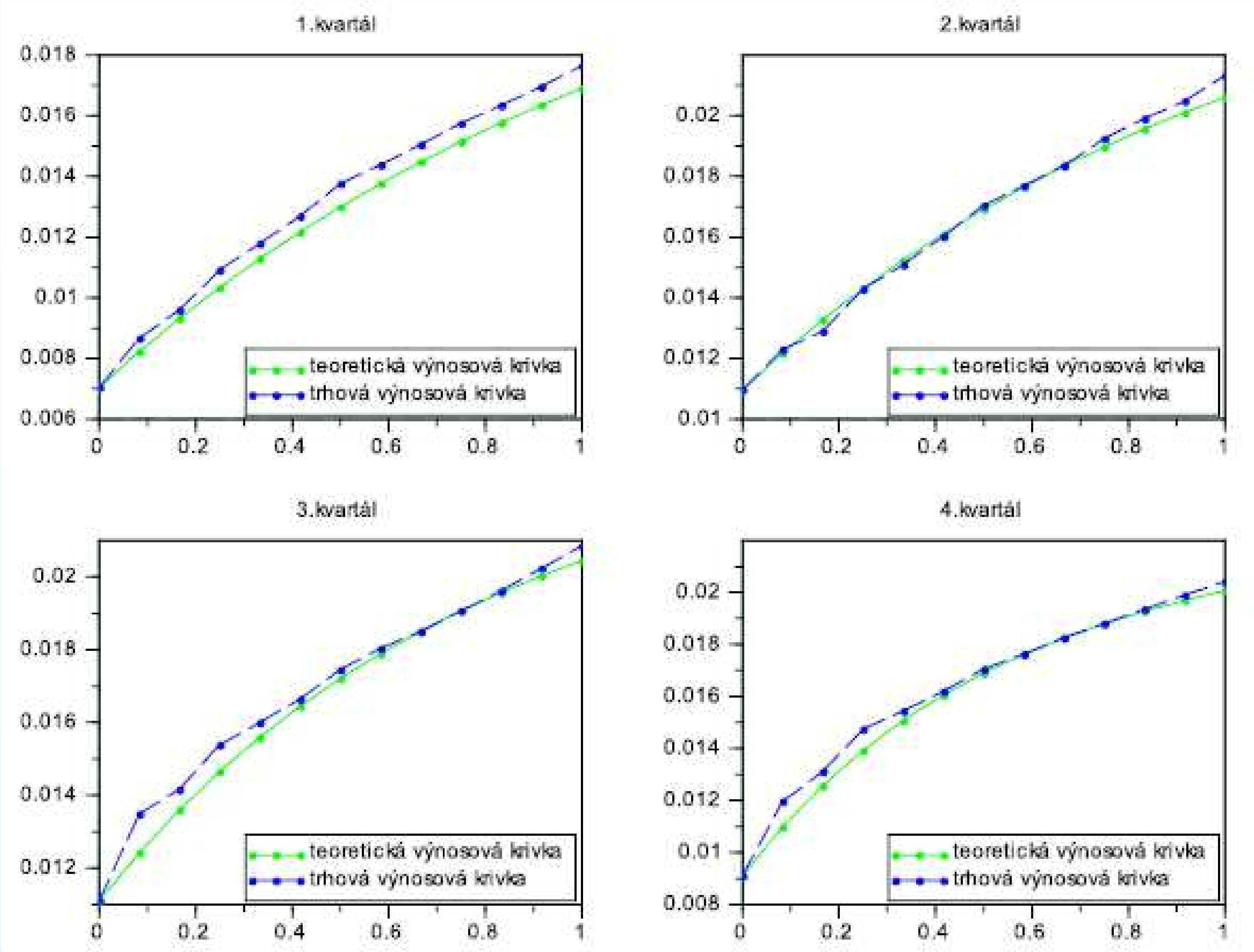
- We have: R_∞ is the limit of term structures as $\tau \to \infty$

Shapes of the term structures



M. Keller-Ressel, T. Steiner: **Yield Curve Shapes and the Asymptotic Short Rate Distribution in Affine One-Factor Models** - shapes in a general 1-factor model

Calibration: Euribor, 2011



Simona Chattová, Master thesis (2013)

CIR model

- System of ODEs:

$$\begin{aligned} \dot{A} + \kappa\theta AB &= 0, \\ \dot{B} + (\kappa + \lambda\sigma)B + \frac{\sigma^2}{2}B^2 - 1 &= 0, \end{aligned}$$

  with initial conditions $A(0) = 1, B(0) = 0$

- Functions A, B:

$$B(\tau) = \frac{2\left(e^{\phi\tau} - 1\right)}{(\psi + \phi)\left(e^{\phi\tau} - 1\right) + 2\phi},$$

$$A(\tau) = \left(\frac{2\phi e^{(\phi+\psi)\tau/2}}{(\phi + \psi)(e^{\phi\tau} - 1) + 2\phi}\right)^{\frac{2\kappa\theta}{\sigma^2}},$$

  where $\psi = \kappa + \lambda\sigma, \quad \phi = \sqrt{\psi^2 + 2\sigma^2} = \sqrt{(\kappa + \lambda\sigma)^2 + 2\sigma^2}.$

General one-factor model

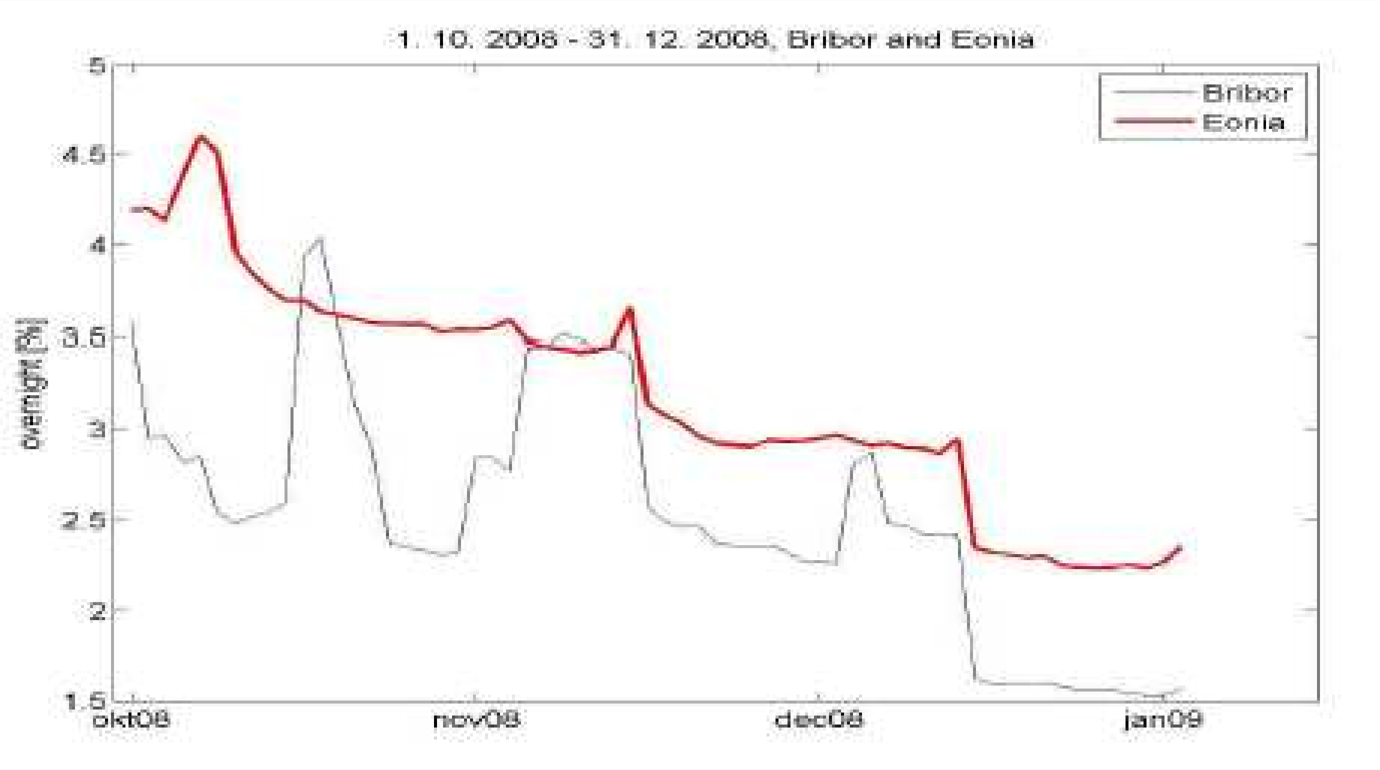
- In general, a closed form solution does not exist
- Numerical solution of the PDE, analytical approximate formulae, Monte Carlo simulations
- Examples (related to my research in this area):
  - Y. Choi, T. Wirjanto, : **An analytic approximation formula for pricing zero- coupon bonds**, Finance Research Letters 4 (2007), pp. 116-126.
  - B. Stehlíková, D. Ševčovič: **Approximate formulae for pricing zero-coupon bonds and their asymptotic analysis.** International Journal of Numerical Analysis and Modeling, 6(2) 2009, 274-283.
  - T. Chernogorova, B. Stehlíková: **A Comparison of Asymptotic Analytical Formulae with Finite-Difference Approximations for Pricing Zero Coupon Bond**. Numerical Algorithms 59 (4), 2012, pp. 571-588.
  - B. Stehlíková, L. Capriotti: **An Effective Approximation for Zero Coupon Bonds and Arrow-Debreu Prices in the Black-Karasinki Model**, International Journal of Theoretical and Applied Finance 17 (6), 2014.

Multi-factor models

- Motivation:
  - term structure is not uniquely determined by the short rate
  - a wider variety term structure shapes
  - modelling the short rate itself (interpretation of the factors)

Convergence models

- Domestic interest rate before entering a monetary union - it is influenced by interest rates in the monetary union
- Example: Slovakia before adopting euro



Convergence models

- First model:

  T. Corzo and E. S. Schwartz: **Convergence within the European Union: Evidence from interest rates**. Econom. Notes 29, 2000, 243-268.

- From the research at our department:

  Z. Zíková, B. Stehlíková: **Convergence model of interest rates of CKLS type**. Kybernetika 48, 2012, 567-586

Short rate as a sum of two factors

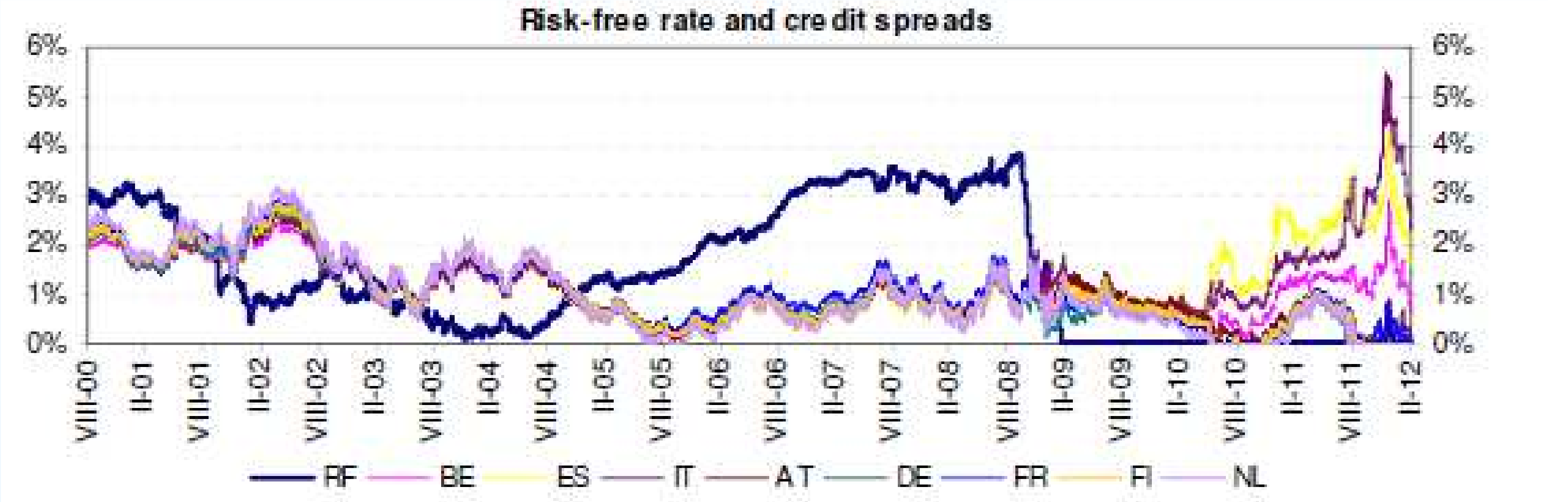
- Short rate as a sum of two factors, each of them modelled by a stochastic differential equation (e.g., of Vasicek or CIR type)
- Application:

  Ľ. Šesták: **Mathematical Analysis and Calibration of a Multifactor Panel Model for Credit Spreads and Risk-free Interest Rate** , Dissertation thesis, FMFI UK, 2012

$$r = r^{rf} + r^{cs},$$

where r^{rf} is risk-free rate (common for all the countries) and r^{cs} is credit spread (individual for each of the countries)

Short rate as a sum of two factors



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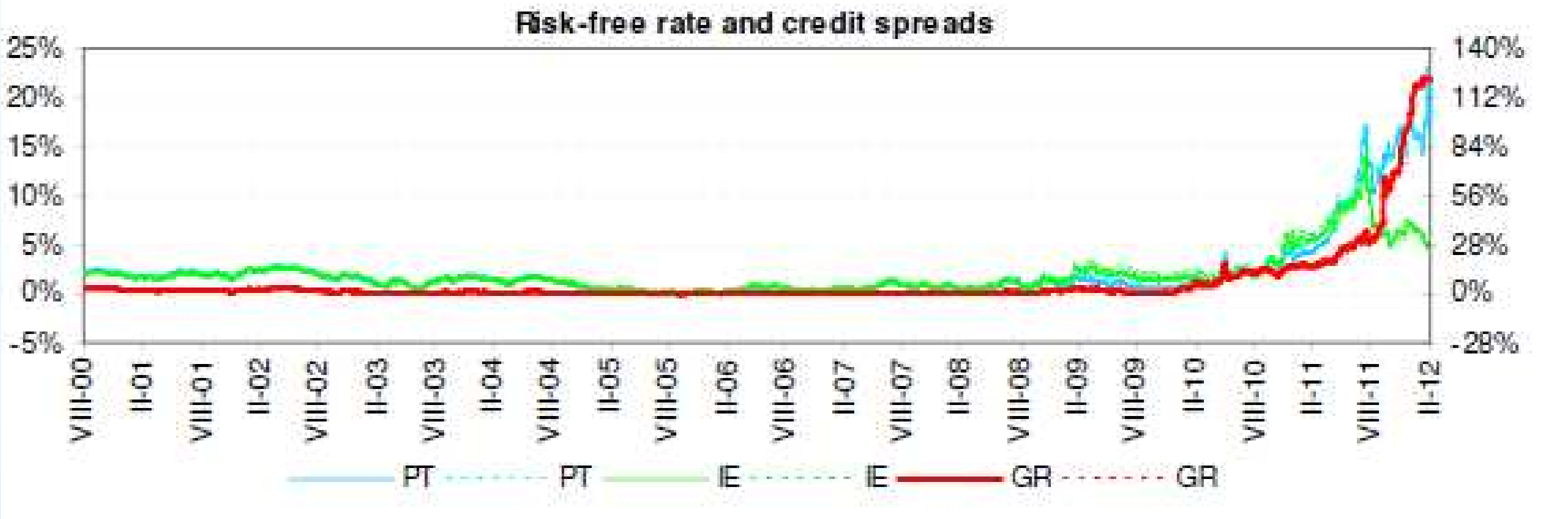

Two-factor models: pricing derivatives

- Basic principle:
  - Model for the short rate: $r = r(x, y)$, where x, y are the factors satisfying the system of SDEs

$$\begin{aligned} dx &= \mu_x(x, y, t)dt + \sigma_x(x, y, t)dw \\ dy &= \mu_y(x, y, t)dt + \sigma_y(x, y, t)dw \end{aligned}$$

    correlation: $\mathbb{E}[dx\, dy] = \rho\, dt$
  - Bond price: $P = P(x, y, t)$
  - PDE for $P(x, y, t)$: again a portfolio containing bonds with different maturities (now three), their amounts chosen such that obtain a riskless portfolio