VI. Leland model: Derivation of the PDE for the price of a derivative

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Leland model

- Taking transaction costs into account
- Original paper:
  Hayne E. Leland: *Option Pricing and Replication with Transactions Costs*, 1985
Assumptions of the model

- Transaction costs are characterized by a constant $c = \frac{S_{\text{ask}} - S_{\text{bid}}}{S}$, where $S$ is the average of bid and ask price of the stock.

- $S$ follows a geometric Brownian motion $dS = \mu S dt + \sigma S dw$.
Example 1:

- Stock:

  - From the data: \( S_{\text{bid}} = 39.85, S_{\text{ask}} = 39.86 \)
  - Average of bid and ask: \( S = 39.855 \)
  - \( c = \frac{0.01}{39.855} = 2.5028 \times 10^{-4} \)
**Example 2:**

- **Stock:**

  ![Stock Data](image)

  - From the data: $S_{bid} = 38.06, S_{ask} = 38.07$
  - Average of bid and ask: $S = 38.065$
  - $c = \frac{0.01}{38.065} = 2.6271 \times 10^{-4}$
**Computation of the constant** $c$

**Example 3:**

- **Stock:**

  ![Stock Data]

  - From the data: $S_{bid} = 372.81$, $S_{ask} = 372.94$
  - Average of bid and ask: $S = 372.875$
  - $c = \frac{0.13}{372.875} = 3.4864 \times 10^{-4}$
Derivation of the PDE

• Portfolio:
  ◦ one option and δ stocks, while the number of stocks is determined by delta hedging, i.e., \( \delta = -\frac{\partial V}{\partial S} \)
  ◦ value of the portfolio: \( P = V + \delta S \)
  ◦ because of the transaction costs, the portfolio cannot be hedged continuously in time → we hedge it discrete times which are \( \Delta t \) [years] apart

• Change of the portfolio value
  ◦ number of transactions with stocks is \( \Delta \delta \)
  ◦ costs for one transaction are \( cS/2 \) ⇒ total costs are equal to \( \frac{cS}{2} |\Delta \delta| \)
  ◦ therefore, change of the portfolio value is:
    \[
    \Delta P = \Delta V + \delta \Delta S - \frac{cS}{2} |\Delta \delta|
    \]
Derivation of the PDE

• Hence we have $\Delta P = \Delta V + \delta \Delta S - \frac{cS}{2} |\Delta \delta|$ and
  
  ◦ $\Delta S = \mu S \Delta t + \sigma S \Delta w$ from the assumptions (GBM)
  ◦ $\Delta V = \left( \frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \right) \Delta t + \sigma S \frac{\partial V}{\partial S} \Delta w$ from Itô lemma
  ◦ what remains, is to derive $\Delta \delta$

• We have $\delta = -\frac{\partial V}{\partial S}$, hence $\frac{\partial \delta}{\partial S} = -\frac{\partial^2 V}{\partial S^2}$, from which:

  \[ \Delta \delta \approx \frac{\partial \delta}{\partial S} \Delta S = -\frac{\partial^2 V}{\partial S^2} \Delta S \]

• Here we substitute $\Delta S$ from the GBM
Derivation of the PDE

- So far we have:

\[ \Delta \delta \approx -\frac{\partial^2 V}{\partial S^2} \mu S \Delta t - \frac{\partial^2 V}{\partial S^2} \sigma S \Delta w \]  

(1)

- Leland has shown:
  - in formula (1), it suffices to consider the terms of the lowest order (i.e. we take only \( \Delta w \approx (\Delta t)^{1/2} \), and \( \Delta t \) is neglected)
  - when computing the absolute value \( |\Delta w| \), it can be replaced by its expected value \( \mathbb{E}[|\Delta w|] = \sqrt{\frac{2}{\pi} \Delta t} \)

- Therefore:

\[ \Delta \delta \approx -\frac{\partial^2 V}{\partial S^2} \sigma S \Delta w \]

\[ |\Delta \delta| \approx \left| \frac{\partial^2 V}{\partial S^2} \right| \sigma S |\Delta w| \approx \left| \frac{\partial^2 V}{\partial S^2} \right| \sigma S \sqrt{\frac{2}{\pi} \sqrt{\Delta t}} \]
Derivation of the PDE

• We substitute everything into the formula for the change of the portfolio value $\Delta P = \Delta V + \delta \Delta S - \frac{cS}{2} |\Delta \delta|:

$$\Delta P = \left( \frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - \frac{c}{2} S \left| \frac{\partial^2 V}{\partial S^2} \right| \sigma S \sqrt{\frac{2}{\pi \Delta t}} \right) \Delta t$$ \quad (2)

• Portfolio is riskless $\Rightarrow$ necessarily (to rule out arbitrage possibilities) $\Delta P = r P \Delta t$

• Portfolio consists of one option and $\delta = -\frac{\partial V}{\partial S}$ stocks $\Rightarrow$ $P = V + \delta S = V - \frac{\partial V}{\partial S}$, and so

$$\Delta P = r (V - \frac{\partial V}{\partial S} S) \Delta t$$ \quad (3)

• Comparing (2) and (3):

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - \frac{c}{2} S \left| \frac{\partial^2 V}{\partial S^2} \right| \sigma S \sqrt{\frac{2}{\pi \Delta t}} = r(V - \frac{\partial V}{\partial S} S)$$
Derivation of the PDE

- We write the PDE into its final form:

\[
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \left[ 1 - \frac{c}{\sigma \sqrt{\Delta t}} \sqrt{\frac{2}{\pi}} \text{sign} \left( \frac{\partial^2 V}{\partial S^2} \right) \right] + \frac{\partial V}{\partial S} S - rV = 0
\]

- The PDE holds for \( S > 0, t \in [0, T] \), we add the terminal condition \( V(S, T) \) depending on the type of the derivative, e.g., \( V(S, T) = \max(0, S - E) \) for \( S > 0 \) when pricing a call option

- Nonlinear PDE because of the term containing the signum function

- However, we will solve it in a closed form for call and put options
Remark on combined strategies

- The price of combined strategies (unlike in the Black-Scholes setting) cannot be found by pricing every option and then adding the prices

**Mathematical point of view:**

- PDE in the Leland model is not linear $\Rightarrow$ for example a sum, difference or some other linear combination is no more a solution
Remark on combined strategies

**FINANCIAL POINT OF VIEW:**

- If we price every option separately, we count transaction costs coming from hedging the portfolio for each of the options separately.
- If the transaction costs are zero, it does not matter that e.g. we have two portfolios, for one of them we buy stock and for the other one we sell stocks (it does not cause any transaction costs).
- In a presence of transaction costs this is no more true. In such a case we need to consider one portfolio, to avoid unnecessary transactions (which would increase transaction costs).
VII.  

Leland model: European call and put options

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Leland PDE

- Recall the Leland PDE for the price of a derivative:

\[
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \left[ 1 - \frac{c}{\sigma \sqrt{\Delta t}} \sqrt{\frac{2}{\pi}} \text{sign} \left( \frac{\partial^2 V}{\partial S^2} \right) \right] + \frac{\partial V}{\partial S} S - rV = 0
\]

- The PDE holds for \( S > 0, t \in [0, T] \), we add the terminal condition \( V(S, T) \) depending on the derivative, e.g., \( V(S, T) = \max(0, S - E) \) for \( S > 0 \) in the case of a call option

- Nonlinear PDE because of the term containing \( \text{signum} \)

- Recall the for the Black-Scholes prices of call and put options we have \( \frac{\partial^2 V}{\partial S^2} > 0 \) (positive gamma) \( \Rightarrow \)

\[ \text{sign} \left( \frac{\partial^2 V}{\partial S^2} \right) = 1 \]
Leland PDE - call and put

- What happens if inserting Black-Scholes price of a call/put with adjusted volatility \( V(S, t; \tilde{\sigma}) \):

\[
\tilde{\sigma}^2 = \sigma^2 \left[ 1 - \frac{c}{\sigma \sqrt{\Delta t}} \sqrt{\frac{2}{\pi}} \right]
\]

into the Leland PDE:

\[
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \left[ 1 - \frac{c}{\sigma \sqrt{\Delta t}} \sqrt{\frac{2}{\pi}} \right] \text{sign} \left( \frac{\partial^2 V}{\partial S^2} \right) + \frac{\partial V}{\partial S} S - rV = \frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \left[ 1 - \frac{c}{\sigma \sqrt{\Delta t}} \sqrt{\frac{2}{\pi}} \right] + \frac{\partial V}{\partial S} S - rV = 0
\]
Leland PDE - call and put

- It means that Black-Scholes price of a call/put with adjusted volatility $V(S, t; \tilde{\sigma})$:

$$\tilde{\sigma}^2 = \sigma^2 \left[ 1 - \frac{c}{\sigma \sqrt{\Delta t} \sqrt{\frac{2}{\pi}}} \right] = \sigma^2 [1 - L_e]$$

is a solution to the Leland PDE for a European call/put.

- $L_e$ is called Leland number

- Term $\tilde{\sigma}^2$ has to be positive $\Rightarrow$ this gives a bound on feasible times $\Delta t$ - i.e. possible times between two changes of the portfolio (parameters $\sigma, c$ are given):

$$\Delta t > \frac{2 c^2}{\pi \sigma^2}$$
Feasible values of $\Delta t$

**GRAPHICALLY:** dependence of $L_e$ on $\Delta t$ for $c = 5 \times 10^{-4}$, $\sigma = 0.2$
Feasible values of $\Delta t$ 

**NUMERICALLY:** what is the borderline of feasible $\Delta t$:

Assume 252 trading days in a year and the market opened 7 hours a day $\Rightarrow \Delta t$ has to be more than approximately 0.42 min.
Computation of the option price I.

- Let us take $\Delta t = 5$ minutes, i.e., $\Delta t = 5/(60 \times 7 \times 252)$
- Leland number is then feasible (less than 1):

```
-->dt=5/(60*7*252);
-->le(dt)
ans =
    0.2902151
```

- Adjusted volatility, to be used in the Black-Scholes formula:

```
-->sigmaTC=sqrt((1-le(dt))*(sigma^2))
sigmaTC =
    0.1684975
```
Computation of the option price I.

- We compute the price of a call option with exercise price $E = 110$ which expires in $\tau = 1$ year, if the interest rate equals $r = 1\%$ and the underlying stock price is $S = 100$

- For a comparison - price in the absence of transaction costs

\[
\text{Call}(100, 110, 0.01, \text{sigmaTC}, 0.5) \\
\text{ans} = 1.6108991
\]

\[
\text{Call}(100, 110, 0.01, \text{sigma}, 0.5) \\
\text{ans} = 2.3394205
\]
Computation of the option price I.

- The same option if $\Delta t = 1/252$, i.e., 1 day:

```matlab
--dt=1/(252);

-->le(dt)
anst =

    0.0316651

-->sigmaTC=sqrt((1-le(dt))*(sigma^2))
sigmaTC =

    0.1968080

-->Call(100,110,0.01,sigmaTC,0.5)
anst =

    2.2630352
```
Bid a ask prices of options in Leland model

- When deriving the Leland PDE, we considered the portfolio: 1 option, \( \delta \) stocks \( \Rightarrow \) the resulting price is bid price
- Let us consider the portfolio minus 1 option, \( \delta \) stocks \( \Rightarrow \) the resulting price will be ask price
- In the same way we obtain that the ask price satisfies

\[
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \left[ 1 + \frac{c}{\sigma \sqrt{\Delta t}} \sqrt{\frac{2}{\pi}} \text{sign} \left( \frac{\partial^2 V}{\partial S^2} \right) \right] + \frac{\partial V}{\partial S} S - rV = 0
\]

- Call a put options: Black-Scholes price with adjusted volatility \( \sigma^2_{TC} = (1 + Le)\sigma^2 \)
Implied parameters

- If we have bid and ask prices of the stock and the option, we can compute:
  - implied volatility
  - implied time between two changes of the portfolio (i.e., the values, for which the theoretical and market bid and ask option prices will coincide)

**INPUTS:**

- Stock - bid and ask prices $S_{bid}, S_{ask}$
- Option - bid and ask prices $V_{bid}, V_{ask}$, exercise price $E$, time $\tau$ remaining to expiration of the option
- Other market parameters: interest rate $r$
Implied parameters

PROCEDURE:

- Using bid and ask prices of the stock we compute
  \[ S = \frac{S_{ask} + S_{bid}}{2} \quad \text{and} \quad c = \frac{S_{ask} - S_{bid}}{S} \]
- Using \( S, E, r, \tau \) and \( V_{bid} \) we compute the Black-Scholes implied volatility,
  then \[ \sqrt{(1 - L e)\sigma^2} := \sigma_{bid} \]
- Using \( S, E, r, \tau \) and \( V_{ask} \) we compute the Black-Scholes implied volatility,
  then \[ \sqrt{(1 + L e)\sigma^2} := \sigma_{ask} \]
- By solving the system of equations \( (1 - L e)\sigma^2 = \sigma^2_{bid} \),
  \( (1 + L e)\sigma^2 = \sigma^2_{ask} \) we compute the implied volatility \( \sigma \) and
  Leland number \( L e \)
- From the definition of the Leland number we compute the implied time \( \Delta t \) between two changes of the hedging portfolio
Implied parameters - example

**EXAMPLE:**

- Data from 8.3.2014 morning
- Stock:

  ![Stock Data](image)

  | Prev Close | 37.54 |
  | Open      | N/A   |
  | Bid       | 37.90 x 1000 |
  | Ask       | 37.94 x 400  |

  **Day's Range:** 37.54 - 38.01
  **52wk Range:** 27.11 - 41.85
  **Volume:** 594,013
  **Avg Vol (3m):** 26,332,800
Implied parameters - example

- Call option:

![Call option chart]

<table>
<thead>
<tr>
<th>Field</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prev Close</td>
<td>0.90</td>
</tr>
<tr>
<td>Open</td>
<td>1.24</td>
</tr>
<tr>
<td>Bid</td>
<td>1.20</td>
</tr>
<tr>
<td>Ask</td>
<td>1.27</td>
</tr>
<tr>
<td>Strike</td>
<td>37.00</td>
</tr>
<tr>
<td>Expire Date</td>
<td>22-Mar-14</td>
</tr>
<tr>
<td>Day's Range</td>
<td>1.00 - 1.24</td>
</tr>
<tr>
<td>Contract Range</td>
<td>N/A - N/A</td>
</tr>
<tr>
<td>Volume</td>
<td>434</td>
</tr>
<tr>
<td>Open Interest</td>
<td>64,168</td>
</tr>
</tbody>
</table>
Implied parameters - example

• Interest rates:

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<thead>
<tr>
<th>Maturity</th>
<th>Yield</th>
<th>Yesterday</th>
<th>Last Week</th>
<th>Last Month</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 Month</td>
<td>0.04</td>
<td>0.04</td>
<td>0.04</td>
<td>0.04</td>
</tr>
<tr>
<td>6 Month</td>
<td>0.07</td>
<td>0.07</td>
<td>0.08</td>
<td>0.04</td>
</tr>
<tr>
<td>2 Year</td>
<td>0.37</td>
<td>0.35</td>
<td>0.31</td>
<td>0.32</td>
</tr>
<tr>
<td>3 Year</td>
<td>0.77</td>
<td>0.71</td>
<td>0.66</td>
<td>0.65</td>
</tr>
<tr>
<td>5 Year</td>
<td>1.64</td>
<td>1.57</td>
<td>1.47</td>
<td>1.49</td>
</tr>
<tr>
<td>10 Year</td>
<td>2.81</td>
<td>2.73</td>
<td>2.65</td>
<td>2.67</td>
</tr>
<tr>
<td>30 Year</td>
<td>3.74</td>
<td>3.69</td>
<td>3.58</td>
<td>3.65</td>
</tr>
</tbody>
</table>
Implied parameters - example

- Hence we have:

\[
\begin{align*}
S_{bid} &= 37.90; S_{ask} = 37.94; \\
V_{bid} &= 1.20; V_{ask} = 1.27; \\
E &= 37; \\
r &= 0.04/100; \\
\tau &= 11/252; \\
S &= (S_{ask} + S_{bid})/2; \\
c &= (S_{ask} - S_{bid})/S;
\end{align*}
\]
Implied parameters - example

- We compute the implied volatilities

\[
\text{implVolCall}(S, E, r, \tau, V_{\text{bid}}) \rightarrow \sigma_{\text{bid}} = 0.2039042
\]

\[
\text{implVolCall}(S, E, r, \tau, V_{\text{ask}}) \rightarrow \sigma_{\text{ask}} = 0.2298972
\]

- Remarks:
  - \( S \) is common (not \( S_{\text{bid}}, S_{\text{ask}} \))
  - implied volatilities are from Black-Scholes model
Implied parameters - example

- From the system of equations

\[(1 - L_e)\sigma^2 = \sigma^2_{bid}, \quad (1 + L_e)\sigma^2 = \sigma^2_{ask}\]

we compute Leland number \(L_e\) and implied volatility \(\sigma\):

```matlab
-->Le=(sigmaAsk^2-sigmaBid^2)/(sigmaAsk^2+sigmaBid^2);

-->sigma=sigmaAsk/sqrt(1+Le)
```

\[
\text{sigma} = 0.2172898
\]
Implied parameters - example

• From the definition of the Leland number we compute the implied time \( \Delta t \):

\[
\Delta t = \frac{2}{\pi} \left( \frac{c}{\sigma \cdot L_e} \right)^2
\]

\[
\Delta t \times 252 = 0.2651599
\]

**Summary:**

• implied volatility \( \sigma_{impl} = 0.217 \)

• implied time between two changes of the portfolio \( \Delta t_{impl} \) is approximately \( 1/4 \) days
VIII. **Nonlinear models for pricing financial derivatives:**
**Basic ideas behind selected models**

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Models

• Selected models:
  ◦ RAPM (risk adjusted pricing methodology) - transaction costs and risk from the volatile (unprotected) portfolio
  ◦ presence of a dominant investor
  ◦ modelling investor’s preferences

• Aim of this lecture - to show a selection of:
  ◦ financial situations which can be modelled
  ◦ mathematical methods which are used in their analysis
  ◦ basic ideas, to obtain an insight about the models, without detailed derivations

- **Transaction costs** as in the Leland model - then we have the portfolio \( P = V + \delta S \) and the change of its value is 
  \[
  \Delta P = \Delta V + \delta \Delta S - r_{TC} S \Delta t,
  \]

  where

  \[
  r_{TC} = \frac{cS口}{2\pi} \left| \frac{\partial^2 V}{\partial S^2} \right| \frac{1}{\Delta t}
  \]

- **Risk from the volatile portfolio** (risk is measured by variance here):

  \[
  r_{VP} = R \frac{Var[\Delta P/S]}{\Delta t},
  \]

  where \( R \) the marginal value of investor’s exposure to a risk
RAPM model

• It can be shown (Itô lemma, computation of variance of a random variable):

\[
    r_{VP} = \frac{1}{2} R \sigma^4 S^2 \left( \frac{\partial^2 V}{\partial S^2} \right)^2 \Delta t
\]

• Risk neutral investor ⇒ wants - by his choice of \( \Delta t \) - to minimize

\[
    r_R = r_{TC} + r_{VP} = \frac{cS \sigma}{\sqrt{2\pi}} \left| \frac{\partial^2 V}{\partial S^2} \right| \frac{1}{\Delta t} + \frac{1}{2} R \sigma^4 S^2 \left( \frac{\partial^2 V}{\partial S^2} \right)^2 \Delta t
\]

⇒ we obtain the optimal length of the time interval \( \Delta t \) between two adjustments of the portfolio
RAPM model

• Finding the optimal $\Delta t_{opt}$:

$$\frac{\text{Cap} \Delta t}{r} + \frac{\text{Cap} \Delta t}{r_{VP}}$$

• For this value of $\Delta t_{opt}$ we have:

$$r_R(\Delta t_{opt}) = \frac{3}{2} \left( \frac{c^2 R}{2\pi} \right)^{1/3} \sigma^2 \left| S \frac{\partial^2 V}{\partial S^2} \right|^{4/3}$$
RAPM model

- For this value of $\Delta t_{opt}$ we obtain the partial differential equation for the price of a derivative:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \left[ 1 + \mu \left(S \frac{\partial^2 V}{\partial S^2} \right)^{1/3} \right] \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial S} S - rV = 0,$$

where:

$$\mu = 3 \left( \frac{c^2 R}{2\pi} \right)^{1/3}$$

is a constant;

$$\Gamma^p$$ for $\Gamma = S \frac{\partial^2 V}{\partial S^2}$ and $p = 1/3$ is computed as $\Gamma^p = |\Gamma|^{p-1} \Gamma$
**RAPM model**

- Solving the PDE for the derivative price:
  - the PDE is a complicated nonlinear PDE
  - firstly standard transformations: $x = \ln(S/E)$, $\tau = T - t$
  - then - since the PDE contains the term $\Gamma = S \frac{\partial^2 V}{\partial S^2}$ - we define a new function
    \[
    H(x, \tau) = S \frac{\partial^2 V}{\partial S^2}
    \]
  - equation for $H(x, \tau)$ is already much simpler quasilinear PDE and an effective numerical method can be derived to solve it numerically
  - computing the derivative price $V(S, t)$ from the auxiliary function $H(x, \tau)$ is not difficult; it leads to a numerical computation of one integral
RAPM model

• Similarly as in the Leland model - also the RAPM model allows a computation of bid and ask option prices

• Example:

(for a comparison: Black-Scholes option price given by dotted lines)
RAPM model

- Computation of implied parameters from the real data - implied volatility $\sigma$ and implied risk parameter $R$:

Left: input data, right: implied parameters
RAPM model

• The PDE

\[
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \left[ 1 + \mu \left( S \frac{\partial^2 V}{\partial S^2} \right)^{1/3} \right] \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial S} S - rV = 0,
\]

can be seen as an equation with nonconstant volatility \( \tilde{\sigma} = \tilde{\sigma}(S, t) \):

\[
\frac{\partial V}{\partial t} + \frac{\tilde{\sigma}^2(S, t)}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial S} S - rV = 0,
\]

where

\[
\tilde{\sigma}(S, t) = \sigma \left[ 1 + \mu \left( S \frac{\partial^2 V}{\partial S^2} \right)^{1/3} \right]
\]
RAPM model

- What is the behaviour of the function $\tilde{\sigma}(S, t)$?

⇒ this model can explain the volatility smile.
Presence of a dominant investor


- Black-Scholes model: we can buy and sell any amount of assets, but it does not have any effect on their price
- In a case of a dominant investor this is not necessarily true - by his strategy he may influence the asset price
- Consider a dominant investor whose strategy for hedging a derivative is characterized by the following variables:
  - $\alpha_t =$ number of stocks at time $t$
  - $\beta_t =$ number of riskless bonds at time $t$ (i.e. cash)
and suppose that his trading the assets influences their market price:

$$dS = \mu S dt + \sigma S dw + \rho S d\alpha$$
Presence of a dominant investor

- Investor’s strategy depends on the time $t$ and on the stock price $S$:

$$\alpha = \Phi(S, t)$$

- Using Itô lemma we compute $d\alpha$ and insert it into the formula for $dS \rightarrow$ we obtain

$$dS = b(S, t)Sdt + \nu(S, t)Sdw,$$

where

$$\nu(S, t) = \frac{\sigma}{1 - \rho S \frac{\partial \Phi}{\partial S}},$$

$$b(S, t) = \frac{1}{1 - \rho S \frac{\partial \Phi}{\partial S}} \left( \mu + \rho \left( \frac{\partial \Phi}{\partial t} + \frac{\nu^2}{2} S^2 \frac{\partial^2 \Phi}{\partial S^2} \right) \right).$$
Presence of a dominant investor

- PDE is derived in the same way as in the case of Black-Scholes model, the only change is, that instead of the constant $\sigma$ there will be the function $\nu(S, t)$:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \nu^2(S, t)S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$$

- Strategy of the dominant investor:
  - analysis of delta hedging based on Black-Scholes price (it is not suitable, it does not replicate the derivative but always leads to higher transaction costs)
  - computation of the correct strategy
  - its qualitative and quantitative analysis
Presence of a dominant investor

- **Numerical solution of the PDE** - the same idea as in the RAPM model:
  - transformation $H(x, \tau) = S \frac{\partial^2 V}{\partial S^2}$
  - numerical solution of the resulting quasilinear PDE
  - the option price is obtained by integration
Modelling investor’s preferences


- Again transaction costs:

\[ S_{ask} = (1 + \mu)S, \quad S_{bid} = (1 - \mu)S, \]

kde \[ S = (S_{bid} + S_{ask})/2 \]

- Consider the portfolio:
  \[ X_t = \text{value of bonds [in dollars]} \]
  \[ Y_t = \text{number of stocks} \]

- Investor has a utility function \( U \) with a constant risk aversion \( \gamma \)
Modelling investor’s preferences

• If it was not possible to trade options:
  ◦ value of the portfolio at time $T$ is $X_T + Y_T S_T$
  ◦ we need to solve a stochastic programming problem

$$v^f(x, y, s, t) = \sup \mathbb{E}[U(X_T + Y_T S_T)]$$

with initial values $X_t = x, Y_t = y, S_t = s$

• If we write $N$ call options:
  ◦ value of the portfolio at the time of options expiration $T$
    is $X_T + Y_T S_T - N(S_T - E)^+$
  ◦ we need to solve a stochastic programming problem

$$v(x, y, s, t) = \sup \mathbb{E}[U(X_T + Y_T S_T - N(S_T - E)^+)]$$

with initial values $X_t = x, Y_t = y, S_t = s$
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• [Hodges, Neuberger]:
  ◦ relationship between these optimization problems

• [Barles, Soner]:
  ◦ construction of the optimal strategies, PDE of the option price
  ◦ mathematical tools: dynamic programming, introducing a small parameter and asymptotic analysis, transformation of the PDE and its numerical solution
  ◦ resulting PDE for the option price has a similar form as in the previous models: instead of a constant volatility (as in the Black-Scholes model) we have a function which depends also on $\frac{\partial^2 V}{\partial S^2}$ ⇒ a similar approach to its solving