VI. Black-Scholes model: Implied volatility

Beáta Stehlíková

Financial derivatives, winter term 2014/2015

Faculty of Mathematics, Physics and Informatics
Comenius University, Bratislava

Market data

- 21.2.2014, shorty after beginning of trading
- General Motors stock:

General Motors Company (GM) - NYSE ★ Follow

36.51 **0.00 (0.00%)** 9:32AM EST - Nasdaq Real Time Price

Prev Close:	36.51	Day's Range:	36.51 - 36.68
Open:	N/A	52wk Range:	26.19 - 41.85
Bid:	36.60 x 500	Volume:	67,337
Ask:	36.65 x 200	Avg Vol (3m):	26,433,200
1y Target Est:	46.44	Market Cap:	58.04B
Beta:	1.76	P/E (ttm):	15.35
Earnings Date:	Apr 28 - May 2 (Est.)	EPS (ttm):	2.38
		Div & Yield:	1.20 (3.30%)

Market data

- Selected options written on this stock:

36.50	GM140314C00036500	0.92	↑0.27
36.50	GM140328C00036500	0.99	↑0.01
37.00	GM140307C00037000	0.51	↓0.03
37.00	GM140314C00037000	0.67	↑0.17
37.00	GM140322C00037000	0.73	↑0.15
37.00	GM140328C00037000	0.77	↑0.09

- How much are these options supposed to cost according to Black-Scholes model?

Black-Scholes model and market data

- Recall Black-Scholes formula for a call option:

$$V(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2),$$

where $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{\xi^2}{2}}\, d\xi$ is the distribution function of a normalized normal distribution $N(0,1)$ and

$$d_1 = \frac{\ln\frac{S}{E} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},\ d_2 = d_1 - \sigma\sqrt{T-t}$$

Black-Scholes model and market data

- Therefore, we need the following values:
  - S = stock price
  - E = exercise price
  - $T - t$ = time remaining to expiration
  - σ = volatility of the stock
  - r = interest rate
- In this case: $S = 36.51$

Black-Scholes model and market data

- Consider the option **GM140322C0037000**
- Exercise price: $E = 37$
- Time remaining to expiration:

February						
Su	Mo	Tu	We	Th	Fr	Sa
						1
2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	**21**	22
23	24	25	26	27	28	

March						
Su	Mo	Tu	We	Th	Fr	Sa
						1
2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	29
30	31					

  - option expire in 21 trading days
  - time has to expressed in years, we assume 252 working days in a year $\rightarrow T - t = 21/252$

Black-Scholes model and market data

- Interest rate (bonds.yahoo.com):

US Treasury Bonds Rates

Maturity	Yield	Yesterday	Last Week	Last Month
3 Month	0.03	0.03	0.03	0.02
6 Month	0.07	0.07	0.05	0.05
2 Year	0.32	0.31	0.32	0.40
3 Year	0.72	0.70	0.67	0.85
5 Year	1.56	1.54	1.53	1.70
10 Year	2.77	2.75	2.72	2.87
30 Year	3.74	3.72	3.69	3.76

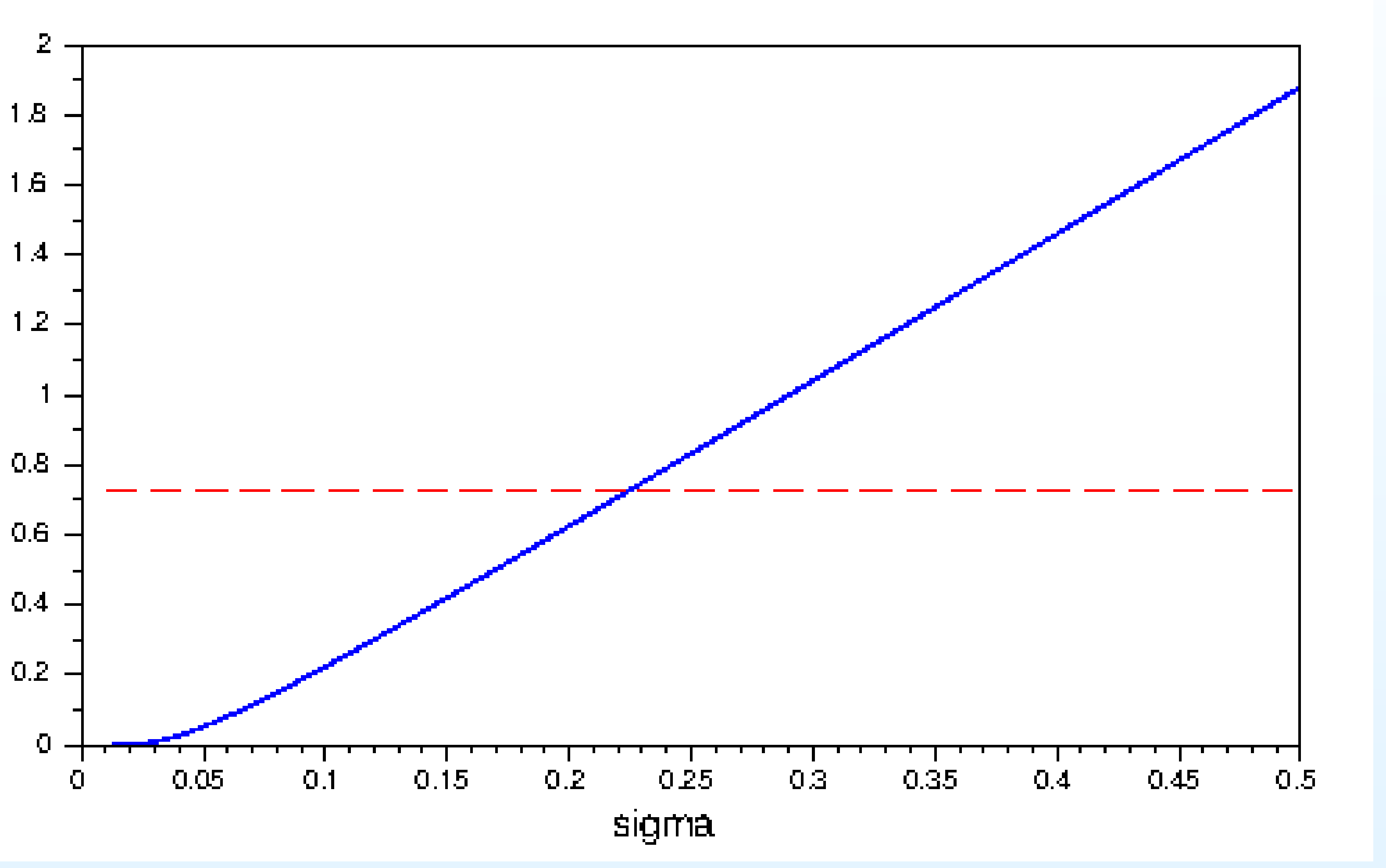
- A common choice: 3-months treasury bills
- Interest rate has to be expressed as a decimal number $\rightarrow$ $r = 0.03/100$

Black-Scholes model and market data

- What is the volatility?
  - Exercises session: computation of the Black-Scholes price using historical volatility
  - Different estimates of volatility, depending on time span of the data
  - Price does not equal the market price
- Question: What value of volatility produces the Black-Scholes price that is equal to the market price?
- This value of volatility is called implied volatility

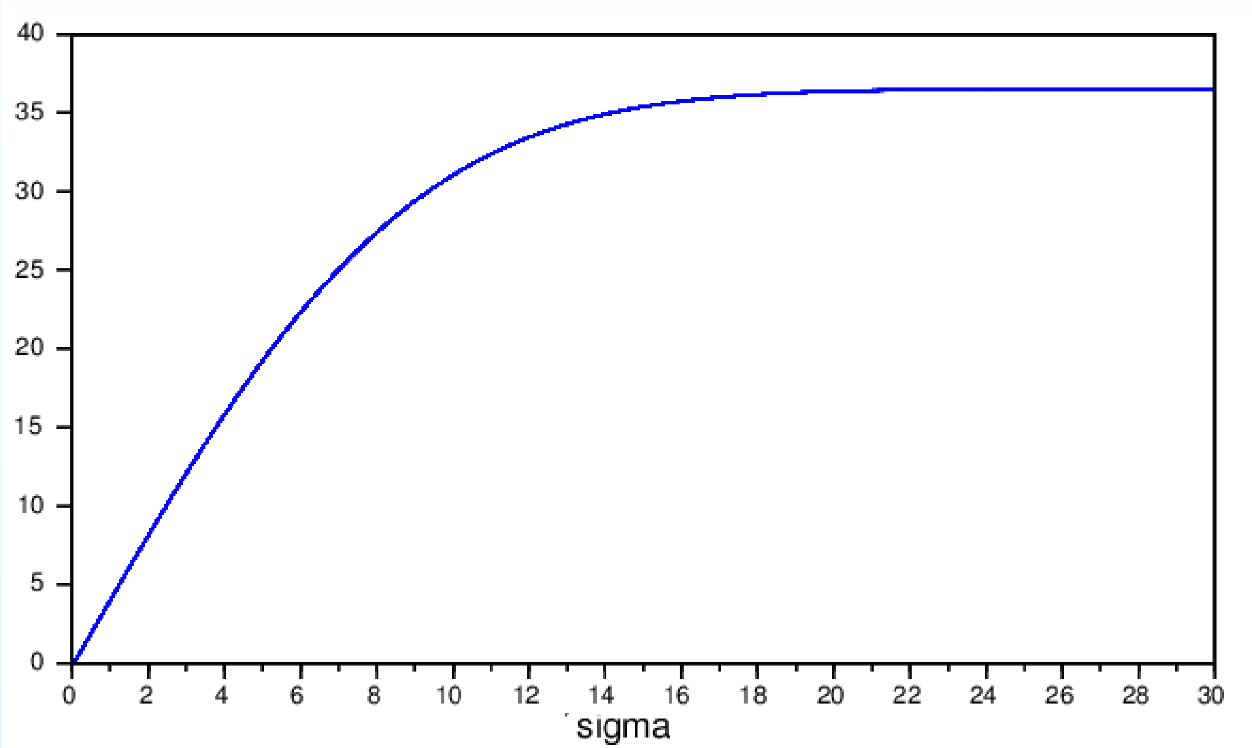
Implied volatility

- Dependence of the Black-Scholes option price on volatility:



Existence of implied volatility

- Dependence of the Black-Scholes option price on volatility - for a wider range of volatility:



Existence of implied volatility

- In general - we show that
  - The Black-Scholes price of a call option is an increasing function of volatility
  - Limits are equal to: $V_0 := \lim_{\sigma \to 0^+} V(S, t; \sigma)$, $V_\infty := \lim_{\sigma \to \infty} V(S, t; \sigma)$
- Then, from continuity of $V \Rightarrow$ for every price from the interval (V_0, V_∞) the implied volatility exists and is uniquely determined
- We do the derivation of a stock which does not pay dividends
- HOMEWORK: call and put option on a stock which pays constinuous dividends

Existence of implied volatility

- To prove that price is an increasing function of volatility:
  - We compute the derivative (using $d_2 = d_1 - \sigma\sqrt{T-t}$):

$$\begin{aligned}\frac{\partial V}{\partial \sigma} &= SN'(d_1)\frac{\partial d_1}{\partial \sigma} - Ee^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial \sigma} \\ &= \left(SN'(d_1) - Ee^{-r(T-t)}N'(d_2)\right)\frac{\partial d_1}{\partial \sigma} \\ &\quad + Ee^{-r(T-t)}N'(d_2)\sqrt{T-t}\end{aligned}$$

  - Derivative of a distribution function is a density function: $N'(x) = \frac{1}{2\pi}e^{-\frac{x^2}{2}}$
  - Useful lemma: $SN'(d_1) - Ee^{-r(T-t)}N'(d_2) = 0$
  - Hence:

$$\frac{\partial V}{\partial \sigma} = Ee^{-r(T-t)}N'(d_2)\sqrt{T-t} > 0$$

Existence of implied volatility

- Limits:
  - We use basic properties of a distribution function:

$$\lim_{x\to-\infty} N(x) = 0, \quad \lim_{x\to+\infty} N(x) = 1$$

  - It follows:

$$\begin{aligned} \lim_{\sigma\to 0^+} V(S,t;\sigma) &= \max(0, S - Ee^{-r(T-t)}) \\ \lim_{\sigma\to\infty} V(S,t;\sigma) &= S \end{aligned}$$

Implied volatility - computation

- A possible implementation:

```
function [s]=ImplVolCall(S,E,r,tau,v)
// implied volatility of a call option on a non-dividend paying stock

    // auxiliary function:
    // differencce between theoretical and market price
    // as a function of 'sigma'
    function [r]=difference(sigma)
        r=Call(S,E,r,sigma,tau)-v;
    endfunction

    // we are looking for an 's'
   // for which this difference is zero
   //(i.e. theoretical price = market price)
    s=fsolve(0.3,difference);
endfunction
```

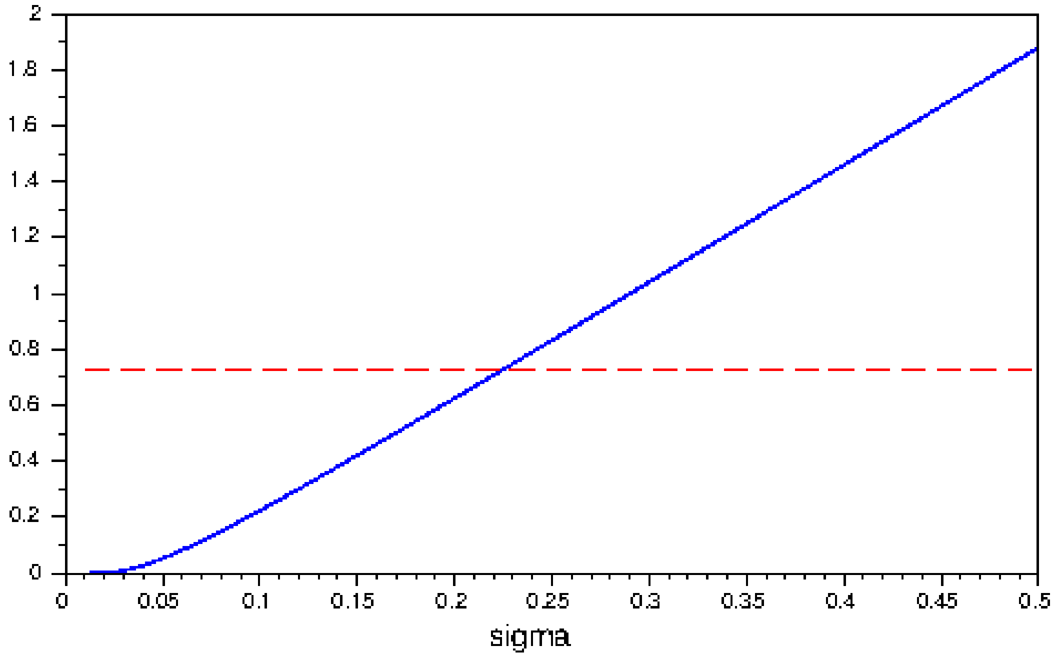
Implied volatility - computation

- Computation:

```
-->ImplVolCall(36.51,37,0.03/100,21/252,0.73)
 ans  =


    0.2255824
```

- Recall:



New finance.yahoo.com web

- Option chains include implied volatilities:

Calls

Strike	Contract Name	Last	· · ·	Implied Volatility
39.00	YHOO141212C00039000	7.95		53.47%
40.00	YHOO141212C00040000	6.30		44.53%
40.50	YHOO141212C00040500	5.95		47.90%
43.50	YHOO141212C00043500	3.80		35.65%
44.00	YHOO141212C00044000	3.90		36.67%
44.50	YHOO141212C00044500	3.16		34.96%
46.00	YHOO141212C00046000	2.27		32.84%
47.00	YHOO141212C00047000	2.05		31.42%