III. Black-Scholes model: Derivation and solution

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Financial derivatives

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Content

- Black-Scholes model:
  - Suppose that stock price S follows a geometric Brownian motion

$$dS = \mu S dt + \sigma S dw$$

    + other assumptions (in a moment)
  - We derive a partial differential equation for the price of a derivative
- Two ways of derivations:
  - due to Black and Scholes
  - due to Merton
- Explicit solution for European call and put options

Assumptions

- Further assumptions (besides GBP):
  - constant riskless interest rate r
  - no transaction costs
  - it is possible to buy/sell any (also fractional) number of stocks; similarly with the cash
  - no restrictions on *short selling*
  - option is of European type
- Firstly, let us consider the case of a non-dividend paying stock

Derivation I. - due to Black and Scholes

- Notation:
  $S =$ stock price, $t =$time
  $V = V(S,t) =$ option price
- Portfolio: 1 option, δ stocks
  $P =$ value of the portfolio: $P = V + \delta S$
- Change in the portfolio value: $dP = dV + \delta dS$
- From the assumptions: $dS = \mu S dt + \sigma S dw,$ From the Itō lemma: $dV = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \sigma S \frac{\partial V}{\partial S} dw$
- Therefore:

$$
\begin{aligned}
dP &= \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \delta \mu S\right) dt \\
&\quad + \left(\sigma S \frac{\partial V}{\partial S} + \delta \sigma S\right) dw
\end{aligned}
$$

Derivation I. - due to Black and Scholes

- We eliminate the randomness: $\delta = -\frac{\partial V}{\partial S}$
- Non-stochastic portfolio $\Rightarrow$ its value has to be the same as if being on a bank account with interest rate r: $dP = rPdt$
- Equality between the two expressions for dP and substituting $P = V + \delta S$:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Dividends in the Black-Scholes' derivation

- We consider continuous divident rate q - holding a stock with value S during the time differential dt brings dividends $qSdt$
- In this case the change in the portfolio value equals $dP = dV + \delta dS + \delta qSdt$
- We proceed in the same way as before and obtain

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S\frac{\partial V}{\partial S} - rV = 0$$

Derivation due to Merton - motivation

- Problem in the previous derivation:
  - we have a portfolio consisting of one option and δ stocks
  - we compute its value and change of its value:

$$\begin{aligned} P &= V + \delta S, \\ dP &= dV + \delta \, dS, \end{aligned}$$

    i.e., treating δ as a constant
  - however, we obtain $\delta = -\frac{\partial V}{\partial S}$

Derivation II. - due to Merton

- Portfolio consisting of options, stocks and cash with the properties:
  - in each time, the portfolio has zero value
  - it is self-financing
- Notation:
  $Q_S =$ number of stocks, each of them has value S
  $Q_V =$ number of options, each of them has value V
  $B =$ cash on the account, which is continuously compounded using the risk-free rate r

  $dQ_S =$ change in the number of stocks
  $dQ_V =$ change in the number of options
  $\delta B =$ change in the cash, caused by buying/selling stocks and options

Derivation II. - due to Merton

- Mathematical formulation of the required properties:
  - zero value $S\,Q_S + V\,Q_V + B = 0$ (1)
  - self-financing: $S\,dQ_S + V\,dQ_V + \delta B = 0$ (2)
- Change in the cash: $dB = r B\,dt + \delta B$
- Differentiating (1):

$$
\begin{aligned}
0 &= d\,(SQ_S + VQ_V + B) = d\,(SQ_S + VQ_V) + \overbrace{dB}^{rB\,dt+\delta B} \\
0 &= \overbrace{SdQ_S + VdQ_V + \delta B}^{=0} + Q_S dS + Q_V dV + rB\,dt \\
0 &= Q_S dS \;+\; Q_V dV \;\overbrace{-\,r(SQ_S + VQ_V)}^{rB}\,dt.
\end{aligned}
$$

Derivation II. - due to Merton

- We divide by Q_V and denote $\Delta = -\frac{Q_S}{Q_V}$:
  $dV - rV\,dt - \Delta(dS - rS\,dt) = 0$
- We have dS from the assumption of GBM and dV from the Itō lemma
- We choose Δ (i.e., the ratio between the number of stocks and options) so that it eliminates the randomness (the coefficient at dw will be zero)
- We obtain the same PDE as before:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Dividends in the Merton's derivation

- Assume continuous dividend rate q.
- Dividents cause an increase in the cash $\Rightarrow$ change in the cash is $dB = rB\,dt + \delta B + qSQ_S dt$
- In the same way we obtain the PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0$$

Black-Scholes PDE: summary

- Matematical formulation of the model:
  Find solution $V(S,t)$ to the partial differential equation (so called Black-Scholes PDE)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

  which holds for $S > 0, t \in [0, T)$.
- So far we have not used the fact that we consider an option $\Rightarrow$ PDE holds for any derivative that pays a payoff at time T depending on the stock price at this time
- Type of the derivative determines the terminal condition at time T
- In general: $V(S,T) =$ payoff of the derivative

Black-Scholes PDE: simple solutions

SOME SIMPLE "DERIVATIVES":

- How to price the derivatives with the following payoffs:
  - $V(S,T) = S$ $\rightarrow$ it is in fact a stock $\rightarrow V(S,t) = S$
  - $V(S,T) = E$ $\rightarrow$ with a certainity we obtain the cash E $\rightarrow V(S,t) = Ee^{-r(T-t)}$

  - by substitution into the PDE we see that they are indeed solutions

EXERCISES:

- Find the price of a derivative with payoff $V(S,T) = S^n$, where $n \in \mathbb{N}$.
  HINT: Look for the solution in the form $V(S,t) = A(t)S^n$
- Find all solutions to the Black-Scholes PDE, which are independent of time, i.e., for which $V(S,t) = V(S)$

Black-Scholes PDE: binary option

- Let us consider a binary option, which pays 1 USD if the stock price is higher that E at expiration time, otherwise its payoff is zero
- In this case

$$V(S,T) = \begin{cases} 1 & \text{if } S > E \\ 0 & \text{otherwise} \end{cases}$$

- The main idea is to transform the Black-Scholes PDE to a heat equation
- Transformations are independent of the derivative type; it affects only the initial condition of the heat equation

Black-Scholes PDE: transformations

Formulation of the problem

- Partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

  which holds for $S > 0, t \in [0, T)$.

- Terminal condition $V(S, T) =$ payoff of the derivative for $S > 0$

Black-Scholes PDE: transformations

STEP 1:

- Transformation $x = \ln(S/E) \in \mathbb{R}$, $\tau = T - t \in [0, T]$ and a new function $Z(x, \tau) = V(Ee^x, T - \tau)$
- PDE for $Z(x, \tau)$, $x \in \mathbb{R}, \tau \in [0, T]$:

$$\frac{\partial Z}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 Z}{\partial x^2} + \left(\frac{\sigma^2}{2} - r\right)\frac{\partial Z}{\partial x} + rZ = 0,$$

$$Z(x, 0) = V(Ee^x, T)$$

STEP 2:

- Transformation to heat equation
- New function $u(x, \tau) = e^{\alpha x + \beta \tau} Z(x, \tau)$, where the constants $\alpha, \beta \in \mathbb{R}$ are chosen so that the PDE for u is the heat equation

Black-Scholes PDE: transformations

- PDE for u:

$$\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + A\frac{\partial u}{\partial x} + Bu = 0\,,$$

$$u(x,0) = e^{\alpha x} Z(x,0) = e^{\alpha x} V(Ee^x, T),$$

where

$$A = \alpha\sigma^2 + \frac{\sigma^2}{2} - r,\; B = (1+\alpha)r - \beta - \frac{\alpha^2\sigma^2 + \alpha\sigma^2}{2}\,.$$

- In order to have $A = B = 0$, we set

$$\alpha = \frac{r}{\sigma^2} - \frac{1}{2},\; \beta = \frac{r}{2} + \frac{\sigma^2}{8} + \frac{r^2}{2\sigma^2}$$

Black-Scholes PDE: transformations

STEP 3:

- Solution $u(x,\tau)$ of the PDE $\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} = 0$ is given by Green formula

$$u(x,\tau) = \frac{1}{\sqrt{2\sigma^2\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{2\sigma^2\tau}} u(s,0)\, ds\,.$$

- We evaluate the integral and perform backward substitutions $u(x,\tau) \to Z(x,\tau) \to V(S,t)$

Black-Scholes PDE: binary option (continued)

- Transformations from the previous slides
- We obtain the heat equation $\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} = 0$ with initial condition

$$u(x,0) = e^{\alpha x} V(Ee^x, T) = \begin{cases} e^{\alpha x} & \text{if } Ee^x > E \\ 0 & \text{otherwise} \end{cases} = \begin{cases} e^{\alpha x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

- Solution $u(x,\tau)$:

$$u(x,\tau) = \frac{1}{\sqrt{2\pi\sigma^2\tau}} \int_0^\infty e^{-\frac{(x-s)^2}{2\sigma^2\tau}} e^{\alpha s} ds = \ldots = e^{\alpha x + \frac{1}{2}\sigma^2\tau\alpha^2} N\left(\frac{x+\sigma^2\tau\alpha}{\sigma\sqrt{\tau}}\right)$$

where $N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{\xi^2}{2}} d\xi$ is the cumulative distribution function of a normalized normal distribution

Black-Scholes PDE: binary option (continued)

- Option price $V(S,t)$:

$$V(S,t) = e^{-r(T-t)}N(d_2),$$

where $d_2 = \frac{\log\left(\frac{S}{E}\right)+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$

Black-Scholes PDE: call option

- In this case

$$V(S,T) = \max(0, S-E) = \begin{cases} S-E & \text{if } S > E \\ 0 & \text{otherwise} \end{cases}$$

- The same sequence of transformations; inital condition for the heat equation:

$$u(x,0) = \begin{cases} e^{\alpha x}(S-E) & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

  and similar evaluation of the integral

- Option price:

$$V(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2),$$

  where N is the distribution function of a normalized normal distribution and $d_1 = \frac{\ln\frac{S}{E} + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$

Black-Scholes PDE: call option

HOMEWORK:
Solve the Black-Scholes PDE for a call option on a stock which pays continuous dividends and write it in the form

$$V(S,t) = Se^{-q(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2),$$

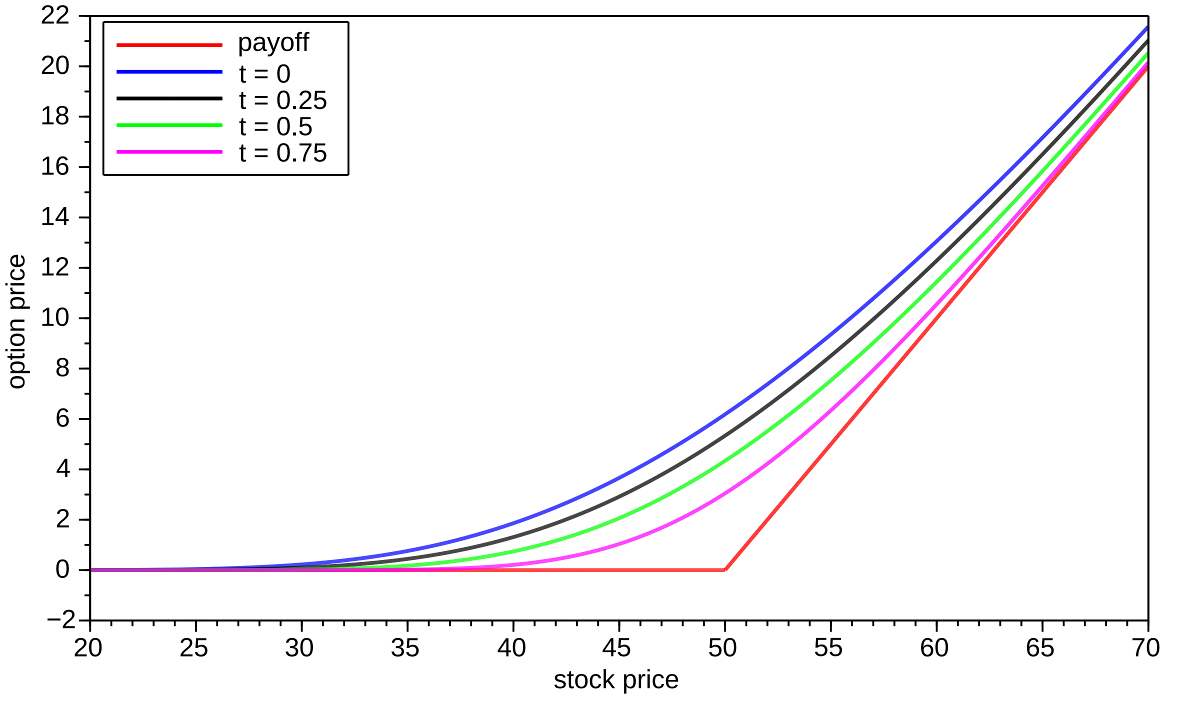
where $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{\xi^2}{2}}\, d\xi$ is the distribution function of a normalized normal distribution $N(0,1)$ and

$$d_1 = \frac{\ln \frac{S}{E} + (r - q + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},\ d_2 = d_1 - \sigma\sqrt{T-t}$$

NOTE: The PDE is different, so the transformations have to be adjusted (do the same steps for the new equation)

Black-Scholes PDE: call option

Payoff (i.e., terminal condition at time $t = T = 1$) and solution $V(S, t)$ for selected times t:



Black-Scholes PDE: put option

Formulation of the problem

- Partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

  which holds for $S > 0, t \in [0, T]$.

- Terminal condition:

$$V(S, T) = \max(0, E - S)$$

  for $S > 0$

Black-Scholes PDE: put option

APPROACH I.

- The same sequence of computations as in the case of a call option

APPROACH II.

- We use the linearity of the Black- Scholes PDE and the solution for a call which we have already found

We show the application of the latter approach.

Black-Scholes PDE: putoption

- Recall that for the payoffs of a call and a put we have

$$-[call\,payoff] + [put\,payoff] + [stock\ price] = E$$

- Hence:

$$[put\ payoff] = [call\ payoff] - S + E$$

- Black-Scholes PDE is linear: a linear combination of solutions is again a solution

Black-Scholes PDE: put option

- Recall the solutions for $V(S,T) = S$ and $V(S,T) = E$ (page 13):

terminal condition	solution
$\max(0, S - E)$	$V^{call}(S,t)$
S	S
E	$Ee^{-r(T-t)}$

- From the linearity:

terminal condition	solution
$\max(0, S - E) - S + E$	$V^{call}(S,t) - S + Ee^{-r(T-t)}$

- Since $[put\ payoff] = \max(0, S - E) - S + E$, we get

$$V^{put}(S,t) = V^{call}(S,t) - S + Ee^{-r(T-t)}$$

Solution for a put option

- The solution

$$V^{put}(S,t) = V^{call}(S,t) - S + Ee^{-r(T-t)}$$

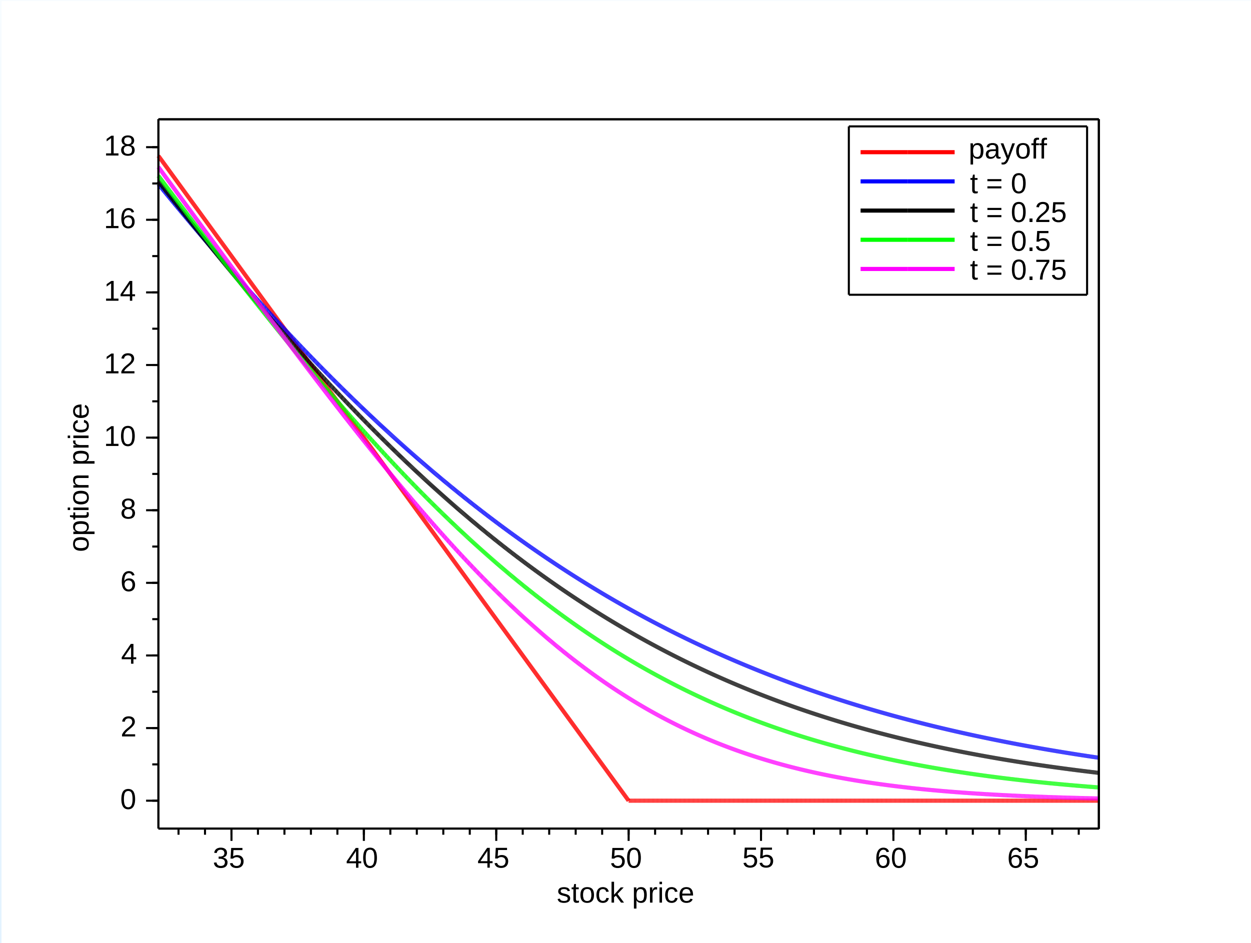
can be written in a similar form as the solution for a call option:

$$V^{ep}(S,t) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1),$$

where N, d_1, d_2 are the same as before

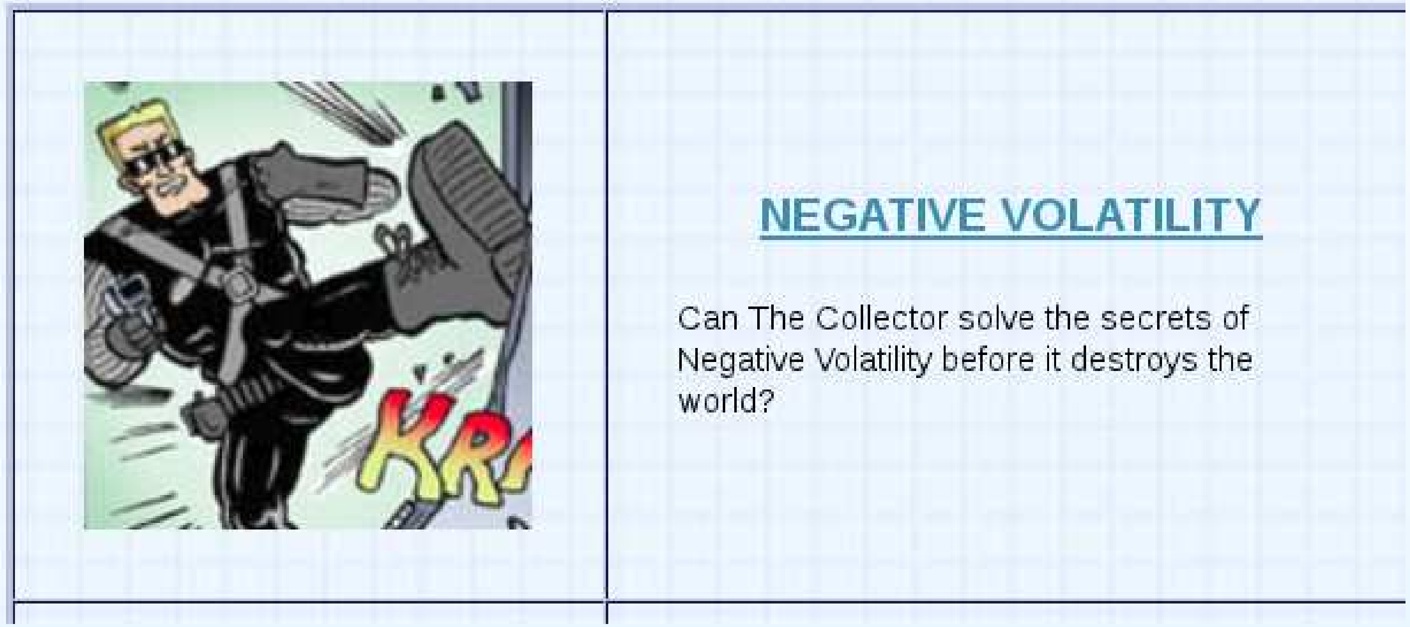
Put option - example

Payoff (i.e. terminal condition at time $t = T = 1$) and solution $V(S,t)$ for selected times t:



Put option - alternative computation

Comics about negative volatility on the webpage of Espen Haug:



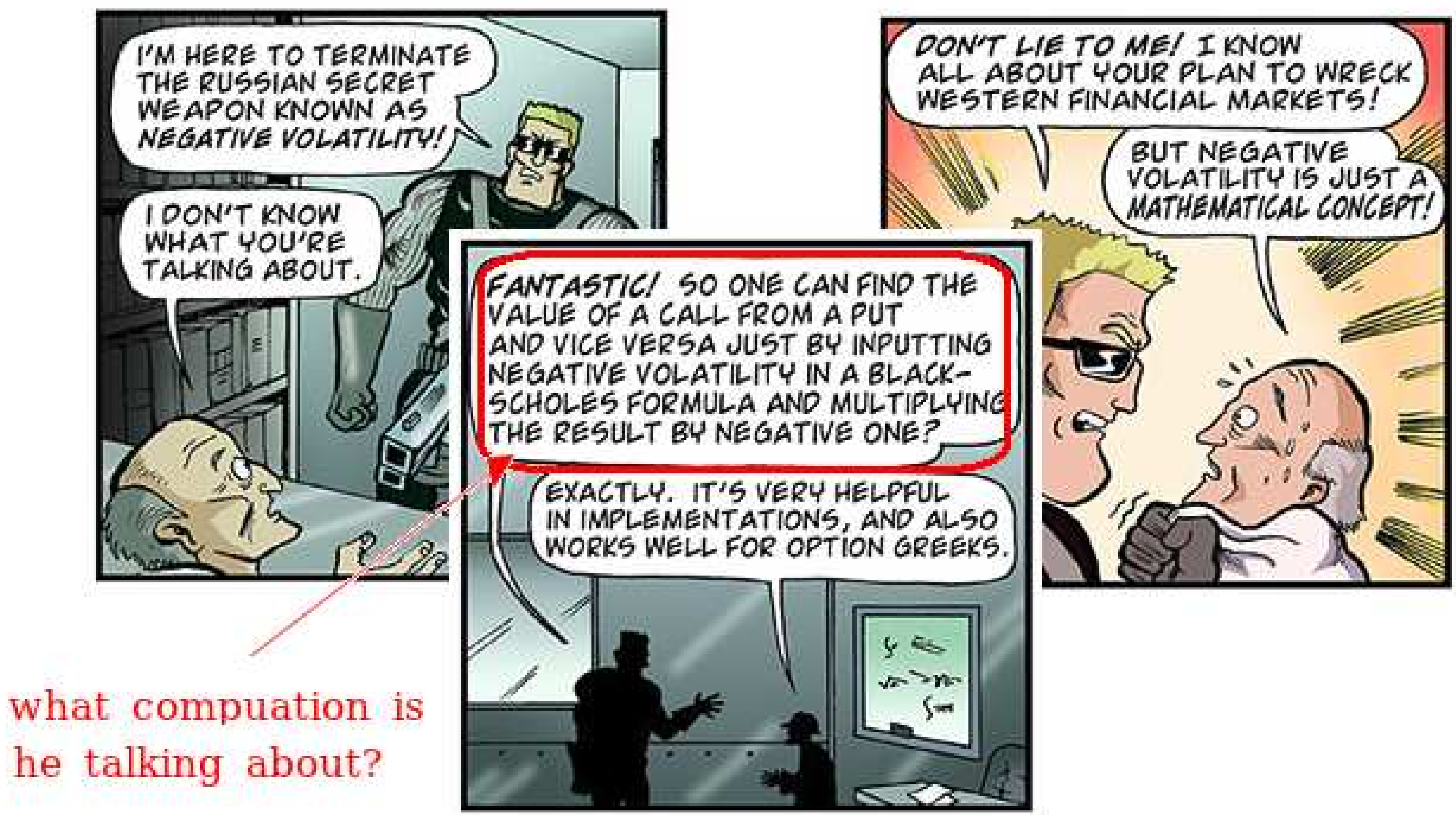
http://www.espenhaug.com/collector/collector.html

Put option - alternative computation

- A nightmare about negative volatility:

- Not only a dream... according to internet, it really exists and is connected with professor Shiryaev from Moscow...

Put option - alternative computation



QUESTION: Why does this computation work?

Stocks paying dividends

- HOMEWORK:
  Solve the Black-Scholes equation for a put option, if the underlying stock pays continuous dividends.

  HINT:
  - In this case, $V(S,t) = S$ is not a solution
  - What is the solution satisfying the terminal condition $V(S,T) = S$? Use financial interpretation and check your answer by substituting it into the PDE

- HOMEWORK:
  Denote $V(S,t;E,r,q)$ the price of an option with exercise price E, if the interest rate is r and the dividend rate is q. Show that

$$V^{put}(S,t;E,r,q) = V^{call}(E,t;S,q,r)$$

  HINT: How do the terms $d_1 d_2$ change when replacing $S \leftrightarrow E$, $r \leftrightarrow q$?

Combined strategies

- From the linearity of the Black-Scholes PDE: if the strategy is a linear combination of call and put options, then its price is the same linear combination of the call and put options prices
- It does not necessarily hold in other models:
  - consider a model with some transaction costs; it is not equivalent
    - whether we hedge the options independenty
    - or we hedge the portfolio - in this case, we might be able to reduce transaction costs

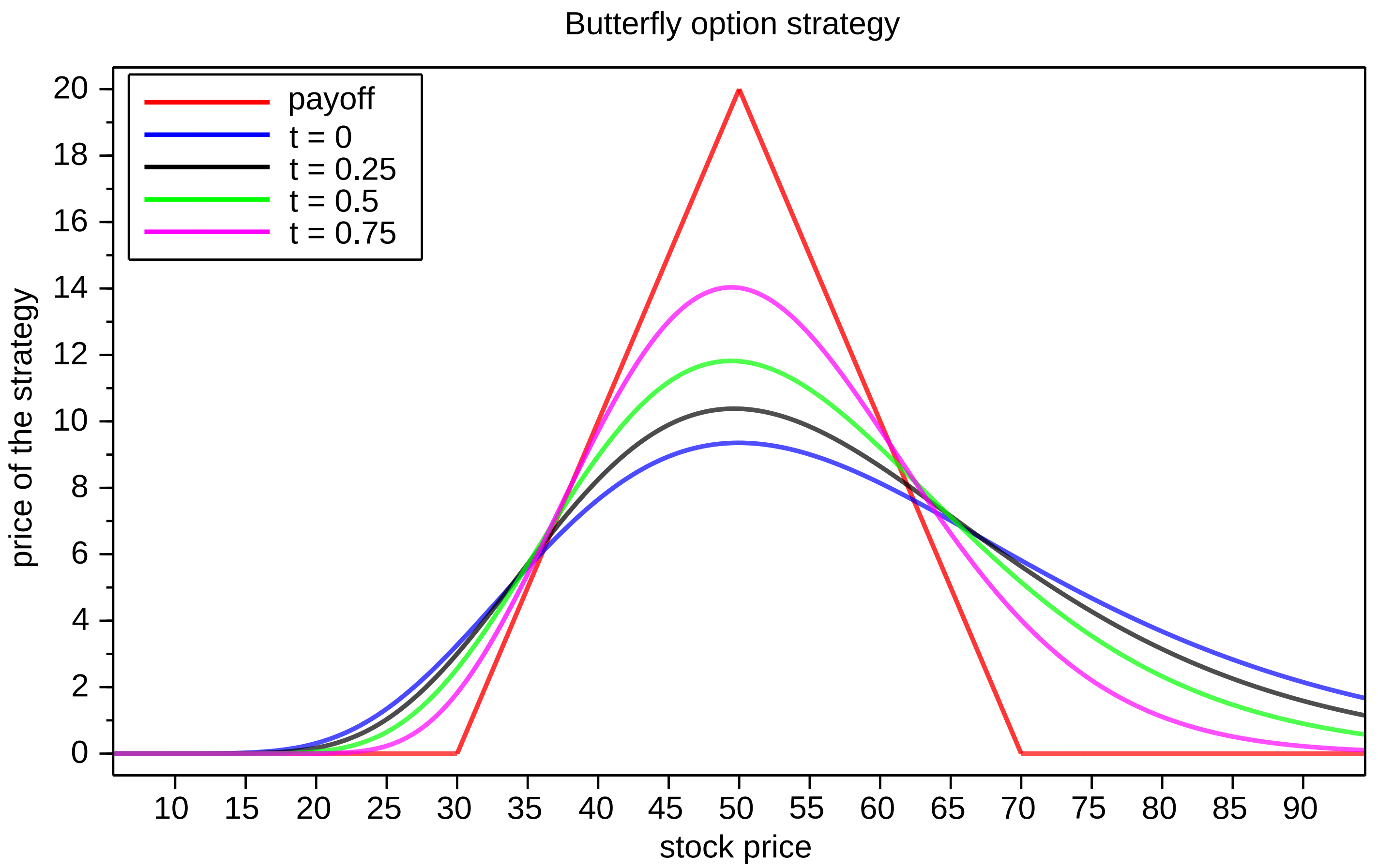
Combined strategies

EXAMPLE:

- we buy call options with exerise prices E_1, E_3 and sell two call options with exercise prices E_2, with exercise prices satisfying $E_1 < E_2 < E_3$ and $E_1 + E_3 = 2E_2$.
- Payoff of the strategy can be written as
$V(S,T) = \max(S - E_1, 0) - 2\max(S - E_2, 0) + \max(S - E_3, 0)$
- Hence its Black-Scholes price is:
$V(S,t) = V^{call}(S,t;E_1) - 2V^{call}(S,t;E_2) + V^{call}(S,t;E_3)$

Combined strategies

- Numerical example - butterfly with $T = 1$:



IV. Black-Scholes model: Implied volatility

Market data

- Stock:

General Motors Company (GM) - NYSE ★ Follow

36.51 **0.00 (0.00%)** 9:32AM EST - Nasdaq Real Time Price

Prev Close:	36.51	Day's Range:	36.51 - 36.68
Open:	N/A	52wk Range:	26.19 - 41.85
Bid:	36.60 x 500	Volume:	67,337
Ask:	36.65 x 200	Avg Vol (3m):	26,433,200
1y Target Est:	46.44	Market Cap:	58.04B
Beta:	1.76	P/E (ttm):	15.35
Earnings Date:	Apr 28 - May 2 (Est.)	EPS (ttm):	2.38
		Div & Yield:	1.20 (3.30%)

Market data

- Selected options:

36.50	GM140314C00036500	0.92	↑0.27
36.50	GM140328C00036500	0.99	↑0.01
37.00	GM140307C00037000	0.51	↓0.03
37.00	GM140314C00037000	0.67	↑0.17
37.00	GM140322C00037000	0.73	↑0.15
37.00	GM140328C00037000	0.77	↑0.09

- How much are these options supposed to cost according to Black-Scholes model?

Black-Scholes model and market data

- Recall Black-Scholes formula for a call option:

$$V(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2),$$

where $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{\xi^2}{2}}\, d\xi$ is the distribution function of a normalized normal distribution $N(0,1)$ and

$$d_1 = \frac{\ln\frac{S}{E} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},\ d_2 = d_1 - \sigma\sqrt{T-t}$$

Black-Scholes model and market data

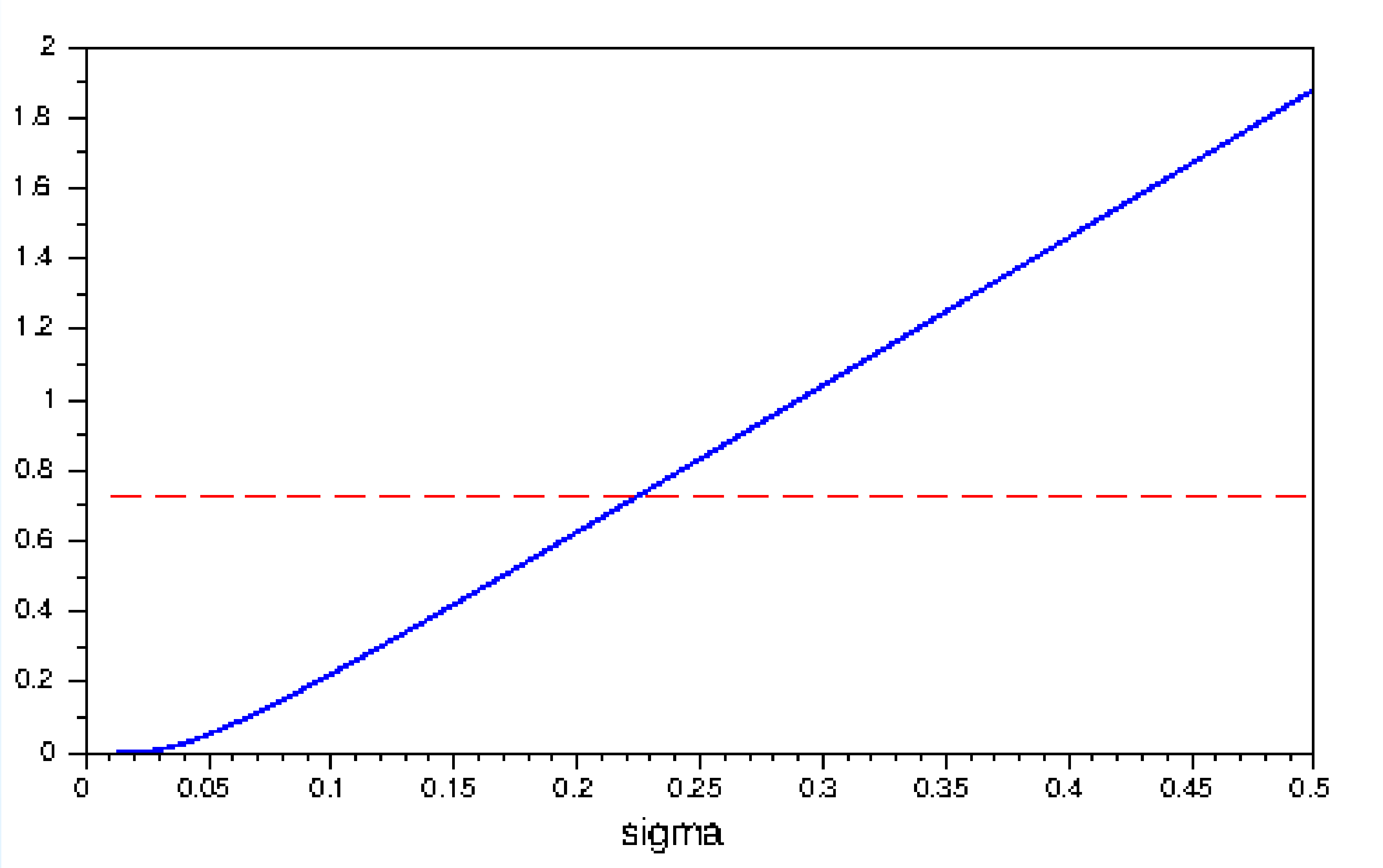
- Therefore, we need the following values:
  - S = stock price
  - E = exercise price
  - $T - t$ = time remaining to expiration
  - σ = volatility of the stock
  - r = interest rate
- What is clear: $S, E, T - t$
- Interest rate (there are different rates on the market):
  - A common choice: 3-months treasury bills
  - Interest rate has to be expressed as a decimal number $\rightarrow$ 0.03 percent is $r = 0.03/100$

Black-Scholes model and market data

- What is the volatility?
  - Exercises session: computation of the Black-Scholes price using historical volatility
  - Different estimates of volatility, depending on time span of the data
  - Price does not equal the market price
- Question: What value of volatility produces the Black-Scholes price that is equal to the market price?
- This value of volatility is called implied volatility

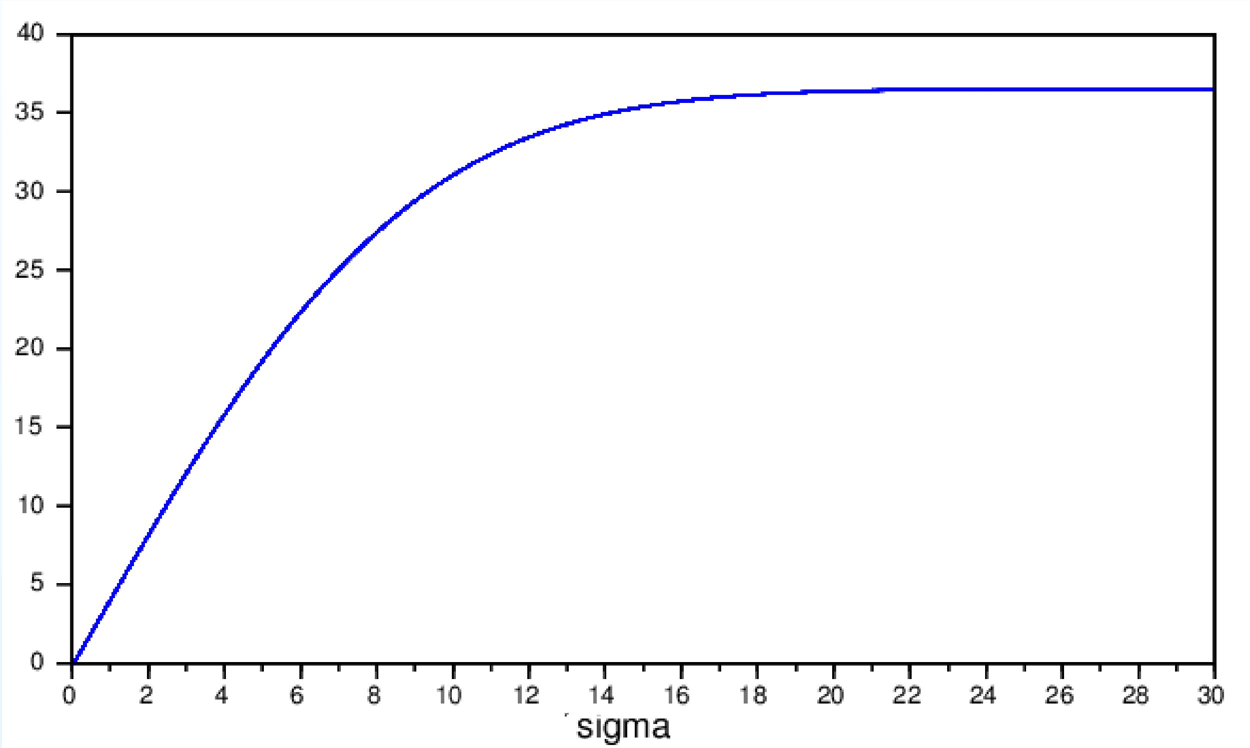
Implied volatility

- Dependence of the Black-Scholes option price on volatility:



Existence of implied volatility

- Dependence of the Black-Scholes option price on volatility - for a wider range of volatility:



Existence of implied volatility

- In general - we show that
  - The Black-Scholes price of a call option is an increasing function of volatility
  - Limits are equal to: $V_0 := \lim_{\sigma \to 0^+} V(S, t; \sigma)$, $V_\infty := \lim_{\sigma \to \infty} V(S, t; \sigma)$
- Then, from continuity of $V \Rightarrow$ for every price from the interval (V_0, V_∞) the implied volatility exists and is uniquely determined

- We do the derivation of a stock which does not pay dividends
- HOMEWORK: call and put option on a stock which pays constinuous dividends

Existence of implied volatility

- To prove that price is an increasing function of volatility:
  - We compute the derivative (using $d_2 = d_1 - \sigma\sqrt{T-t}$):

$$\begin{aligned} \frac{\partial V}{\partial \sigma} &= SN'(d_1)\frac{\partial d_1}{\partial \sigma} - Ee^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial \sigma} \\ &= \left(SN'(d_1) - Ee^{-r(T-t)}N'(d_2)\right)\frac{\partial d_1}{\partial \sigma} \\ &\quad + Ee^{-r(T-t)}N'(d_2)\sqrt{T-t} \end{aligned}$$

  - Derivative of a distribution function is a density function: $N'(x) = \frac{1}{2\pi}e^{-\frac{x^2}{2}}$
  - Useful lemma: $SN'(d_1) - Ee^{-r(T-t)}N'(d_2) = 0$
  - Hence:

$$\frac{\partial V}{\partial \sigma} = Ee^{-r(T-t)}N'(d_2)\sqrt{T-t} > 0$$

Existence of implied volatility

- Limits:
  - We use basic properties of a distribution function:

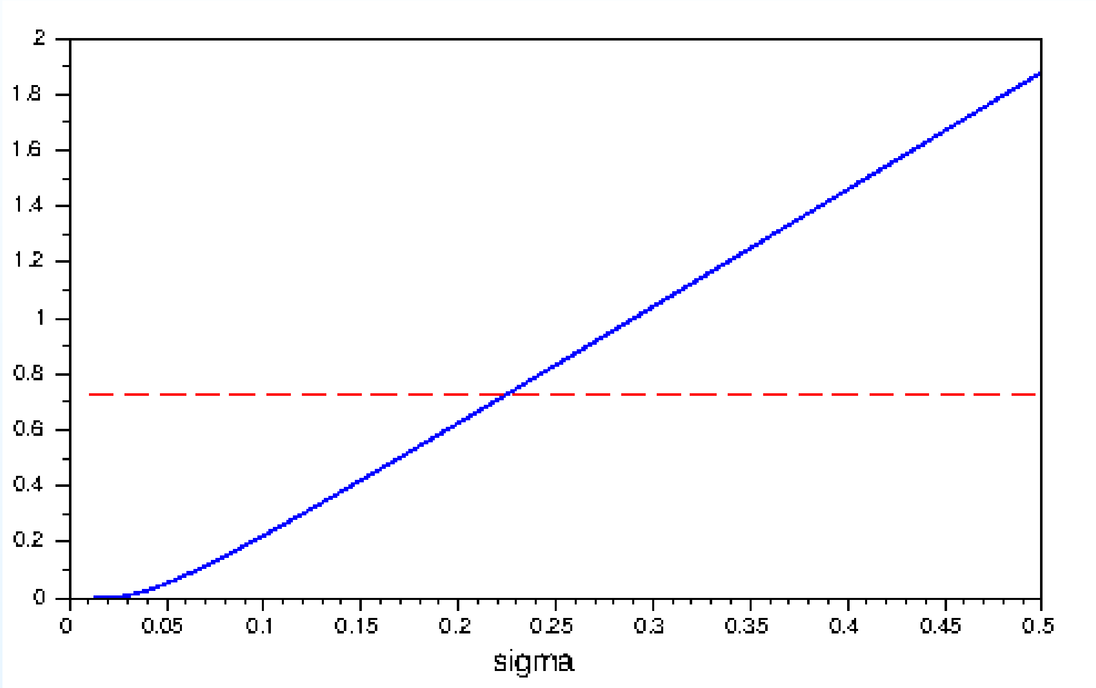
$$\lim_{x\to-\infty} N(x) = 0, \quad \lim_{x\to+\infty} N(x) = 1$$

  - It follows:

$$\begin{aligned} \lim_{\sigma\to 0^+} V(S,t;\sigma) &= \max(0, S - Ee^{-r(T-t)}) \\ \lim_{\sigma\to\infty} V(S,t;\sigma) &= S \end{aligned}$$

Implied volatility - computation

- In our case:



- We get the implied volatility 0.22558

Website finance.yahoo.com

- Option chains include implied volatilities:

Calls

Strike	Contract Name	Last	. . .	Implied Volatility
39.00	YHOO141212C00039000	7.95		53.47%
40.00	YHOO141212C00040000	6.30		44.53%
40.50	YHOO141212C00040500	5.95		47.90%
43.50	YHOO141212C00043500	3.80		35.65%
44.00	YHOO141212C00044000	3.90		36.67%
44.50	YHOO141212C00044500	3.16		34.96%
46.00	YHOO141212C00046000	2.27		32.84%
47.00	YHOO141212C00047000	2.05		31.42%

V. Black-Scholes model: Greeks - sensitivity analysis

Greeks

- Greeks:
  - derivatives of the option price with respect to parameters
  - they measure the sensitivity of the option price to these parameters
- We have already computed $\frac{\partial V_{call}}{\partial \sigma} = Ee^{-r(T-t)}N'(d_2)\sqrt{T-t}$, it is denoted by Υ (vega)
- Others: (Remark: P is a Greek letter rho)

$$\Delta = \frac{\partial V}{\partial S}, \ \Gamma = \frac{\partial^2 V}{\partial S^2}, \ P = \frac{\partial V}{\partial r}, \ \Theta = \frac{\partial V}{\partial t}$$

- Notation: V^{ec} = price of a European call, V^{ep} = price of a European put; in the same way their American counterparts V^{ac}, V^{ap}

Delta

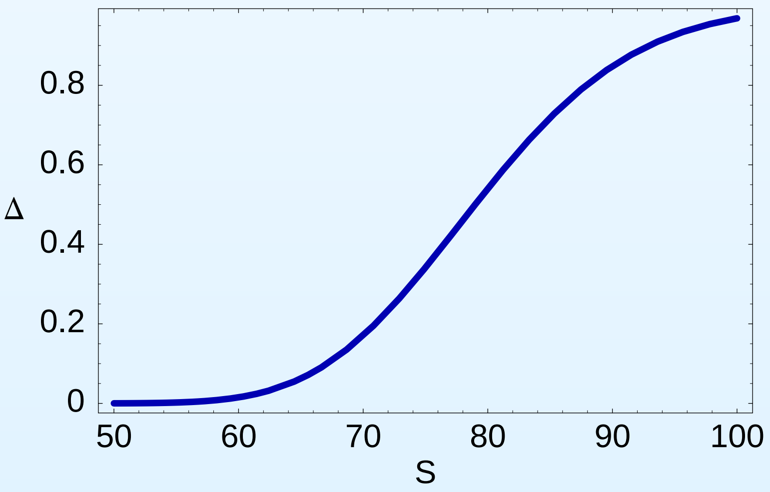
- Call option - from Black-Scholes formula, we use the same lemma as in the case of volatility:

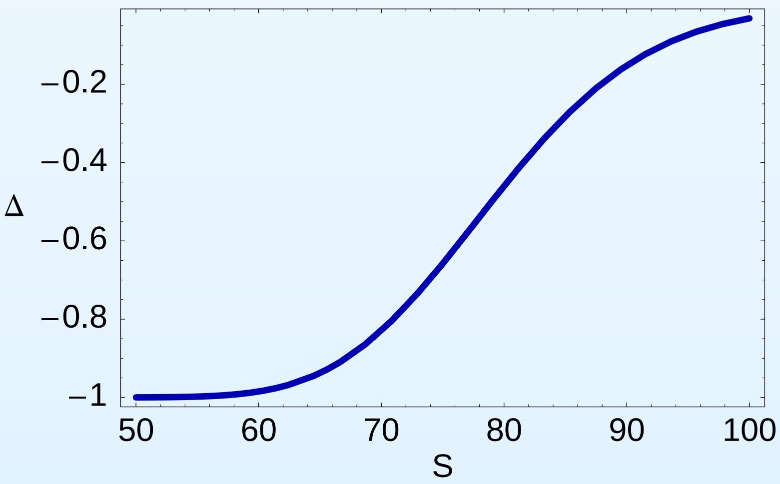
$$\Delta^{ec} = \frac{\partial V^{ec}}{\partial S} = N(d_1) \in (0,1)$$

- Put option - we do not need to compute the derivative, we can use the put-call parity:

$$\Delta^{ep} = \frac{\partial V^{ep}}{\partial S} = -N(-d_1) \in (-1,0)$$

- Example: call(left), put (right)





Delta - delta hedging

- Recall the derivation of the Black-Scholes model and contruction of a riskless portfolio:

$$\frac{Q_S}{Q_V} = -\frac{\partial V}{\partial S} = -\Delta$$

where Q_V, Q_S are the numbers of options and stock in the portfolio

- Construction of such a portfolio is call delta hedging (hedge = protection, transaction that reduces risk)

Delta - example of delta hedging

- Real data example - call option on IBM stock, 21st May 2002, 5-minute ticks
- At time t:
  - we have option price $V_{real}(t)$ and stock price $S_{real}(t)$
  - we compute the impled volatility, i.e., we solve the equation

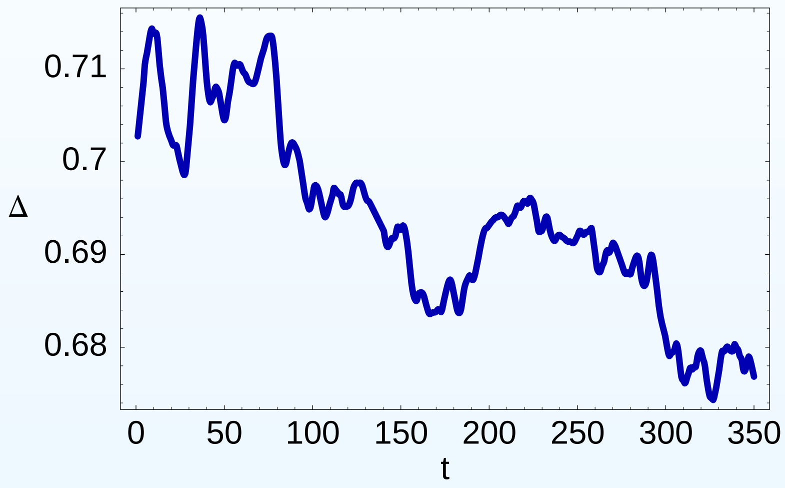
$$V_{real}(t) = V^{ec}(S_{real}(t), t; \sigma_{impl}(t)).$$

  - implied volatility $\sigma_{impl}(t)$ is used in the call option price formula:

$$\Delta^{ec}(t) = \frac{\partial V^{ec}}{\partial S}(S_{real}(t), t; \sigma_{impl}(t))$$

Delta - example of delta hedging

- Delta during the day:



- We wrote one option - then, this is the number of stocks in our portfolio

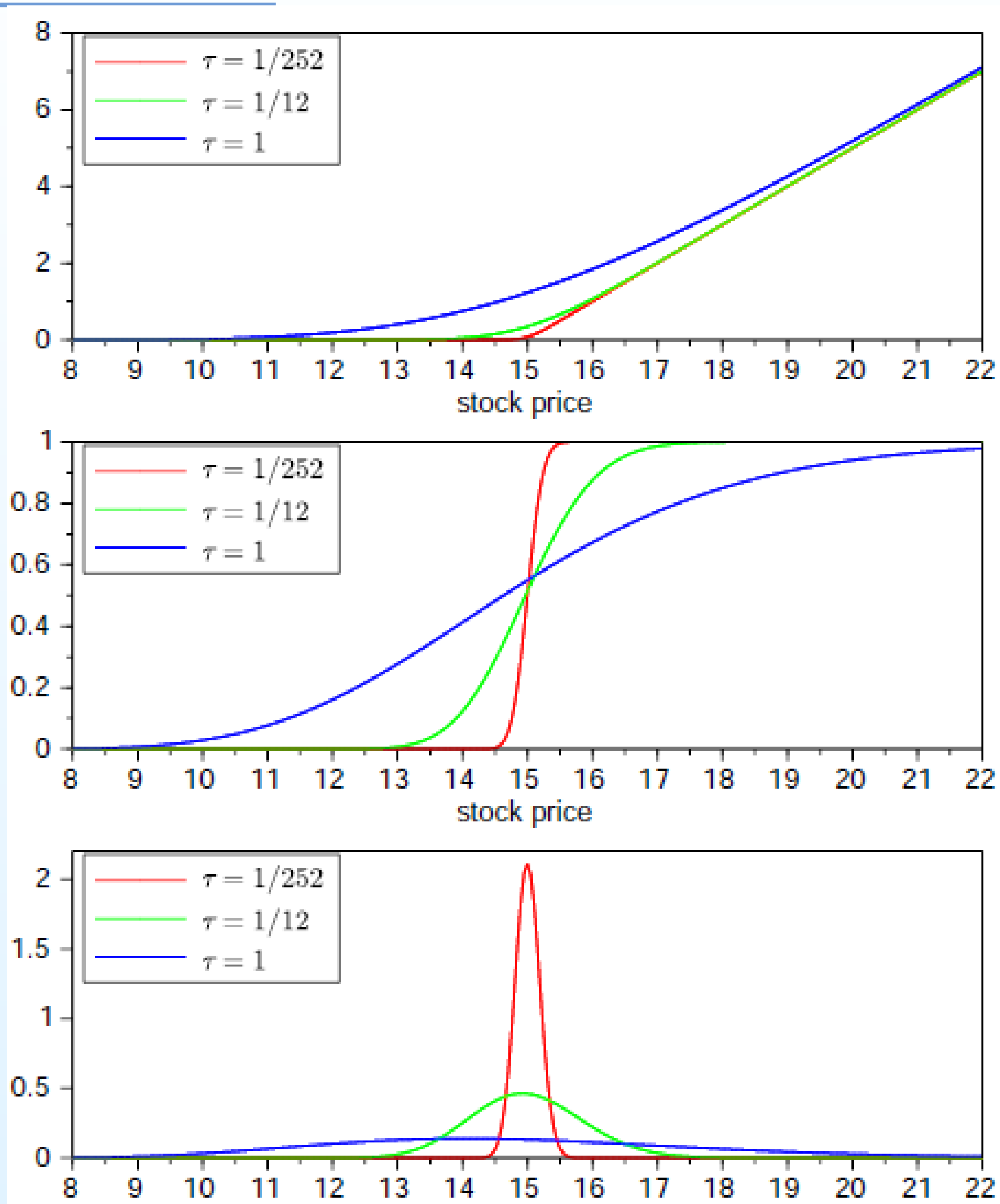
Gamma

- Computation:

$$
\begin{aligned}
\Gamma^{ec} &= \frac{\partial \Delta^{ec}}{\partial S} = N'(d_1)\frac{\partial d_1}{\partial S} = \frac{\exp(-\frac{1}{2}d_1^2)}{\sigma\sqrt{2\pi(T-t)}S} > 0 \\
\Gamma^{ep} &= \Gamma^{ec}
\end{aligned}
$$

- Measures a sensitivity of delta to a change in stock price

Price, delta, gamma



Price, delta, gamma

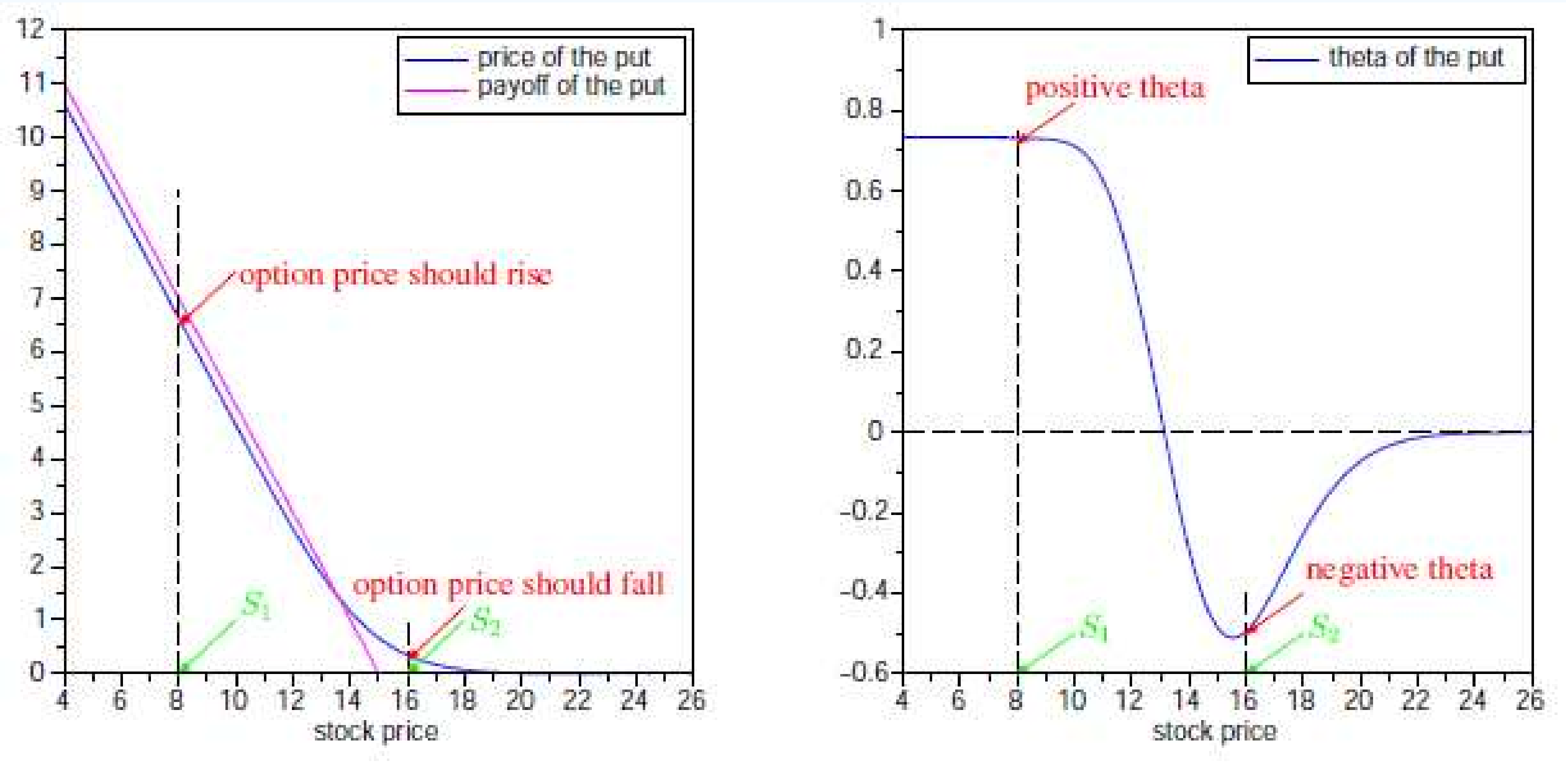
- Simultaneously:
  - the option price is "almost a straight line"
  - delta does not change much with a small change in the stock price
  - gamma is almost zero
- Also:
  - graph of the option price has a big curvature
  - delta significantly changes with a small change in the stock price
  - gamma is significantly nonzero

Vega, rho, theta

- Vega
  - we have already computed:
    $\Upsilon^{ec} = \frac{\partial V^{ec}}{\partial \sigma} = Ee^{-r(T-t)} N'(d_2)\sqrt{T-t} > 0$
  - from put-call parity: $\Upsilon^{ep} = \Upsilon^{ec}$
  - higher volatility $\Rightarrow$ higher probability of high profit, while a possible loss is bounded $\Rightarrow$ positive vega
- Rho
  - call: $P^{ec} = \frac{\partial V^{ec}}{\partial r} = E(T-t)e^{-r(T-t)} N(d_2) > 0$
  - put: $P^{ep} = \frac{\partial V^{ep}}{\partial r} = -E(T-t)e^{-r(T-t)} N(-d_2) < 0$
- Theta:
  - call: from financial mathematics we know that if a stock does not pay dividends, it is not optimal to exercise an American option prior to its expiry $\Rightarrow$ prices of European and American options are equal $\Rightarrow$ $\Theta^{ec} < 0$

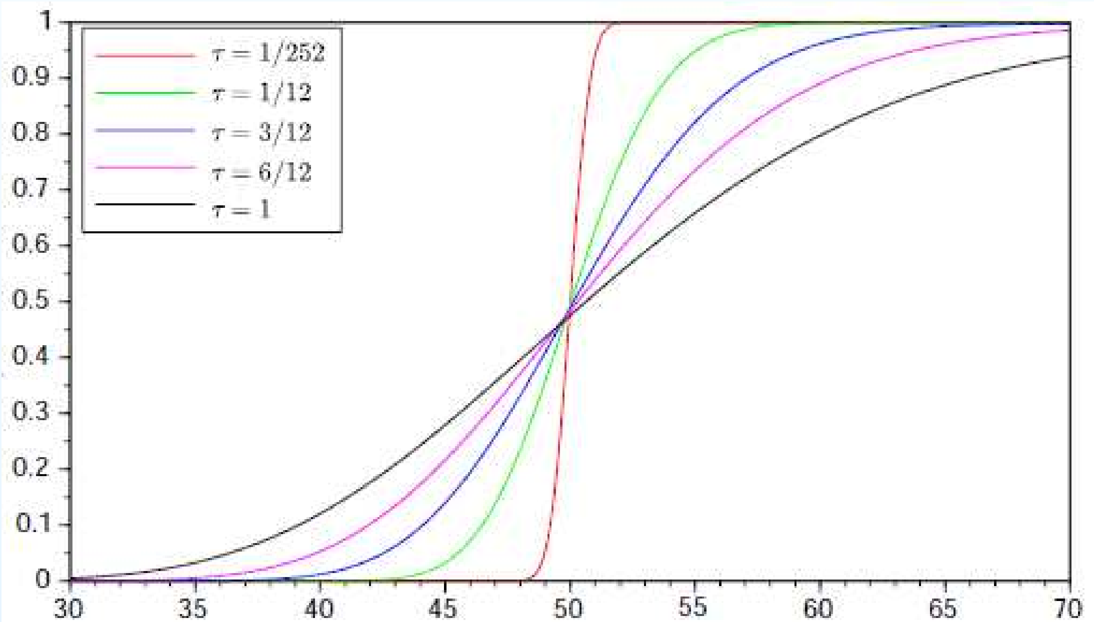
Vega, rho, theta

- Theta
  - put: the sign may be different for different sets of parameters



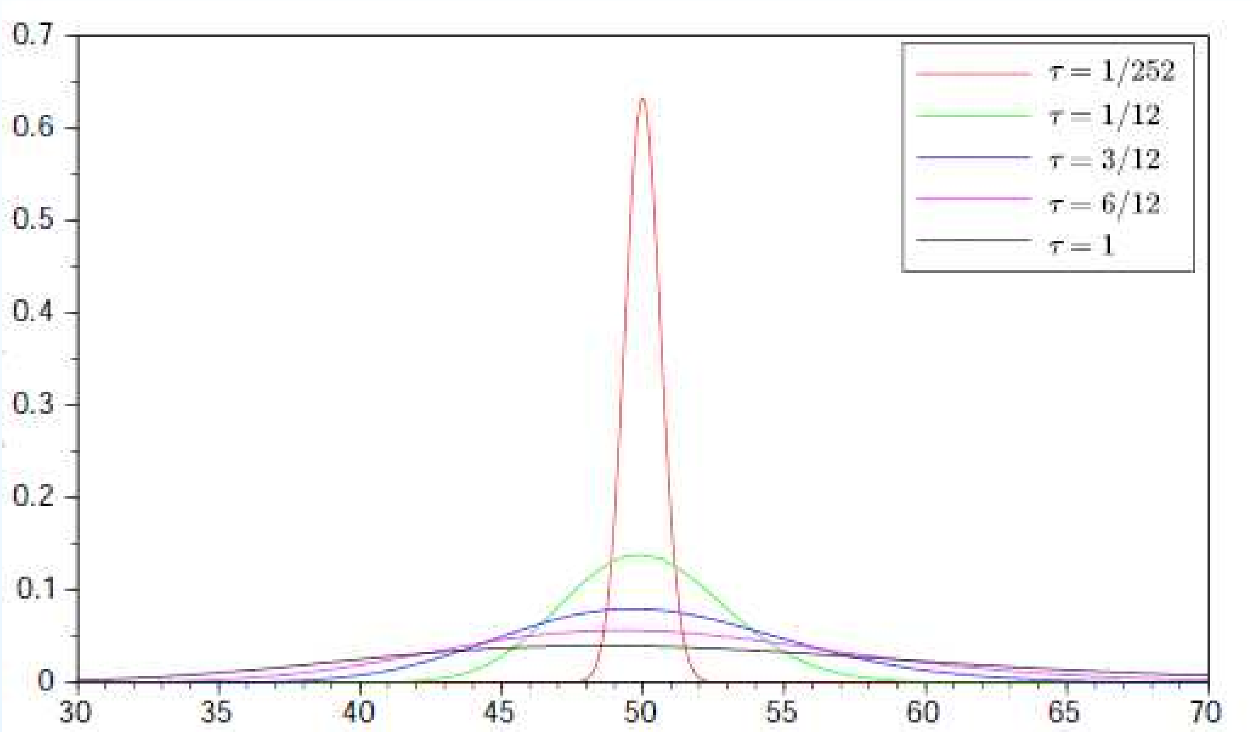
Exercise: "cash-or-nothing" option

- "Cash-or-nothing" opcia: pays 1 USD if the stock exceeds the value E at the expiration time; otherwise 0.
- Option price:

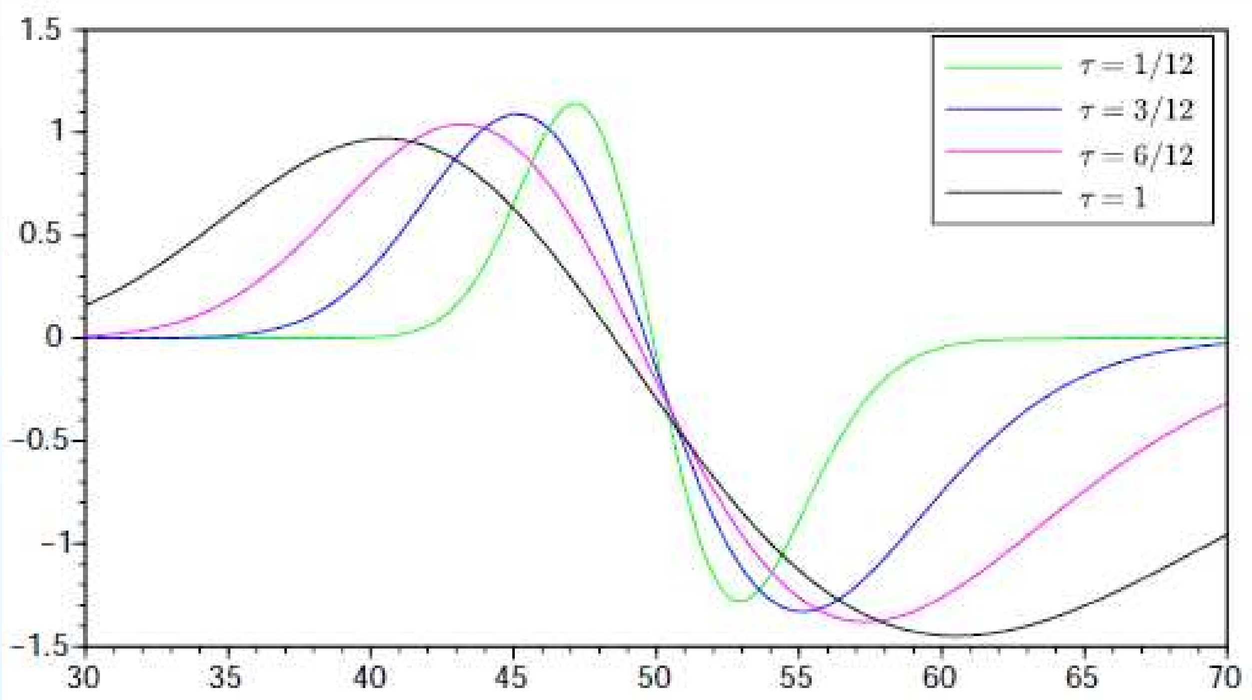


- Using the interpretation of the *greeks* - sketch delta and vega as function of the stock price

Exercise: "cash-or-nothing" delta



Exercise: "cash-or-nothing" vega



Exercise: sensitivity of delta to volatility

- Espen Haug in the paper *Know your weapon*:

> One fine day in the dealing room my risk manager asked me to get into his office. He asked me why I had a big outright position in some stock index futures - I was supposed to do "arbitrage trading". That was strange as I believed I was delta neutral: long call options hedged with short index futures. I knew the options I had were far out-of-the-money and that their DdeltaDvol was very high. So I immediately asked what volatility the risk management used to calculate their delta. As expected, the volatility in the risk-management-system was considerable below the market and again was leading to (2) a very low delta for the options. This example is just to illustrate how a feeling of your DdeltaDvol can be useful. If you have a high DdeltaDvol the volatility you use to compute your deltas becomes very important. (1)

- Questions:
  1. What is the dependence of delta on volatility which is used in its computation?
  2. Low volatility led to low delta - why?
- More $\rightarrow$ exercises session