VI. Leland model: Derivation of the PDE for the price of a derivative

Beáta Stehlíková

Financial derivatives

Faculty of Mathematics, Physics and Informatics
Comenius University, Bratislava

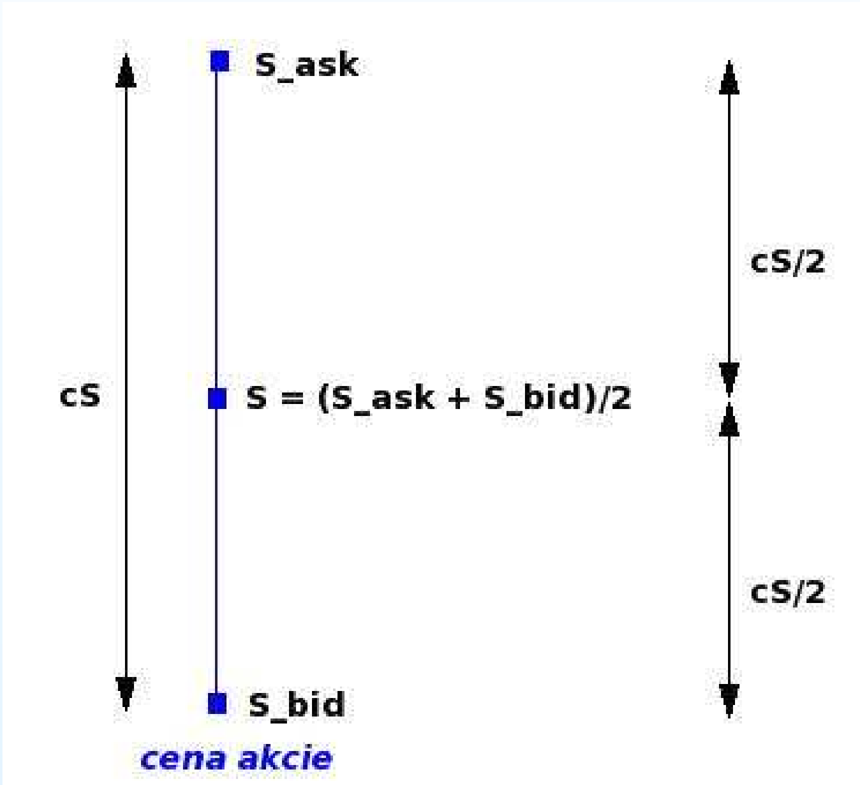
Leland model

- Taking transaction costs into account
- Original paper:

  Hayne E. Leland: **Option Pricing and Replication with Transactions Costs**, 1985

Assumptions of the model

- Transaction costs are characterized by a constant $c = \frac{S_{ask} - S_{bid}}{S}$, where S is the average of bid and ask price of the stock



- S follows a geometric Brownian motion $dS = \mu S dt + \sigma S dw$

Computation of the constant c

Example 1:

- Stock:

Yahoo! Inc. (YHOO) - NasdaqGS

39.82 ↑ 0.32(0.81%) 12:09PM EST - Nasdaq Real Time Price

Prev Close:	**39.50**	Day's Range:	**39.50 - 39.98**
Open:	**39.50**	52wk Range:	**21.87 - 41.72**
Bid:	**39.85** x 1700	Volume:	**5,105,244**
Ask:	**39.86** x 1100	Avg Vol (3m):	**16,828,600**
1y Target Est:	**41.16**	Market Cap:	**40.19B**
Beta:	**1.12**	P/E (ttm):	**31.63**
Earnings Date:	**Apr 14 - Apr 18 (Est.)**	EPS (ttm):	**1.26**
		Div & Yield:	**N/A (N/A)**

- From the data: $S_{bid} = 39.85, S_{ask} = 39.86$
- Average of bid and ask: $S = 39.855$
- $c = \dfrac{0.01}{39.855} = 2.5028 \times 10^{-4}$

Computation of the constant c

EXAMPLE 2:

- Stock:

Microsoft Corporation (MSFT) - NasdaqGS ★ Follow

38.09 ↓ **0.02(0.05%)** 12:12PM EST - Nasdaq Real Time Price

Prev Close:	38.11	Day's Range:	37.89 - 38.18
Open:	38.11	52wk Range:	27.64 - 38.98
Bid:	38.06 x 3900	Volume:	8,045,030
Ask:	38.07 x 6200	Avg Vol (3m):	37,930,700
1y Target Est:	38.60	Market Cap:	316.20B
Beta:	0.71	P/E (ttm):	14.10
Next Earnings Date:	24-Apr-14	EPS (ttm):	2.70
		Div & Yield:	1.12 (2.90%)

- From the data: $S_{bid} = 38.06, S_{ask} = 38.07$
- Average of bid and ask: $S = 38.065$
- $c = \frac{0.01}{38.065} = 2.6271 \times 10^{-4}$

Computation of the constant c

EXAMPLE 3:

- Stock:

Amazon.com Inc. (AMZN) - NasdaqGS

372.27 ↓ 0.10(0.03%) 12:12PM EST - Nasdaq Real Time Price

Prev Close:	372.37	Day's Range:	368.90 - 375.33
Open:	374.08	52wk Range:	245.75 - 408.06
Bid:	372.81 x 100	Volume:	1,576,414
Ask:	372.94 x 200	Avg Vol (3m):	3,606,330
1y Target Est:	433.05	Market Cap:	170.97B
Beta:	0.77	P/E (ttm):	631.76
Earnings Date:	Apr 21 - Apr 25 (Est.)	EPS (ttm):	0.59
		Div & Yield:	N/A (N/A)

- From the data: $S_{bid} = 372.81, S_{ask} = 372.94$
- Average of bid and ask: $S = 372.875$
- $c = \frac{0.13}{372.875} = 3.4864 \times 10^{-4}$

Derivation of the PDE

- Portfolio:
  - one option and δ stocks, while the number of stocks is determined by delta hedging, i.e., $\delta = -\partial V/\partial S$
  - value of the portfolio: $P = V + \delta S$
  - because of the transaction costs, the portfolio cannot be hedged continuously in time $\rightarrow$ we hedge it discrete times which are Δt [years] apart
- Change of the portfolio value
  - number of transactions with stocks is $\Delta\delta$
  - costs for one transaction are $cS/2 \Rightarrow$ total costs are equal to $\frac{cS}{2}|\Delta\delta|$
  - therefore, change of the portfolio value is:

$$\Delta P = \Delta V + \delta \Delta S - \frac{cS}{2}|\Delta\delta|$$

Derivation of the PDE

- Hence we have $\Delta P = \Delta V + \delta \Delta S - \frac{cS}{2}|\Delta\delta|$ and
  - $\Delta S = \mu S \Delta t + \sigma S \Delta w$ from the assumptions (GBM)
  - $\Delta V = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2}\right) \Delta t + \sigma S \frac{\partial V}{\partial S} \Delta w$ from Itō lemma
  - what remains, is to derive $\Delta\delta$
- We have $\delta = -\frac{\partial V}{\partial S}$, hence $\frac{\partial \delta}{\partial S} = -\frac{\partial^2 V}{\partial S^2}$, from which:

$$\Delta\delta \approx \frac{\partial \delta}{\partial S} \Delta S = -\frac{\partial^2 V}{\partial S^2} \Delta S$$

- Here we substitute ΔS from the GBM

Derivation of the PDE

- So far we have:

$$\Delta\delta \approx -\frac{\partial^2 V}{\partial S^2}\mu S\Delta t - \frac{\partial^2 V}{\partial S^2}\sigma S\Delta w \qquad (1)$$

- Leland has shown:
  - in formula (1), it suffices to consider the terms of the lowest order (i.e. we take only $\Delta w \approx (\Delta t)^{1/2}$, and Δt is neglected)
  - when computing the absolute value $|\Delta w|$, it can be replaced by its expected value $\mathbb{E}[|\Delta w|] = \sqrt{\frac{2}{\pi}\Delta t}$
- Therefore:

$$\begin{aligned}\Delta\delta &\approx -\frac{\partial^2 V}{\partial S^2}\sigma S\Delta w \\ |\Delta\delta| &\approx \left|\frac{\partial^2 V}{\partial S^2}\right|\sigma S|\Delta w| \approx \left|\frac{\partial^2 V}{\partial S^2}\right|\sigma S\sqrt{\frac{2}{\pi}}\sqrt{\Delta t}\end{aligned}$$

Derivation of the PDE

- We substitute everything into the formula for the change of the portfolio value $\Delta P = \Delta V + \delta \Delta S - \frac{cS}{2}|\Delta\delta|$:

$$\Delta P = \left(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - \frac{c}{2} S \left| \frac{\partial^2 V}{\partial S^2} \right| \sigma S \sqrt{\frac{2}{\pi \Delta t}} \right) \Delta t \quad (2)$$

- Portfolio is riskless $\Rightarrow$ necessarily (to rule out arbitrage possibilities) $\Delta P = rP\Delta t$
- Portfolio consists of one option and $\delta = -\partial V / \partial S$ stocks $\Rightarrow$ $P = V + \delta S = V - \frac{\partial V}{\partial S}$, and so

$$\Delta P = r(V - \frac{\partial V}{\partial S} S)\Delta t \quad (3)$$

- Comparing (2) and (3):

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - \frac{c}{2} S \left| \frac{\partial^2 V}{\partial S^2} \right| \sigma S \sqrt{\frac{2}{\pi \Delta t}} = r(V - \frac{\partial V}{\partial S} S)$$

Derivation of the PDE

- We write the PDE into its final form:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \left[1 - \frac{c}{\sigma\sqrt{\Delta t}} \sqrt{\frac{2}{\pi}} sign\left(\frac{\partial^2 V}{\partial S^2} \right) \right] + \frac{\partial V}{\partial S} S - rV = 0$$

- The PDE holds for $S > 0, t \in [0, T]$, we add the terminal condition $V(S,T)$ depending on the type of the derivative, e.g., $V(S,T) = \max(0, S - E)$ for $S > 0$ when pricing a call option
- Nonliner PDE because of the term containing the *signum* function
- However, we will solve it in a closed form for call and put options

Remark on combined strategies

- The price of combined strategies (unlike in the Black-Scholes setting) cannot be found be pricing every option and then adding the prices

MATHEMATICAL POINT OF VIEW:

- PDE in the Leland model is not linear $\Rightarrow$ for example a sum, difference or some other linear combination is no more a solution

Remark on combined strategies

FINANCIAL POINT OF VIEW:

- If we price every option separately, we count transaction costs coming from hedging the portfolio for each of the options separately
- If the transaction costs are zero, it does not matter that e.g. we have two portfolios, for one of them we buy stock and for the other one we sell stocks (it does not cause any transaction costs)
- In a presence of transaction costs this is no more true. In such a case we need to consider one portfolio, to avoid unnecessary transactions (which would increase transaction costs)

VII. Leland model: European call and put options

Beáta Stehlíková
Financial derivatives,

Faculty of Mathematics, Physics and Informatics
Comenius University, Bratislava

Leland PDE

- Recall the Leland PDE for the price of a derivative:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \left[1 - \frac{c}{\sigma\sqrt{\Delta t}} \sqrt{\frac{2}{\pi}} sign \left(\frac{\partial^2 V}{\partial S^2} \right) \right] + \frac{\partial V}{\partial S} S - rV = 0$$

- The PDE holds for $S > 0, t \in [0, T]$, we add the terminal condition $V(S,T)$ depending on the derivative, e.g., $V(S,T) = \max(0, S - E)$ for $S > 0$ in the case of a call option
- Nonlinear PDE because of the term containing *signum*
- Recall the for the Black-Scholes prices of call and put options we have $\frac{\partial^2 V}{\partial S^2} > 0$ (positive gamma) $\Rightarrow$ $sign\left(\frac{\partial^2 V}{\partial S^2}\right) = 1$

Leland PDE - call and put

- What happens if inserting Black-Scholes price of a call/put with adjusted volatility $V(S,t;\tilde\sigma)$:

$$\tilde\sigma^2 = \sigma^2\left[1 - \frac{c}{\sigma\sqrt{\Delta t}}\sqrt{\frac{2}{\pi}}\right]$$

into the Leland PDE:

$$\begin{aligned}
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}\left[1 - \frac{c}{\sigma\sqrt{\Delta t}}\sqrt{\frac{2}{\pi}}sign\left(\frac{\partial^2 V}{\partial S^2}\right)\right] + \frac{\partial V}{\partial S}S - rV &= \\
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}\left[1 - \frac{c}{\sigma\sqrt{\Delta t}}\sqrt{\frac{2}{\pi}}\right] + \frac{\partial V}{\partial S}S - rV & \\
\frac{\partial V}{\partial t} + \frac{\tilde\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial S}S - rV &= 0
\end{aligned}$$

Leland PDE - call and put

- It means that Black-Scholes price of a call/put with adjusted volatility $V(S,t;\tilde\sigma)$:

$$\tilde\sigma^2 = \sigma^2 \left[1 - \frac{c}{\sigma\sqrt{\Delta t}}\sqrt{\frac{2}{\pi}}\right] = \sigma^2\,[1 - Le]$$

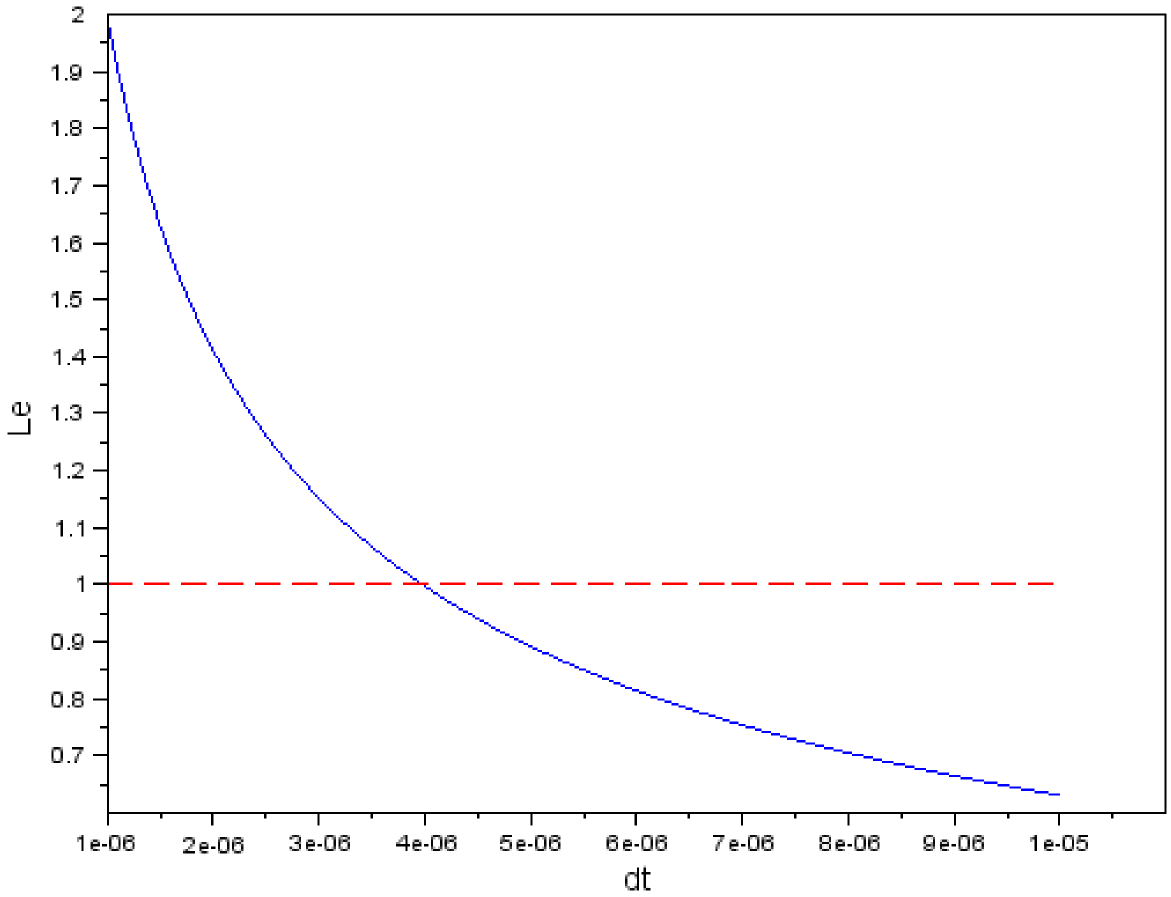
  is a solution to the Leland PDE for a European call/put.

- Le is called Leland number
- Term $\tilde\sigma^2$ has to be positive $\Rightarrow$ this gives a bound on feasible times Δt - i.e. possible times between two changes of the portfolio (parameters σ, c are given):

$$\Delta t > \frac{2}{\pi}\frac{c^2}{\sigma^2}$$

Feasible values of Δt

GRAPHICALLY: dependence of Le on Δt for $c = 5 \times 10^{-4}$, $\sigma = 0.2$



Feasible values of Δt

NUMERICALLY: what is the borderline of feasible Δt:
Assume 252 trading days in a year and the market opened 7 hours a day $\Rightarrow$ Δt has to be more than approximately 0.42 min.

Computation of the option price I.

- Let us take $\Delta t = 5$ minutes, i.e., $\Delta t = 5/(60 * 7 * 252)$
- Leland number is then feasible (less than 1):

```
-->dt=5/(60*7*252);

-->le(dt)
 ans  =

    0.2902151
```

- Adjusted volatility, to be used in the Black-Scholes formula:

```
-->sigmaTC=sqrt((1-le(dt))*(sigma^2))
 sigmaTC  =

    0.1684975
```

Computation of the option price I.

- We compute the price of a call option with exercise price $E = 110$ which expires in $\tau = 1$ year, if the interest rate equals $r = 1\%$ and the underlying stock price is $S = 100$
- For a comparison - price in the absence of transaction costs

```
-->Call(100,110,0.01,sigmaTC,0.5)
 ans  =


    1.6108991

-->Call(100,110,0.01,sigma,0.5)
 ans  =


    2.3394205
```

Computation of the option price I.

- The same option if $\Delta t = 1/252$, i.e., 1 day:

```
-->dt=1/(252);

-->le(dt)
 ans  =

    0.0316651

-->sigmaTC=sqrt((1-le(dt))*(sigma^2))
 sigmaTC  =

    0.1968080

-->Call(100,110,0.01,sigmaTC,0.5)
 ans  =

    2.2630352
```

Bid a ask prices of options in Leland model

- When deriving the Leland PDE, we considered the portfolio: 1 option, δ stocks $\Rightarrow$ the resulting price is *bid* price
- Let us consider the portfolio minus 1 option, δ stocks $\Rightarrow$ the resulting price will be *ask* price
- In the same way we obtain that the ask price satisfies

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \left[1 + \frac{c}{\sigma \sqrt{\Delta t}} \sqrt{\frac{2}{\pi}} sign \left(\frac{\partial^2 V}{\partial S^2} \right) \right] + \frac{\partial V}{\partial S} S - rV = 0$$

- Call a put options: Black-Scholes price with adjusted volatility $\sigma_{TC}^2 = (1 + Le)\sigma^2$

Implied parameters

- If we have bid and ask prices of the stock and the option, we can compute:
  - implied volatility
  - implied time between two changes of the portfolio

  (i.e., the values, for which the theoretical and market bid and ask option prices will coincide)

INPUTS:

- Stock - bid and ask prices S_{bid}, S_{ask}
- Option - bid and ask prices V_{bid}, V_{ask}, exercise price E, time τ remaining to expiration of the option
- Other market parameters: interest rate r

Implied parameters

PROCEDURE:

- Using bid and ask prices of the stock we compute $S = (S_{ask} + S_{bid})/2$ and $c = (S_{ask} - S_{bid})/S$
- Using S, E, r, τ and
  - V_{bid} we compute the Black-Scholes implied volatility, then $\sqrt{(1 - Le)\sigma^2} := \sigma_{bid}$
  - V_{ask} we compute the Black-Scholes implied volatility, then $\sqrt{(1 + Le)\sigma^2} := \sigma_{ask}$
- By solving the system of equations $(1 - Le)\sigma^2 = \sigma_{bid}^2$, $(1 + Le)\sigma^2 = \sigma_{ask}^2$ we compute the implied volatility σ and Leland number Le
- From the definition of the Leland number we compute the implied time Δt between two changes of the hedging portfolio

Implied parameters - example

EXAMPLE:

- Data from 8.3.2014 morning
- Stock:

General Motors Company (GM) - NYSE ★ Follow

37.65 ↑0.11 (0.29%) 9:44AM EST - Nasdaq Real Time Price

Prev Close:	**37.54**	Day's Range:	**37.54 - 38.01**
Open:	**N/A**	52wk Range:	**27.11 - 41.85**
Bid:	**37.90 x 1000**	Volume:	**594,013**
Ask:	**37.94 x 400**	Avg Vol (3m):	**26,332,800**

Implied parameters - example

- Call option:

GM Mar 2014 37.000 call (GM140322C00037000) - OPR

1.00 ↑0.10(11.11%) Mar 6

Prev Close:	0.90	Day's Range:	1.00 - 1.24
Open:	1.24	Contract Range:	N/A - N/A
Bid:	1.20	Volume:	434
Ask:	1.27	Open Interest:	64,168
Strike:	37.00		
Expire Date:	22-Mar-14		

Implied parameters - example

- Interest rates:

US Treasury Bonds Rates

Maturity	Yield	Yesterday	Last Week	Last Month
3 Month	0.04	0.04	0.04	0.04
6 Month	0.07	0.07	0.08	0.04
2 Year	0.37	0.35	0.31	0.32
3 Year	0.77	0.71	0.66	0.65
5 Year	1.64	1.57	1.47	1.49
10 Year	2.81	2.73	2.65	2.67
30 Year	3.74	3.69	3.58	3.65

Implied parameters - example

- Hence we have:

```
Sbid=37.90; Sask=37.94;
Vbid=1.20; Vask=1.27;
E=37;
r=0.04/100;

tau=11/252;


S=(Sask+Sbid)/2;
c=(Sask-Sbid)/S;
```

Implied parameters - example

- We compute the implied volatilities

```
->sigmaBid=ImplVolCall(S,E,r,tau,Vbid)
sigmaBid  =

   0.2039042

->sigmaAsk=ImplVolCall(S,E,r,tau,Vask)
sigmaAsk  =

   0.2298972
```

- Remarks:
  - S is common (not S_{bid}, S_{ask})
  - implied volatilities are from Black-Scholes model

Implied parameters - example

- From the system of equations

$$(1 - Le)\sigma^2 = \sigma^2_{bid}, \ (1 + Le)\sigma^2 = \sigma^2_{ask}$$

we compute Leland number Le and implied volatility σ :

```
-->Le=(sigmaAsk^2-sigmaBid^2)/(sigmaAsk^2+sigmaBid^2);
```

```
->sigma=sigmaAsk/sqrt(1+Le)
sigma  =

   0.2172898
```

Implied parameters - example

- From the definition of the Leland number we compute the implied time Δt:

```
->dt=(2/%pi)*(c/(sigma*Le))^2;

->dt*252
ans  =

   0.2651599   dt in days
```

SUMMARY:

- implied volatility $\sigma_{impl} = 0.217$
- implied time between two changes of the portfolio Δt_{impl} is approximately $1/4$ days

VIII. Nonlinear models for pricing financial derivatives: Basic ideas behind selected models

Beáta Stehlíková
Financial derivatives

Faculty of Mathematics, Physics and Informatics
Comenius University, Bratislava

Models

- Selected models:
  - RAPM (risk adjusted pricing methodology) - transaction costs and risk from the volatile (unprotected) portfolio)
  - presence of a dominant investor
  - modelling investor's preferences
- Aim of this lecture - to show a selection of:
  - financial situations which can be modelled
  - mathematical methods which are used in their analysis
  - basic ideas, to obtain an insight about the models, without detailed derivations

RAPM model

M. Jandačka, D. Ševčovič: **On the risk adjusted pricing methodology based valuation of vanilla options and explanation of the volatility smile**, Journal of Applied Mathematics, 3, 2005, 235-258

- Transaction costs as in the Leland model - then we have the portfolio $P = V + \delta S$ and the change of its value is $\Delta P = \Delta V + \delta \Delta S - r_{TC} S \Delta t$, where

$$r_{TC} = \frac{c S \sigma}{\sqrt{2\pi}} \left| \frac{\partial^2 V}{\partial S^2} \right| \frac{1}{\Delta t}$$

- Risk from the volatile portfolio (risk is measured by variance here):

$$r_{VP} = R \frac{Var[\Delta P / S]}{\Delta t},$$

where R the marginal value of investor's exposure to a risk

RAPM model

- It can be shown (Itō lemma, computation of variance of a random variable):

$$r_{VP} = \frac{1}{2} R\sigma^4 S^2 \left(\frac{\partial^2 V}{\partial S^2} \right)^2 \Delta t$$

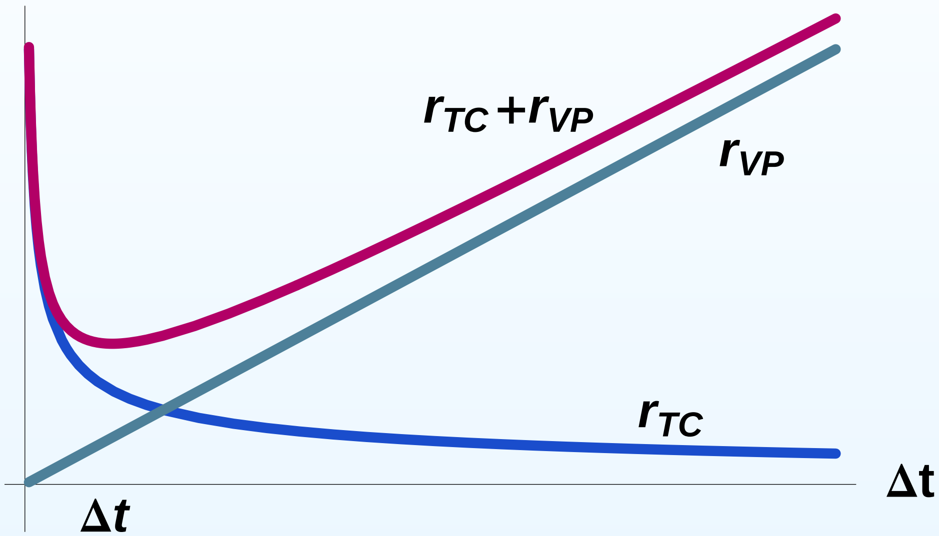
- Risk neutral investor $\Rightarrow$ wants - by his choice of Δt - to minimize

$$r_R = r_{TC} + r_{VP} = \frac{cS\sigma}{\sqrt{2\pi}} \left| \frac{\partial^2 V}{\partial S^2} \right| \frac{1}{\Delta t} + \frac{1}{2} R\sigma^4 S^2 \left(\frac{\partial^2 V}{\partial S^2} \right)^2 \Delta t$$

$\Rightarrow$ we obtain the optimal length of the time interval Δt between two adjustments of the portfolio

RAPM model

- Finding the optimal Δt_{opt}:



- For this value of Δt_{opt} we have:

$$r_R(\Delta t_{opt}) = \frac{3}{2}\left(\frac{c^2 R}{2\pi}\right)^{1/3} \sigma^2 \left| S \frac{\partial^2 V}{\partial S^2} \right|^{4/3}$$

RAPM model

- For this value of Δt_{opt} we obtain the partial differential equation for the price of a derivative :

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \left[1 + \mu \left(S \frac{\partial^2 V}{\partial S^2} \right)^{1/3} \right] \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial S} S - rV = 0,$$

where:

$\mu = 3 \left(\frac{c^2 R}{2\pi} \right)^{1/3}$ is a constant;

Γ^p for $\Gamma = S \frac{\partial^2 V}{\partial S^2}$ and $p = 1/3$ is computed as $\Gamma^p = |\Gamma|^{p-1} \Gamma$

RAPM model

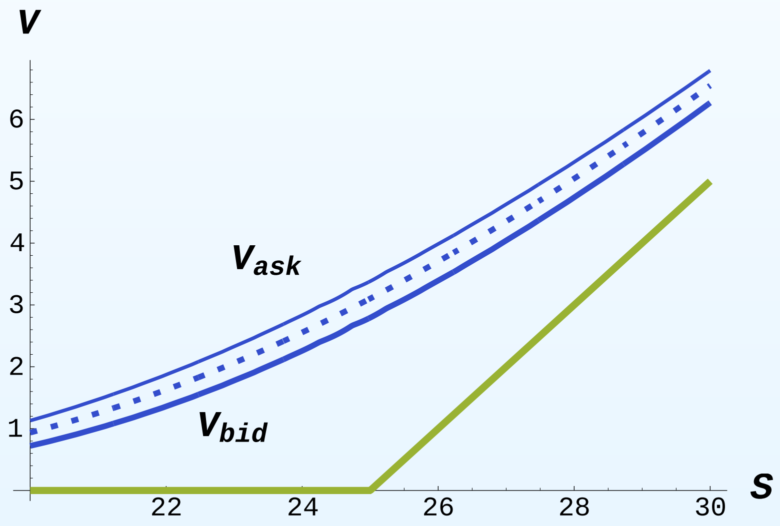
- Solving the PDE for the derivative price:
  - the PDE is a complicated nonlinear PDE
  - firstly standard transformations: $x = \ln(S/E)$, $\tau = T - t$
  - then - since the PDE contains the term $\Gamma = S\frac{\partial^2 V}{\partial S^2}$ - we define a new function

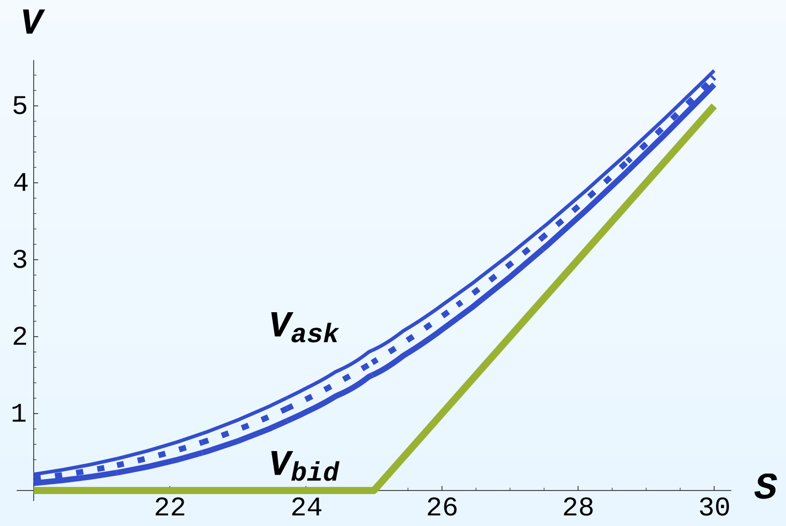
$$H(x,\tau) = S\frac{\partial^2 V}{\partial S^2}$$

  - equation for $H(x,\tau)$ is already much simpler quasilinear PDE and an effective numerical method can be derived to solve it numerically
  - computing the derivative price $V(S,t)$ from the auxiliary function $H(x,\tau)$ is not difficlut; it leads to a numerical computation of one integral

RAPM model

- Similarly as in the Leland model - also the RAPM model allows a computation of bid and ask option prices
- Example:

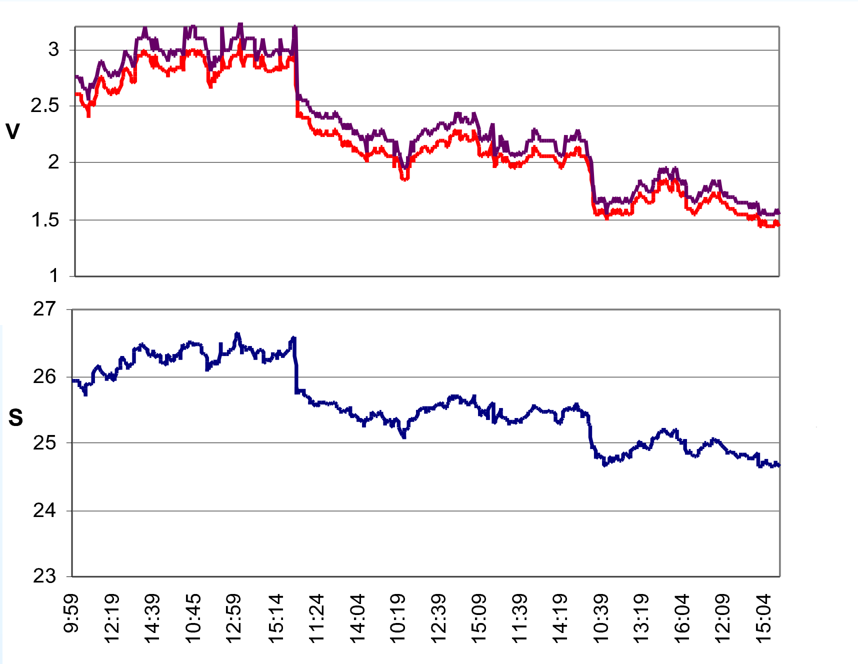


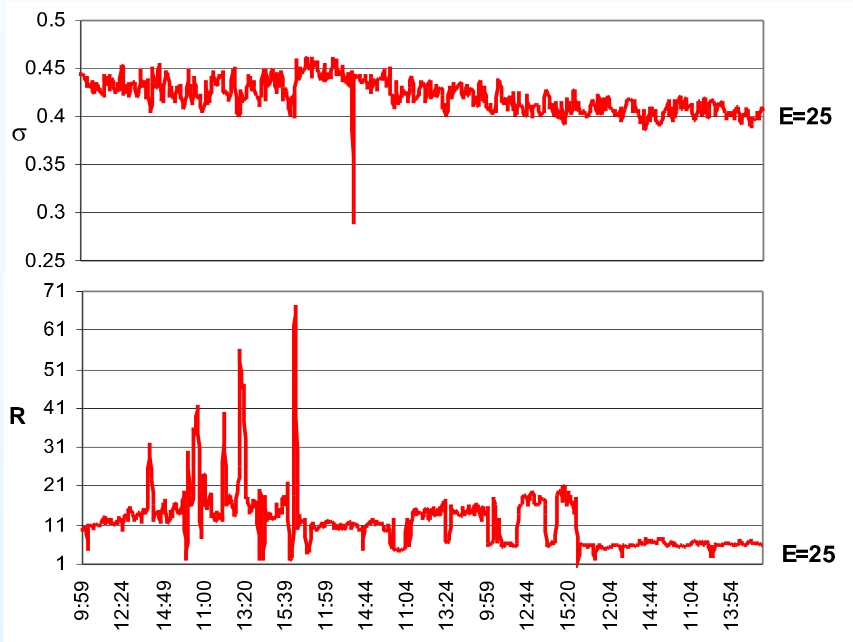


(for a comparison: Black-Scholes option price given by dotted lines)

RAPM model

- Computation of implied parameters from the real data - implied volatility σ and implied risk parameter R:





Left: input data, right: implied parameters

RAPM model

- The PDE

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \left[1 + \mu \left(S \frac{\partial^2 V}{\partial S^2} \right)^{1/3} \right] \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial S} S - rV = 0,$$

can be seen as an equation with nonconstant volatility $\tilde{\sigma} = \tilde{\sigma}(S,t)$:

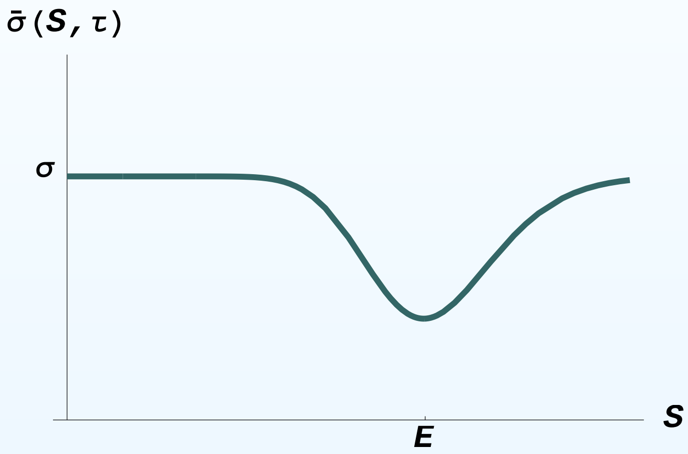
$$\frac{\partial V}{\partial t} + \frac{\tilde{\sigma}^2(S,t)}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial S} S - rV = 0,$$

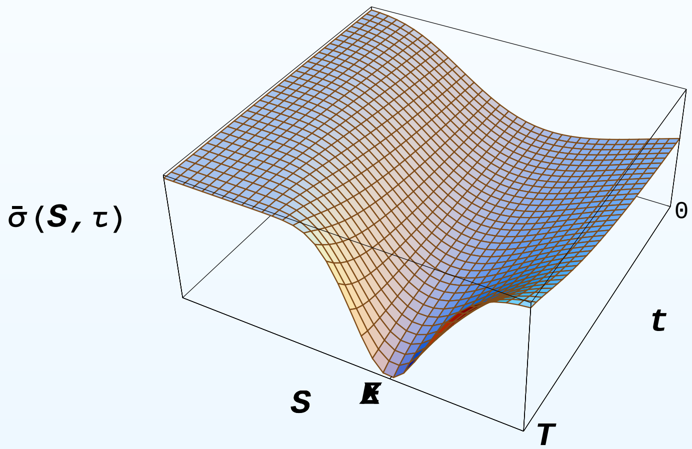
where

$$\tilde{\sigma}(S,t) = \sigma \left[1 + \mu \left(S \frac{\partial^2 V}{\partial S^2} \right)^{1/3} \right]$$

RAPM model

- What is the behaviour of the function $\tilde{\sigma}(S, t)$:





$\Rightarrow$ this model can explain the volatility smile

Presence of a dominant investor

R. Frey: **Market illiquidity as a source of the model risk in dynamic hedging**, RISK publications, R. Gibson Ed., London, 2000.

- Black-Scholes model: we can buy and sell any amount of assets, but it does not have any effect on their price
- In a case of a dominant investor this is not necessarily true - by his strategy he may influence the asset price
- Consider a dominant investor whose strategy for hedging a derivative is characterized by the following variables:
  - α_t = number of stocks at time t
  - β_t = number of riskless bonds at time t (i.e. cash)

  and suppose that his trading the assets influences their market price:

$$dS = \mu S dt + \sigma S dw + \rho S d\alpha$$

Presence of a dominant investor

- Investor's strategy depends on the time t and on the stock price S:

$$\alpha = \Phi(S, t)$$

- Using Itō lemma we compute $d\alpha$ and insert it into the formula for $dS \rightarrow$ we obtain

$$dS = b(S,t)Sdt + \nu(S,t)Sdw,$$

where

$$\begin{aligned}
\nu(S,t) &= \frac{\sigma}{1 - \rho S \frac{\partial \Phi}{\partial S}}, \\
b(S,t) &= \frac{1}{1 - \rho S \frac{\partial \Phi}{\partial S}} \left(\mu + \rho \left(\frac{\partial \Phi}{\partial t} + \frac{\nu^2}{2} S^2 \frac{\partial^2 \Phi}{\partial S^2} \right) \right).
\end{aligned}$$

Presence of a dominant investor

- PDE is derived in the same way as in the case of Black-Scholes model, the only change is, that instead of the constant σ there will be the function $\nu(S,t)$:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\nu^2(S,t)S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

- Strategy of the dominant investor:
  - analysis of delta hedging based on Black-Scholes pricce (it is not suitable, it does not replicate the derivative but always leads to higher transaction costs)
  - computation of the correct strategy
  - its qualitative and quantitative analysis

Presence of a dominant investor

- Numerical solution of the PDE - the same idea as in the RAPM model:
  - transformation $H(x,\tau) = S\frac{\partial^2 V}{\partial S^2}$
  - numerical solution of the resulting quasilinear PDE
  - the option price is obtained by integration

Modelling investor's preferences

S. D. Hodges, A. Neuberger: **Optimal replication of contingent claims under transaction costs**, Advances o Futures and Options Research(1994), 21-35.

G. Barles, H.M. Soner: **Option Pricing with transaction costs and a nonlinear Black-Scholes equation**, Finance Stochast. 2 (1998) 369-397.

- Again transaction costs:

$$S_{ask} = (1 + \mu)S, \; S_{bid} = (1 - \mu)S,$$

  kde $S = (S_{bid} + S_{ask})/2$

- Consider the portfolio:
  $X_t =$ value of bonds [in dollars]
  $Y_t =$ number of stocks
- Investor has a utility function U with a constant risk aversion γ

Modelling investor's preferences

- If it was not possible to trade options::
  - value of the portfolio at time T is $X_T + Y_T S_T$
  - we need to solve a stochastic programming problem

$$v^f(x, y, s, t) = \sup \mathbb{E}[U(X_T + Y_T S_T)]$$

with initial values $X_t = x$, $Y_t = y$, $S_t = s$

- If we write N call options:
  - value of the portfolio at the time of options expiration T is $X_T + Y_T S_T - N(S_T - E)^+$
  - we need to solve a stochastic programming problem

$$v(x, y, s, t) = \sup \mathbb{E}[U(X_T + Y_T S_T - N(S_T - E)^+)]$$

with initial values $X_t = x$, $Y_t = y$, $S_t = s$

Modelling investor's preferences

- [Hodges, Neuberger]:
  - relationship between these optimization problems
- [Barles, Soner]:
  - construction of the optimal strantegies, PDE of the option price
  - matematical tools: dynamic programming, introducing a small parameter and asymptotic analysis, transformation of the PDE and its numerical solution
  - resulting PDE for the option price has a similar form as in the previous models: instead of a constant volatility (as in the Black-Scholes model) we have a function which depends also on $\frac{\partial^2 V}{\partial S^2}$ $\Rightarrow$ a similar approach to its solving