XIV: Numerical methods for pricing American options - PSOR algorithm

Beáta Stehlíková

Financial derivatives, winter term 2014/2015

Faculty of Mathematics, Physics and Informatics
Comenius University, Bratislava

Numerical solution

- Recall:

$$\begin{aligned} \left(\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} \right) (u(x,\tau) - g(x,\tau)) &= 0, \\ \frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} \geq 0, \quad u(x,\tau) - g(x,\tau) &\geq 0 \end{aligned}$$

- Discretization - in the same way as in the implicit scheme for European options:

$$\mathbf{A} u^{j+1} \geq u^j + b^j, \quad u^{j+1} \geq g^{j+1} \quad \text{for } j = 0, 1, ..., m-1,$$
$$(\mathbf{A} u^{j+1} - u^j - b^j)_i (u^{j+1} - g^{j+1})_i = 0 \quad \forall \ i$$

Numerical solution

- Matrix $\mathbf{A}$ and vector b remain the same:

$$\mathbf{A} = \begin{pmatrix} 1+2\gamma & -\gamma & 0 & \cdots & 0 \\ -\gamma & 1+2\gamma & -\gamma & & \vdots \\ 0 & \cdot & \cdot & \cdot & 0 \\ \vdots & & -\gamma & 1+2\gamma & -\gamma \\ 0 & \cdots & 0 & -\gamma & 1+2\gamma \end{pmatrix},$$

$$b^j = (\gamma\phi^{j+1}, 0, \ldots, 0, \gamma\psi^{j+1})^T$$

where $\gamma = \frac{\sigma^2 k}{2h^2}$

PSOR method

- On each time level we solve a problem of the form

$$\mathbf{A}u \geq b, \quad u \geq g,$$
$$(\mathbf{A}u - b)_i(u_i - g_i) = 0 \quad \forall\, i.$$

- Define the sequence

$$u^0 = 0, \quad u^{p+1} = \max\left(\mathbf{T}_\omega u^p + c_\omega,\ g\right) \quad \text{for} \ \ p = 1, 2, ...,$$

 where T_ω, c_ω come from the classical SOR method and maaximum is taken componentwise

- Projected SOR $\rightarrow$ known as PSOR method or PSOR algorithm

PSOR method

- Components of the approximate solution:

$$u_i^{p+1} = \max\left[\frac{\omega}{A_{ii}}\left(b_i - \sum_{j<i} A_{ij}u_j^{p+1} - \sum_{j>i} A_{ij}u_j^{p}\right) + (1-\omega)u_i^p,\ g_i\right]$$

Convergence of the algorithm to the solution

- The sequence u^p converges to some limit u - proof uses Banach fixed point theorem [Ševčovič, Stehlíková, Mikula: **Analytical and numerical methods for pricing financial derivatives**, pp. 156-157]
- This limit is a solution:
  - $u_i^{p+1} \geq g_i \Rightarrow$ also the limit satisfies $u_i \geq g_i$
  - $u_i^{p+1} \geq \frac{\omega}{A_{ii}} \left(b_i - \sum_{j<i} A_{ij} u_j^{p+1} - \sum_{j>i} A_{ij} u_j^p \right)$ $+(1-\omega) u_i^p \Rightarrow$ also the limit satisfies $u_i \geq \frac{\omega}{A_{ii}} \left(b_i - \sum_{j<i} A_{ij} u_j - \sum_{j>i} A_{ij} u_j^p \right) + (1-\omega) u_i$ we use that $A_{ii} > 0, \omega > 0 \rightarrow$ we obtain $(Au)_i \geq b_i$
  - if $u_i > g_i$, then starting with some index p_0 we have $u_i^p > g_i$; for these indices:

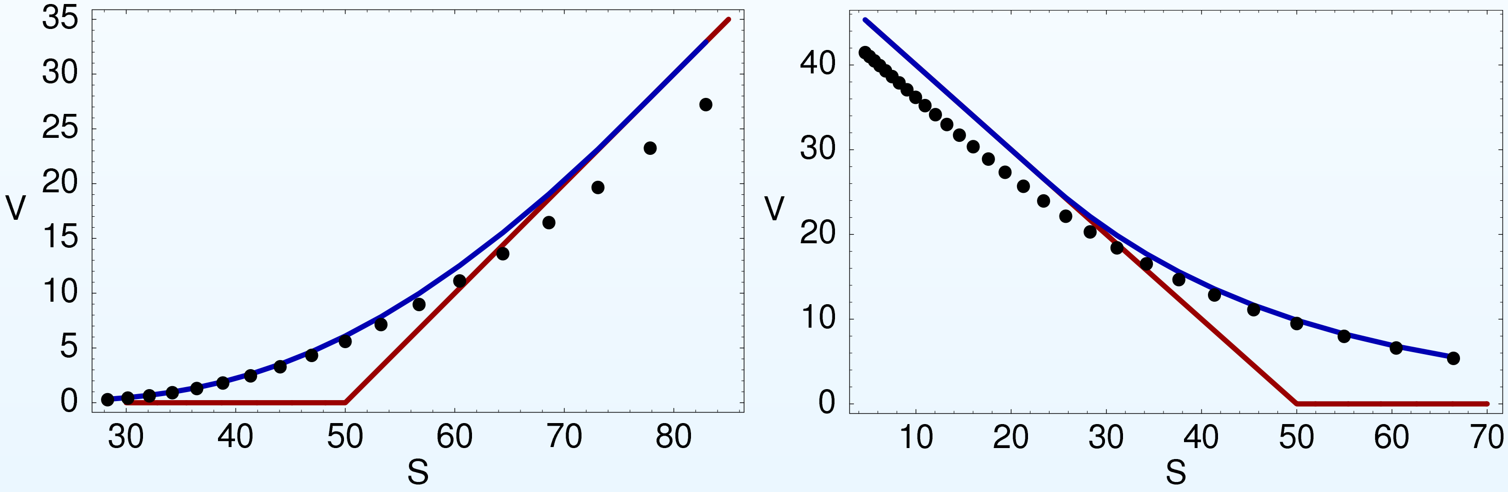
Convergence of the algorithm to the solution

$u_i^{p+1} = \frac{\omega}{A_{ii}} \left(b_i - \sum_{j<i} A_{ij} u_j^{p+1} - \sum_{j>i} A_{ij} u_j^p \right) + (1-\omega) u_i^p,$

taking limit as $p \to \infty$ we get $(Au)_i = b_i \Rightarrow$ condition $(Au - b)_i (u_i - g_i) = 0$ is satisfied

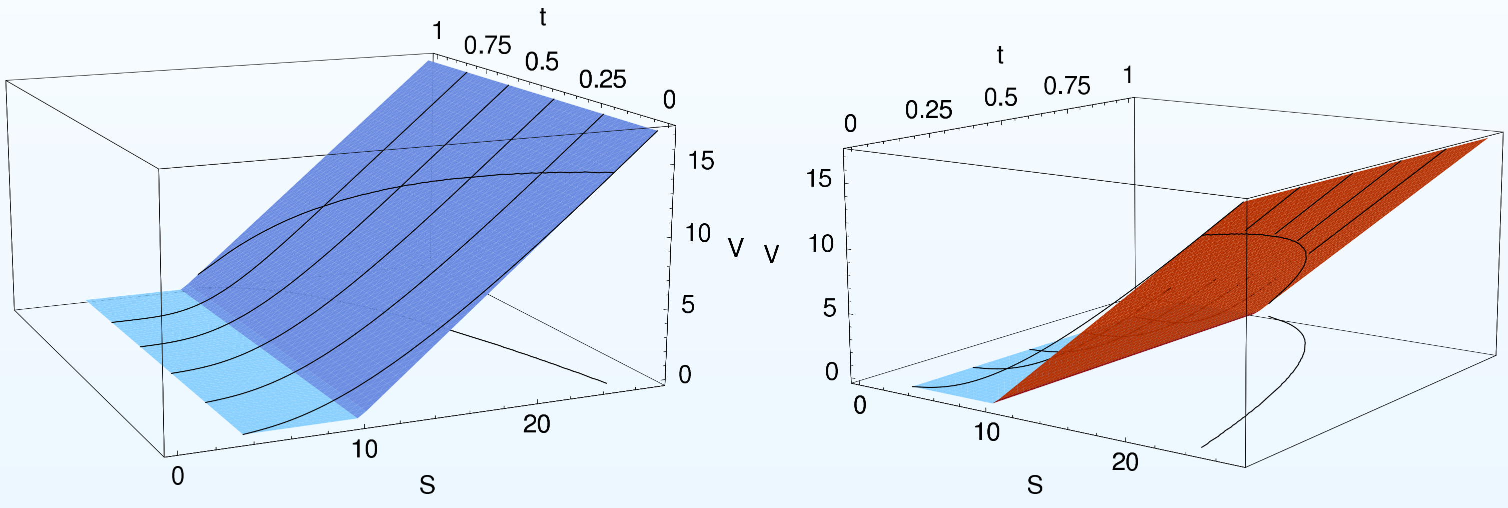
Numerical examples

- Pricing American call and put options (for a comparison: price of a European option - dotted line)



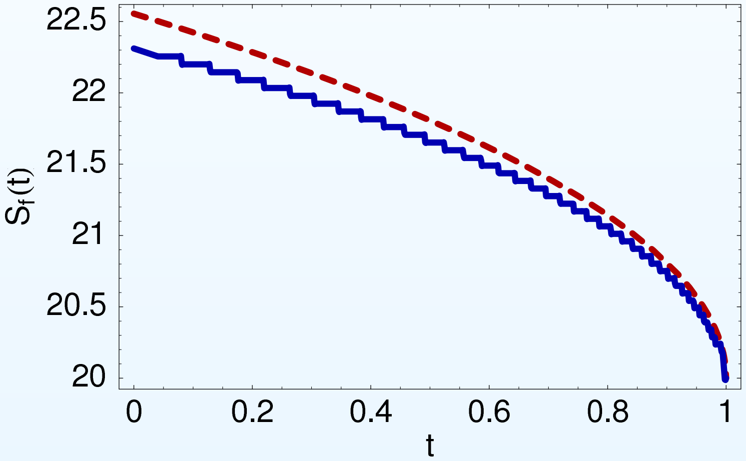
Numerical examples

- 3D graph for a call option, the free is depicted:



Numerical examples

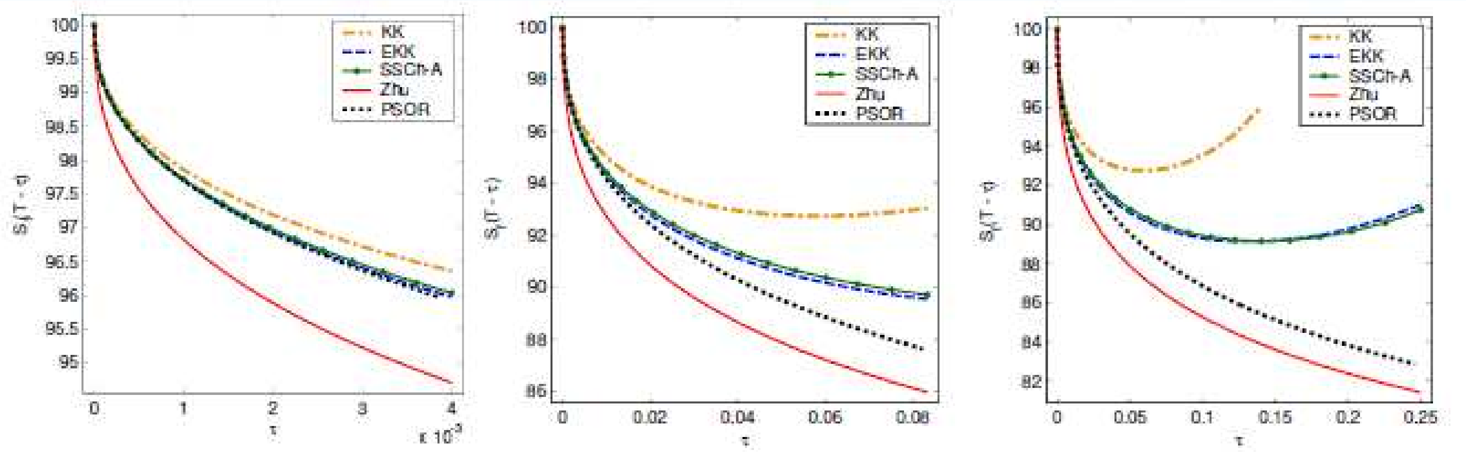
- Numerical computation of the free boundary and its comparison with the "square root approximation formula"



Numerical examples

- M. Lauko, D. Ševčovič: **Comparison of numerical and analytical approximations of the early exercise boundary of American put options**, ANZIAM journal 51, 2010, 430-448.

Comparison of approximation formulae for put options:



$T = 4 \times 10^{-3}$ (1 day) $T = 0.08$ (1 month) $T = 0.25$ (3 months)