III. Black-Scholes model: Derivation and solution

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Financial derivatives

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• Black-Scholes model:
  ◦ Suppose that stock price $S$ follows a geometric Brownian motion

\[ dS = \mu S dt + \sigma S dw \]

+ other assumptions (in a moment)
  ◦ We derive a partial differential equation for the price of a derivative

• Two ways of derivations:
  ◦ due to Black and Scholes
  ◦ due to Merton

• Explicit solution for European call and put options
Assumptions

• Further assumptions (besides GBP):
  ◦ constant riskless interest rate $r$
  ◦ no transaction costs
  ◦ it is possible to buy/sell any (also fractional) number of stocks; similarly with the cash
  ◦ no restrictions on short selling
  ◦ option is of European type

• Firstly, let us consider the case of a non-dividend paying stock
Derivation I. - due to Black and Scholes

- Notation:
  \( S \) = stock price, \( t \) = time
  \( V = V(S, t) \) = option price

- Portfolio: 1 option, \( \delta \) stocks
  \( P = \) value of the portfolio: \( P = V + \delta S \)

- Change in the portfolio value: \( dP = dV + \delta dS \)

- From the assumptions: \( dS = \mu S dt + \sigma S dw \), From the Itô lemma: \( dV = \left( \frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dw \)

- Therefore:

\[
    dP = \left( \frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + \delta \mu S \right) dt \\
    + \left( \sigma S \frac{\partial V}{\partial S} + \delta \sigma S \right) dw
\]
Derivation I. - due to Black and Scholes

• We eliminate the randomness: \( \delta = -\frac{\partial V}{\partial S} \)

• Non-stochastic portfolio \( \Rightarrow \) its value has to be the same as if being on a bank account with interest rate \( r \): \( dP = rPdt \)

• Equality between the two expressions for \( dP \) and substituting \( P = V + \delta S \):

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0
\]
Dividends in the Black-Scholes’ derivation

- We consider **continuous dividend rate** $q$ - holding a stock with value $S$ during the time differential $dt$ brings dividends $qSdt$

- In this case the change in the portfolio value equals
  $dP = dV + \delta dS + \delta qSdt$

- We proceed in the same way as before and obtain

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S \frac{\partial V}{\partial S} - rV = 0$$
Derivation due to Merton - motivation

- Problem in the previous derivation:
  - we have a portfolio consisting of one option and $\delta$ stocks
  - we compute its value and change of its value:
    \[
    P = V + \delta S,
    \]
    \[
    dP = dV + \delta dS,
    \]
    i.e., treating $\delta$ as a constant
  - however, we obtain $\delta = -\frac{\partial V}{\partial S}$
Derivation II. - due to Merton

• Portfolio consisting of options, stocks and cash with the properties:
  ◦ in each time, the portfolio has zero value
  ◦ it is self-financing

• Notation:
  \( Q_S = \) number of stocks, each of them has value \( S \)
  \( Q_V = \) number of options, each of them has value \( V \)
  \( B = \) cash on the account, which is continuously compounded using the risk-free rate \( r \)

  \( dQ_S = \) change in the number of stocks
  \( dQ_V = \) change in the number of options
  \( \delta B = \) change in the cash, caused by buying/selling stocks and options
Derivation II. - due to Merton

- Mathematical formulation of the required properties:
  - zero value: \( SQ_S + V Q_V + B = 0 \) \( (1) \)
  - self-financing: \( S dQ_S + V dQ_V + \delta B = 0 \) \( (2) \)

- Change in the cash: \( dB = rB \, dt + \delta B \)

- Differentiating (1):

\[
0 = d(SQ_S + VQ_V + B) = d(SQ_S + VQ_V) + dB
\]

\[
0 = SdQ_S + VdQ_V + \delta B + Q_SdS + Q_VdV + rB \, dt
\]

\[
0 = Q_SdS + Q_VdV - r(SQ_S + VQ_V) \, dt.
\]
Derivation II. - due to Merton

- We divide by $Q_V$ and denote $\Delta = -\frac{Q_S}{Q_V}$:
  $$dV - rV \, dt - \Delta (dS - rS \, dt) = 0$$

- We have $dS$ from the assumption of GBM and $dV$ from the Itô lemma.

- We choose $\Delta$ (i.e., the ratio between the number of stocks and options) so that it eliminates the randomness (the coefficient at $dw$ will be zero).

- We obtain the same PDE as before:
  $$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$$
Dividends in the Merton’s derivation

- Assume continuous dividend rate $q$.
- Dividends cause an increase in the cash $\Rightarrow$ change in the cash is $dB = rB dt + \delta B + qSQ_S dt$
- In the same way we obtain the PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S \frac{\partial V}{\partial S} - rV = 0$$
Black-Scholes PDE: summary

• Mathematical formulation of the model:
  Find solution \( V(S, t) \) to the partial differential equation (so called Black-Scholes PDE)

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0
\]

which holds for \( S > 0, t \in [0, T) \).

• So far we have not used the fact that we consider an option
  \( \Rightarrow \) PDE holds for any derivative that pays a payoff at time \( T \)
  depending on the stock price at this time

• Type of the derivative determines the terminal condition at
  time \( T \)

• In general: \( V(S, T) = \) payoff of the derivative
Black-Scholes PDE: simple solutions

**Some simple "derivatives":**

- How to price the derivatives with the following payoffs:
  - $V(S, T) = S$ → it is in fact a stock → $V(S, t) = S$
  - $V(S, T) = E$ → with a certainty we obtain the cash $E$
    → $V(S, t) = E e^{-r(T-t)}$

- by substitution into the PDE we see that they are indeed solutions

**Exercises:**

- Find the price of a derivative with payoff $V(S, T) = S^n$, where $n \in \mathbb{N}$.
  **Hint:** Look for the solution in the form $V(S, t) = A(t)S^n$

- Find all solutions to the Black-Scholes PDE, which are independent of time, i.e., for which $V(S, t) = V(S)$
Black-Scholes PDE: binary option

- Let us consider a binary option, which pays 1 USD if the stock price is higher than $E$ at expiration time, otherwise its payoff is zero.
- In this case:

$$V(S, T) = \begin{cases} 
1 & \text{if } S > E \\
0 & \text{otherwise}
\end{cases}$$

- The main idea is to transform the Black-Scholes PDE to a heat equation.
- Transformations are independent of the derivative type; it affects only the initial condition of the heat equation.
Black-Scholes PDE: transformations

FORMULATION OF THE PROBLEM

- Partial differential equation

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0
\]

which holds for \( S > 0, t \in [0, T) \).

- Terminal condition \( V(S, T) = \text{payoff of the derivative} \) for \( S > 0 \)
Black-Scholes PDE: transformations

**Step 1:**
- Transformation $x = \ln(S/E) \in \mathbb{R}$, $\tau = T - t \in [0, T]$ and a new function $Z(x, \tau) = V(\text{e}^{x}, T - \tau)$
- PDE for $Z(x, \tau)$, $x \in \mathbb{R}$, $\tau \in [0, T]$:
  \[
  \frac{\partial Z}{\partial \tau} - \frac{1}{2} \sigma^2 \frac{\partial^2 Z}{\partial x^2} + \left( \frac{\sigma^2}{2} - r \right) \frac{\partial Z}{\partial x} + rZ = 0,
  \]
  
  $Z(x, 0) = V(\text{e}^{x}, T)$

**Step 2:**
- Transformation to heat equation
- New function $u(x, \tau) = e^{\alpha x + \beta \tau} Z(x, \tau)$, where the constants $\alpha, \beta \in \mathbb{R}$ are chosen so that the PDE for $u$ is the heat equation
### Black-Scholes PDE: transformations

- **PDE for $u$:**

  $$\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} + A \frac{\partial u}{\partial x} + Bu = 0,$$

  $$u(x, 0) = e^{\alpha x} Z(x, 0) = e^{\alpha x} V(EE^x, T),$$

  where

  $$A = \alpha \sigma^2 + \frac{\sigma^2}{2} - r, \quad B = (1 + \alpha)r - \beta - \frac{\alpha^2 \sigma^2 + \alpha \sigma^2}{2}.$$  

- **In order to have $A = B = 0$, we set**

  $$\alpha = \frac{r}{\sigma^2} - \frac{1}{2}, \quad \beta = \frac{r}{2} + \frac{\sigma^2}{8} + \frac{r^2}{2 \sigma^2}$$
**Black-Scholes PDE: transformations**

**STEP 3:**

- Solution $u(x, \tau)$ of the PDE $\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} = 0$ is given by Green formula

  $$u(x, \tau) = \frac{1}{\sqrt{2\pi^{2}\tau}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{2\sigma^2\tau}} u(s, 0) \, ds .$$

- We evaluate the integral and perform backward substitutions $u(x, \tau) \rightarrow Z(x, \tau) \rightarrow V(S, t)$
Black-Scholes PDE: binary option (continued)

- Transformations from the previous slides

- We obtain the heat equation \( \frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} = 0 \) with initial condition

\[ u(x, 0) = e^{\alpha x} V(Ee^x, T) = \begin{cases} 
  e^{\alpha x} & \text{if } Ee^x > E \\
  0 & \text{otherwise}
\end{cases} = \begin{cases} 
  e^{\alpha x} & \text{if } x > 0 \\
  0 & \text{otherwise}
\end{cases} \]

- Solution \( u(x, \tau) \):

\[ u(x, \tau) = \frac{1}{\sqrt{2\pi\sigma^2\tau}} \int_0^\infty e^{-\frac{(x-s)^2}{2\sigma^2\tau}} e^{\alpha s} \, ds = \ldots = e^{\alpha x + \frac{1}{2} \sigma^2 \tau \alpha^2} N\left( \frac{x + \sigma^2 \tau \alpha}{\sigma \sqrt{\tau}} \right) \]

where \( N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{\xi^2}{2}} \, d\xi \) is the cumulative distribution function of a normalized normal distribution
Black-Scholes PDE: binary option (continued)

- Option price $V(S, t)$:

$$V(S, t) = e^{-r(T-t)} N(d_2),$$

where $d_2 = \frac{\log\left(\frac{S}{E}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$.
Black-Scholes PDE: call option

- In this case
  \[ V(S, T) = \max(0, S - E) = \begin{cases} 
  S - E & \text{if } S > E \\ 
  0 & \text{otherwise} 
\end{cases} \]

- The same sequence of transformations; initial condition for the heat equation:
  \[ u(x, 0) = \begin{cases} 
  e^{\alpha x}(S - E) & \text{if } x > 0 \\ 
  0 & \text{otherwise} 
\end{cases} \]
  and similar evaluation of the integral

- Option price:
  \[ V(S, t) = SN(d_1) - EE^{-r(T-t)}N(d_2), \]
  where \( N \) is the distribution function of a normalized normal distribution and
  \[ d_1 = \frac{\ln \frac{S}{E} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}}, \quad d_2 = d_1 - \sigma \sqrt{T-t} \]
Black-Scholes PDE: call option

**HOMEWORK:**
Solve the Black-Scholes PDE for a call option on a stock which pays continuous dividends and write it in the form

\[ V(S, t) = S e^{-q(T-t)} N(d_1) - E e^{-r(T-t)} N(d_2), \]

where \( N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{\xi^2}{2}} d\xi \) is the distribution function of a normalized normal distribution \( N(0, 1) \) and

\[ d_1 = \frac{\ln \frac{S}{E} + (r - q + \frac{\sigma^2}{2})(T - t)}{\sigma \sqrt{T - t}}, \quad d_2 = d_1 - \sigma \sqrt{T - t} \]

**NOTE:** The PDE is different, so the transformations have to be adjusted (do the same steps for the new equation)
Payoff (i.e., terminal condition at time $t = T = 1$) and solution $V(S, t)$ for selected times $t$: 

![Diagram showing payoff and option price for Black-Scholes model]
Black-Scholes PDE: put option

FORMULATION OF THE PROBLEM

• Partial differential equation

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0
\]

which holds for \( S > 0, t \in [0, T] \).

• Terminal condition:

\[
V(S, T) = \max(0, E - S)
\]

for \( S > 0 \)
Black-Scholes PDE: put option

**APPROACH I.**
- The same sequence of computations as in the case of a call option

**APPROACH II.**
- We use the linearity of the Black-Scholes PDE and the solution for a call which we have already found

We show the application of the latter approach.
Black-Scholes PDE: put option

- Recall that for the payoffs of a call and a put we have
  
  \[-[\text{call payoff}] + [\text{put payoff}] + [\text{stock price}] = E\]

- Hence:
  
  \([\text{put payoff}] = [\text{call payoff}] - S + E\]

- Black-Scholes PDE is linear: a linear combination of solutions is again a solution
Black-Scholes PDE: put option

- Recall the solutions for $V(S, T) = S$ and $V(S, T) = E$ (page 13):

<table>
<thead>
<tr>
<th>terminal condition</th>
<th>solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\max(0, S - E)$</td>
<td>$V^{\text{call}}(S, t)$</td>
</tr>
<tr>
<td>$S$</td>
<td>$S$</td>
</tr>
<tr>
<td>$E$</td>
<td>$Ee^{-r(T-t)}$</td>
</tr>
</tbody>
</table>

- From the linearity:

<table>
<thead>
<tr>
<th>terminal condition</th>
<th>solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\max(0, S - E) - S + E$</td>
<td>$V^{\text{call}}(S, t) - S + Ee^{-r(T-t)}$</td>
</tr>
</tbody>
</table>

- Since $[\text{put payoff}] = \max(0, S - E) - S + E$, we get

$$V^{\text{put}}(S, t) = V^{\text{call}}(S, t) - S + Ee^{-r(T-t)}$$
Solution for a put option

- The solution

\[ V_{\text{put}}(S, t) = V_{\text{call}}(S, t) - S + E e^{-r(T-t)} \]

can be written in a similar form as the solution for a call option:

\[ V_{\text{ep}}(S, t) = E e^{-r(T-t)} N(-d_2) - S N(-d_1), \]

where \( N, d_1, d_2 \) are the same as before
Put option - example

Payoff (i.e. terminal condition at time $t = T = 1$) and solution $V(S, t)$ for selected times $t$:
Put option - alternative computation

Comics about negative volatility on the webpage of Espen Haug:

http://www.espenhaug.com/collector/collector.html
Put option - alternative computation

- A nightmare about negative volatility:

- Not only a dream... according to internet, it really exists and is connected with professor Shiryaev from Moscow...
Put option - alternative computation

**QUESTION:** Why does this computation work?
Stocks paying dividends

- **HOMEWORK:**
  Solve the Black-Scholes equation for a put option, if the underlying stock pays continuous dividends.

**HINT:**
- In this case, $V(S,t) = S$ is not a solution
- What is the solution satisfying the terminal condition $V(S,T) = S$? Use financial interpretation and check your answer by substituting it into the PDE

- **HOMEWORK:**
  Denote $V(S,t; E, r, q)$ the price of an option with exercise price $E$, if the interest rate is $r$ and the dividend rate is $q$.
  Show that

  \[ V^{\text{put}}(S,t; E, r, q) = V^{\text{call}}(E, t; S, q, r) \]

  **HINT:** How do the terms $d_1d_2$ change when replacing $S \leftrightarrow E$, $r \leftrightarrow q$?
Combined strategies

- From the linearity of the Black-Scholes PDE: if the strategy is a linear combination of call and put options, then its price is the same linear combination of the call and put options prices.

- It does not necessarily hold in other models:
  - consider a model with some transaction costs; it is not equivalent
    - whether we hedge the options independently
    - or we hedge the portfolio - in this case, we might be able to reduce transaction costs.
Combined strategies

**Example:**

- we buy call options with exercise prices $E_1$, $E_3$ and sell two call options with exercise prices $E_2$, with exercise prices satisfying $E_1 < E_2 < E_3$ and $E_1 + E_3 = 2E_2$.

- Payoff of the strategy can be written as
  \[ V(S,T) = \max(S - E_1, 0) - 2\max(S - E_2, 0) + \max(S - E_3, 0) \]

- Hence its Black-Scholes price is:
  \[ V(S,t) = V^{\text{call}}(S,t; E_1) - 2V^{\text{call}}(S,t; E_2) + V^{\text{call}}(S,t; E_3) \]
Combined strategies

- Numerical example - butterfly with $T = 1$:

![Butterfly option strategy graph]

III. Black-Scholes model: Derivation and solution – p.36/36
IV. Black-Scholes model: Implied volatility
Market data

- Stock:

![General Motors Company Stock Chart](image-url)
Market data

- Selected options:

<table>
<thead>
<tr>
<th>Strike</th>
<th>Option ID</th>
<th>IV</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>36.50</td>
<td>GM140314C00036500</td>
<td>0.92</td>
<td>+0.27</td>
</tr>
<tr>
<td>36.50</td>
<td>GM140328C00036500</td>
<td>0.99</td>
<td>+0.01</td>
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<tr>
<td>37.00</td>
<td>GM140307C00037000</td>
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<td>-0.03</td>
</tr>
<tr>
<td>37.00</td>
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<td>0.67</td>
<td>+0.17</td>
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<tr>
<td>37.00</td>
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<tr>
<td>37.00</td>
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<td>0.77</td>
<td>+0.09</td>
</tr>
</tbody>
</table>

- How much are these options supposed to cost according to Black-Scholes model?
Recall Black-Scholes formula for a call option:

\[ V(S, t) = SN(d_1) - Ee^{-r(T-t)}N(d_2), \]

where \( N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{\xi^2}{2}} d\xi \) is the distribution function of a normalized normal distribution \( N(0, 1) \) and

\[ d_1 = \frac{\ln \frac{S}{E} + (r + \frac{\sigma^2}{2})(T - t)}{\sigma \sqrt{T - t}}, \quad d_2 = d_1 - \sigma \sqrt{T - t} \]
Black-Scholes model and market data

- Therefore, we need the following values:
  - $S$ = stock price
  - $E$ = exercise price
  - $T - t$ = time remaining to expiration
  - $\sigma$ = volatility of the stock
  - $r$ = interest rate

- What is clear: $S, E, T - t$

- Interest rate (there are different rates on the market):
  - A common choice: 3-months treasury bills
  - Interest rate has to be expressed as a decimal number
    $\Rightarrow$ 0.03 percent is $r = 0.03/100$
Black-Scholes model and market data

• What is the volatility?
  ◦ Exercises session: computation of the Black-Scholes price using historical volatility
  ◦ Different estimates of volatility, depending on time span of the data
  ◦ Price does not equal the market price

• Question: What value of volatility produces the Black-Scholes price that is equal to the market price?

• This value of volatility is called implied volatility
Implied volatility

- Dependence of the Black-Scholes option price on volatility:
Existence of implied volatility

- Dependence of the Black-Scholes option price on volatility - for a wider range of volatility:

![Graph showing the dependence of the Black-Scholes option price on volatility](image-url)
Existence of implied volatility

- In general - we show that
  - The Black-Scholes price of a call option is an increasing function of volatility
  - Limits are equal to: $V_0 := \lim_{\sigma \to 0^+} V(S, t; \sigma)$, $V_\infty := \lim_{\sigma \to \infty} V(S, t; \sigma)$

- Then, from continuity of $V \Rightarrow$ for every price from the interval $(V_0, V_\infty)$ the implied volatility exists and is uniquely determined

- We do the derivation of a stock which does not pay dividends

- HOMEWORK: call and put option on a stock which pays continuous dividends
Existence of implied volatility

• To prove that price is an increasing function of volatility:
  ◦ We compute the derivative (using $d_2 = d_1 - \sigma \sqrt{T - t}$):

$$\frac{\partial V}{\partial \sigma} = SN'(d_1) \frac{\partial d_1}{\partial \sigma} - Ee^{-r(T-t)} N'(d_2) \frac{\partial d_2}{\partial \sigma}$$

$$= \left( SN'(d_1) - Ee^{-r(T-t)} N'(d_2) \right) \frac{\partial d_1}{\partial \sigma}$$

$$+ Ee^{-r(T-t)} N'(d_2) \sqrt{T - t}$$

◦ Derivative of a distribution function is a density function: $N'(x) = \frac{1}{2\pi} e^{-\frac{x^2}{2}}$

◦ Useful lemma: $SN'(d_1) - Ee^{-r(T-t)} N'(d_2) = 0$

◦ Hence:

$$\frac{\partial V}{\partial \sigma} = Ee^{-r(T-t)} N'(d_2) \sqrt{T - t} > 0$$
Existence of implied volatility

• Limits:
  ◦ We use basic properties of a distribution function:
    \[ \lim_{x \to -\infty} N(x) = 0, \quad \lim_{x \to +\infty} N(x) = 1 \]
  ◦ It follows:
    \[ \lim_{\sigma \to 0^+} V(S, t; \sigma) = \max(0, S - E e^{-r(T-t)}) \]
    \[ \lim_{\sigma \to \infty} V(S, t; \sigma) = S \]
Implied volatility - computation

• In our case:

• We get the implied volatility 0.22558
Website finance.yahoo.com

- Option chains include implied volatilities:

<table>
<thead>
<tr>
<th>Strike</th>
<th>Contract Name</th>
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<td>31.42%</td>
</tr>
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V. Black-Scholes model: Greeks - sensitivity analysis
Greeks

- Greeks:
  - derivatives of the option price with respect to parameters
  - they measure the sensitivity of the option price to these parameters

- We have already computed
  \[
  \frac{\partial V_{\text{call}}}{\partial \sigma} = Ee^{-r(T-t)}N'(d_2)\sqrt{T-t}, \text{ it is denoted by } \Upsilon \text{ (vega)}
  \]

- Others: (Remark: \( P \) is a Greek letter rho)
  \[
  \Delta = \frac{\partial V}{\partial S}, \quad \Gamma = \frac{\partial^2 V}{\partial S^2}, \quad P = \frac{\partial V}{\partial r}, \quad \Theta = \frac{\partial V}{\partial t}
  \]

- Notation: \( V^{\text{ec}} \) = price of a European call, \( V^{\text{ep}} \) = price of a European put; in the same way their American counterparts \( V^{\text{ac}}, V^{\text{ap}} \)
Delta

- **Call option** - from Black-Scholes formula, we use the same lemma as in the case of volatility:

\[ \Delta^{ec} = \frac{\partial V^{ec}}{\partial S} = N(d_1) \in (0, 1) \]

- **Put option** - we do not need to compute the derivative, we can use the put-call parity:

\[ \Delta^{ep} = \frac{\partial V^{ep}}{\partial S} = -N(-d_1) \in (-1, 0) \]

- **Example**: call (left), put (right)
Delta - delta hedging

• Recall the derivation of the Black-Scholes model and construction of a riskless portfolio:

\[
\frac{Q_S}{Q_V} = - \frac{\partial V}{\partial S} = - \Delta
\]

where \( Q_V, Q_S \) are the numbers of options and stock in the portfolio

• Construction of such a portfolio is call delta hedging (hedge = protection, transaction that reduces risk)
Delta - example of delta hedging

- Real data example - call option on IBM stock, 21st May 2002, 5-minute ticks
- At time $t$:
  - we have option price $V_{\text{real}}(t)$ and stock price $S_{\text{real}}(t)$
  - we compute the implied volatility, i.e., we solve the equation
    \[
    V_{\text{real}}(t) = V^{ec}(S_{\text{real}}(t), t; \sigma_{impl}(t)).
    \]
  - implied volatility $\sigma_{impl}(t)$ is used in the call option price formula:
    \[
    \Delta^{ec}(t) = \frac{\partial V^{ec}}{\partial S}(S_{\text{real}}(t), t; \sigma_{impl}(t)).
    \]
Delta - example of delta hedging

- Delta during the day:

- We wrote one option - then, this is the number of stocks in our portfolio
Gamma

• Computation:

$$\Gamma^{ec} = \frac{\partial \Delta^{ec}}{\partial S} = N'(d_1) \frac{\partial d_1}{\partial S} = \frac{\exp\left(-\frac{1}{2}d_1^2\right)}{\sigma \sqrt{2\pi(T-t)S}} > 0$$

$$\Gamma^{ep} = \Gamma^{ec}$$

• Measures a sensitivity of delta to a change in stock price
Price, delta, gamma
Price, delta, gamma

• Simultaneously:
  ◦ the option price is "almost a straight line"
  ◦ delta does not change much with a small change in the stock price
  ◦ gamma is almost zero

• Also:
  ◦ graph of the option price has a big curvature
  ◦ delta significantly changes with a small change in the stock price
  ◦ gamma is significantly nonzero
Vega, rho, theta

- **Vega**
  - we have already computed: 
    \[ \Upsilon^{ec} = \frac{\partial V^{ec}}{\partial \sigma} = E e^{-r(T-t)} N'(d_2) \sqrt{T-t} > 0 \]
  - from put-call parity: \( \Upsilon^{ep} = \Upsilon^{ec} \)
  - higher volatility \( \Rightarrow \) higher probability of high profit, while a possible loss is bounded \( \Rightarrow \) positive vega

- **Rho**
  - call: \( P^{ec} = \frac{\partial V^{ec}}{\partial \rho} = E(T-t)e^{-r(T-t)} N(d_2) > 0 \)
  - put: \( P^{ep} = \frac{\partial V^{ep}}{\partial \rho} = -E(T-t)e^{-r(T-t)} N(-d_2) < 0 \)

- **Theta**
  - call: from financial mathematics we know that if a stock does not pay dividends, it is not optimal to exercise an American option prior to its expiry \( \Rightarrow \) prices of European and American options are equal \( \Rightarrow \) \( \Theta^{ec} < 0 \)
Vega, rho, theta

- **Theta**
  - put: the sign may be different for different sets of parameters
Exercise: "cash-or-nothing" option

- "Cash-or-nothing" opcia: pays 1 USD if the stock exceeds the value $E$ at the expiration time; otherwise 0.
- Option price:

Using the interpretation of the greeks - sketch delta and vega as function of the stock price
Exercise: "cash-or-nothing" delta
Exercise: "cash-or-nothing" vega
Exercise: sensitivity of delta to volatility

- Espen Haug in the paper *Know your weapon*:

  One fine day in the dealing room my risk manager asked me to get into his office. He asked me why I had a big outright position in some stock index futures - I was supposed to do "arbitrage trading". That was strange as I believed I was delta neutral: long call options hedged with short index futures. I knew the options I had were far out-of-the-money and that their $\Delta\sigma$ was very high. So I immediately asked what volatility the risk management system was estimating. As expected, the volatility in the risk-management system was considerably below the market and again was leading to a very low delta for the options. This example is just to illustrate how a feeling of your $\Delta\sigma$ can be useful. If you have a high $\Delta\sigma$ the volatility you use to compute your deltas becomes very important.

- Questions:
  1. What is the dependence of delta on volatility which is used in its computation?
  2. Low volatility led to low delta - why?

- More → exercises session