X. Nonlinear models for pricing financial derivatives: Basic ideas behind selected models

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Models

- Selected models:
  - RAPM (risk adjusted pricing methodology) - transaction costs and risk from the volatile (unprotected) portfolio)
  - presence of a dominant investor
  - modelling investor's preferences
- Aim of this lecture - to show a selection of:
  - financial situations which can be modelled
  - mathematical methods which are used in their analysis
  - basic ideas, to obtain an insight about the models, without detailed derivations

RAPM model

M. Jandačka, D. Ševčovič: **On the risk adjusted pricing methodology based valuation of vanilla options and explanation of the volatility smile**, Journal of Applied Mathematics, 3, 2005, 235-258

- Transaction costs as in the Leland model - then we have the portfolio $P = V + \delta S$ and the change of its value is $\Delta P = \Delta V + \delta \Delta S - r_{TC} S \Delta t$, where

$$r_{TC} = \frac{cS\sigma}{\sqrt{2\pi}} \left| \frac{\partial^2 V}{\partial S^2} \right| \frac{1}{\Delta t}$$

- Risk from the volatile portfolio (risk is measured by variance here):

$$r_{VP} = R \frac{Var[\Delta P / S]}{\Delta t},$$

where R the marginal value of investor's exposure to a risk

RAPM model

- It can be shown (Itō lemma, computation of variance of a random variable):

$$r_{VP} = \frac{1}{2} R\sigma^4 S^2 \left(\frac{\partial^2 V}{\partial S^2}\right)^2 \Delta t$$

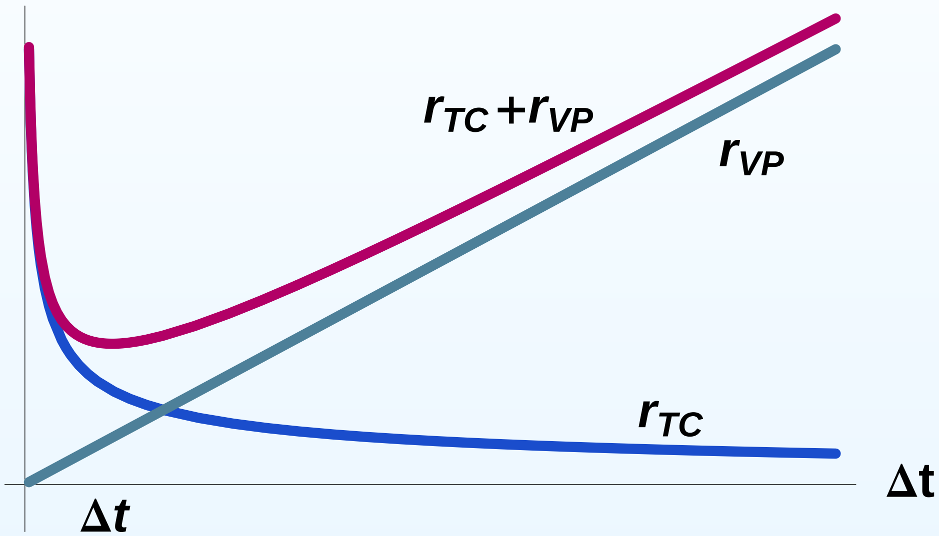
- Risk neutral investor $\Rightarrow$ wants - by his choice of Δt - to minimize

$$r_R = r_{TC} + r_{VP} = \frac{cS\sigma}{\sqrt{2\pi}} \left|\frac{\partial^2 V}{\partial S^2}\right| \frac{1}{\Delta t} + \frac{1}{2} R\sigma^4 S^2 \left(\frac{\partial^2 V}{\partial S^2}\right)^2 \Delta t$$

$\Rightarrow$ we obtain the optimal length of the time interval Δt between two adjustments of the portfolio

RAPM model

- Finding the optimal Δt_{opt}:



- For this value of Δt_{opt} we have:

$$r_R(\Delta t_{opt}) = \frac{3}{2}\left(\frac{c^2 R}{2\pi}\right)^{1/3} \sigma^2 \left| S \frac{\partial^2 V}{\partial S^2} \right|^{4/3}$$

RAPM model

- For this value of Δt_{opt} we obtain the partial differential equation for the price of a derivative :

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \left[1 + \mu \left(S \frac{\partial^2 V}{\partial S^2} \right)^{1/3} \right] \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial S} S - rV = 0,$$

where:

$\mu = 3 \left(\frac{c^2 R}{2\pi} \right)^{1/3}$ is a constant;

Γ^p for $\Gamma = S \frac{\partial^2 V}{\partial S^2}$ and $p = 1/3$ is computed as $\Gamma^p = |\Gamma|^{p-1} \Gamma$

RAPM model

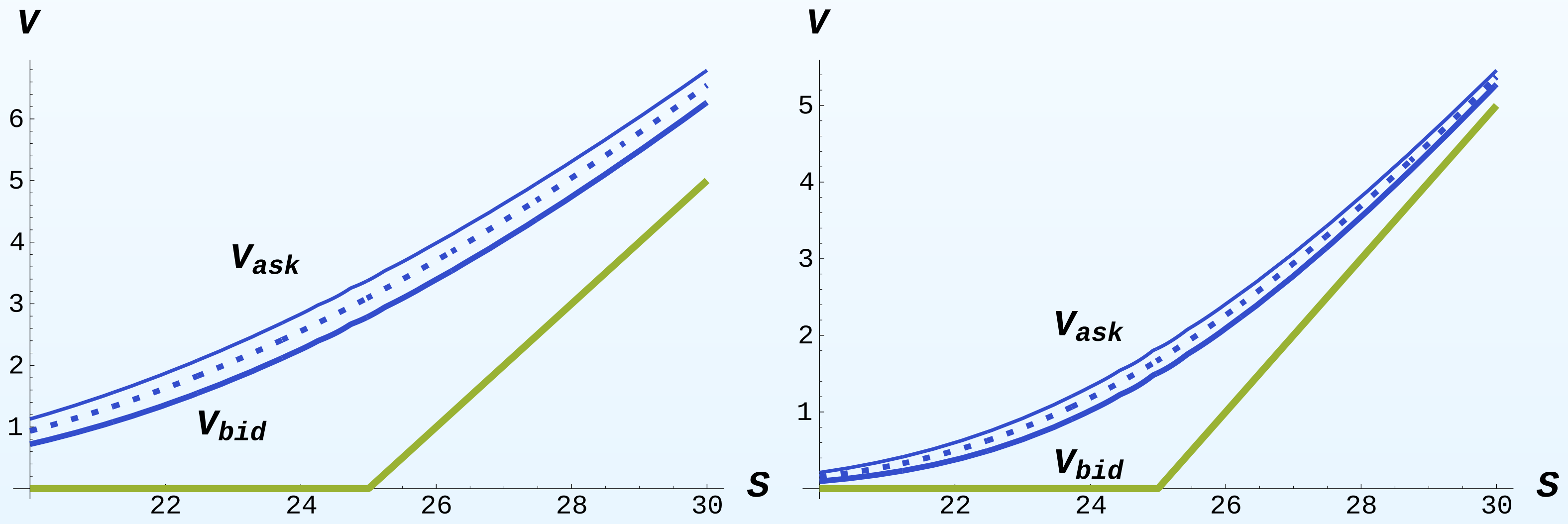
- Solving the PDE for the derivative price:
  - the PDE is a complicated nonlinear PDE
  - firstly standard transformations: $x = \ln(S/E)$, $\tau = T - t$
  - then - since the PDE contains the term $\Gamma = S\frac{\partial^2 V}{\partial S^2}$ - we define a new function

$$H(x,\tau) = S\frac{\partial^2 V}{\partial S^2}$$

  - equation for $H(x,\tau)$ is already much simpler quasilinear PDE and an effective numerical method can be derived to solve it numerically
  - computing the derivative price $V(S,t)$ from the auxiliary function $H(x,\tau)$ is not difficlut; it leads to a numerical computation of one integral

RAPM model

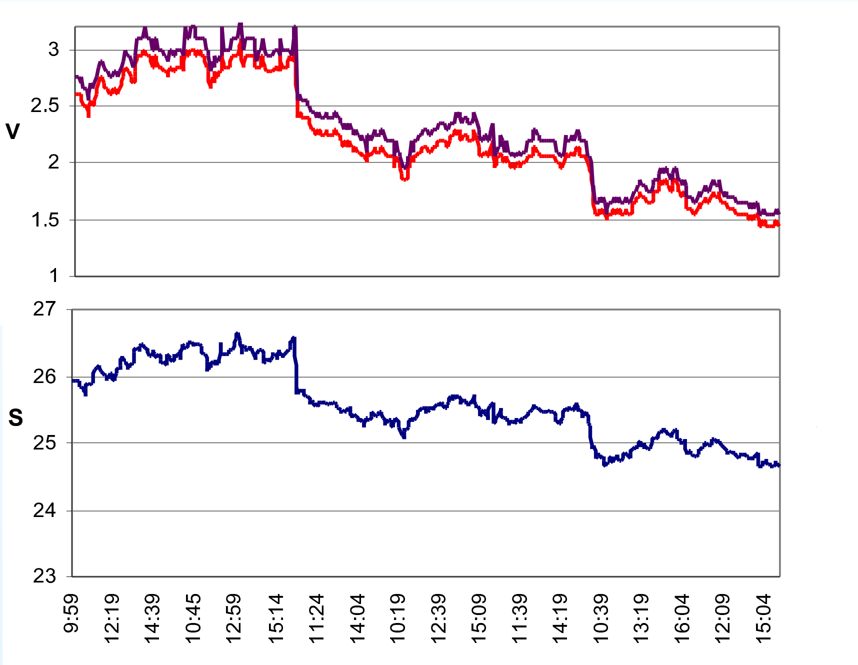
- Similarly as in the Leland model - also the RAPM model allows a computation of bid and ask option prices
- Example:

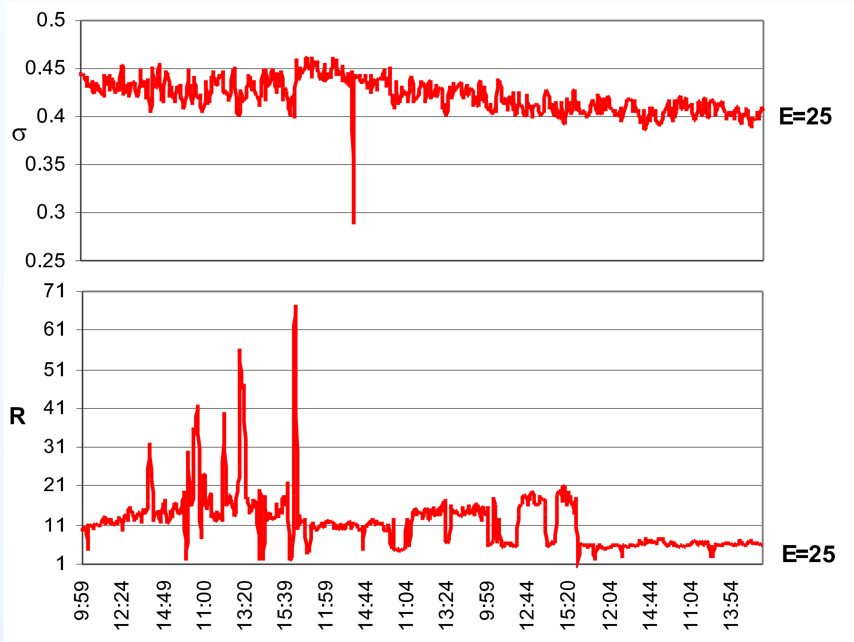


(for a comparison: Black-Scholes option price given by dotted lines)

RAPM model

- Computation of implied parameters from the real data - implied volatility σ and implied risk parameter R:





Left: input data, right: implied parameters

RAPM model

- The PDE

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \left[1 + \mu \left(S \frac{\partial^2 V}{\partial S^2} \right)^{1/3} \right] \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial S} S - rV = 0,$$

can be seen as an equation with nonconstant volatility $\tilde{\sigma} = \tilde{\sigma}(S, t)$:

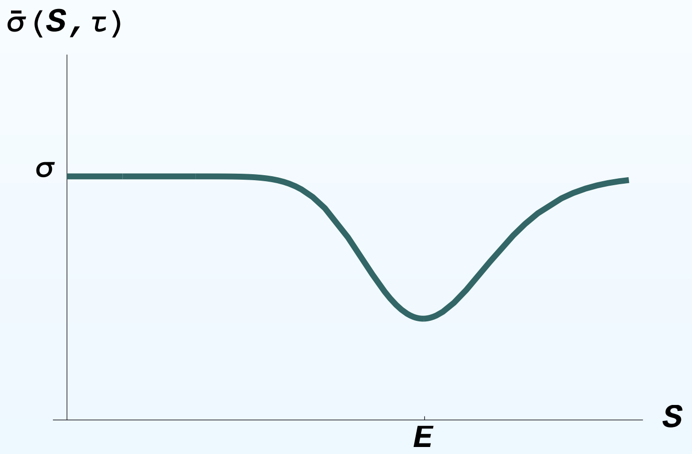
$$\frac{\partial V}{\partial t} + \frac{\tilde{\sigma}^2(S,t)}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial S} S - rV = 0,$$

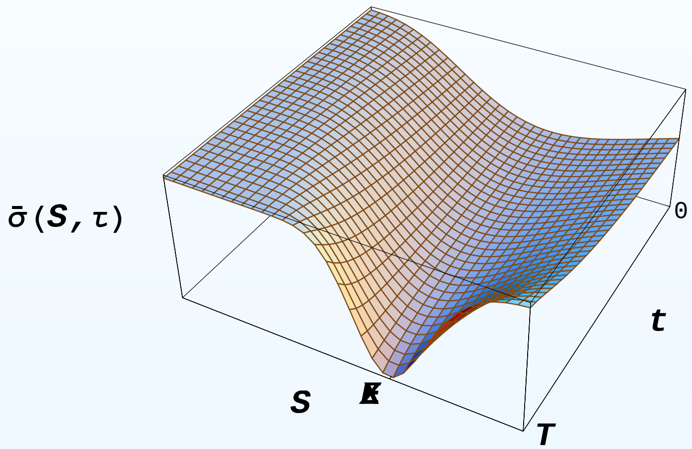
where

$$\tilde{\sigma}(S,t) = \sigma \left[1 + \mu \left(S \frac{\partial^2 V}{\partial S^2} \right)^{1/3} \right]$$

RAPM model

- What is the behaviour of the function $\tilde{\sigma}(S, t)$:





$\Rightarrow$ this model can explain the volatility smile

Presence of a dominant investor

R. Frey: **Market illiquidity as a source of the model risk in dynamic hedging**, RISK publications, R. Gibson Ed., London, 2000.

- Black-Scholes model: we can buy and sell any amount of assets, but it does not have any effect on their price
- In a case of a dominant investor this is not necessarily true - by his strategy he may influence the asset price
- Consider a dominant investor whose strategy for hedging a derivative is characterized by the following variables:
  - $\alpha_t =$ number of stocks at time t
  - $\beta_t =$ number of riskless bonds at time t (i.e. cash)

  and suppose that his trading the assets influences their market price:

$$dS = \mu S dt + \sigma S dw + \rho S d\alpha$$

Presence of a dominant investor

- Investor's strategy depends on the time t and on the stock price S:

$$\alpha = \Phi(S,t)$$

- Using Itō lemma we compute $d\alpha$ and insert it into the formula for dS $\rightarrow$ we obtain

$$dS = b(S,t)Sdt + \nu(S,t)Sdw,$$

where

$$\begin{aligned} \nu(S,t) &= \frac{\sigma}{1-\rho S\frac{\partial\Phi}{\partial S}}, \\ b(S,t) &= \frac{1}{1-\rho S\frac{\partial\Phi}{\partial S}}\left(\mu+\rho\left(\frac{\partial\Phi}{\partial t}+\frac{\nu^2}{2}S^2\frac{\partial^2\Phi}{\partial S^2}\right)\right). \end{aligned}$$

Presence of a dominant investor

- PDE is derived in the same way as in the case of Black-Scholes model, the only change is, that instead of the constant σ there will be the function $\nu(S,t)$:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\nu^2(S,t)S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

- Strategy of the dominant investor:
  - analysis of delta hedging based on Black-Scholes pricce (it is not suitable, it does not replicate the derivative but always leads to higher transaction costs)
  - computation of the correct strategy
  - its qualitative and quantitative analysis

Presence of a dominant investor

- Numerical solution of the PDE - the same idea as in the RAPM model:
  - transformation $H(x,\tau) = S\frac{\partial^2 V}{\partial S^2}$
  - numerical solution of the resulting quasilinear PDE
  - the option price is obtained by integration

Modelling investor's preferences

S. D. Hodges, A. Neuberger: **Optimal replication of contingent claims under transaction costs**, Advances o Futures and Options Research(1994), 21-35.

G. Barles, H.M. Soner: **Option Pricing with transaction costs and a nonlinear Black-Scholes equation**, Finance Stochast. 2 (1998) 369-397.

- Again transaction costs:

$$S_{ask} = (1+\mu)S,\ S_{bid} = (1-\mu)S,$$

kde $S = (S_{bid} + S_{ask})/2$

- Consider the portfolio:
  $X_t =$ value of bonds [in dollars]
  $Y_t =$ number of stocks
- Investor has a utility function U with a constant risk aversion γ

Modelling investor's preferences

- If it was not possible to trade options::
  - value of the portfolio at time T is $X_T + Y_T S_T$
  - we need to solve a stochastic programming problem

$$v^f(x, y, s, t) = \sup \mathbb{E}[U(X_T + Y_T S_T)]$$

  with initial values $X_t = x$, $Y_t = y$, $S_t = s$

- If we write N call options:
  - value of the portfolio at the time of options expiration T is $X_T + Y_T S_T - N(S_T - E)^+$
  - we need to solve a stochastic programming problem

$$v(x, y, s, t) = \sup \mathbb{E}[U(X_T + Y_T S_T - N(S_T - E)^+)]$$

  with initial values $X_t = x$, $Y_t = y$, $S_t = s$

Modelling investor's preferences

- [Hodges, Neuberger]:
  - relationship between these optimization problems
- [Barles, Soner]:
  - construction of the optimal strantegies, PDE of the option price
  - matematical tools: dynamic programming, introducing a small parameter and asymptotic analysis, transformation of the PDE and its numerical solution
  - resulting PDE for the option price has a similar form as in the previous models: instead of a constant volatility (as in the Black-Scholes model) we have a function which depends also on $\frac{\partial^2 V}{\partial S^2}$ $\Rightarrow$ a similar approach to its solving