IV. **Short rate models: Pricing bonds in short rate models**

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Interest rates

• So far we modelled the instantaneous interest rate
• Now: interest rates with different maturities

http://www.euribor-ebf.eu/
Bonds

• Bond
  ◦ a security that in the predetermined time (called maturity of the bond) pays a predetermined amount of money - WLOG we take 1 USD
  ◦ $P(t, T) =$ bond price at time $t$, if its maturity is at time $T$
  ◦ $R(t, T) =$ interest rate with maturity $T$ at time $t$
  ◦ Relation between them:

  $$P(t, T) = e^{-R(t, T)(T-t)} \Rightarrow R(t, T) = -\frac{\log P(t, T)}{T-t}$$

• In short rate models: bond price $P$ is a solution to a PDE, $P = P(r, t, T)$
PDE for the bond price: derivation

- SDE for the short rate:

\[ dr = \mu(t, r)dt + \sigma(t, r)dw \]

- Consider a bond with maturity \( T \), then from Itô:

\[
\begin{align*}
    dP &= \left( \frac{\partial P}{\partial t} + \mu \frac{\partial P}{\partial r} + \frac{\sigma^2}{2} \frac{\partial^2 P}{\partial r^2} \right) dt + \sigma \frac{\partial P}{\partial r} dw \\
        &= \left( \mu_B(t, r) dt + \sigma_B(t, r) dw \right)
\end{align*}
\]

- Portfolio: 1 bond with maturity \( T_1 \) and \( \Delta \) bonds with maturity \( T_2 \); its value:

\[ \Pi = P(r, t, T_1) + \Delta P(r, t, T_2) \]
PDE for the bond price: derivation

• Change in the portfolio value:

\[ d\Pi = dP(r, t, T_1) + \Delta dP(r, t, T_2) \]
\[ = (\mu_B(r, t, T_1) + \Delta \mu_B(r, t, T_2)) \, dt \]
\[ + (\sigma_B(r, t, T_1) + \Delta \sigma_B(r, t, T_2)) \, dw \]

• We eliminate the randomness - by taking

\[ \Delta = -\frac{\sigma_B(t, r, T_1)}{\sigma_B(t, r, T_2)}, \]

then

\[ d\Pi = \left(\mu_B(t, r, T_1) - \frac{\sigma_B(t, r, T_1)}{\sigma_B(t, r, T_2)} \mu_B(t, r, T_2)\right) \, dt \]
PDE for the bond price: derivation

- Yield of a riskless portfolio has to be \( r \) (instantaneous interest rate), i.e. \( d\Pi = r\Pi dt \):

\[
\mu_B(t, r, T_1) - \frac{\sigma_B(t, r, T_1)}{\sigma_B(t, r, T_2)} \mu_B(t, r, T_2) = r\Pi
\]

- Substituting:

\[
\mu_B(t, r, T_1) - \frac{\sigma_B(t, r, T_1)}{\sigma_B(t, r, T_2)} \mu_B(t, r, T_2) = r \left( P(t, r, T_1) - \frac{\sigma_B(t, r, T_1)}{\sigma_B(t, r, T_2)} P(t, r, T_2) \right)
\]
PDE for the bond price: derivation

- Maturities $T_1, T_2$ were arbitrary, hence there must be $\lambda = \lambda(r, t)$ such that for all $t$:

$$\lambda(r, t) = \frac{\mu_B(r, t, T) - rP(r, t, T)}{\sigma_B(r, t, T)}$$

- Function $\lambda = \lambda(r, t)$ does not depend on the maturity $T$; it is called market price of risk

**Conclusion:** PDE for the bond price $P = P(r, t)$ is

$$\frac{\partial P}{\partial t} + (\mu(r, t) - \lambda(r, t)\sigma(r, t))\frac{\partial P}{\partial r} + \frac{\sigma^2(r, t)}{2} \frac{\partial^2 P}{\partial r^2} - rP = 0.$$  

for $r \in (0, \infty), t \in (0, T)$ with terminal condition $P(r, T) = 1$ for $r \in (0, \infty)$
Closed form solutions

- Closed form solutions for the bond price
  - Vasicek model with market price of risk \( \lambda(r, t) = \lambda \)
  - CIR model with market price of risk \( \lambda(r, t) = \lambda \sqrt{r} \)

- We are looking for a solution in the form

\[
P(r, \tau) = A(\tau)e^{-B(\tau)r},
\]

where \( \tau = T - t \)

- Substituting into the PDE \( \Rightarrow \) we obtain a system of ordinary differential equations for the functions \( A(\tau), B(\tau) \) \( \Rightarrow \) this system can be solved explicitly
Vasicek model

• System of ODEs:

\[- \dot{A} + \frac{\sigma^2}{2} AB^2 - (\kappa \theta - \lambda \sigma) AB = 0\]

\[\dot{B} + \kappa B - 1 = 0\]

with initial conditions \(A(0) = 1, B(0) = 0\)

• Functions \(A, B\):

\[B(\tau) = \frac{1 - e^{-\kappa \tau}}{\kappa},\]

\[\log A(\tau) = \left[\frac{1}{\kappa}(1 - e^{-\kappa \tau}) - \tau\right] R_\infty - \frac{\sigma^2}{4\kappa^3} \left(1 - e^{-\kappa \tau}\right)^2,\]

where \(R_\infty = \theta - \frac{\lambda \sigma}{\kappa} - \frac{\sigma^2}{2\kappa^2}\)

• We have: \(R_\infty\) is the limit of term structures as \(\tau \rightarrow \infty\)
Shapes of the term structures

M. Keller-Ressel, T. Steiner: Yield Curve Shapes and the Asymptotic Short Rate Distribution in Affine One-Factor Models - shapes in a general 1-factor model
Calibration: Euribor, 2011

Simona Chattová, Master thesis (2013)
CIR model

- System of ODEs:
  \[
  \dot{A} + \kappa \theta AB = 0, \\
  \dot{B} + (\kappa + \lambda \sigma)B + \frac{\sigma^2}{2} B^2 - 1 = 0, 
  \]
  with initial conditions \(A(0) = 1, B(0) = 0\)
- Functions \(A, B\):
  \[
  B(\tau) = \frac{2 \left(e^{\phi \tau} - 1\right)}{(\psi + \phi)(e^{\phi \tau} - 1) + 2\phi}, \\
  A(\tau) = \left(\frac{2\phi e^{(\phi + \psi)\tau/2}}{(\phi + \psi)(e^{\phi \tau} - 1) + 2\phi}\right)^{\frac{2\kappa \theta}{\sigma^2}}, 
  \]
  where \(\psi = \kappa + \lambda \sigma\), \(\phi = \sqrt{\psi^2 + 2\sigma^2} = \sqrt{(\kappa + \lambda \sigma)^2 + 2\sigma^2}\).
General one-factor model

- In general, a closed form solution does not exist
- Numerical solution of the PDE, analytical approximate formulae, Monte Carlo simulations
- Examples (related to my research in this area):
Multi-factor models

• Motivation:
  ◦ term structure is not uniquely determined by the short rate
  ◦ a wider variety term structure shapes
  ◦ modelling the short rate itself (interpretation of the factors)
Convergence models

- Domestic interest rate before entering a monetary union - it is influenced by interest rates in the monetary union

- Example: Slovakia before adopting euro
Convergence models

• First model:

• From the research at our department:
Short rate as a sum of two factors

- Short rate as a sum of two factors, each of them modelled by a stochastic differential equation (e.g., of Vasicek or CIR type)

- Application:
  

\[ r = r^{rf} + r^{cs}, \]

where \( r^{rf} \) is risk-free rate (common for all the countries) and \( r^{cs} \) is credit spread (individual for each of the countries)
Short rate as a sum of two factors
Two-factor models: pricing derivatives

• Basic principle:
  ◦ Model for the short rate: \( r = r(x, y) \), where \( x, y \) are the factors satisfying the system of SDEs

\[
\begin{align*}
    dx &= \mu_x(x, y, t)\,dt + \sigma_x(x, y, t)\,dw \\
    dy &= \mu_y(x, y, t)\,dt + \sigma_y(x, y, t)\,dw
\end{align*}
\]

  correlation: \( \mathbb{E}[dx\,dy] = \rho\,dt \)
  ◦ Bond price: \( P = P(x, y, t) \)
  ◦ PDE for \( P(x, y, t) \): again a portfolio containing bonds with different maturities (now three), their amounts chosen such that obtain a riskless portfolio