XV. Pricing exotic derivatives I.

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Financial derivatives, winter term 2014/2015

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Exotic options

- Path-dependent options - payoff depends not only on the value of the underlying asset at the expiration time, but also on its evolution (path) before the expiration time
- Lowers the risk coming from sudden changes in prices
- Extra credit 2013, the price rose during the last day before the expiration of the options:

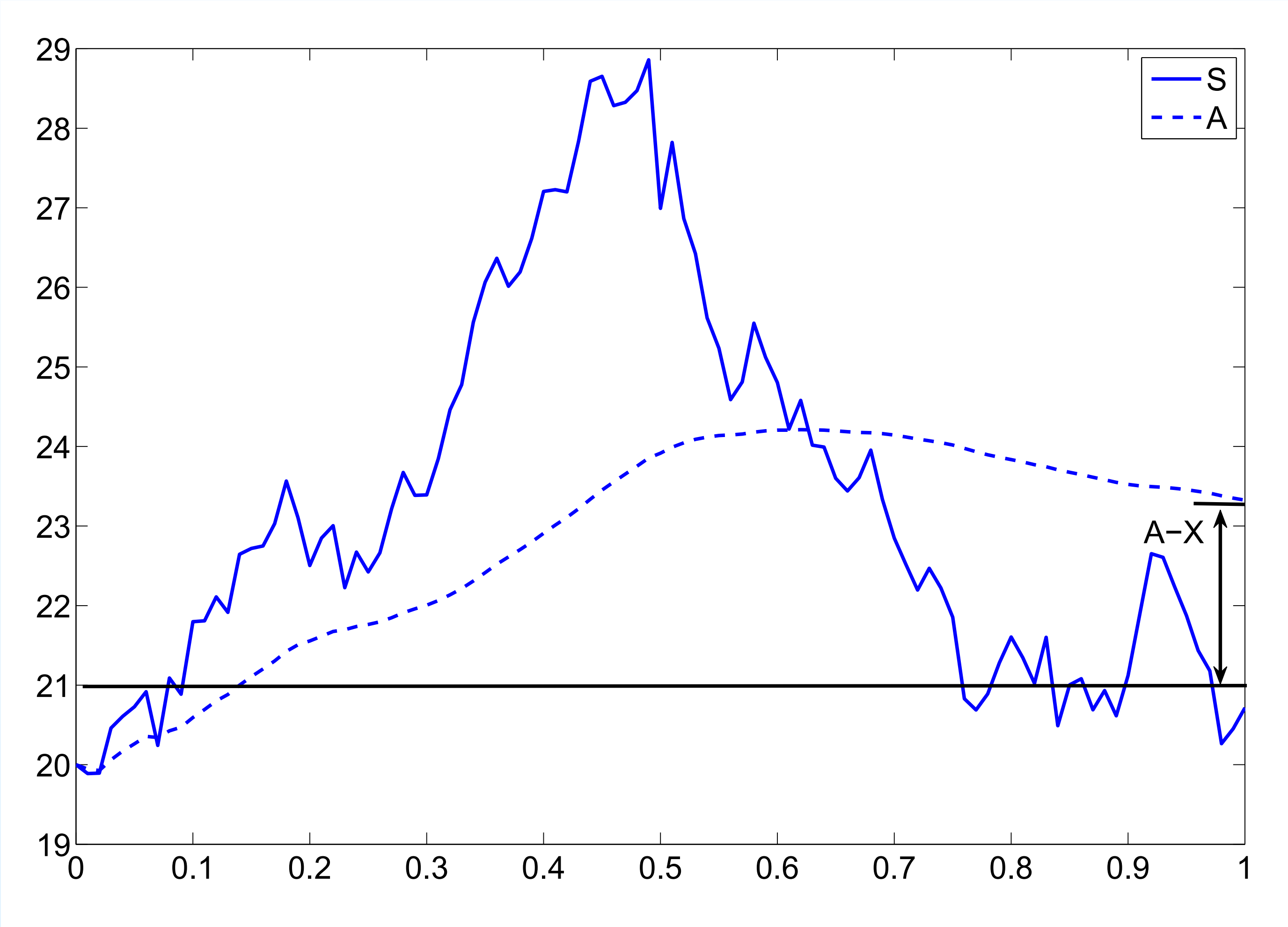
http://finance.google.com/

Asian options

- Payoff depends on historical average of stock prices
- Classification of Asian options:
  - based on averaging - arithmetic or geometric average
  - based on position of the average in the payoff - can take the role of the stock price or the exercise price
- Average:
  - arithmetic:
    discrete case $A_t = \frac{1}{n} \sum_{i=1}^{n} S_{t_i}$
    continuous case $A_t = \frac{1}{t} \int_0^t S_\tau d\tau$
  - geometric:
    discrete case $\ln A_t = \frac{1}{n} \sum_{i=1}^{n} \ln S_{t_i}$
    continuous case $\ln A_t = \frac{1}{t} \int_0^t \ln S_\tau d\tau$

Asian options

Stock price and its average (dashed line):



Asian options

- Position of the average in the payoff
  - the average price A enters the payoff taking the role of the stock price - the option is called average rate call, resp. put :

$$V(S,A,T) = \max(A-E,0) \quad \text{for a call}$$
$$V(S,A,T) = \max(E-A,0) \quad \text{for a put}$$

  - the average price A enters the payoff taking the role of the exercise price - the option is called average strike call, resp. put :

$$V(S,A,T) = \max(S-A,0) \quad \text{for a call}$$
$$V(S,A,T) = \max(A-S,0) \quad \text{for a put}$$

- So we have, for example:
  - *Asian arithmetically averaged average rate call option*,
  - *Asian geometrically averaged average strike put option*

Differential of the averaged price

- We will use continuous time
- Arithmetic average:

$$\frac{dA}{dt} = -\frac{1}{t^2}\int_0^t S_\tau d\tau + \frac{1}{t}S_t = \frac{S_t - A_t}{t}$$

- Geometric average:

$$\frac{dA}{dt} = A_t\left[-\frac{1}{t^2}\int_0^t \ln S_\tau d\tau + \frac{1}{t}\ln S_t\right] = A_t\,\frac{\ln S_t - \ln A_t}{t}$$

- In both cases:

$$dA = A\,f\left(\frac{S}{A}, t\right)dt,$$

where $f(x,t) = (x-1)/t$, resp. $f(x,t) = (\ln x)/t$

PDE for the Asian option price

- Geometric Brownian motion for the stock price $dS = \mu S dt + \sigma S dw$, stock pays continuous dividends with rate D
- Option price $V = V(S, A, t)$:

$$dV = \frac{\partial V}{\partial S} dS + \left(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial A} A f\left(\frac{S}{A}, t \right) \right) dt$$

- As in the case of the Black-Scholeso model:
  - portfolio: option + stocks
  - elimination the random part of the SDE for the portfolio value
  - yield of a riskless portfolio has to be equal to r (riskless instantaneous interest rate)

PDE for the Asian option price

- Resulting PDE for the Asian option price $V(S,A,t)$:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r-D) S \frac{\partial V}{\partial S} + A f\left(\frac{S}{A}, t\right) \frac{\partial V}{\partial A} - rV = 0$$

  for $S \in (0,\infty)$, $A \in (0,\infty)$, $t \in (0,T)$

- Terminal condition depends on the option, e.g.,

$$V(S,A,T) = \max(S-A, 0)$$

  for $S \in (0,\infty)$, $A \in (0,\infty)$

- Three variables, but only one derivative of the second order $\rightarrow$ the PDE is not in a suitable form for finding a numerical scheme $\rightarrow$ for average strike option we perform a transformation

Transformation for average strike options

- Transformation:

$$V(S, A, t) = AW(x, t), \quad x = \frac{S}{A}$$

- PDE for the function $W(x, t)$:

$$\frac{\partial W}{\partial t} + \frac{\sigma^2}{2} x^2 \frac{\partial^2 W}{\partial x^2} + (r - D)x \frac{\partial W}{\partial x} + f(x, t)\left(W - x\frac{\partial W}{\partial x}\right) - rW = 0$$

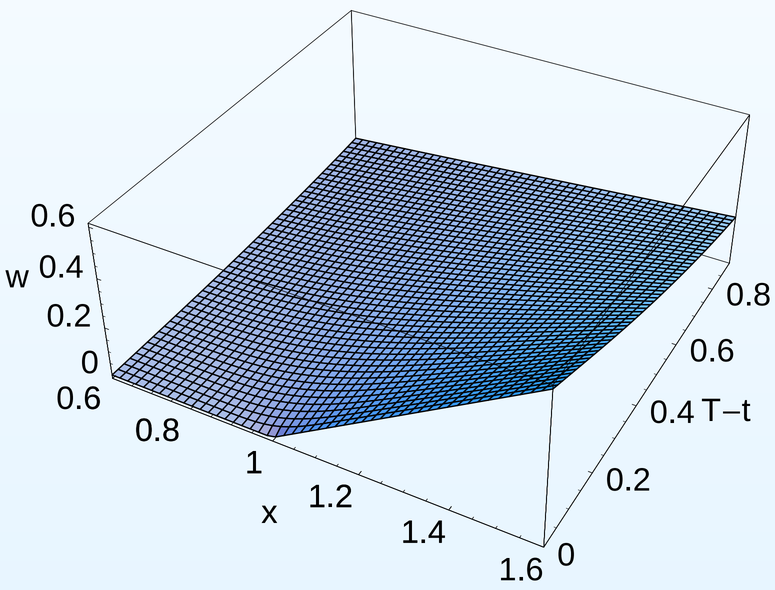
for $x \in (0, \infty)$, $t \in (0, T)$

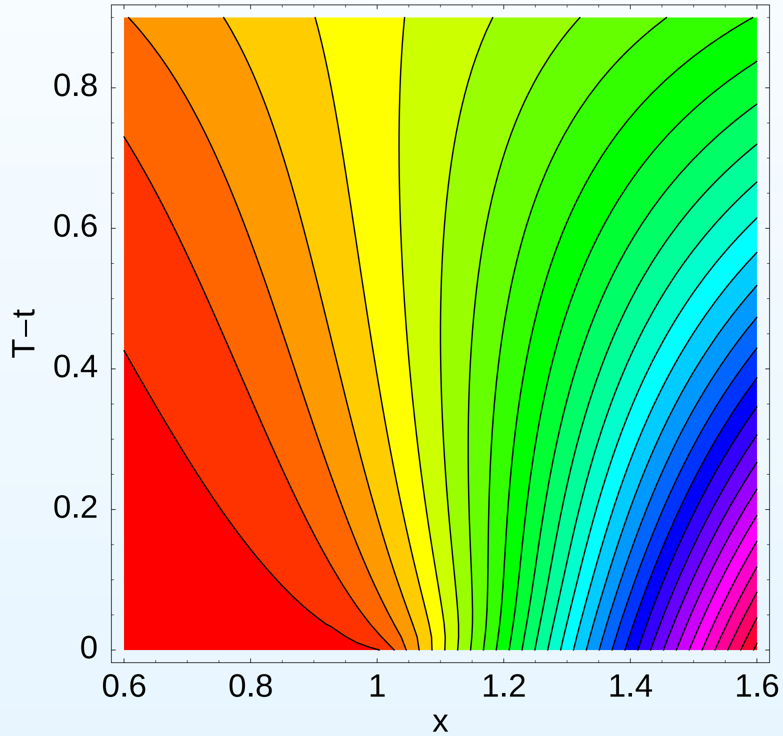
- Terminal condition for $x \in (0, \infty)$:

$$W^{call}(x, T) = \max(x - 1, 0), \quad W^{put}(x, T) = \max(1 - x, 0)$$

Average strike option - example

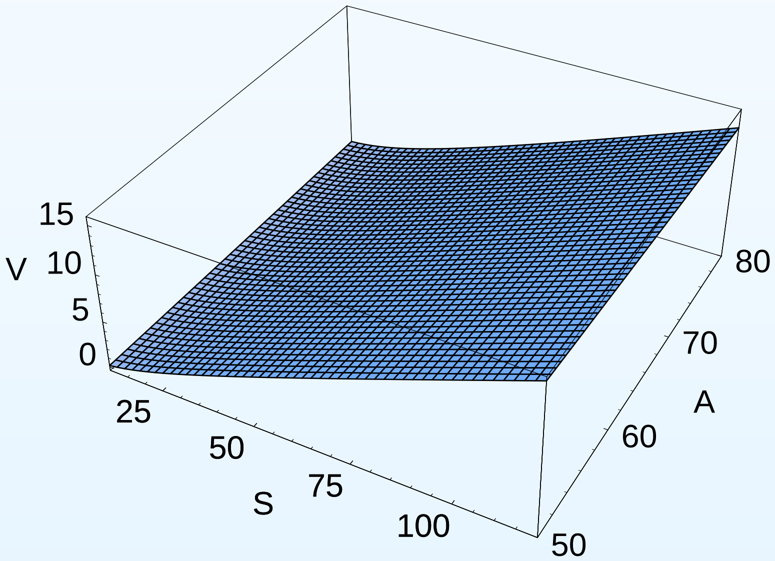
- Auxiliary function $W(x, t)$:

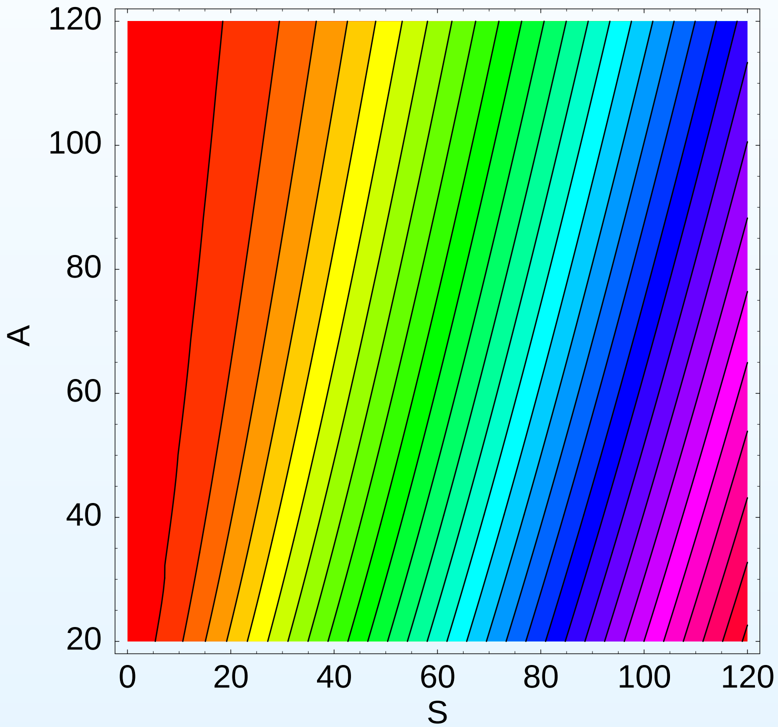




Average strike option - example

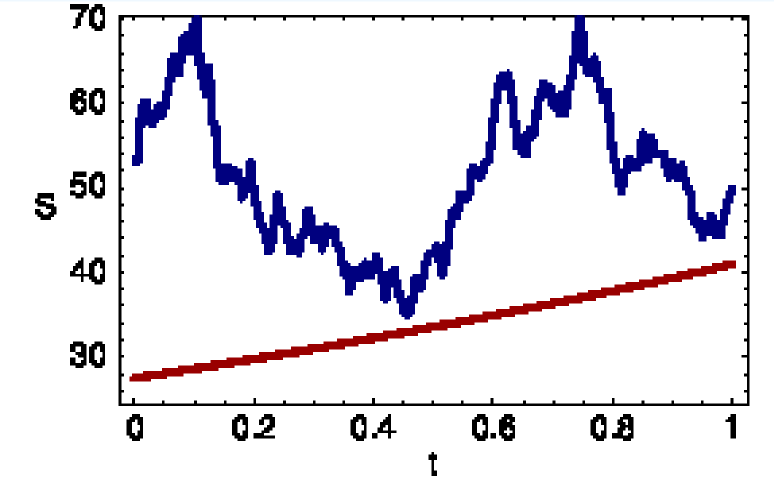
- Option price $V(S, A, t)$ for selected t:

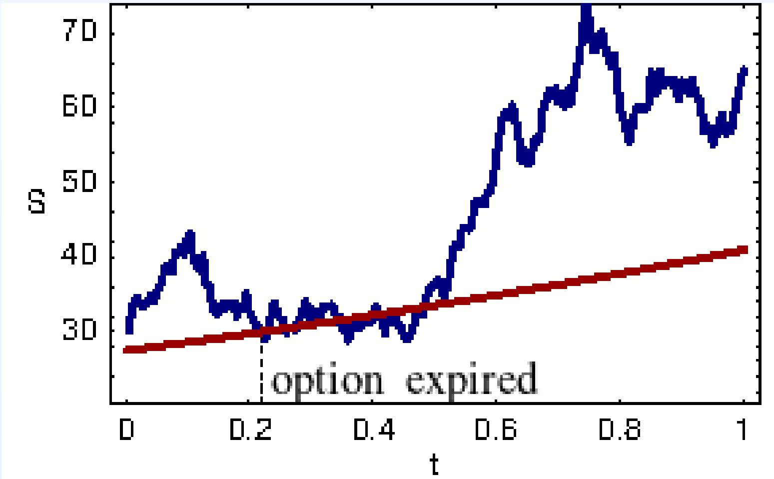




Barrier options

- Similar to classical call and put options
- The difference: if at some time of the life of the options the stock price hits the given barrier, then:
  - the option is no longer valid
  - the option holder receives rebate from the writer
- Example: stock price (blue), barrier (brown)





Barrier options - barrier and rebate

- Classification of barriers:
  - down-and-out: if the stock price hits the barrier from above
  - up-and-out: if the stock price hits the barrier from below
- A typical example of a barrier:

$$B(t) = bEe^{-\alpha(T-t)},$$

  where $0 < b \leq 1, \alpha \geq 0$ are constants
- Example of a rebate:

$$R(t) = E\left(1 - e^{-\beta(T-t)}\right),$$

  where $\beta \geq 0$ is a constant - satisfies $R(T) = 0$

PDE for a down-and-out option

- Option is valid in the domain $S > B(t)$ - here, the Black-Scholes PDE holds:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV = 0$$

  for $S \in (B(t), \infty), t \in (0, T)$.

- On the boudary, i.e., for $S = B(t)$ - the option is cancelled and its value equals the rebate:

$$V(B(t), t) = R(t)$$

  for $t \in (0, T)$.

- Terminal condition for $S \in (B(t), \infty), t = T$ depends on the option type:

$$V^{call}(S, T) = \max(0, S - E), \quad V^{put}(S, T) = \max(0, E - S)$$

PDE for a down-and-out option

- Transformation to a fixed domain $x \in (0, \infty)$:

$$V(S,t) = W(x,t), \quad x = \ln\left(\frac{S}{B(t)}\right),$$

- PDE for the function W:

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 W}{\partial x^2} + \left(r - D - \frac{\sigma^2}{2} - \alpha\right)\frac{\partial W}{\partial x} - rW = 0$$

  for $x \in (0, \infty), t \in (0, T)$.

- Boundary condition: $W(0,t) = R(t)$ pre $t \in (0, T)$.
- Terminal condition:

$$V^{call}(x,T) = E\max(0, be^x - 1)$$
$$V^{put}(x,T) = E\max(0, 1 - be^x)$$

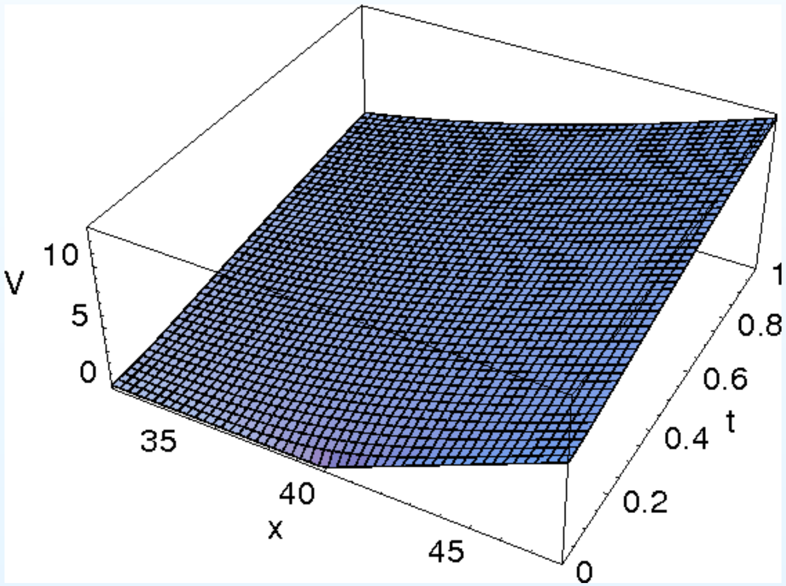
  for $x \in (0, \infty)$.

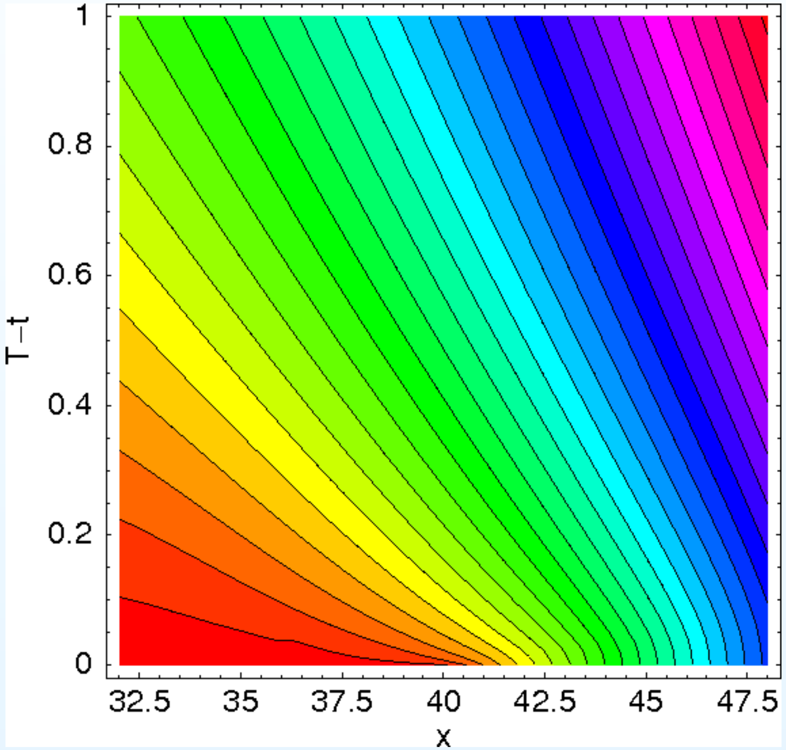
Up-and-out option: homework

- Write the mathematical formulation of pricing an up-and-out option: PDE (and its domain), boundary condition, terminal condition
- Transform it to a PDE on a fixed domain

Barrier options: example

- Price of a barrier option:





Barrier options: interactive

- Web page:
  http://demonstrations.wolfram.com/BarrierOptionPricingWithinTheBlackScholesModel/
- Requires the player, available at:
  http://demonstrations.wolfram.com/download-cdf-player.html