VI. **Black-Scholes model: Implied volatility**

Beáta Stehlíková  
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Faculty of Mathematics, Physics and Informatics  
Comenius University, Bratislava
Market data

- 21.2.2014, shortly after beginning of trading
- General Motors stock:
Market data

• Selected options written on this stock:

<table>
<thead>
<tr>
<th>Strike Price</th>
<th>Option ID</th>
<th>Price</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>36.50</td>
<td>GM140314C00036500</td>
<td>0.92</td>
<td>0.27</td>
</tr>
<tr>
<td>36.50</td>
<td>GM140328C00036500</td>
<td>0.99</td>
<td>0.01</td>
</tr>
<tr>
<td>37.00</td>
<td>GM140307C00037000</td>
<td>0.51</td>
<td>0.03</td>
</tr>
<tr>
<td>37.00</td>
<td>GM140314C00037000</td>
<td>0.67</td>
<td>0.17</td>
</tr>
<tr>
<td>37.00</td>
<td>GM140322C00037000</td>
<td>0.73</td>
<td>0.15</td>
</tr>
<tr>
<td>37.00</td>
<td>GM140328C00037000</td>
<td>0.77</td>
<td>0.09</td>
</tr>
</tbody>
</table>

• How much are these options supposed to cost according to Black-Scholes model?
Black-Scholes model and market data

• Recall Black-Scholes formula for a call option:

\[ V(S, t) = SN(d_1) - Ee^{-r(T-t)}N(d_2), \]

where \( N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{\xi^2}{2}} d\xi \) is the distribution function of a normalized normal distribution \( N(0, 1) \) and

\[ d_1 = \frac{\ln \frac{S}{E} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}}, \quad d_2 = d_1 - \sigma \sqrt{T-t} \]
Therefore, we need the following values:

- $S$ = stock price
- $E$ = exercise price
- $T - t$ = time remaining to expiration
- $\sigma$ = volatility of the stock
- $r$ = interest rate

In this case: $S = 36.51$
Black-Scholes model and market data

• Consider the option \textbf{GM140322C0037000}
• Exercise price: $E = 37$
• Time remaining to expiration:

\begin{itemize}
  \item option expire in 21 trading days
  \item time has to expressed in years, we assume 252 working days in a year \( T - t = 21/252 \)
\end{itemize}
Black-Scholes model and market data

- **Interest rate** (bonds.yahoo.com):

  ![US Treasury Bonds Rates Table]

  - A common choice: 3-months treasury bills
  - Interest rate has to be expressed as a decimal number →
  
  \[ r = 0.03/100 \]
Black-Scholes model and market data

• What is the volatility?
  ◦ Exercises session: computation of the Black-Scholes price using historical volatility
  ◦ Different estimates of volatility, depending on time span of the data
  ◦ Price does not equal the market price

• Question: What value of volatility produces the Black-Scholes price that is equal to the market price?

• This value of volatility is called implied volatility
Implied volatility

- Dependence of the Black-Scholes option price on volatility:
Existence of implied volatility

- Dependence of the Black-Scholes option price on volatility - for a wider range of volatility:
Existence of implied volatility

- In general - we show that
  - The Black-Scholes price of a call option is an increasing function of volatility
  - Limits are equal to: 
    \[ V_0 := \lim_{\sigma \to 0^+} V(S, t; \sigma), \]
    \[ V_\infty := \lim_{\sigma \to \infty} V(S, t; \sigma) \]

- Then, from continuity of \( V \Rightarrow \) for every price from the interval \((V_0, V_\infty)\) the implied volatility exists and is uniquely determined

- We do the derivation of a stock which does not pay dividends

- HOMEWORK: call and put option on a stock which pays continuous dividends
Existence of implied volatility

• To prove that price is an increasing function of volatility:
  ◦ We compute the derivative (using $d_2 = d_1 - \sigma \sqrt{T-t}$):

$$\frac{\partial V}{\partial \sigma} = SN'(d_1) \frac{\partial d_1}{\partial \sigma} - Ee^{-r(T-t)}N'(d_2) \frac{\partial d_2}{\partial \sigma}$$

$$= \left( SN'(d_1) - Ee^{-r(T-t)}N'(d_2) \right) \frac{\partial d_1}{\partial \sigma} + Ee^{-r(T-t)}N'(d_2) \sqrt{T-t}$$

  ◦ Derivative of a distribution function is a density function: $N'(x) = \frac{1}{2\pi}e^{-\frac{x^2}{2}}$

  ◦ Useful lemma: $SN'(d_1) - Ee^{-r(T-t)}N'(d_2) = 0$

  ◦ Hence:

$$\frac{\partial V}{\partial \sigma} = Ee^{-r(T-t)}N'(d_2) \sqrt{T-t} > 0$$
Existence of implied volatility

• Limits:
  ◦ We use basic properties of a distribution function:

    \[
    \lim_{x \to -\infty} N(x) = 0, \quad \lim_{x \to +\infty} N(x) = 1
    \]

  ◦ It follows:

    \[
    \lim_{\sigma \to 0^+} V(S, t; \sigma) = \max(0, S - E e^{-r(T-t)})
    \]

    \[
    \lim_{\sigma \to \infty} V(S, t; \sigma) = S
    \]
Implied volatility - computation

- A possible implementation:

```matlab
function [s] = ImplVolCall(S,E,r,tau,v)
% implied volatility of a call option on a non-dividend paying stock

% auxiliary function:
% difference between theoretical and market price
% as a function of 'sigma'
function [r] = difference(sigma)
    r = Call(S,E,r,sigma,tau) - v;
endfunction

% we are looking for an 's'
% for which this difference is zero
% (i.e. theoretical price = market price)
s = fsolve(0.3, difference);
endfunction
```
Implied volatility - computation

• Computation:

\[
\text{ImplVolCall}(36.51, 37, 0.03/100, 21/252, 0.73)
\]

\[
\text{ans} = 0.2255824
\]

• Recall:
New finance.yahoo.com web

- Option chains include implied volatilities: