VII. Black-Scholes model: Greeks - sensitivity analysis

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Greeks

• Greeks:
  ◦ derivatives of the option price with respect to parameters
  ◦ they measure the sensitivity of the option price to these parameters

• We have already computed
  \[ \frac{\partial V_{\text{call}}}{\partial \sigma} = E e^{-r(T-t)} N'(d_2) \sqrt{T-t} \]
  it is denoted by \( \Upsilon \) (vega)

• Others: (Remark: \( P \) is a Greek letter rho)

  \[ \Delta = \frac{\partial V}{\partial S}, \quad \Gamma = \frac{\partial^2 V}{\partial S^2}, \quad P = \frac{\partial V}{\partial r}, \quad \Theta = \frac{\partial V}{\partial t} \]

• Notation: \( V^{ec} \) = price of a European call, \( V^{ep} \) = price of a European put; in the same way their American counterparts \( V^{ac}, V^{ap} \)
Delta

- Call option - from Black-Scholes formula, we use the same lemma as in the case of volatility:
  \[ \Delta^{ec} = \frac{\partial V^{ec}}{\partial S} = N(d_1) \in (0, 1) \]

- Put option - we do not need to compute the derivative, we can use the put-call parity:
  \[ \Delta^{ep} = \frac{\partial V^{ep}}{\partial S} = -N(-d_1) \in (-1, 0) \]

- Example: call (left), put (right)
Delta - delta hedging

• Recall the derivation of the Black-Scholes model and construction of a riskless portfolio:

\[ \frac{Q_S}{Q_V} = - \frac{\partial V}{\partial S} = -\Delta \]

where \( Q_V, Q_S \) are the numbers of options and stock in the portfolio

• Construction of such a portfolio is call delta hedging (hedge = protection, transaction that reduces risk)
Delta - example of delta hedging

- Real data example - call option on IBM stock, 21st May 2002, 5-minute ticks
- At time $t$:
  - we have option price $V_{\text{real}}(t)$ and stock price $S_{\text{real}}(t)$
  - we compute the implied volatility, i.e., we solve the equation
    \[
    V_{\text{real}}(t) = V^{ec}(S_{\text{real}}(t), t; \sigma_{\text{impl}}(t)).
    \]
  - implied volatility $\sigma_{\text{impl}}(t)$ is used in the call option price formula:
    \[
    \Delta^{ec}(t) = \frac{\partial V^{ec}}{\partial S}(S_{\text{real}}(t), t; \sigma_{\text{impl}}(t))
    \]
Delta - example of delta hedging

- Delta during the day:

- We wrote one option - then, this is the number of stocks in our portfolio
Gamma

- Computation:

\[ \Gamma^{ec} = \frac{\partial \Delta^{ec}}{\partial S} = N'(d_1) \frac{\partial d_1}{\partial S} = \frac{\exp\left(-\frac{1}{2}d_1^2\right)}{\sigma \sqrt{2\pi(T-t)S}} > 0 \]

\[ \Gamma^{ep} = \Gamma^{ec} \]

- Measures a sensitivity of delta to a change in stock price
Price, delta, gamma
Price, delta, gamma

• Simultaneously:
  ◦ the option price is "almost a straight line"
  ◦ delta does not change much with a small change in the stock price
  ◦ gamma is almost zero

• Also:
  ◦ graph of the option price has a big curvature
  ◦ delta significantly changes with a small change in the stock price
  ◦ gamma is significantly nonzero
Vega, rho, theta

- **Vega**
  - we have already computed:
    \[ \Upsilon_{ec} = \frac{\partial V_{ec}}{\partial \sigma} = E e^{-r(T-t)} N'(d_2) \sqrt{T-t} > 0 \]
  - from put-call parity: \( \Upsilon_{ep} = \Upsilon_{ec} \)
  - higher volatility \( \Rightarrow \) higher probability of high profit, while a possible loss is bounded \( \Rightarrow \) positive vega

- **Rho**
  - call: \( P_{ec} = \frac{\partial V_{ec}}{\partial r} = E (T-t) e^{-r(T-t)} N(d_2) > 0 \)
  - put: \( P_{ep} = \frac{\partial V_{ep}}{\partial r} = -E (T-t) e^{-r(T-t)} N(-d_2) < 0 \)

- **Theta:**
  - call: from financial mathematics we know that if a stock does not pay dividends, it is not optimal to exercise an American option prior to its expiry \( \Rightarrow \) prices of European and American options are equal \( \Rightarrow \) \( \Theta_{ec} < 0 \)
Vega, rho, theta

- **Theta**
  - put: the sign may be different for different sets of parameters
Exercise: "cash-or-nothing" option

- "Cash-or-nothing" opcja: pays 1 USD if the stock exceeds the value $E$ at the expiration time; otherwise 0.

- Option price:

- Using the interpretation of the greeks - sketch delta and vega as function of the stock price
Exercise: "cash-or-nothing" delta
Exercise: "cash-or-nothing" vega
Exercise: sensitivity of delta to volatility

- Espen Haug in the paper *Know your weapon*:

  One fine day in the dealing room my risk manager asked me to get into his office. He asked me why I had a big outright position in some stock index futures - I was supposed to do "arbitrage trading". That was strange as I believed I was delta neutral: long call options hedged with short index futures. I knew the options I had were far out-of-the-money and that their DdeltaDvvol was very high. So I immediately asked what volatility the risk management system was considering below the market and again was leading to a very low delta for the options. This example is just to illustrate how a feeling of your DdeltaDvvol can be useful. If you have a high DdeltaDvvol the volatility you use to compute your deltas becomes very important.

- Questions:
  1. What is the dependence of delta on volatility which is used in its computation?
  2. Low volatility led to low delta - why?

- More → exercises session