V. **Black-Scholes model: Derivation and solution**

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• Black-Scholes model:
  ◦ Suppose that stock price $S$ follows a geometric Brownian motion

$$dS = \mu S dt + \sigma S dw$$

+ other assumptions (in a moment)
  ◦ We derive a partial differential equation for the price of a derivative

• Two ways of derivations:
  ◦ due to Black and Scholes
  ◦ due to Merton

• Explicit solution for European call and put options
Assumptions

• Further assumptions (besides GBP):
  ◦ constant riskless interest rate $r$
  ◦ no transaction costs
  ◦ it is possible to buy/sell any (also fractional) number of stocks; similarly with the cash
  ◦ no restrictions on short selling
  ◦ option is of European type

• Firstly, let us consider the case of a non-dividend paying stock
Derivation I. - due to Black and Scholes

- Notation:
  \( S \) = stock price, \( t \) = time
  \( V = V(S, t) \) = option price

- Portfolio: 1 option, \( \delta \) stocks
  \( P = \) value of the portfolio: \( P = V + \delta S \)

- Change in the portfolio value:
  \( dP = dV + \delta dS \)

- From the assumptions:
  \( dS = \mu S dt + \sigma S dw \), From the Itô lemma:
  \[ dV = \left( \frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dw \]

- Therefore:
  \[ dP = \left( \frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \delta \mu S \right) dt + \left( \sigma S \frac{\partial V}{\partial S} + \delta \sigma S \right) dw \]
Derivation I. - due to Black and Scholes

- We eliminate the randomness: \( \delta = -\frac{\partial V}{\partial S} \)

- Non-stochastic portfolio \( \Rightarrow \) its value has to be the same as if being on a bank account with interest rate \( r \): \( dP = rP dt \)

- Equality between the two expressions for \( dP \) and substituting \( P = V + \delta S \):

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0
\]
Dividends in the Black-Scholes’ derivation

- We consider continuous dividend rate $q$ - holding a stock with value $S$ during the time differential $dt$ brings dividends $qSdt$.

- In this case the change in the portfolio value equals $dP = dV + \delta dS + \delta qSdt$.

- We proceed in the same way as before and obtain

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q) S \frac{\partial V}{\partial S} - rV = 0$$
Derivation due to Merton - motivation

- Problem in the previous derivation:
  - we have a portfolio consisting of one option and \( \delta \) stocks
  - we compute its value and change of its value:

\[
P = V + \delta S, \\
dP = dV + \delta dS,
\]

i.e., treating \( \delta \) as a constant

- however, we obtain \( \delta = -\frac{\partial V}{\partial S} \)
Derivation II. - due to Merton

- Portfolio consisting of options, stocks and cash with the properties:
  - in each time, the portfolio has zero value
  - it is self-financing

- Notation:
  \[ Q_S = \text{number of stocks, each of them has value } S \]
  \[ Q_V = \text{number of options, each of them has value } V \]
  \[ B = \text{cash on the account, which is continuously compounded using the risk-free rate } r \]

\[ dQ_S = \text{change in the number of stocks} \]
\[ dQ_V = \text{change in the number of options} \]
\[ \delta B = \text{change in the cash, caused by buying/selling stocks and options} \]
Derivation II. - due to Merton

- Mathematical formulation of the required properties:
  - zero value: \( S Q_S + V Q_V + B = 0 \)  \( (1) \)
  - self-financing: \( S dQ_S + V dQ_V + \delta B = 0 \)  \( (2) \)
- Change in the cash: \( dB = rB \; dt + \delta B \)
- Differentiating (1):

\[
0 = d(SQ_S + VQ_V + B) = d(SQ_S + VQ_V) + dB \\
= 0 \\
0 = SdQ_S + VdQ_V + \delta B + Q_SdS + Q_VdV + rB \; dt \\
0 = Q_SdS + Q_VdV - r(SQ_S + VQ_V) \; dt.
\]
Derivation II. - due to Merton

- We divide by $Q_V$ and denote $\Delta = -\frac{Q_S}{Q_V}$:
  $$dV - rV \, dt - \Delta(dS - rS \, dt) = 0$$

- We have $dS$ from the assumption of GBM and $dV$ from the Itô lemma

- We choose $\Delta$ (i.e., the ratio between the number of stocks and options) so that it eliminates the randomness (the coefficient at $dw$ will be zero)

- We obtain the same PDE as before:
  $$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$$
Dividends in the Merton’s derivation

- Assume continuous dividend rate $q$.
- Dividends cause an increase in the cash $\Rightarrow$ change in the cash is $dB = rBdt + \delta B + qSQSdt$
- In the same way we obtain the PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S^2\frac{\partial^2V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0$$
Black-Scholes PDE: summary

• Mathematical formulation of the model:
  Find solution $V(S, t)$ to the partial differential equation (so called Black-Scholes PDE)

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$$

which holds for $S > 0, t \in [0, T)$.

• So far we have not used the fact that we consider an option
  $\Rightarrow$ PDE holds for any derivative that pays a payoff at time $T$
depending on the stock price at this time

• Type of the derivative determines the terminal condition at
time $T$

• In general: $V(S, T) = \text{payoff of the derivative}$
Black-Scholes PDE: simple solutions

**Some simple "derivatives":**

- How to price the derivatives with the following payoffs:
  - $V(S, T) = S \rightarrow$ it is in fact a stock $\rightarrow V(S, t) = S$
  - $V(S, T) = E \rightarrow$ with a certainty we obtain the cash $E$
    $\rightarrow V(S, t) = E e^{-r(T-t)}$

  - by substitution into the PDE we see that they are indeed solutions

**Exercises:**

- Find the price of a derivative with payoff $V(S, T) = S^n$, where $n \in \mathbb{N}$.
  **Hint:** Look for the solution in the form $V(S, t) = A(t) S^n$

- Find all solutions to the Black-Scholes PDE, which are independent of time, i.e., for which $V(S, t) = V(S)$
Black-Scholes PDE: binary option

- Let us consider a binary option, which pays 1 USD if the stock price is higher than $E$ at expiration time, otherwise its payoff is zero.

- In this case

$$V(S, T) = \begin{cases} 
1 & \text{if } S > E \\
0 & \text{otherwise}
\end{cases}$$

- The main idea is to transform the Black-Scholes PDE to a heat equation.

- Transformations are independent of the derivative type; it affects only the initial condition of the heat equation.
Black-Scholes PDE: transformations

**FORMULATION OF THE PROBLEM**

- **Partial differential equation**

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0
\]

which holds for \( S > 0, t \in [0, T) \).

- **Terminal condition** \( V(S, T) = \text{payoff of the derivative} \) for \( S > 0 \)
Black-Scholes PDE: transformations

**Step 1:**

- **Transformation** \( x = \ln(S/E) \in \mathbb{R}, \tau = T - t \in [0, T] \) and a new function \( Z(x, \tau) = V(Ee^x, T - \tau) \)
- **PDE for** \( Z(x, \tau), x \in \mathbb{R}, \tau \in [0, T] \):
  \[
  \frac{\partial Z}{\partial \tau} - \frac{1}{2} \sigma^2 \frac{\partial^2 Z}{\partial x^2} + \left( \frac{\sigma^2}{2} - r \right) \frac{\partial Z}{\partial x} + r Z = 0,
  \]
  \[
  Z(x, 0) = V(Ee^x, T)
  \]

**Step 2:**

- **Transformation to heat equation**
- **New function** \( u(x, \tau) = e^{\alpha x + \beta \tau} Z(x, \tau), \) where the constants \( \alpha, \beta \in \mathbb{R} \) are chosen so that the PDE for \( u \) is the heat equation
Black-Scholes PDE: transformations

• PDE for $u$:

$$
\frac{\partial u}{\partial \tau} - \sigma^2 \frac{\partial^2 u}{2 \partial x^2} + A \frac{\partial u}{\partial x} + Bu = 0,
$$

$$
\begin{align*}
\alpha & = \frac{r}{\sigma^2} - \frac{1}{2}, \\
\beta & = \frac{r}{2} + \frac{\sigma^2}{8} + \frac{r^2}{2\sigma^2}.
\end{align*}
$$

$u(x, 0) = e^{\alpha x} Z(x, 0) = e^{\alpha x} V(Ee^x, T),$

where

$$
A = \alpha \sigma^2 + \frac{\sigma^2}{2} - r, \quad B = (1 + \alpha)r - \beta - \frac{\alpha^2 \sigma^2 + \alpha \sigma^2}{2}.
$$

• In order to have $A = B = 0$, we set
Black-Scholes PDE: transformations

**Step 3:**

- Solution \( u(x, \tau) \) of the PDE \( \frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} = 0 \) is given by the Green formula

\[
u(x, \tau) = \frac{1}{\sqrt{2\sigma^2 \pi \tau}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{2\sigma^2 \tau}} u(s, 0) \, ds.
\]

- We evaluate the integral and perform backward substitutions \( u(x, \tau) \to Z(x, \tau) \to V(S, t) \).
Black-Scholes PDE: binary option (continued)

• Transformations from the previous slides

• We obtain the heat equation \( \frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} = 0 \) with initial condition

\[
   u(x, 0) = e^{\alpha x} V(Ee^x, T) = \begin{cases} 
   e^{\alpha x} & \text{if } Ee^x > E \\
   0 & \text{otherwise}
   \end{cases}
\]

• Solution \( u(x, \tau) \):

\[
   u(x, \tau) = \frac{1}{\sqrt{2\pi \sigma^2 \tau}} \int_0^\infty e^{-\frac{(x-s)^2}{2\sigma^2 \tau}} e^{\alpha s} ds = \ldots = e^{\alpha x + \frac{1}{2} \sigma^2 \tau \alpha^2} N\left(\frac{x + \sigma^2 \tau \alpha}{\sigma \sqrt{\tau}}\right)
\]

where \( N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{\xi^2}{2}} d\xi \) is the cumulative distribution function of a normalized normal distribution.
Black-Scholes PDE: binary option (continued)

- Option price $V(S, t)$:

$$V(S, t) = e^{-r(T-t)}N(d_2),$$

where $d_2 = \frac{\log\left(\frac{S}{E}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma \sqrt{T-t}}$.
Black-Scholes PDE: call option

• In this case

\[ V(S, T) = \max(0, S - E) = \begin{cases} S - E & \text{if } S > E \\ 0 & \text{otherwise} \end{cases} \]

• The same sequence of transformations; initial condition for the heat equation:

\[ u(x, 0) = \begin{cases} e^{\alpha x} (S - E) & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} \]

and similar evaluation of the integral

• Option price:

\[ V(S, t) = SN(d_1) - Ee^{-r(T-t)}N(d_2), \]

where \( N \) is the distribution function of a normalized normal distribution and \( d_1 = \frac{\ln \frac{S}{E} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}} \), \( d_2 = d_1 - \sigma \sqrt{T-t} \)
Black-Scholes PDE: call option

**HOMEWORK:**
Solve the Black-Scholes PDE for a call option on a stock which pays continuous dividends and write it in the form

\[ V(S, t) = Se^{-q(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2), \]

where \( N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{\xi^2}{2}} d\xi \) is the distribution function of a normalized normal distribution \( N(0, 1) \) and

\[
d_1 = \frac{\ln \frac{S}{E} + (r - q + \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}}, \quad d_2 = d_1 - \sigma \sqrt{T-t}
\]

**NOTE:** The PDE is different, so the transformations have to be adjusted (do the same steps for the new equation)
Black-Scholes PDE: call option

Payoff (i.e., terminal condition at time $t = T = 1$) and solution $V(S, t)$ for selected times $t$: 
Black-Scholes PDE: put option

**FORMULATION OF THE PROBLEM**

- Partial differential equation

\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \]

which holds for \( S > 0, t \in [0, T] \).

- Terminal condition:

\[ V(S, T) = \max(0, E - S) \]

for \( S > 0 \)
Black-Scholes PDE: put option

**APPROACH I.**
- The same sequence of computations as in the case of a call option

**APPROACH II.**
- We use the linearity of the Black-Scholes PDE and the solution for a call which we have already found

We show the application of the latter approach.
Black-Scholes PDE: putoption

- Recall that for the payoffs of a call and a put we have
  
  \[-[\text{call payoff}] + [\text{put payoff}] + [\text{stock price}] = E\]

- Hence:

  \[[\text{put payoff}] = [\text{call payoff}] - S + E\]

- Black-Scholes PDE is linear: a linear combination of solutions is again a solution
Black-Scholes PDE: put option

- Recall the solutions for $V(S,T) = S$ and $V(S,T) = E$ (page 13):

<table>
<thead>
<tr>
<th>terminal condition</th>
<th>solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>max($0, S - E$)</td>
<td>$V^{\text{call}}(S,t)$</td>
</tr>
<tr>
<td>$S$</td>
<td>$S$</td>
</tr>
<tr>
<td>$E$</td>
<td>$Ee^{-r(T-t)}$</td>
</tr>
</tbody>
</table>

- From the linearity:

<table>
<thead>
<tr>
<th>terminal condition</th>
<th>solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>max($0, S - E$) − $S + E$</td>
<td>$V^{\text{call}}(S,t) − S + Ee^{-r(T-t)}$</td>
</tr>
</tbody>
</table>

- Since [put payoff] = max($0, S - E$) − $S + E$, we get

$$V^{\text{put}}(S,t) = V^{\text{call}}(S,t) − S + Ee^{-r(T-t)}$$
Solution for a put option

- The solution

\[ V_{\text{put}}(S, t) = V_{\text{call}}(S, t) - S + Ee^{-r(T-t)} \]

can be written in a similar form as the solution for a call option:

\[ V_{\text{ep}}(S, t) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1), \]

where \( N, d_1, d_2 \) are the same as before
Put option - example

Payoff (i.e. terminal condition at time $t = T = 1$) and solution $V(S, t)$ for selected times $t$:

![Graph showing the payoff and option price for different times $t$.]
Put option - alternative computation

Comics about negative volatility on the webpage of Espen Haug:

http://www.espenhaug.com/collector/collector.html
Put option - alternative computation

- A nightmare about negative volatility:

- Not only a dream... according to internet, it really exists and is connected with professor Shiryaev from Moscow...
Put option - alternative computation

QUESTION: Why does this computation work?
Stocks paying dividends

- **HOMEWORK:** Solve the Black-Scholes equation for a put option, if the underlying stock pays continuous dividends.

  **HINT:**
  - In this case, $V(S, t) = S$ is not a solution
  - What is the solution satisfying the terminal condition $V(S, T) = S$? Use financial interpretation and check your answer by substituting it into the PDE

- **HOMEWORK:** Denote $V(S, t; E, r, q)$ the price of an option with exercise price $E$, if the interest rate is $r$ and the dividend rate is $q$. Show that

  $$V_{\text{put}}(S, t; E, r, q) = V_{\text{call}}(E, t; S, q, r)$$

  **HINT:** How do the terms $d_1 d_2$ change when replacing $S \leftrightarrow E$, $r \leftrightarrow q$?
Combined strategies

- From the linearity of the Black-Scholes PDE: if the strategy is a linear combination of call and put options, then its price is the same linear combination of the call and put options prices.
- It does not necessarily hold in other models:
  - consider a model with some transaction costs; it is not equivalent
  - whether we hedge the options independently
  - or we hedge the portfolio - in this case, we might be able to reduce transaction costs
Combined strategies

**Example:**

- we buy call options with exercise prices $E_1$, $E_3$ and sell two call options with exercise prices $E_2$, with exercise prices satisfying $E_1 < E_2 < E_3$ and $E_1 + E_3 = 2E_2$.

- Payoff of the strategy can be written as
  $$V(S,T) = \max(S - E_1, 0) - 2 \max(S - E_2, 0) + \max(S - E_3, 0)$$

- Hence its Black-Scholes price is:
  $$V(S,t) = V^{\text{call}}(S,t;E_1) - 2V^{\text{call}}(S,t;E_2) + V^{\text{call}}(S,t;E_3)$$
Combined strategies

- Numerical example - butterfly with $T = 1$: