XII. Numerical methods: Pricing European options

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Transformation to a heat equation

- Transformation

$$V(S,t) = e^{-\alpha x - \beta\tau} u(x,\tau),$$

$$\alpha = \frac{r-q}{\sigma^2} - \frac{1}{2}, \beta = \frac{r+q}{2} + \frac{\sigma^2}{8} + \frac{(r-q)^2}{2\sigma^2}, \tau = T - t, x = \ln(S/E),$$

  transforms the Black-Scholes equation to the following heat equation:

$$\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} = 0$$

  for $x \in \mathbb{R}, \tau \in [0,T]$

- Initial condition: $u(x,0) = g(x)$
  - call option: $g(x) = Ee^{\alpha x + \beta\tau}\max(e^x - 1, 0)$
  - put option: $g(x) = Ee^{\alpha x + \beta\tau}\max(1 - e^x, 0)$

Boundary conditions

- For a numerical scheme we also need boundary condition - we need to think of the option value for very small and very large stock prices
- Call option:
  - $V(0, t) = 0$
  - for $S \to \infty$ we have: $V(S, t) \sim Se^{-q(T-t)}$, more precisely: $V(S, t) \sim Se^{-q(T-t)} - Ee^{-r(T-t)}$
- Put option:
  - $V(0, t) = Ee^{-r(T-t)}$
  - $V(S, t) \to 0$ for $S \to \infty$

Approximation of the solution

- Numerical solution on a bounded space interval $x \in [-L, L]$
- Grid points - in time and space:

$$x_i = ih, \;\; i = -n, ..., -2, -1, 0, 1, 2, ...n,$$

$$\tau_j = jk, \; j = 0, 1, ..., m.$$

where $h = L/n, k = T/m$

- Approximation of the solution u in the point (x_i, τ_j) will be denoted by

$$u_i^j \approx u(x_i, \tau_j), \qquad g_i^j \approx g(x_i, \tau_j)$$

Approximation of the solution

- Boundary conditions:
  - call option:

$$\begin{aligned} \phi^j &:= u^j_{-N} = 0 \\ \psi^j &:= u^j_N = Ee^{(\alpha+1)Nh+(\beta-q)jk} \end{aligned}$$

  - put option:

$$\begin{aligned} \phi^j &:= u^j_{-N} = Ee^{-\alpha Nh+(\beta-r)jk} \\ \psi^j &:= u^j_N = 0 \end{aligned}$$

Implicit scheme

- Recall from the numerical methods course: explicit and implicit scheme for a heat equation
- Implicit scheme - can be written as: $-\gamma u_{i-1}^{j} + (1+2\gamma)u_i^j - \gamma u_{i+1}^j = u_i^{j-1}$, where $\gamma = \frac{\sigma^2 k}{2h^2}$,
- In a matrix form: $\mathbf{A}u^j = u^{j-1} + b^{j-1}$ for $j = 1, 2, \ldots, m$ where

$$\mathbf{A} = \begin{pmatrix} 1+2\gamma & -\gamma & 0 & \cdots & 0 \\ -\gamma & 1+2\gamma & -\gamma & & \vdots \\ 0 & \cdot & \cdot & \cdot & 0 \\ \vdots & & -\gamma & 1+2\gamma & -\gamma \\ 0 & \cdots & 0 & -\gamma & 1+2\gamma \end{pmatrix},$$

$$b^j = (\gamma\phi^{j+1}, 0, \ldots, 0, \gamma\psi^{j+1})^T$$

Solving the linear system

- The system $\mathbf{A}x = b$ with the matrix

$$\mathbf{A} = \begin{pmatrix} 1+2\gamma & -\gamma & 0 & \cdots & 0 \\ -\gamma & 1+2\gamma & -\gamma & & \vdots \\ 0 & \cdot & \cdot & \cdot & 0 \\ \vdots & & -\gamma & 1+2\gamma & -\gamma \\ 0 & \cdots & 0 & -\gamma & 1+2\gamma \end{pmatrix}$$

- Firstly - we solve it using Gauss-Seidel method
- Then we show its generalization - SOR method (its modification will be used in a scheme for American options)