VIII. Leland model: Derivation of the PDE for the price of a derivative

Beáta Stehlíková

Financial derivatives, winter term 2014/2015

Faculty of Mathematics, Physics and Informatics
Comenius University, Bratislava

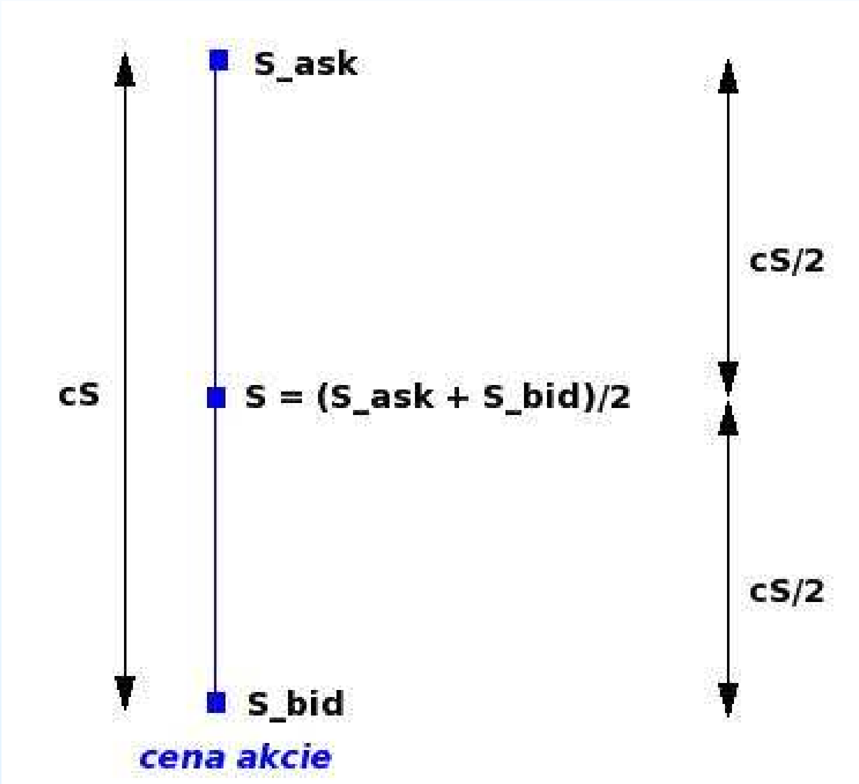
Leland model

- Taking transaction costs into account
- Original paper:

  Hayne E. Leland: **Option Pricing and Replication with Transactions Costs**, 1985

Assumptions of the model

- Transaction costs are characterized by a constant $c = \frac{S_{ask} - S_{bid}}{S}$, where S is the average of bid and ask price of the stock



- S follows a geometric Brownian motion $dS = \mu S dt + \sigma S dw$

Computation of the constant c

EXAMPLE 1:

- Stock:

Yahoo! Inc. (YHOO) - NasdaqGS

39.82 ↑ **0.32(0.81%)** 12:09PM EST - Nasdaq Real Time Price

Prev Close:	**39.50**	Day's Range:	**39.50 - 39.98**
Open:	**39.50**	52wk Range:	**21.87 - 41.72**
Bid:	**39.85** x 1700	Volume:	**5,105,244**
Ask:	**39.86** x 1100	Avg Vol (3m):	**16,828,600**
1y Target Est:	**41.16**	Market Cap:	**40.19B**
Beta:	**1.12**	P/E (ttm):	**31.63**
Earnings Date:	**Apr 14 - Apr 18 (Est.)**	EPS (ttm):	**1.26**
		Div & Yield:	**N/A (N/A)**

- From the data: $S_{bid} = 39.85, S_{ask} = 39.86$
- Average of bid and ask: $S = 39.855$
- $c = \frac{0.01}{39.855} = 2.5028 \times 10^{-4}$

Computation of the constant c

EXAMPLE 2:

- Stock:

Microsoft Corporation (MSFT) - NasdaqGS ★ Follow

38.09 ↓ 0.02(0.05%) 12:12PM EST - Nasdaq Real Time Price

Prev Close:	38.11	Day's Range:	37.89 - 38.18
Open:	38.11	52wk Range:	27.64 - 38.98
Bid:	38.06 x 3900	Volume:	8,045,030
Ask:	38.07 x 6200	Avg Vol (3m):	37,930,700
1y Target Est:	38.60	Market Cap:	316.20B
Beta:	0.71	P/E (ttm):	14.10
Next Earnings Date:	24-Apr-14	EPS (ttm):	2.70
		Div & Yield:	1.12 (2.90%)

- From the data: $S_{bid} = 38.06$, $S_{ask} = 38.07$
- Average of bid and ask: $S = 38.065$
- $c = \dfrac{0.01}{38.065} = 2.6271 \times 10^{-4}$

Computation of the constant c

EXAMPLE 3:

- Stock:

Amazon.com Inc. (AMZN) - NasdaqGS

372.27 ↓ 0.10(0.03%) 12:12PM EST - Nasdaq Real Time Price

Prev Close:	372.37	Day's Range:	368.90 - 375.33
Open:	374.08	52wk Range:	245.75 - 408.06
Bid:	372.81 x 100	Volume:	1,576,414
Ask:	372.94 x 200	Avg Vol (3m):	3,606,330
1y Target Est:	433.05	Market Cap:	170.97B
Beta:	0.77	P/E (ttm):	631.76
Earnings Date:	Apr 21 - Apr 25 (Est.)	EPS (ttm):	0.59
		Div & Yield:	N/A (N/A)

- From the data: $S_{bid} = 372.81, S_{ask} = 372.94$
- Average of bid and ask: $S = 372.875$
- $c = \frac{0.13}{372.875} = 3.4864 \times 10^{-4}$

Derivation of the PDE

- Portfolio:
  - one option and δ stocks, while the number of stocks is determined by delta hedging, i.e., $\delta = -\partial V/\partial S$
  - value of the portfolio: $P = V + \delta S$
  - because of the transaction costs, the portfolio cannot be hedged continuously in time $\rightarrow$ we hedge it discrete times which are Δt [years] apart
- Change of the portfolio value
  - number of transactions with stocks is $\Delta\delta$
  - costs for one transaction are $cS/2 \Rightarrow$ total costs are equal to $\frac{cS}{2}|\Delta\delta|$
  - therefore, change of the portfolio value is:

$$\Delta P = \Delta V + \delta \Delta S - \frac{cS}{2}|\Delta\delta|$$

Derivation of the PDE

- Hence we have $\Delta P = \Delta V + \delta \Delta S - \frac{cS}{2}|\Delta\delta|$ and
  - $\Delta S = \mu S \Delta t + \sigma S \Delta w$ from the assumptions (GBM)
  - $\Delta V = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2}\right) \Delta t + \sigma S \frac{\partial V}{\partial S} \Delta w$ from Itō lemma
  - what remains, is to derive $\Delta\delta$
- We have $\delta = -\frac{\partial V}{\partial S}$, hence $\frac{\partial \delta}{\partial S} = -\frac{\partial^2 V}{\partial S^2}$, from which:

$$\Delta\delta \approx \frac{\partial \delta}{\partial S} \Delta S = -\frac{\partial^2 V}{\partial S^2} \Delta S$$

- Here we substitute ΔS from the GBM

Derivation of the PDE

- So far we have:

$$\Delta\delta \approx -\tfrac{\partial^2 V}{\partial S^2}\mu S \Delta t - \tfrac{\partial^2 V}{\partial S^2}\sigma S \Delta w \qquad (1)$$

- Leland has shown:
  - in formula (1), it suffices to consider the terms of the lowest order (i.e. we take only $\Delta w \approx (\Delta t)^{1/2}$, and Δt is neglected)
  - when computing the absolute value $|\Delta w|$, it can be replaced by its expected value $\mathbb{E}[|\Delta w|] = \sqrt{\frac{2}{\pi}\Delta t}$
- Therefore:

$$\begin{aligned} \Delta\delta &\approx -\frac{\partial^2 V}{\partial S^2}\sigma S \Delta w \\ |\Delta\delta| &\approx \left|\frac{\partial^2 V}{\partial S^2}\right| \sigma S |\Delta w| \approx \left|\frac{\partial^2 V}{\partial S^2}\right| \sigma S \sqrt{\frac{2}{\pi}}\sqrt{\Delta t} \end{aligned}$$

Derivation of the PDE

- We substitute everything into the formula for the change of the portfolio value $\Delta P = \Delta V + \delta \Delta S - \frac{cS}{2}|\Delta\delta|$:

$$\Delta P = \left(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - \frac{c}{2} S \left| \frac{\partial^2 V}{\partial S^2} \right| \sigma S \sqrt{\frac{2}{\pi \Delta t}} \right) \Delta t \quad (2)$$

- Portfolio is riskless $\Rightarrow$ necessarily (to rule out arbitrage possibilities) $\Delta P = r P \Delta t$
- Portfolio consists of one option and $\delta = -\partial V / \partial S$ stocks $\Rightarrow$ $P = V + \delta S = V - \frac{\partial V}{\partial S}$, and so

$$\Delta P = r(V - \frac{\partial V}{\partial S} S) \Delta t \quad (3)$$

- Comparing (2) and (3):

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - \frac{c}{2} S \left| \frac{\partial^2 V}{\partial S^2} \right| \sigma S \sqrt{\frac{2}{\pi \Delta t}} = r(V - \frac{\partial V}{\partial S} S)$$

Derivation of the PDE

- We write the PDE into its final form:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \left[1 - \frac{c}{\sigma \sqrt{\Delta t}} \sqrt{\frac{2}{\pi}} sign \left(\frac{\partial^2 V}{\partial S^2} \right) \right] + \frac{\partial V}{\partial S} S - rV = 0$$

- The PDE holds for $S > 0, t \in [0, T]$, we add the terminal condition $V(S, T)$ depending on the type of the derivative, e.g., $V(S, T) = \max(0, S - E)$ for $S > 0$ when pricing a call option
- Nonliner PDE because of the term containing the *signum* function
- However, we will solve it in a closed form for call and put options

Remark on combined strategies

- The price of combined strategies (unlike in the Black-Scholes setting) cannot be found be pricing every option and then adding the prices

MATHEMATICAL POINT OF VIEW:

- PDE in the Leland model is not linear $\Rightarrow$ for example a sum, difference or some other linear combination is no more a solution

Remark on combined strategies

FINANCIAL POINT OF VIEW:

- If we price every option separately, we count transaction costs coming from hedging the portfolio for each of the options separately
- If the transaction costs are zero, it does not matter that e.g. we have two portfolios, for one of them we buy stock and for the other one we sell stocks (it does not cause any transaction costs)
- In a presence of transaction costs this is no more true. In such a case we need to consider one portfolio, to avoid unnecessary transactions (which would increase transaction costs)