XVI. Pricing exotic derivatives II.

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Basket options, options on indices, etc.

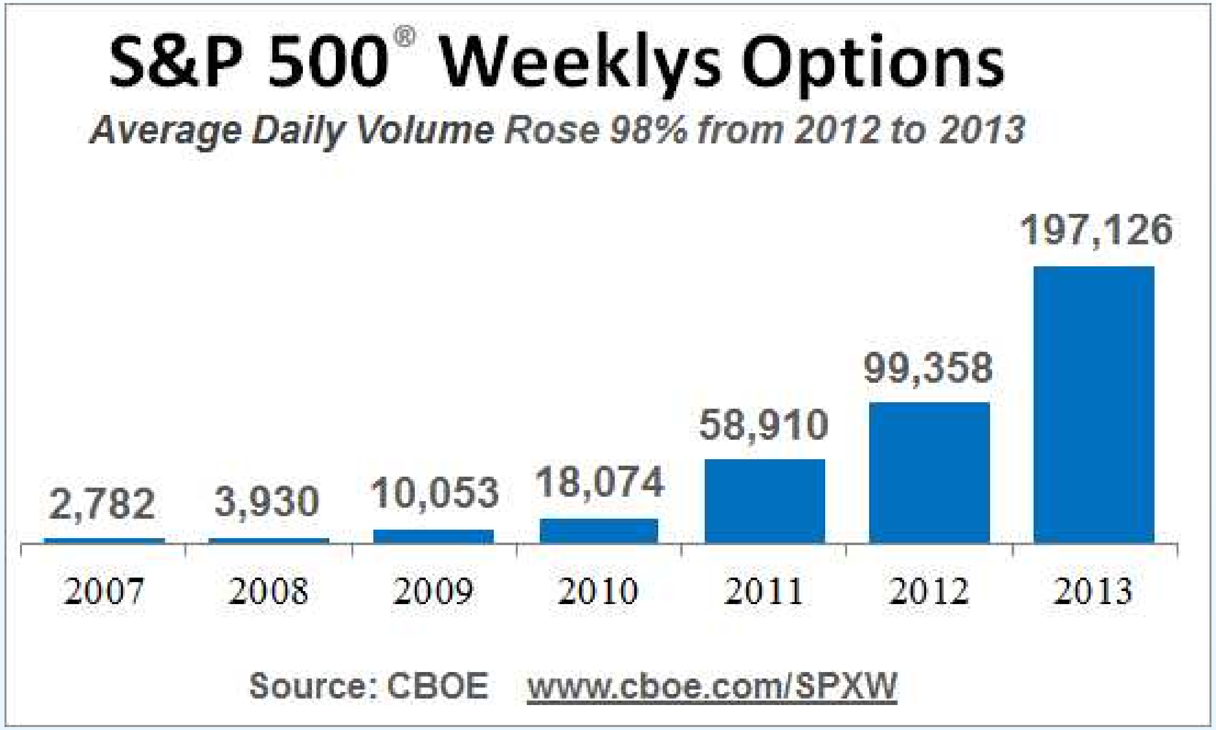
- Payoff of the option depends on the value of several assets or on the value of an index
- EXAMPLE 1: spread options - payoff depends on a difference between values of two assets at the expiration time, e.g.,

$$V(S_1, S_2, T) = \max((S_1 - S_2) - E, 0)$$

  - useful for example for commodities (prices of an input and an output)
- EXAMPLE 2: options on indices - for example S&P 500, NYSE, ...
  If each stock follows a GBM, we obtain n-dimensional Black-Scholes equation ($n =$ number of stock in the index)

Basket options, options on indices, etc.

- Trading S&P 500 options:



http://www.cboe.com/

Lookback options

- Lookback options - payoff depends on the maximal price of the underlying asset during the given period

$$M = M_{T_0}^{T} = \max(S_t, t \in [T_0, T]),$$

where $T \geq 0$

- For example: maximum M instead of the stock price on the payoff:

$$V^{call}(S, M, T) = \max(0, M - E)$$
$$V^{put}(S, M, T) = \max(0, E - M)$$

Spread options: Margrabe formula

- Recall: spread options

$$V(S_1, S_2, T) = \max((S_1 - S_2) - E, 0)$$

- Suppose that the stocks do not pay dividends and that

$$\begin{aligned} dS_1 &= \mu_1 S_1 dt + \sigma_1 S_1 dw_1 \\ dS_2 &= \mu_2 S_2 dt + \sigma_2 S_2 dw_2 \end{aligned}$$

  where $\mathbb{E}[dw_1 dw_2] = \rho dt$

- For the case of $E = 0$ there is an explicit formula for the option price - co called Margrabe formula
- We derive the PDE for the option price and find its solution

Spread options: Margrabe formula

- Similarly as in the derivation of the Black-Scholes model
- Portfolio:
  - one option V
  - $-\Delta_1$ stocks S_1
  - $-\Delta_2$ stocks S_2

  Portfolio value: $P = V - \Delta_1 S_1 - \Delta_2 S_2$
- Change in the pofolio value $P = dV - \Delta_1 dS_1 - \Delta_2 dS_2$, where
  - dS_1, dS_2 are in the assumptions
  - dV is given by the multidimensional Itō lemma (since $V = V(S_1, S_2, t)$)
- We eliminate randomness (terms dw_1, dw_2) - by setting $\Delta_1 = \frac{\partial V}{\partial S_1}$, $\Delta_2 = \frac{\partial V}{\partial S_2}$
- Yield of a riskless portfolio has to be r

Spread options: Margrabe formula

- The resulting PDE:

$$\frac{\partial V}{\partial t} + rS_1\frac{\partial V}{\partial S_1} + rS_2\frac{\partial V}{\partial S_2} + \frac{1}{2}\sigma_1^2 S_1^2\frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2\frac{\partial^2 V}{\partial S_2^2}$$

$$+\rho\sigma_1\sigma_2 S_1 S_2\frac{\partial^2 V}{\partial S_1 S_2} - rV = 0$$

with terminal condition

$$V(S_1, S_2, T) = \max(S_1 - S_2, 0)$$

- Transformation:

$$V(S_1, S_2, t) = S_2 f(x, t), \;\; x = \frac{S_1}{S_2}$$

Spread options: Margrabe formula

- PDE for the function $f(x,t)$:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\tilde{\sigma}^2 x^2 \frac{\partial^2 f}{\partial x^2} = 0,$$

  kde $\tilde{\sigma}^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$
- Terminal condition $f(x,T) = \max(x-1,0)$
- This is the Black-Scholes PDE for a call, where
  - the variable x corresponds to the stock price S
  - exercise price $E = 1$
  - interest rate is zero
- Hence, the solution is: $f(x,t) = xN(d_1) - N(d_2)$, where
  $d_1 = \frac{\log x + \frac{\tilde{\sigma}^2}{2}\tau}{\tilde{\sigma}\sqrt{\tau}}$, $d_2 = d_1 - \tilde{\sigma}\sqrt{\tau}$

Spread options: Margrabe formula

- Solution in the original variables (i.e., the spread option price):

$$V(S_1, S_2, t) = S_1 N(d_1) - S_2 N(d_2),$$

where

$$d_1 = \frac{\log \frac{S_1}{S_2} + \frac{\tilde{\sigma}^2}{2}\tau}{\tilde{\sigma}\sqrt{\tau}}, d_2 = d_1 - \tilde{\sigma}\sqrt{\tau}$$

- this is known as Margrabe formula

- HOMEWORK: Derive the spread option price, if the stocks pay continuous dividends with rates q_1, q_2.