Homework: individual assignments
Financial derivatives 2015/2016

- Homework can be solved independently or in grops of two or three students.
- Everyone/every group of students has a unique assignment, consisting of one problem, chosen by them from this list.
- Organize way for reservation of the problems that us convenient for you and send me a link to the list.
- Solutions should be e-mailed to **beata.ulohy@gmail.com** as a **pdf file** (typed or scanned) with subject **derivatives 2016 - homework - name/names**. If the solution involves computations in R, the code should be attached but the results should be provided iin the pdf (so that it canbe read independently, without having to check the code).
- Deadline: **18th May 2016**

The Wiener process is denoted by w. If not stated otherwise, we assume the Black-Scholes model and a stock which does not pay dividends.

List of problems:

1. Compute the value of a *stock-or-nothing option* in the Black-Scholes framework, i.e., the solution $V = V(S, \tau)$ to the Black-Scholes PDE

$$-\frac{\partial V}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0$$

   ($S > 0, \tau \in (0, T]$) with initial condition $V(S, 0)$ which is equal to S if $S > S_0$ (where S_0 is a predeterminted value) and zero otherwise.

2. Compute the value of a *power option* n the Black-Scholes framework, i.e., the solution $V = V(S, \tau)$ to the Black-Scholes PDE

$$-\frac{\partial V}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0$$

$(S > 0, \tau \in (0, T])$ with initial condition $V(S, 0) = \max(S^n - E, 0)$ where $n \in \mathbb{N}$ and $E > 0$ are given constants.
HINT: Make a transformation $V(S, \tau) = W(Y, \tau)$, where $Y = S^n$ and derive the PDE for $W(Y, \tau)$. You can solve this equation for W more easily by comparing with to the classical Black-Scholes equation and using the Black-Scholes formula.

3. *Sharpe ratio* of a derivative (of an underlying stock S) is given by

$$\frac{\text{expected return - risk free rate}}{\text{return volatility}}$$

which is equal to $\frac{\tilde{\mu}-r}{\tilde{\sigma}}$, if r is the risk free rate and the value x of the derivative satisfies the stochastic differential equation $dx/x = \tilde{\mu}(S, t)dt + \tilde{\sigma}(S, t)$. Now, consider the Black-Scholes setting, the stock which does not pay dividends and the derivative with the payoff $\bar{V}(S) = S^2$

- Find the price $V(S, t)$ of the derivative.
HINT: See lecture slides for a form of the solution.
- Use Itō lemma to compute dV and use it to compute the Sharpe ratio of this derivative. Show that it is same as the Sharpe ratio for the underlying stock.

4. *Sharpe ratio* of a derivative (of an underlying stock S) is given by

$$\frac{\text{expected return - risk free rate}}{\text{return volatility}}$$

which is equal to $\frac{\tilde{\mu}-r}{\tilde{\sigma}}$, if r is the risk free rate and the value x of the derivative satisfies the stochastic differential equation $dx/x = \tilde{\mu}(S, t)dt + \tilde{\sigma}(S, t)$. Now, consider the Black-Scholes setting and some general European-style derivative.

Show that the Sharpe ratio of this derivative is same as the Sharpe ratio for the underlying stock.
HINT: Use Itō lemma to compute dV and connect it to Black-Scholes PDE which is satisfied by the derivative price.

5. *Source: Society of Actuaries exams.* Consider a 3-month American call option on a nondividend-paying stock with strike price 41.5. You are given:

- The Black-Scholes framework holds.
- The stock is currently selling for 40.
- The stock's volatility is 30 % (i.e. 0.3)
- The current call option delta is 0.5.

Determine the current price of the option.
HINT: The underlying stock does not pay dividends, so the value of the American call option equals to the value of the European call option with the same parameters. Recall the formula for the delta of such an option and the graph of the CDF of $\mathcal{N}(0, 1)$ distribution.

6. Let $A, K > 0$ be given constants. Consider the Black-Scholes framework and a European option with payoff equal to $(\min(S-K, A))^+$, where $x^+ = \max(x, 0)$ and S is a stock price. Prove that delta of this option is positive and not greater than

$$\frac{\ln\left(1 + \frac{a}{K}\right)}{\sigma\sqrt{2\pi(T-t)}}.$$

7. Consider the Black-Scholes price of a call of option $V(S,t)$. Show that

$$\frac{V(S_1,t)}{V(S_2,t)} > \frac{S_1}{S_2}$$

for $S_1 > S_2$

8. Derivative of the vega of an option with respect to volatility is sometimes denoted as *Vomma*. For call and put option, the webpage `http://en.wikipedia.org/wiki/Greeks_(finance)#Vomma` states:

> Vomma is positive for options away from the money, and initially increases with distance from the money (but drops off as vega drops off). (Specifically, vomma is positive where the usual d_1 and d_2 terms are of the same sign, which is true when $d_2 < 0$ or $d_1 > 0$.)

   (a) Illustrate the first sentence numerically (clearly explain what our graphs show).

   (b) Prove the conditions on the positive sign for the parameter *Vomma*.

9. According to so called **Paley-Wiener representation of a Wiener process**, if $\{Z_n\}_{n=0}^{\infty}$ is a sequence of independent random variables with distribution $\mathcal{N}(0,1)$, then

$$W(t,\omega) = Z_0(\omega)\frac{t}{\sqrt{2\pi}} + \frac{2}{\sqrt{\pi}}\sum_{n=1}^{\infty} Z_n(\omega)\frac{\sin(nt/2)}{n}, \quad t \in [0, 2\pi]$$

is a Wiener process.

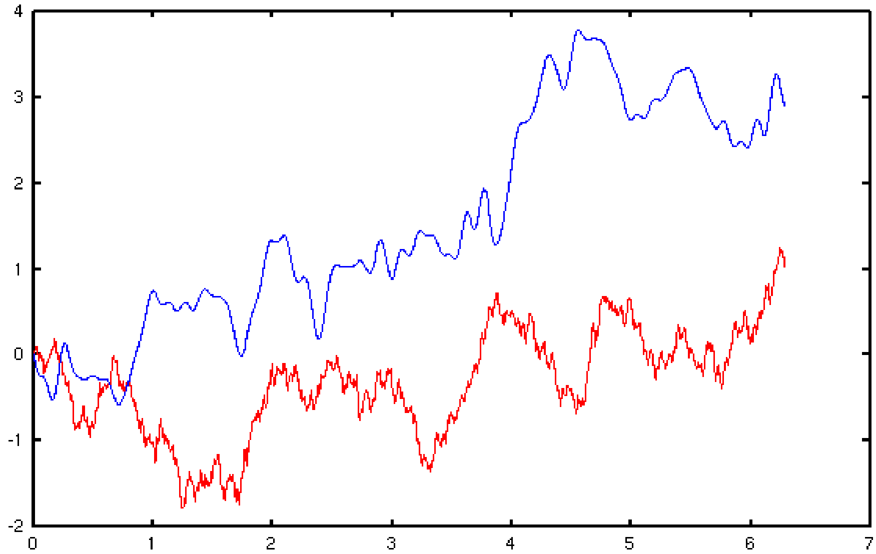
When performing simulations, we can simulate the values at times $t_j = \frac{2\pi j}{N}$ pre $j = 0, 1, \ldots, N$, if in the infinite sum we take the first M terms:

$$Z_0(\omega)\frac{t_j}{\sqrt{2\pi}} + \frac{2}{\sqrt{\pi}}\sum_{n=1}^{M} Z_n(\omega)\frac{\sin(nt_j/2)}{n},$$

where we need to choose M, N. Too small M causes the resulting trajectory to be "too smooth", see Figure 1.

   (a) Select sufficiently large M, N and plot some sample paths into a graph.

   (b) Simulate (for example) 100 or 1000 trajectories. Choose two non-overlapping time subintervals of $[0, 2\pi]$ and test (using a suitable statistical test) if the increments of the process on these subintervals are uncorrelated (which holds for an exact Wiener process).



Obr. 1: AApproximation of a Wiener process using the Paley-Wiener representation. Parmeter $N = 100$ is common to both trajectories, parameter M equals 100 for the blue line and 1000 for the red line.

10. Find the Black-Scholes price of an option, which has a payoff equal to the second power of a traditiona call option, i.e., $[(S-E)^+]^2$.

11. Find the Black-Scholes price of an option, which has a payoff equal to $\left(S + \frac{1}{S}\right)^2$.

12. Find the Black-Scholes price of an option, which has a payoff equal to $1 + S \ln S$.

13. Bachelier at the beginning of the 20th century modelled the stock price S as a Brownian motion, i.e.
$$dS = \mu dt + \sigma dw,$$
where $\mu \in \mathbb{R}, \sigma > 0$ are constatnts. Consider the Black-Scholes framework with this assumption about the stock price behaviour. Furthermore, assume the zero interest rate.

    (a) Derive the partial differential equation of the price of a derivaive.

    (b) Find the price of a call option (When solving the PDE, it is necessary to take into account that now the domain for the stock price is $S \in \mathbb{R}$.)

14. Consider the Black-Scholes model. Let $a, b > 0$ be constants, such that $a < b$ Consider an option which pays 1 USD if the stock price at expiration is in the interval (a, b) and otherwise pays zerp.

    (a) Derive the price of this option. Plot a graph, which has the stock price on the x-axis and option price for several times at y-axis, together with the payoff.

    (b) Derive the stock price (for general values of parameters a, b for which the option price is maximal.

15. Compute the correlation of the values of the process X at times t and s, if X is
    (a) a geometric Brownian motion b$X(t) = x_0 e^{\mu t + \sigma w(t)}$ for $t \geq 0$, where $\mu \in \mathbb{R}, \sigma > 0$ are given constants,
    (b) Brownian bridge $X(t) = w(t) - tw(1)$ for $t \in [0, 1]$.

    For selected values of parameters and time t plot a graph of correlation of $X(t)$ and $X(s)$ as a function of s (for $s \geq 0$ in the case of a geometric Brownian motion and $s \in [0, 1]$ in the case of a Brownian bridge.).

16. In this example we do not assume any particular model for stock prices, so we cannot use for example Black-Scholes prices. From the exercises we know that also in this general case the price of a put option has to be a convex function of the strike price. Prove that it needs to be a convex function also of the stock price.

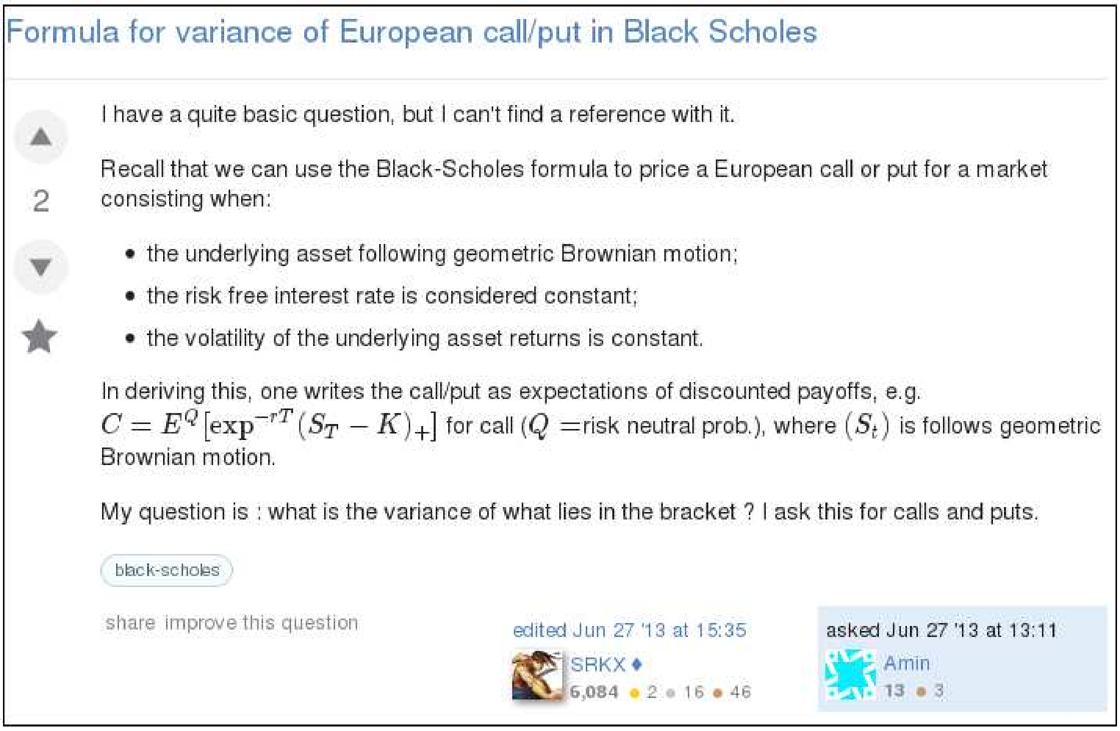
    Hint: Denote by $p(S, E)$ the price of a put option with strike E, if the stock option price is S. Since the option price cannot depend on units, in which we measure the price, we have $p(hS, hE) = hp(S, E)$ for $h > 0$. Write the stock prices S_1, S_2 from the definition of convexity in the form $S_1 = h_1 E$, $S_2 = h_2 E$ and use the convexity of p with respect to E.

17. (a) Using the "no arbitrage" principle prove the followig bounds on the price of an American put option price:
    $$c(S, E, \tau) - S + Ee^{-r\tau} \leq P(S, E, \tau) \leq c(S, E, \tau) - S + E,$$
    where $c(S, E, \tau)$ is the price of a European call option with strike E and expiration in τ years, when the current price of the stock is S and $P(S, E, \tau)$ is the price of an American put option with the same parameters.
    (b) Illustrate these bounds graphically on an example of an American put option with expiration in 3 months, if the current stock price is 100 USD and its volatility is 0.3, and the interest rate equals 0.5 percent. Consider strike prices from the interval $(90, 110)$. The x-axis of the graph should have these values of the strike and y-xis the bounds for the American put option price.

18. Ont the website `quant.stackexchange.com` we can find also a question shown in Figure 2.

    Let us make the question precise: The stock price follows a geometric Brownian motion
    $$dS = rSdt + \sigma Sdw$$
    (since we consider the risk-neutral measure, the drift is equal to the riskless interest rate). Define
    $$X_t = e^{-rT}(S_t - K)^+.$$
    The time today is $t = 0$, so we know S_0. Compute the variance of the process X at time T.

    Hint: On the page with the question, there is a hint as an answer: `http://quant.stackexchange.com/questions/8346/formula-for-variance-of-european-call-put-in-black-scholes?rq=1` *The expression denoted as S_T is X_T, which you will see after expressing the solutioin of the stochastic differential equation for the stock price. Write your solution for a general S_0.*


Formula for variance of European call/put in Black Scholes

2

I have a quite basic question, but I can't find a reference with it.

Recall that we can use the Black-Scholes formula to price a European call or put for a market consisting when:

- the underlying asset following geometric Brownian motion;
- the risk free interest rate is considered constant;
- the volatility of the underlying asset returns is constant.

In deriving this, one writes the call/put as expectations of discounted payoffs, e.g. $C = E^Q[\exp^{-rT}(S_T - K)_+]$ for call (Q =risk neutral prob.), where (S_t) is follows geometric Brownian motion.

My question is : what is the variance of what lies in the bracket ? I ask this for calls and puts.

black-scholes

share improve this question

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Obr. 2: Question.

19. Consider a cash-or-nothing option, which pays 1 USD at expiration time if the stock price is greater than the strike E, otherwise it pays nothing. Suppose that we want to compute implied volatility from the data.

    (a) What is the possible range of option prices, if we change the parameters σ and other parameters remain the same?

    (b) For a feasible option price (in the sense of the values that are possible to obtain, according to the previous point), is implied volatility necessarily unique?

    Give mathematical proofs (for general values of parameters) and add illustrative graphs for selected parameter values.

20. Suppose that European calls of all strike prices are available. Regarding the stock price S as fixed and the strike price E as variable, show that the price $V(E,t)$ satisfies the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 E^2 \frac{\partial^2 V}{\partial E^2} - rE\frac{\partial V}{\partial E} = 0.$$

21. In the paper

    Zhang, Jin E., Shoujun Huang, and Tiecheng Li. "The Intersection Between European Put Price And Its Payoff Function." International Journal of Theoretical and Applied Finance 16.04 (2013).

the authors study the stock price, for which the graph of the put option price intersects the payoff, see Figure 3. The stock can pay continuous dividends.

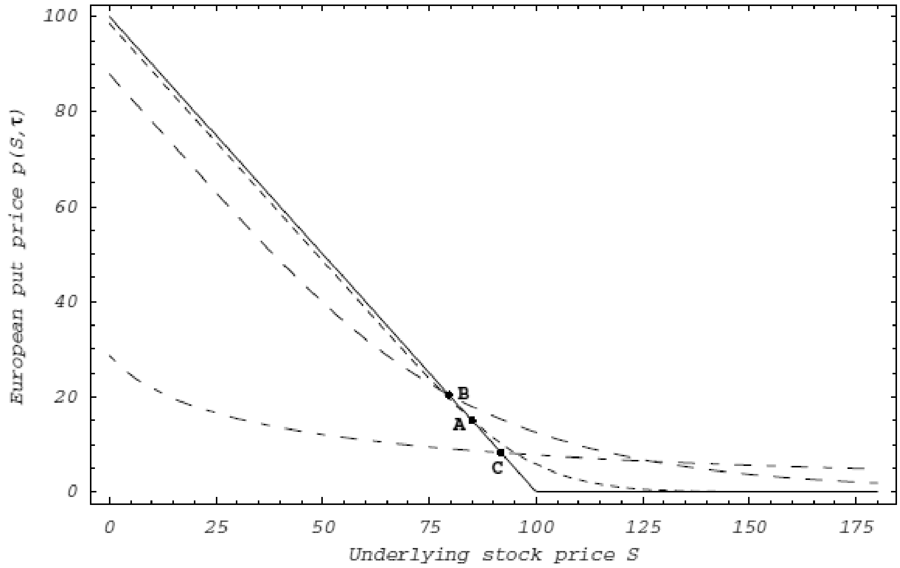

Fig. 1. The intersection between the price of a European put and its payoff function shown as A, B, and C for three different times to maturity. The solid line is the payoff function of the European option, $\max(K - S, 0)$, with $K = 100$. The short dashed line is the price of the European put, $p(S, \tau)$, at time to maturity, $\tau_A = 0.29$. The long dashed line is the option price at $\tau_B = 2.58$. The uneven dashed line is the option price at $\tau_C = 25$. The parameters are $r = 0.05$, $q = 0$ and $\sigma = 0.3$. The horizontal coordinates of the three intersection points are $S_A(\tau) = 85.00$, $S_B(\tau) = 79.55$, and $S_C(\tau) = 91.80$.

Obr. 3: Intersection of put option price and payoff.

They compute this intersection numerically (so that they can compare it with their approximations). A sample result is shown in Figure 4, with the price corresponding to the intersection (y axis) displayed as a dunction of time remaining to expiration (x axis), while other parameters of the option are constant.

Reconstruct Figure 4 using your own computation. Also make a table, show stock prices corresponding to times $1, 2, 3, \dots 10$ for each of the curves.

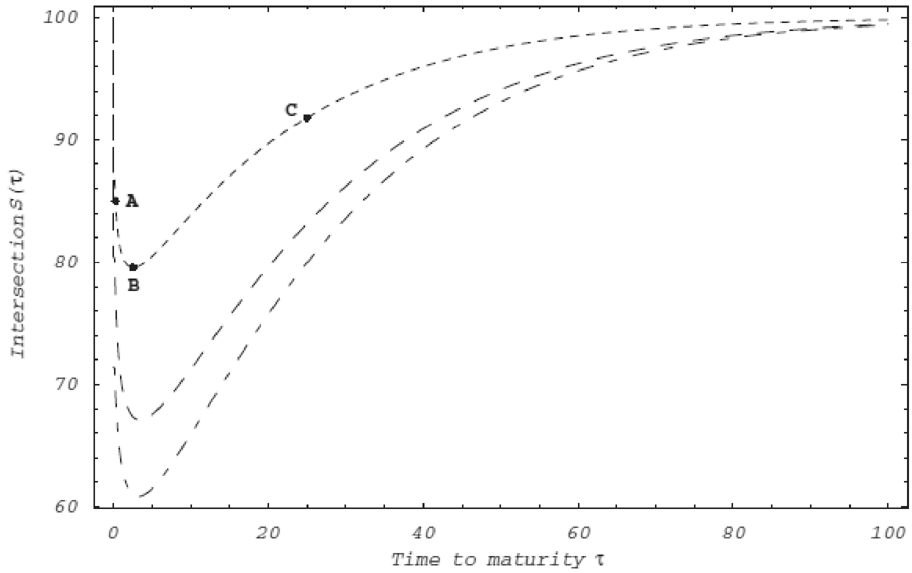

Fig. 2. The intersection, $S(\tau)$, between the price of a European put and its payoff as a function of time to maturity, τ, for $\sigma = 0.3$. The short dashed line starting from $S(0) = K = 100$ is for the case of $0.05 = r > q = 0$. The points $A(0.29, 85.00)$, $B(2.58, 79.55)$ and $C(25, 91.80)$ on the short dashed line correspond to the three intersections in Fig. 1, where $S(\tau)$ reaches the minimum at point B. The long dashed line starting from $S(0) = K = 100$ as well is for the case of $r = q = 0.05$. The uneven dashed line starting from $S(0) = \frac{r}{q}K = 71.4286$ is for the case of $0.05 = r < q = 0.07$.

Obr. 4: Intersection of the put option price and the payoff: dependence of the stock price corresponding to the intersectino on time remaining to expiration.