VII. Black-Scholes model: Greeks - sensitivity analysis

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Greeks

- Greeks:
  - derivatives of the option price with respect to parameters
  - they measure the sensitivity of the option price to these parameters
- We have already computed $\frac{\partial V_{call}}{\partial \sigma} = Ee^{-r(T-t)} N'(d_2)\sqrt{T-t}$, it is denoted by Υ (vega)
- Others: (Remark: P is a Greek letter rho)

$$\Delta = \frac{\partial V}{\partial S}, \; \Gamma = \frac{\partial^2 V}{\partial S^2}, \; P = \frac{\partial V}{\partial r}, \; \Theta = \frac{\partial V}{\partial t}$$

- Notation: V^{ec} = price of a European call, V^{ep} = price of a European put; in the same way their American counterparts V^{ac}, V^{ap}

Delta

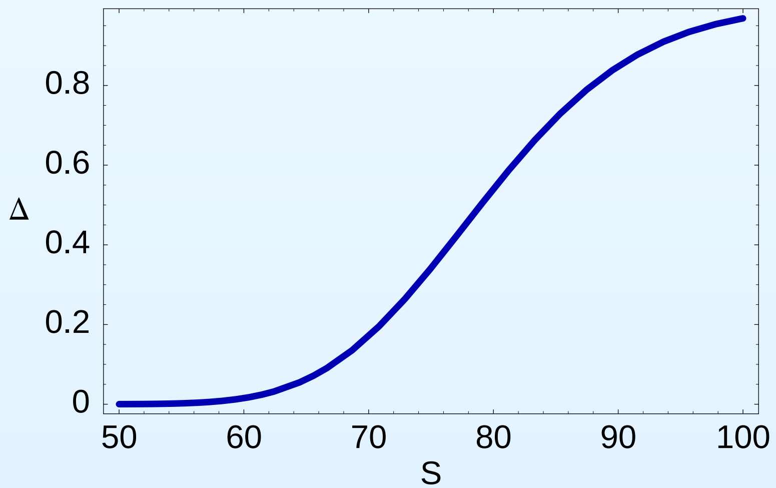
- Call option - from Black-Scholes formula, we use the same lemma as in the case of volatility:

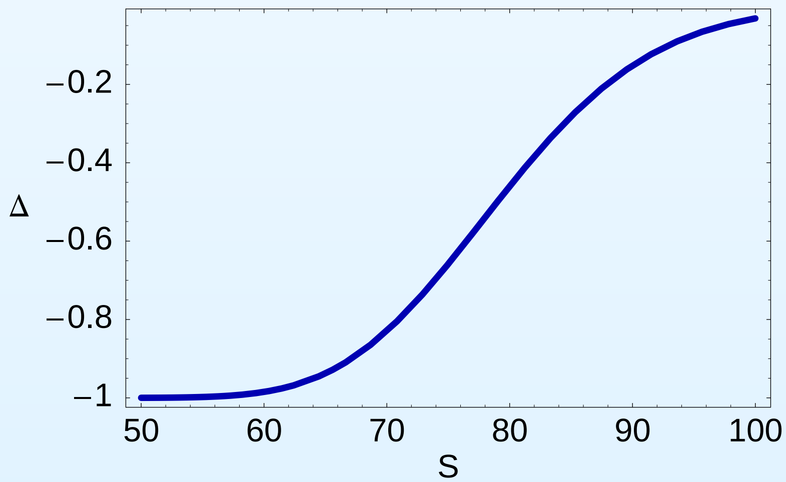
$$\Delta^{ec} = \frac{\partial V^{ec}}{\partial S} = N(d_1) \in (0, 1)$$

- Put option - we do not need to compute the derivative, we can use the put-call parity:

$$\Delta^{ep} = \frac{\partial V^{ep}}{\partial S} = -N(-d_1) \in (-1, 0)$$

- Example: call(left), put (right)





Delta - delta hedging

- Recall the derivation of the Black-Scholes model and contruction of a riskless portfolio:

$$\frac{Q_S}{Q_V} = -\frac{\partial V}{\partial S} = -\Delta$$

  where Q_V, Q_S are the numbers of options and stock in the portfolio

- Construction of such a portfolio is call delta hedging (hedge = protection, transaction that reduces risk)

Delta - example of delta hedging

- Real data example - call option on IBM stock, 21st May 2002, 5-minute ticks
- At time t:
  - we have option price $V_{real}(t)$ and stock price $S_{real}(t)$
  - we compute the impled volatility, i.e., we solve the equation

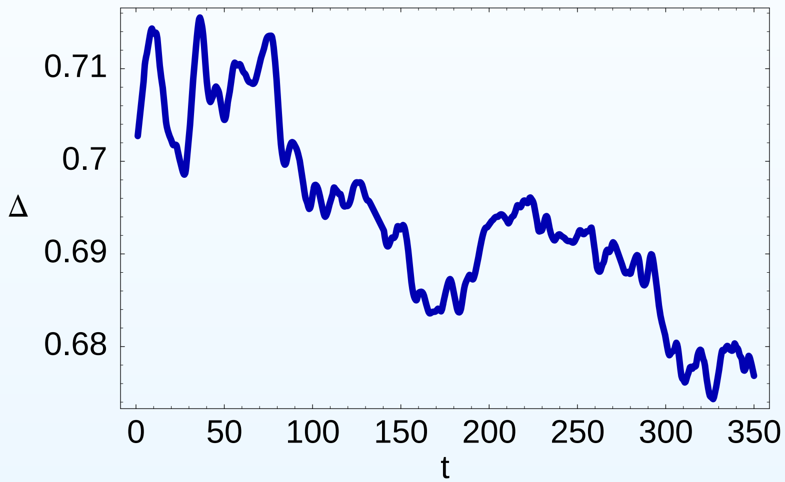
$$V_{real}(t) = V^{ec}(S_{real}(t), t; \sigma_{impl}(t)).$$

  - implied volatility $\sigma_{impl}(t)$ is used in the call option price formula:

$$\Delta^{ec}(t) = \frac{\partial V^{ec}}{\partial S}(S_{real}(t), t; \sigma_{impl}(t))$$

Delta - example of delta hedging

- Delta during the day:



- We wrote one option - then, this is the number of stocks in our portfolio

Gamma

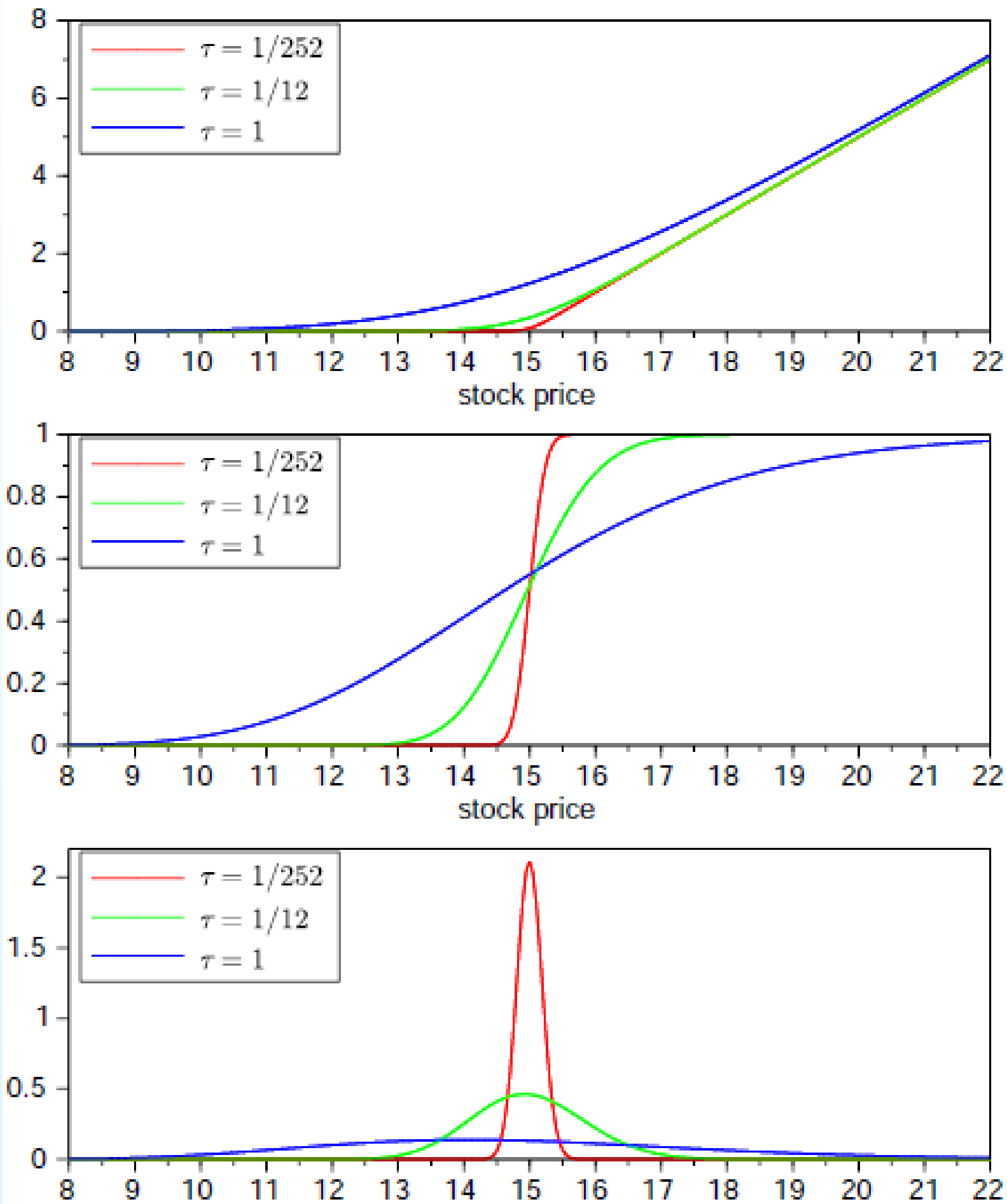
- Computation:

$$\Gamma^{ec} = \frac{\partial \Delta^{ec}}{\partial S} = N'(d_1)\frac{\partial d_1}{\partial S} = \frac{\exp(-\frac{1}{2}d_1^2)}{\sigma\sqrt{2\pi(T-t)}S} > 0$$

$$\Gamma^{ep} = \Gamma^{ec}$$

- Measures a sensitivity of delta to a change in stock price

Price, delta, gamma



Price, delta, gamma

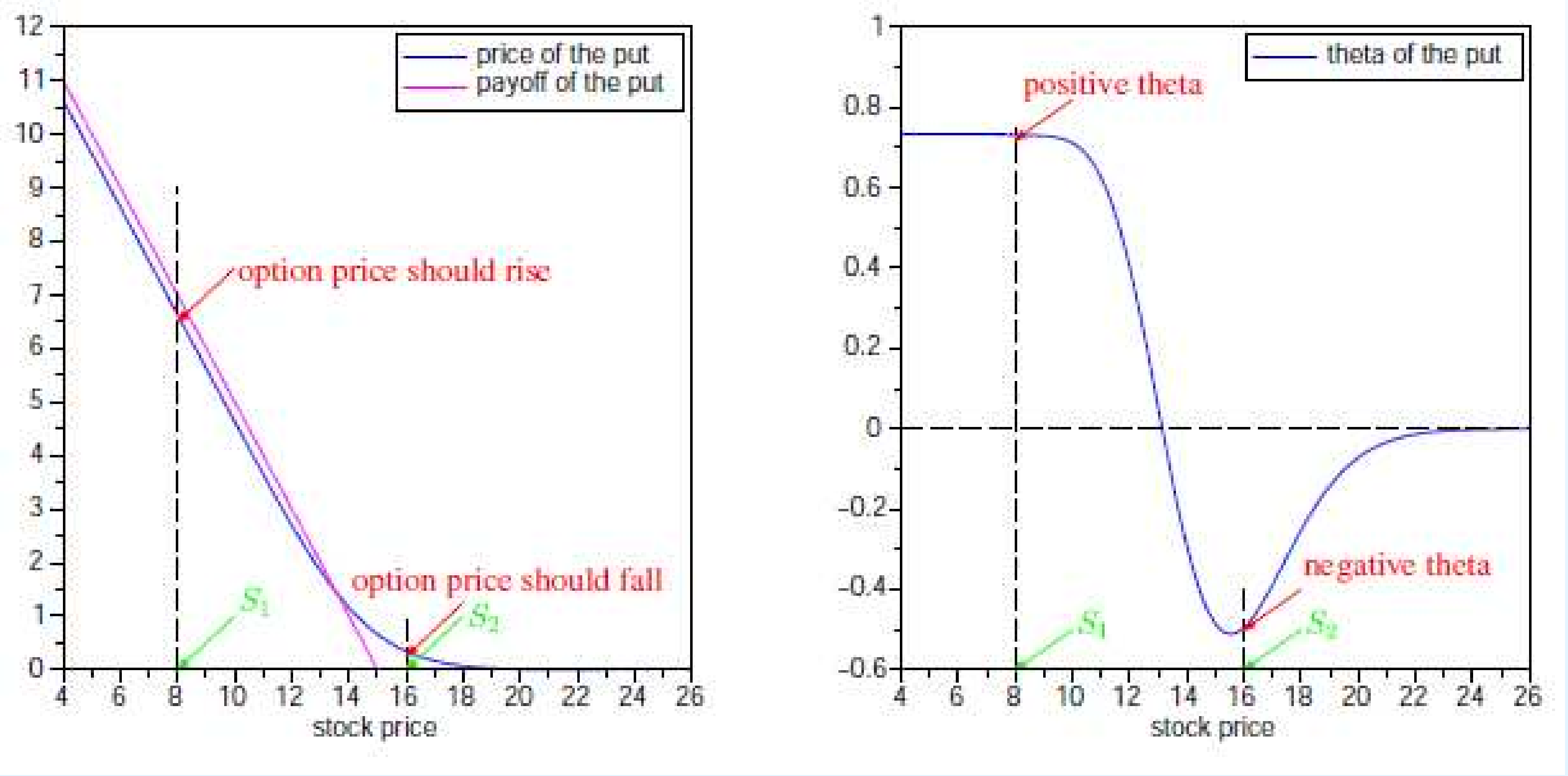
- Simultaneously:
  - the option price is "almost a straight line"
  - delta does not change much with a small change in the stock price
  - gamma is almost zero
- Also:
  - graph of the option price has a big curvature
  - delta significantly changes with a small change in the stock price
  - gamma is significantly nonzero

Vega, rho, theta

- Vega
  - we have already computed:
    $\Upsilon^{ec} = \frac{\partial V^{ec}}{\partial \sigma} = Ee^{-r(T-t)}N'(d_2)\sqrt{T-t} > 0$
  - from put-call parity: $\Upsilon^{ep} = \Upsilon^{ec}$
  - higher volatility $\Rightarrow$ higher probability of high profit, while a possible loss is bounded $\Rightarrow$ positive vega
- Rho
  - call: $P^{ec} = \frac{\partial V^{ec}}{\partial r} = E(T-t)e^{-r(T-t)}N(d_2) > 0$
  - put: $P^{ep} = \frac{\partial V^{ep}}{\partial r} = -E(T-t)e^{-r(T-t)}N(-d_2) < 0$
- Theta:
  - call: from financial mathematics we know that if a stock does not pay dividends, it is not optimal to exercise an American option prior to its expiry $\Rightarrow$ prices of European and American options are equal $\Rightarrow$ $\Theta^{ec} < 0$

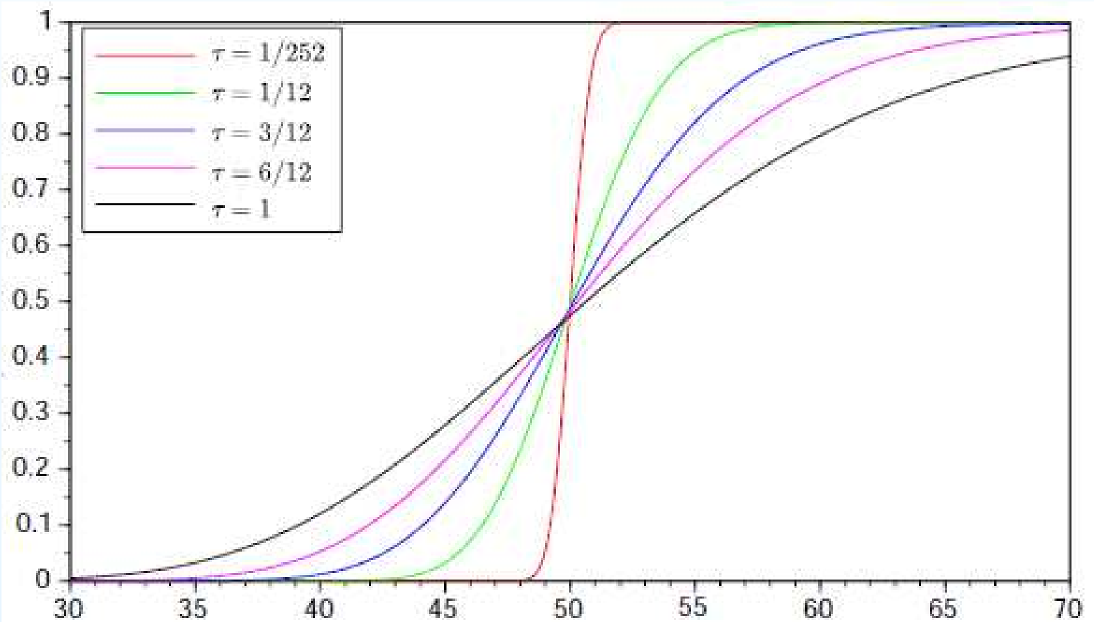
Vega, rho, theta

- Theta
  - put: the sign may be different for different sets of parameters



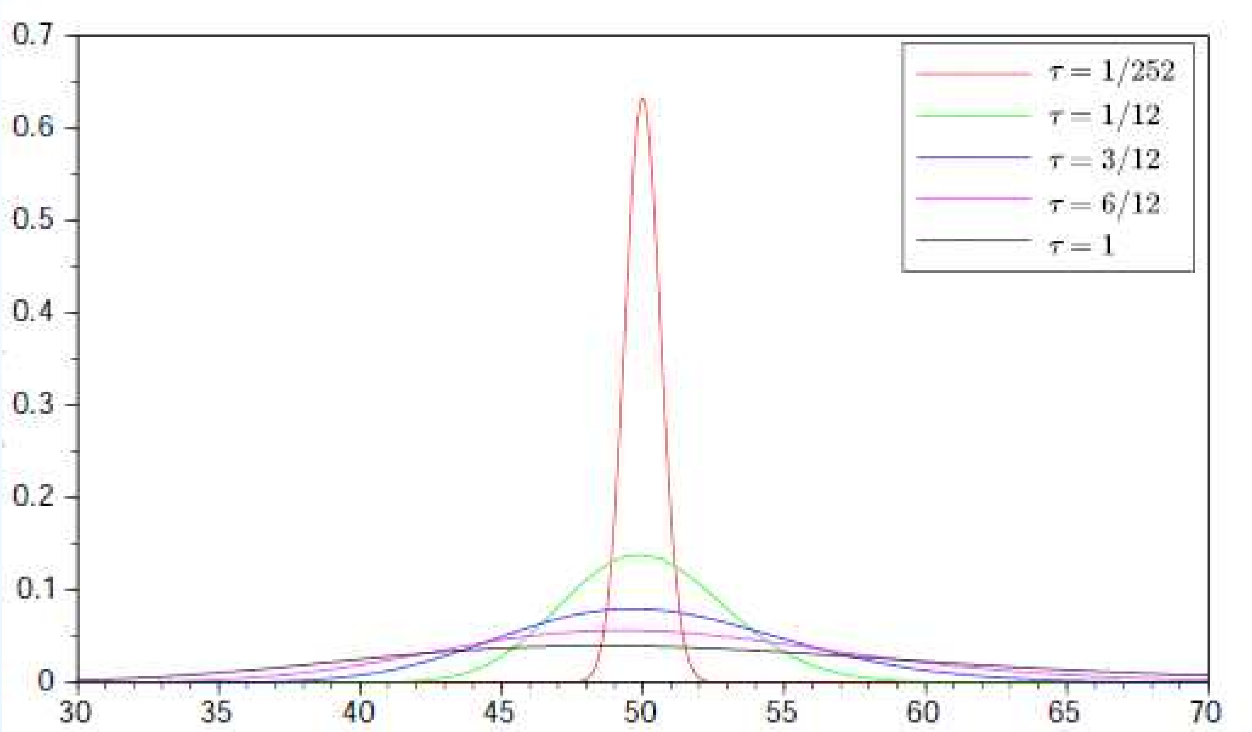
Exercise: "cash-or-nothing" option

- "Cash-or-nothing" opcia: pays 1 USD if the stock exceeds the value E at the expiration time; otherwise 0.
- Option price:

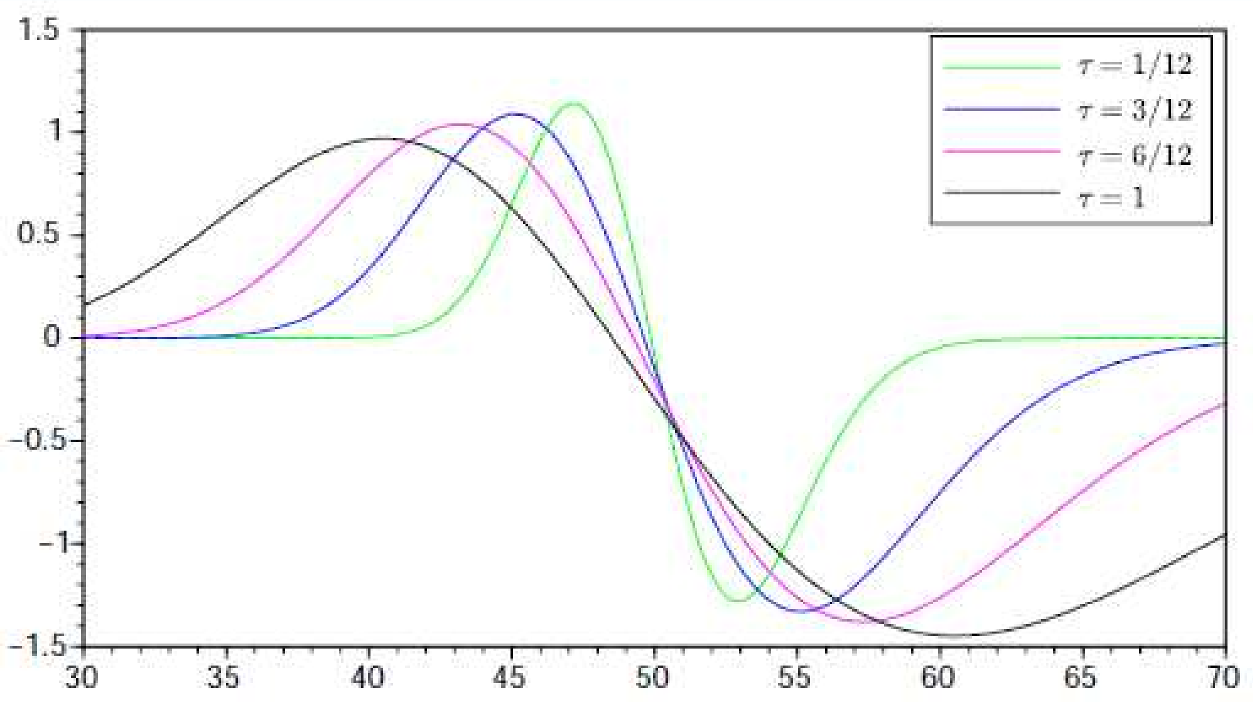


- Using the interpretation of the *greeks* - sketch delta and vega as function of the stock price

Exercise: ”cash-or-nothing” delta



Exercise: "cash-or-nothing" vega



Exercise: sensitivity of delta to volatility

- Espen Haug in the paper *Know your weapon*:

> One fine day in the dealing room my risk manager asked me to get into his office. He asked me why I had a big outright position in some stock index futures - I was supposed to do "arbitrage trading". That was strange as I believed I was delta neutral: long call options hedged with short index futures. I knew the options I had were far out-of-the-money and that their DdeltaDvol was very high. So I immediately asked what volatility the risk management used to calculate their delta. As expected, the volatility in the risk-management-system was considerable below the market and again was leading to (2) a very low delta for the options. This example is just to illustrate how a feeling of your DdeltaDvol can be useful. If you have a high DdeltaDvol the volatility you use to compute your deltas becomes very important. (1)

- Questions:
  1. What is the dependence of delta on volatility which is used in its computation?
  2. Low volatility led to low delta - why?
- More $\rightarrow$ exercises session