Financial derivatives 2014/2015 - individual homework

- Homework can be solved independently or in pairs.
- Everyone/every pair of students has a unique assignment, consisting of one problem, chosen by them from this list. Reservation of the problems: `http://users2.smartgb.com/g/g.php?a=s&i=g26-32343-26`
- Solutions should be e-mailed as a **pdf file** (typed or scanned) to **beata.ulohy@gmail.com** with subject **derivatives 2014 - homework - name/names**
- Deadline: 2 working days before the exam

List of problems:

1. Compute the value of a *stock-or-nothing option* in the Black-Scholes framework, i.e., the solution $V = V(S, \tau)$ to the Black-Scholes PDE

$$-\frac{\partial V}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0$$

   ($S > 0, \tau \in (0, T]$) with initial condition $V(S, 0)$ which is equal to S if $S > S_0$ (where S_0 is a predeterminted value) and zero otherwise.

2. Compute the value of a *power option* n the Black-Scholes framework, i.e., the solution $V = V(S, \tau)$ to the Black-Scholes PDE

$$-\frac{\partial V}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0$$

   ($S > 0, \tau \in (0, T]$) with initial condition $V(S, 0) = \max(S^n - E, 0)$ where $n \in \mathbb{N}$ and $E > 0$ are given constants.
   Hint: Make a transformation $V(S, \tau) = W(Y, \tau)$, where $Y = S^n$ and derive the PDE for $W(Y, \tau)$. You can solve this equation for W more easily by comparing with to the classical Black-Scholes equation and using the Black-Scholes formula.

3. *Source: beanactuary.com, acturial exams.* You are using the Vasicek one-factor interest rate model with the short rate process calibrated as

$$dr = 0.6(b - r)dt + \sigma dw.$$

   For $t \leq T$ let $P(r, t, T)$ be the price at time t of a zero-coupon bond that pays 1 USD at time T, if the short-rate at time t is r. The price of each zero-coupon bond in the Vasicek model follow an Itô process

$$\frac{dP(r, t, T)}{P(r, t, T)} = \alpha(r, t, T)dt - q(r, t, T)dw$$

   for $t \leq T$. You are given that $\alpha(0.04, 0, 2) = 0.04139761$. Find $\alpha(0.05, 1, 4)$.
   Hint: Recall the definition of the market price of risk and the fact that it is taken to be constant in Vasicek model

4. Consider the Cox-Ingersoll-Ross model of interest rates and the PDE for the bond prices $P = P(r, \tau)$:

$$-\frac{\partial P}{\partial \tau} + (\kappa(\theta - r) - \lambda\sigma r)\frac{\partial P}{\partial r} + \frac{\sigma^2 r}{2}\frac{\partial^2 P}{\partial r^2} - rP = 0$$

for $r > 0, \tau \in (0, T]$ and initial condition $P(r, 0) = 1$. We are looking for a solution in the form $P(r, \tau) = A(\tau)e^{-B(\tau)r}$. Derive the system of ordinary differential equations for the functions A, B and their initial conditions. Using the solution to this system (you do not need to derive it; you can use the formulae given in the lectures), for selected parameters plot the term structures of interest rates showing their three possible shapes - increasing, humped and decreasing.

5. *Chooser options* are a type of exotic option that, at some pre-specified time T_1 in the future, can be converted into either a put or call option with expiry $T_2 > T_1$ and strike K. We consider these options in the Black-Sholes setting. See the website `http://demonstrations.wolfram.com/ChooserOptions/` and the interactive demonstrations there (you will need to download a free player). Then, solve the following problems.

   - Explain the put-call parity argument ("It can be shown using general put-call parity considerations...") and derive the price of a chooser option. You can check your solution for example here: `http://www.haas.berkeley.edu/groups/finance/WP/rpf220.pdf`, pp. 56-57 (but you need to provide a more detailed explanation).
   - Compute its delta and gamma and plot their graphs as functions of the stock price for some selected times. Note the behaviour for t approaching T_1, as mentioned also on the website.

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   - Read the paper[1]

     Martinkute-Kauliene, Raimonda. "Exotic Options: a Chooser Option and its Pricing." Business, Management and Education 2 (2012): 289-301.

     and, for selected values of parameters, make similar plots as those in Figure 2 (dependence of chooser option price on the time T_1 when it is converted into put or call; note that in general this dependence is not necessarily so close to linear, as claimed in the paper) and Figure 3 (dependence of chooser option price on the strike price K). Give a financial interpretation for their increasing/decreasing behaviour.

[1] It can be downloaded at `http://www.bme.vgtu.lt/index.php/bme/article/download/101/116`

7. *Sharpe ratio* of a derivative (of an underlying stock S) is given by

$$\frac{\text{expected return - risk free rate}}{\text{return volatility}}$$

which is equal to $\frac{\tilde{\mu}-r}{\tilde{\sigma}}$, if r is the risk free rate and the value x of the derivative satisfies the stochastic differential equation $dx/x = \tilde{\mu}(S,t)dt + \tilde{\sigma}(S,t)dw$. Now, consider the Black-Scholes setting, the stock which does not pay dividends and the derivative with the payoff $\bar{V}(S) = S^2$

- Find the price $V(S,t)$ of the derivative.
  HINT: See lecture slides for a form of the solution.
- Use Itō lemma to compute dV and use it to compute the Sharpe ratio of this derivative. Show that it is same as the Sharpe ratio for the underlying stock.

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Show that the Sharpe ratio of this derivative is same as the Sharpe ratio for the underlying stock.
HINT: Use Itō lemma to compute dV and connect it to Black-Scholes PDE which is satisfied by the derivative price.

9. *Source: beanactuary.com, acturial exams.* Consider a 3-month American call option on a nondividend-paying stock with strike price 41.5. You are given:

- The Black-Scholes framework holds.
- The stock is currently selling for 40.
- The stock's volatility is 30 % (i.e. 0.3)
- The current call option delta is 0.5.

Determine the current price of the option.
HINT: The underlying stock does not pay dividends, so the value of the American call option equals to the value of the European call option with the same parameters. Recall the formula for the delta of such an option and the graph of the CDF of $\mathcal{N}(0,1)$ distribution.

10. Let $A, K > 0$ be given constants. Consider the Black-Scholes framework and a European option with payoff equal to $(\min(S-K, A))^+$, where $x^+ = \max(x, 0)$ and S is a stock price. Prove that delta of this option is positive.

11. Consider the Black-Scholes price of a call of option $V(S,t)$. Show that

$$\frac{V(S_1,t)}{V(S_2,t)} > \frac{S_1}{S_2}$$

for $S_1 > S_2$

12. Consider geometrical Brownian motion $S(t) = S_0 e^{\mu t + \sigma w(t)}$ as a model for the stock price, with $\mu = 0$ and $\sigma = 0.3$. The stock price today is 150 USD. Compute the expected time when the price leaves the region beween 140 and 170 USD for the first time.