V. Black-Scholes model: Derivation and solution

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Content

- Black-Scholes model:
  - Suppose that stock price S follows a geometric Brownian motion

$$dS = \mu S dt + \sigma S dw$$

    + other assumptions (in a moment)
  - We derive a partial differential equation for the price of a derivative
- Two ways of derivations:
  - due to Black and Scholes
  - due to Merton
- Explicit solution for European call and put options

Assumptions

- Further assumptions (besides GBP):
  - constant riskless interest rate r
  - no transaction costs
  - it is possible to buy/sell any (also fractional) number of stocks; similarly with the cash
  - no restrictions on *short selling*
  - option is of European type
- Firstly, let us consider the case of a non-dividend paying stock

Derivation I. - due to Black and Scholes

- Notation:
  $S =$ stock price, $t =$time
  $V = V(S,t) =$ option price
- Portfolio: 1 option, δ stocks
  $P =$ value of the portfolio: $P = V + \delta S$
- Change in the portfolio value: $dP = dV + \delta dS$
- From the assumptions: $dS = \mu S dt + \sigma S dw,$ From the Itō lemma: $dV = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \sigma S \frac{\partial V}{\partial S} dw$
- Therefore:

$$
\begin{aligned}
dP &= \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \delta \mu S\right) dt \\
&\quad + \left(\sigma S \frac{\partial V}{\partial S} + \delta \sigma S\right) dw
\end{aligned}
$$

Derivation I. - due to Black and Scholes

- We eliminate the randomness: $\delta = -\frac{\partial V}{\partial S}$
- Non-stochastic portfolio $\Rightarrow$ its value has to be the same as if being on a bank account with interest rate r: $dP = rPdt$
- Equality between the two expressions for dP and substituting $P = V + \delta S$:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Dividends in the Black-Scholes' derivation

- We consider continuous divident rate q - holding a stock with value S during the time differential dt brings dividends $qSdt$
- In this case the change in the portfolio value equals $dP = dV + \delta dS + \delta qSdt$
- We proceed in the same way as before and obtain

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S\frac{\partial V}{\partial S} - rV = 0$$

Derivation due to Merton - motivation

- Problem in the previous derivation:
  - we have a portfolio consisting of one option and δ stocks
  - we compute its value and change of its value:

$$
\begin{aligned}
P &= V + \delta S, \\
dP &= dV + \delta\, dS,
\end{aligned}
$$

    i.e., treating δ as a constant
  - however, we obtain $\delta = -\frac{\partial V}{\partial S}$

Derivation II. - due to Merton

- Portfolio consisting of options, stocks and cash with the properties:
  - in each time, the portfolio has zero value
  - it is self-financing
- Notation:
  $Q_S =$ number of stocks, each of them has value S
  $Q_V =$ number of options, each of them has value V
  $B =$ cash on the account, which is continuously compounded using the risk-free rate r

  $dQ_S =$ change in the number of stocks
  $dQ_V =$ change in the number of options
  $\delta B =$ change in the cash, caused by buying/selling stocks and options

Derivation II. - due to Merton

- Mathematical formulation of the required properties:
  - zero value $S\,Q_S + V\,Q_V + B = 0$ (1)
  - self-financing: $S\,dQ_S + V\,dQ_V + \delta B = 0$ (2)
- Change in the cash: $dB = r B\,dt + \delta B$
- Differentiating (1):

$$
\begin{aligned}
0 &= d\,(SQ_S + VQ_V + B) = d\,(SQ_S + VQ_V) + \overbrace{dB}^{rB\,dt+\delta B} \\
0 &= \overbrace{SdQ_S + VdQ_V + \delta B}^{=0} + Q_S dS + Q_V dV + rB\,dt \\
0 &= Q_S dS \;+\; Q_V dV \;\overbrace{-\,r(SQ_S + VQ_V)}^{rB}\,dt.
\end{aligned}
$$

Derivation II. - due to Merton

- We divide by Q_V and denote $\Delta = -\frac{Q_S}{Q_V}$:
  $dV - rV\,dt - \Delta(dS - rS\,dt) = 0$
- We have dS from the assumption of GBM and dV from the Itō lemma
- We choose Δ (i.e., the ratio between the number of stocks and options) so that it eliminates the randomness (the coefficient at dw will be zero)
- We obtain the same PDE as before:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Dividends in the Merton's derivation

- Assume continuous dividend rate q.
- Dividents cause an increase in the cash $\Rightarrow$ change in the cash is $dB = rB\,dt + \delta B + qSQ_S dt$
- In the same way we obtain the PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0$$

Black-Scholes PDE: summary

- Matematical formulation of the model:
  Find solution $V(S,t)$ to the partial differential equation (so called Black-Scholes PDE)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

  which holds for $S > 0, t \in [0, T)$.
- So far we have not used the fact that we consider an option $\Rightarrow$ PDE holds for any derivative that pays a payoff at time T depending on the stock price at this time
- Type of the derivative determines the terminal condition at time T
- In general: $V(S,T) =$ payoff of the derivative

Black-Scholes PDE: simple solutions

SOME SIMPLE "DERIVATIVES":

- How to price the derivatives with the following payoffs:
  - $V(S,T) = S$ $\rightarrow$ it is in fact a stock $\rightarrow V(S,t) = S$
  - $V(S,T) = E$ $\rightarrow$ with a certainity we obtain the cash E $\rightarrow V(S,t) = Ee^{-r(T-t)}$

  - by substitution into the PDE we see that they are indeed solutions

EXERCISES:

- Find the price of a derivative with payoff $V(S,T) = S^n$, where $n \in \mathbb{N}$.
  HINT: Look for the solution in the form $V(S,t) = A(t)S^n$
- Find all solutions to the Black-Scholes PDE, which are independent of time, i.e., for which $V(S,t) = V(S)$

Black-Scholes PDE: binary option

- Let us consider a binary option, which pays 1 USD if the stock price is higher that E at expiration time, otherwise its payoff is zero
- In this case

$$V(S,T) = \begin{cases} 1 & \text{if } S > E \\ 0 & \text{otherwise} \end{cases}$$

- The main idea is to transform the Black-Scholes PDE to a heat equation
- Transformations are independent of the derivative type; it affects only the initial condition of the heat equation

Black-Scholes PDE: transformations

FORMULATION OF THE PROBLEM

- Partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

  which holds for $S > 0, t \in [0, T)$.

- Terminal condition $V(S, T) =$ payoff of the derivative for $S > 0$

Black-Scholes PDE: transformations

STEP 1:

- Transformation $x = \ln(S/E) \in \mathbb{R}$, $\tau = T - t \in [0, T]$ and a new function $Z(x, \tau) = V(Ee^x, T - \tau)$
- PDE for $Z(x, \tau)$, $x \in \mathbb{R}, \tau \in [0, T]$:

$$\frac{\partial Z}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 Z}{\partial x^2} + \left(\frac{\sigma^2}{2} - r\right)\frac{\partial Z}{\partial x} + rZ = 0,$$

$$Z(x, 0) = V(Ee^x, T)$$

STEP 2:

- Transformation to heat equation
- New function $u(x, \tau) = e^{\alpha x + \beta \tau} Z(x, \tau)$, where the constants $\alpha, \beta \in \mathbb{R}$ are chosen so that the PDE for u is the heat equation

Black-Scholes PDE: transformations

- PDE for u:

$$\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + A\frac{\partial u}{\partial x} + Bu = 0\,,$$

$$u(x,0) = e^{\alpha x} Z(x,0) = e^{\alpha x} V(Ee^{x},T),$$

where

$$A = \alpha\sigma^2 + \frac{\sigma^2}{2} - r,\ B = (1+\alpha)r - \beta - \frac{\alpha^2\sigma^2 + \alpha\sigma^2}{2}\,.$$

- In order to have $A = B = 0$, we set

$$\alpha = \frac{r}{\sigma^2} - \frac{1}{2},\ \beta = \frac{r}{2} + \frac{\sigma^2}{8} + \frac{r^2}{2\sigma^2}$$

Black-Scholes PDE: transformations

STEP 3:

- Solution $u(x,\tau)$ of the PDE $\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} = 0$ is given by Green formula

$$u(x,\tau) = \frac{1}{\sqrt{2\sigma^2\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{2\sigma^2\tau}} u(s,0)\, ds\,.$$

- We evaluate the integral and perform backward substitutions $u(x,\tau) \to Z(x,\tau) \to V(S,t)$

Black-Scholes PDE: binary option (continued)

- Transformations from the previous slides
- We obtain the heat equation $\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} = 0$ with initial condition

$$u(x,0) = e^{\alpha x} V(Ee^x, T) = \begin{cases} e^{\alpha x} & \text{if } Ee^x > E \\ 0 & \text{otherwise} \end{cases} = \begin{cases} e^{\alpha x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

- Solution $u(x,\tau)$:

$$u(x,\tau) = \frac{1}{\sqrt{2\pi\sigma^2\tau}} \int_0^\infty e^{-\frac{(x-s)^2}{2\sigma^2\tau}} e^{\alpha s} ds = \ldots = e^{\alpha x + \frac{1}{2}\sigma^2\tau\alpha^2} N\left(\frac{x+\sigma^2\tau\alpha}{\sigma\sqrt{\tau}}\right)$$

where $N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{\xi^2}{2}} d\xi$ is the cumulative distribution function of a normalized normal distribution

Black-Scholes PDE: binary option (continued)

- Option price $V(S,t)$:

$$V(S,t) = e^{-r(T-t)}N(d_2),$$

where $d_2 = \frac{\log\left(\frac{S}{E}\right)+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$

Black-Scholes PDE: call option

- In this case

$$V(S,T) = \max(0, S-E) = \begin{cases} S-E & \text{if } S > E \\ 0 & \text{otherwise} \end{cases}$$

- The same sequence of transformations; inital condition for the heat equation:

$$u(x,0) = \begin{cases} e^{\alpha x}(S-E) & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

  and similar evaluation of the integral

- Option price:

$$V(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2),$$

  where N is the distribution function of a normalized normal distribution and $d_1 = \frac{\ln\frac{S}{E} + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$

Black-Scholes PDE: call option

HOMEWORK:
Solve the Black-Scholes PDE for a call option on a stock which pays continuous dividends and write it in the form

$$V(S,t) = Se^{-q(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2),$$

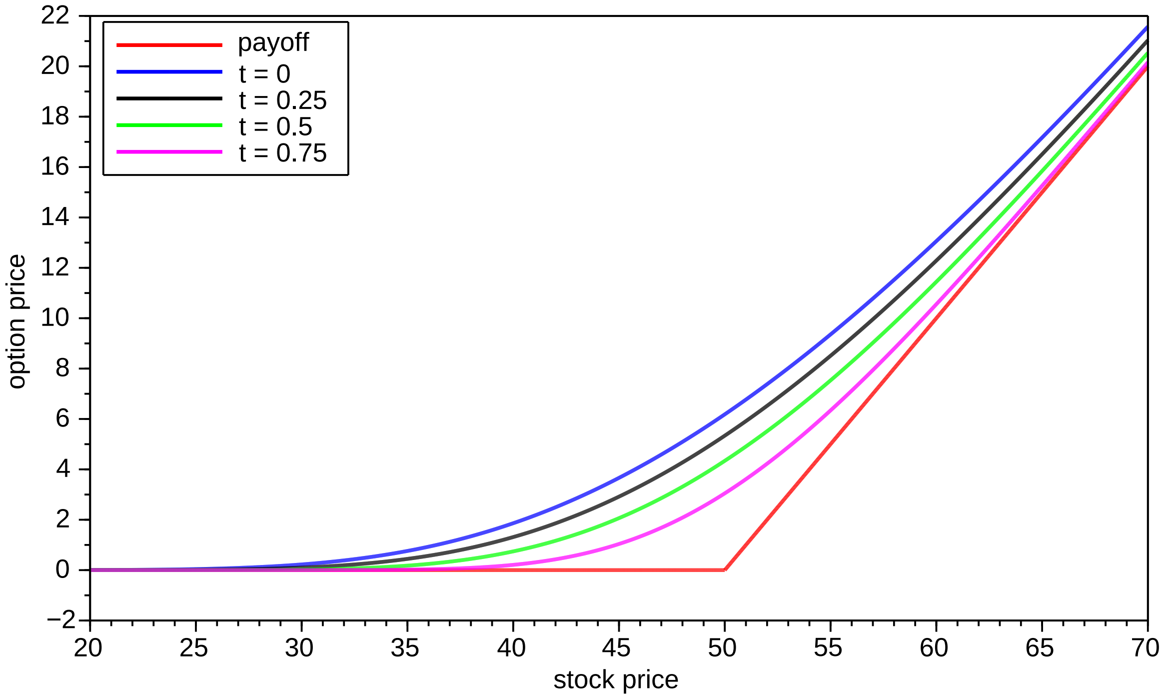
where $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{\xi^2}{2}}\, d\xi$ is the distribution function of a normalized normal distribution $N(0,1)$ and

$$d_1 = \frac{\ln\frac{S}{E} + (r - q + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},\ d_2 = d_1 - \sigma\sqrt{T-t}$$

NOTE: The PDE is different, so the transformations have to be adjusted (do the same steps for the new equation)

Black-Scholes PDE: call option

Payoff (i.e., terminal condition at time $t = T = 1$) and solution $V(S,t)$ for selected times t:



Black-Scholes PDE: put option

Formulation of the problem

- Partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

  which holds for $S > 0, t \in [0, T]$.

- Terminal condition:

$$V(S, T) = \max(0, E - S)$$

  for $S > 0$

Black-Scholes PDE: put option

APPROACH I.

- The same sequence of computations as in the case of a call option

APPROACH II.

- We use the linearity of the Black- Scholes PDE and the solution for a call which we have already found

We show the application of the latter approach.

Black-Scholes PDE: putoption

- Recall that for the payoffs of a call and a put we have

$$-[call\,payoff] + [put\,payoff] + [stock\ price] = E$$

- Hence:

$$[put\ payoff] = [call\ payoff] - S + E$$

- Black-Scholes PDE is linear: a linear combination of solutions is again a solution

Black-Scholes PDE: put option

- Recall the solutions for $V(S,T) = S$ and $V(S,T) = E$ (page 13):

terminal condition	solution
$\max(0, S-E)$	$V^{call}(S,t)$
S	S
E	$Ee^{-r(T-t)}$

- From the linearity:

terminal condition	solution
$\max(0, S-E) - S + E$	$V^{call}(S,t) - S + Ee^{-r(T-t)}$

- Since $[put\ payoff] = \max(0, S-E) - S + E$, we get

$$V^{put}(S,t) = V^{call}(S,t) - S + Ee^{-r(T-t)}$$

Solution for a put option

- The solution

$$V^{put}(S,t) = V^{call}(S,t) - S + Ee^{-r(T-t)}$$

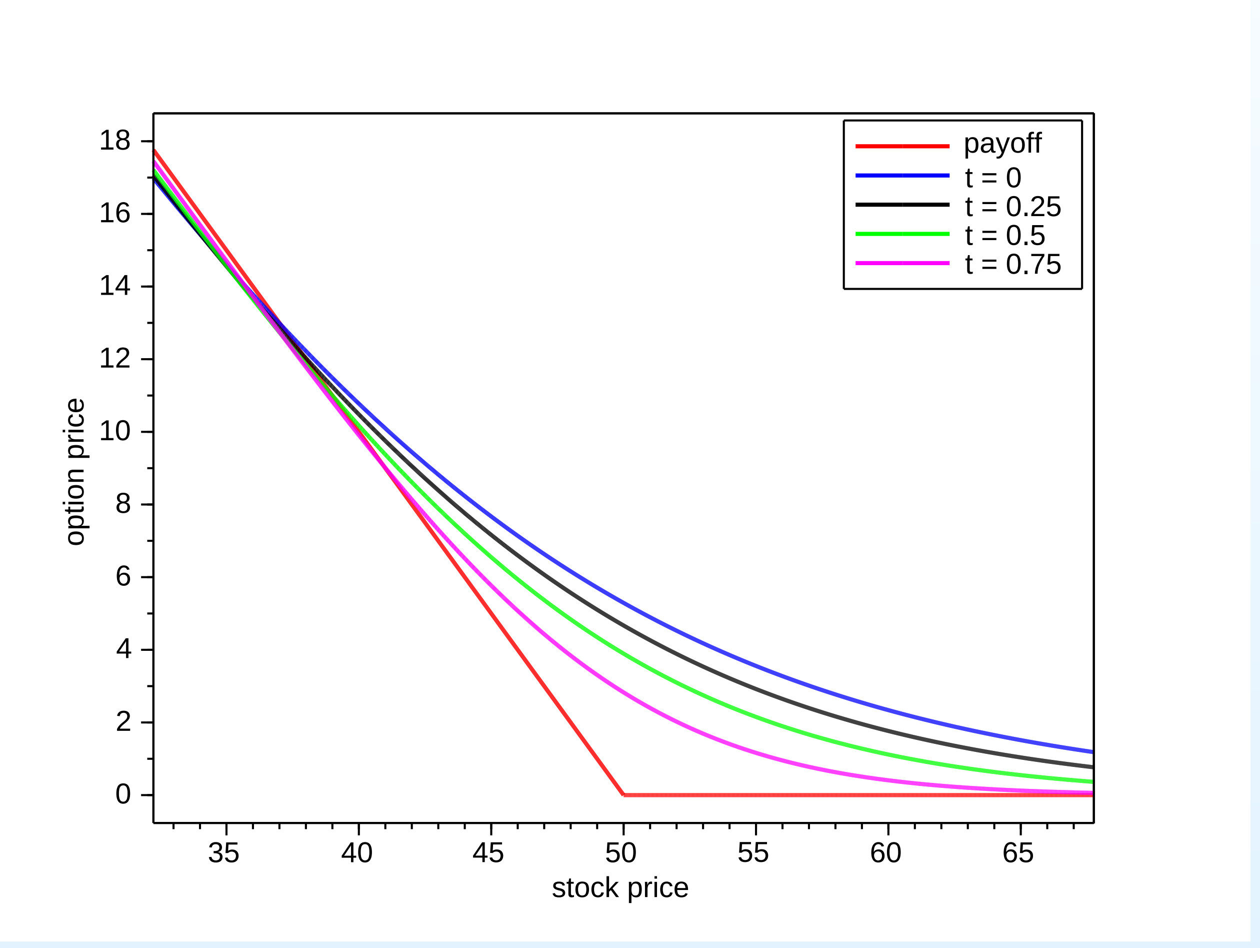
can be written in a similar form as the solution for a call option:

$$V^{ep}(S,t) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1),$$

where N, d_1, d_2 are the same as before

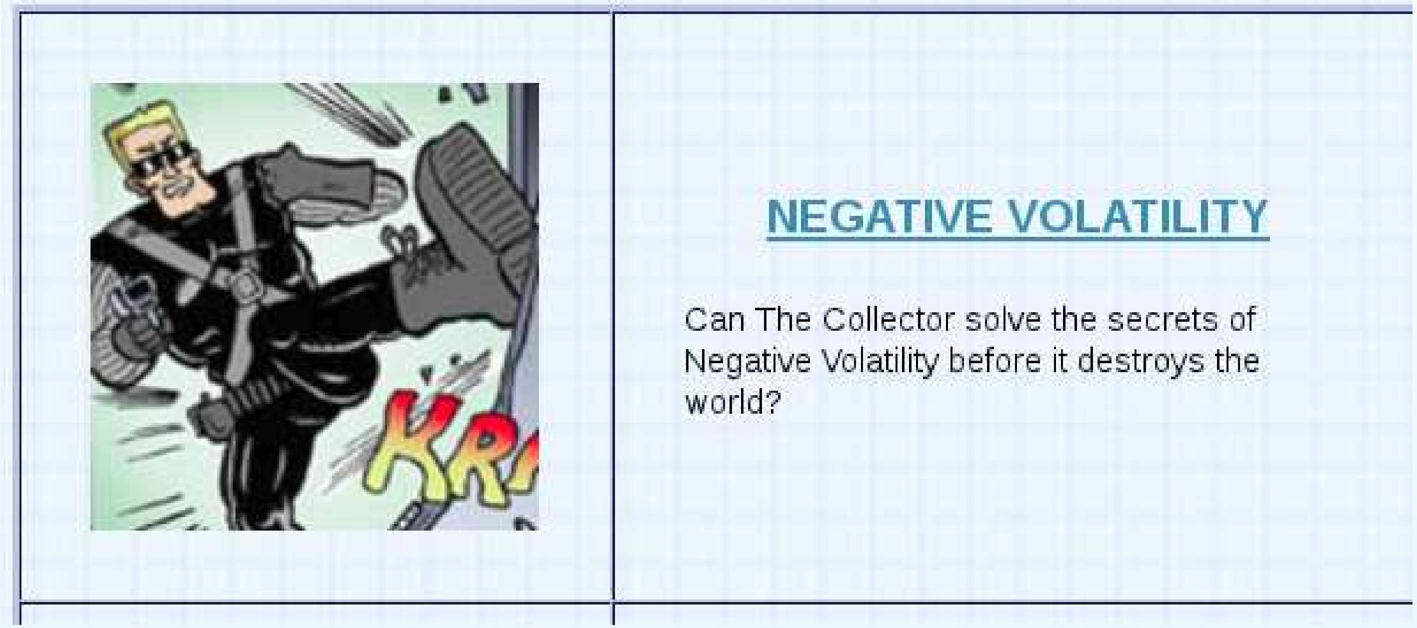
Put option - example

Payoff (i.e. terminal condition at time $t = T = 1$) and solution $V(S, t)$ for selected times t:



Put option - alternative computation

Comics about negative volatility on the webpage of Espen Haug:



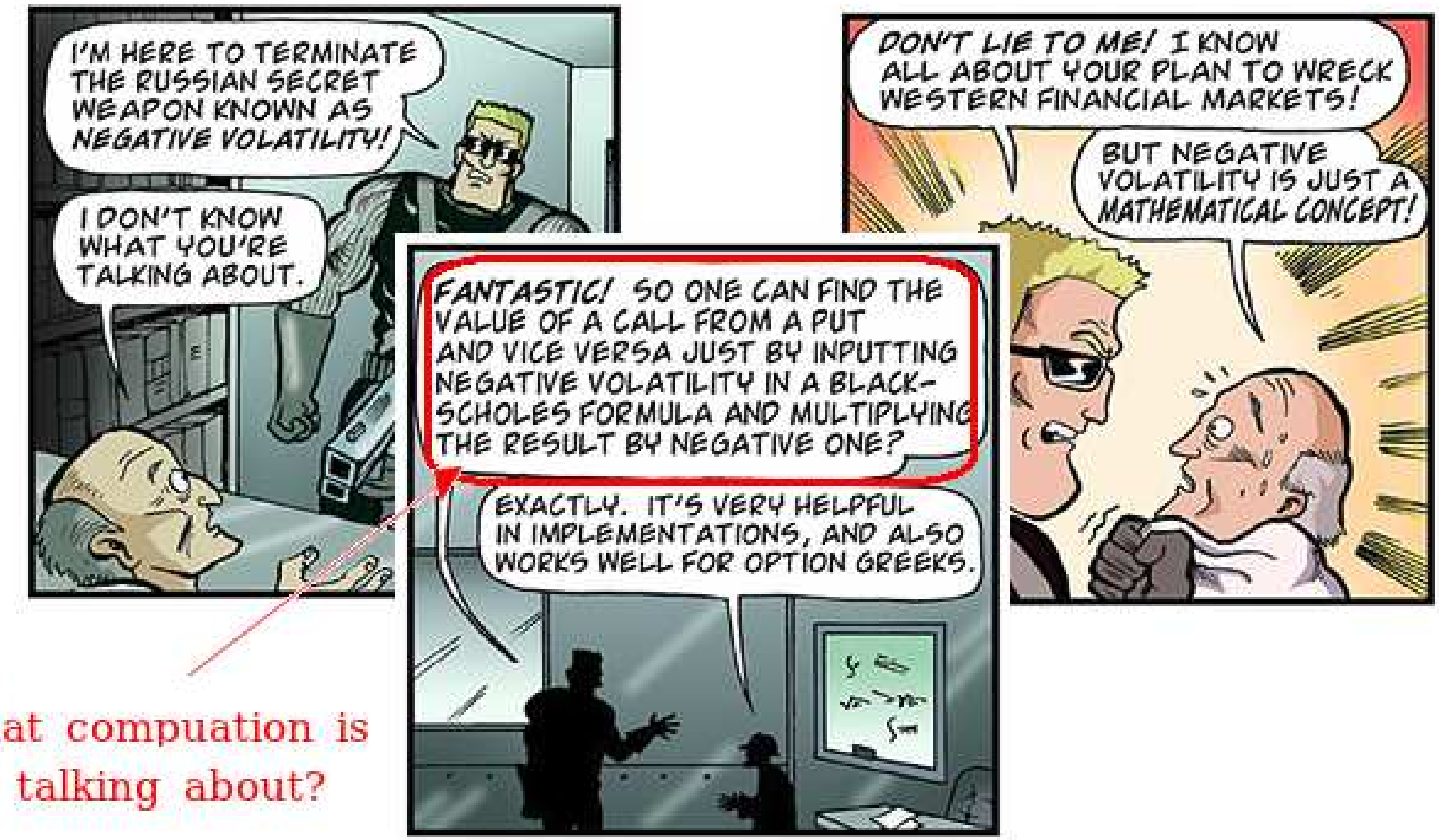
http://www.espenhaug.com/collector/collector.html

Put option - alternative computation

- A nightmare about negative volatility:

- Not only a dream... according to internet, it really exists and is connected with professor Shiryaev from Moscow...

Put option - alternative computation



what compuation is he talking about?

QUESTION: Why does this computation work?

Stocks paying dividends

- HOMEWORK:
  Solve the Black-Scholes equation for a put option, if the underlying stock pays continuous dividends.

  HINT:
  - In this case, $V(S,t) = S$ is not a solution
  - What is the solution satisfying the terminal condition $V(S,T) = S$? Use financial interpretation and check your answer by substituting it into the PDE

- HOMEWORK:
  Denote $V(S,t;E,r,q)$ the price of an option with exercise price E, if the interest rate is r and the dividend rate is q. Show that

$$V^{put}(S,t;E,r,q) = V^{call}(E,t;S,q,r)$$

  HINT: How do the terms $d_1 d_2$ change when replacing $S \leftrightarrow E$, $r \leftrightarrow q$?

Combined strategies

- From the linearity of the Black-Scholes PDE: if the strategy is a linear combination of call and put options, then its price is the same linear combination of the call and put options prices
- It does not necessarily hold in other models:
  - consider a model with some transaction costs; it is not equivalent
    - whether we hedge the options independenty
    - or we hedge the portfolio - in this case, we might be able to reduce transaction costs

Combined strategies

EXAMPLE:

- we buy call options with exerise prices E_1, E_3 and sell two call options with exercise prices E_2, with exercise prices satisfying $E_1 < E_2 < E_3$ and $E_1 + E_3 = 2E_2$.
- Payoff of the strategy can be written as $V(S,T) = \max(S-E_1, 0) - 2\max(S-E_2, 0) + \max(S-E_3, 0)$
- Hence its Black-Scholes price is: $V(S,t) = V^{call}(S,t;E_1) - 2V^{call}(S,t;E_2) + V^{call}(S,t;E_3)$

Combined strategies

- Numerical example - butterfly with $T = 1$: