IX. Pricing American options

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Financial derivatives

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European and American types of derivatives

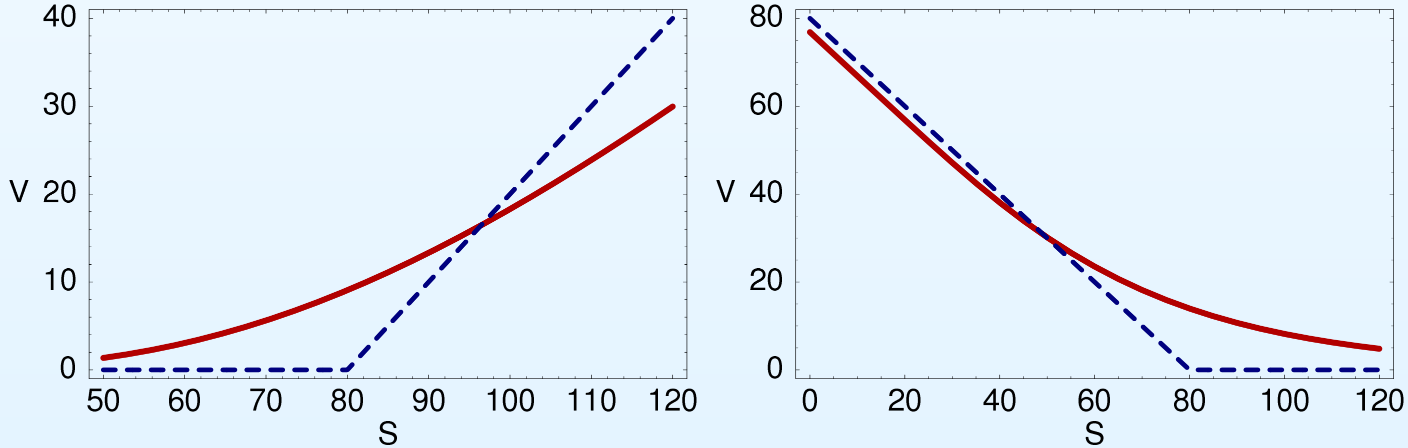
- Consider a put option with exercise price of 150 USD, i.e., a right to sell the underlying stock for 150 USD. Suppose that the option expires in one month.
  - if it is a European option: this right can be exercised at the time of expiry
  - if it is an American options: this right can be exercised at any time prior to the expiry
- If the option costs 20 USD and the stock price today is 110 USD:
  - if it is a European option:: our profit depends on future evolution of the stock price
  - if it is an American option: we buy the option and exercise it immediately $\rightarrow$ instantaneous riskless profit
- EXERCISE: Create a similar example for a call option.

European and American types of derivatives

- Bound on possible option price which has to hold to rule out possibilities of riskless profit (i.e., arbitrage):

price of an American derivative cannot lie under its payoff

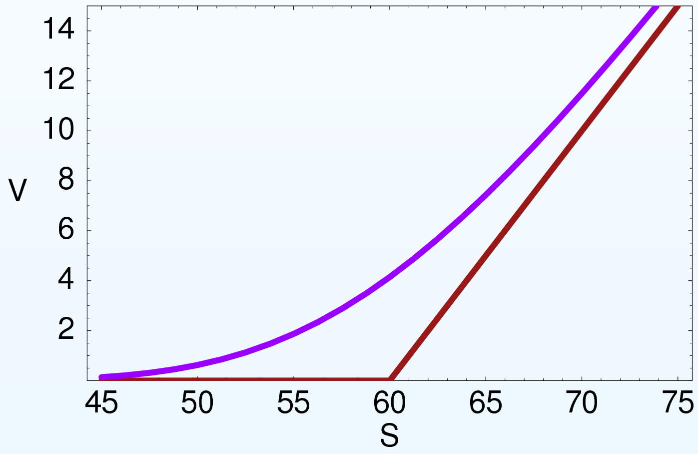
- European derivatives: call on a stock which pays dividends, put on an arbitrary stock



$\Rightarrow$ price of American-type derivative is not equal to its Euopean counterpart

European and American types of derivatives

- European derivatives - continued: call on a stock that does not pay dividends



the price is always above the payoff $\Rightarrow$ agrees with our knowledge from the financial mathematics lectures: price of an American call equals to its European counterpart

European and American types of derivatives

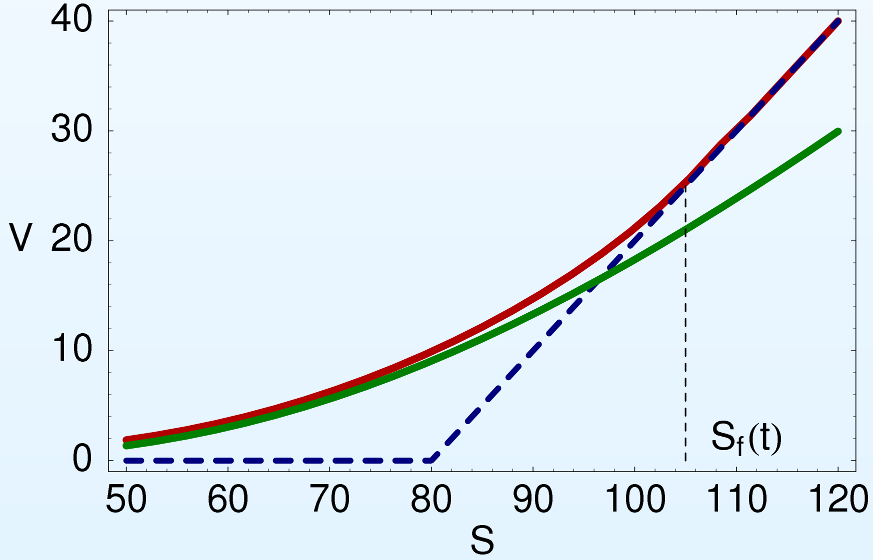
- This holds in general - we show that:
  - If the underlying stock does not pay dividends, the call option price always lies above the payoff.
  - If the underlying stock pays dividends, the call option price always intersects the payoff.
  - The call option price always intersects the payoff.

  Basic idea: we compute the limit of $V(S,t)/(S-E)$, as $S \to \infty$ (call), resp. $S \to 0^+$ (put)
- Therefore:
  - The price of an American call on a stock that does not pay dividends equals to the price of a European option with the same parameters.
  - The prices of American calls on a stock that pays dividends and of all puts needs to computed in another way.

American derivatives

- We cannot take: price = max (European option, payoff) - the price has to be a **smooth function**
- Why smooth - matematical derivation:

  Ševčovič, Stehlíková, Mikula: **Analytical and numerical methods for pricing financial derivatives**, pp. 132-133
- Sketch of the solution (smooth pasting at $S_f(t)$):



American derivatives

- Solution (for a call):
  - if $S < S_f(t)$: price satisfies the Black-Scholes PDE, we keep the option (do not exercise it)
  - if $S > S_f(t)$: price equals payoff, we exercise the option
  - if $S = S_f(t)$: price has the same value (continuity condition) and the same derivative (smoothness condition) as the payoff
- $S_f(t)$ - early exercise boundary, from a mathematical point of view it is a free boundary

Mathematical formulation of the problem

- For a call option:
  - fuction $V(S,t)$ is a solution to the Black-Scholes PDE

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r-q) S \frac{\partial V}{\partial S} - rV = 0$$

    on a time-dependent domain $0 < t < T, 0 < S < S_f(t)$.

  - Terminal condition:

$$V(S,T) = \max(S-E, 0).$$

  - Boundary conditions on the boundary $S = 0$ and $S = S_f(t)$ for $0 < t < T$:

$$V(0,t) = 0, \quad V(S_f(t),t) = S_f(t) - E, \quad \frac{\partial V}{\partial S}(S_f(t),t) = 1$$

Mathematical formulation of the problem

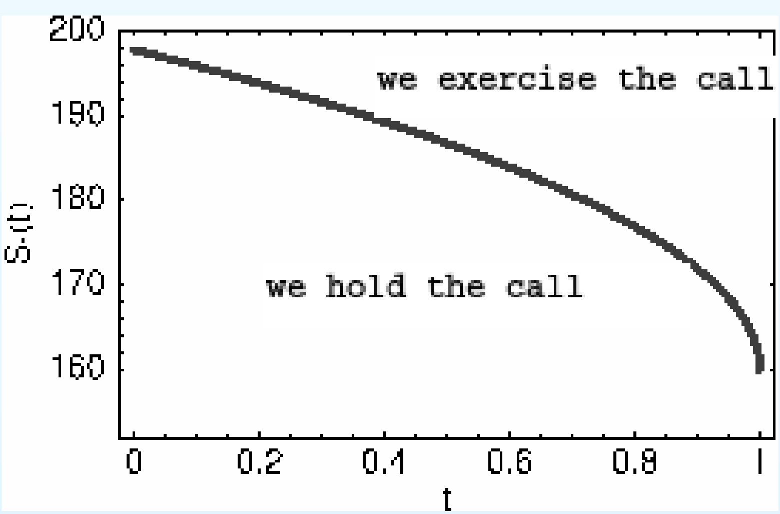
- Changes in case of a put option:
  - time-dependent domain is $0 < t < T$, $S > S_f(t)$
  - terminal condition is $V(S,T) = \max(E - S, 0)$
  - boundary condition are

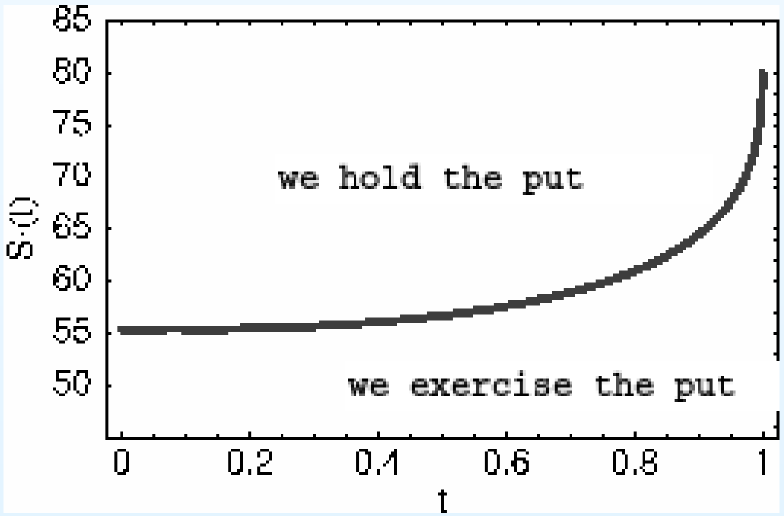
$$V(+\infty, t) = 0,$$

$$V(S_f(t), t) = E - S_f(t), \frac{\partial V}{\partial S}(S_f(t), t) = -1$$

Output

- We obtain
  - the option price $V(S,t)$ as function of the underlying stock price and time
  - early exercise boundary - i.e., an information whether for the given time and stock price we exercise the option or not





Analysis of the free boundary $S_f(t)$

- A research topic in financial mathematics
- We show one inequality for $S_f(t)$ in the case of a call option
- We will need it to derive numerical scheme for pricing American options

Inequality for $S_f(t)$

- Consider the Black-Scholes PDE for $S < S_f$:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0$$

- We take the limit as $S \to S_f$ and use:
    - a lemma, which we prove first: $\frac{\partial V}{\partial t}(S_f(t), t) = 0$
    - convexity of V with respect to S
    - boundary condition

  We obtain:

$$(r-q)S_f(t) - r(S_f(t) - E) \leq 0 \Rightarrow S_f(t) \geq \frac{r}{q}E$$

- Hence:

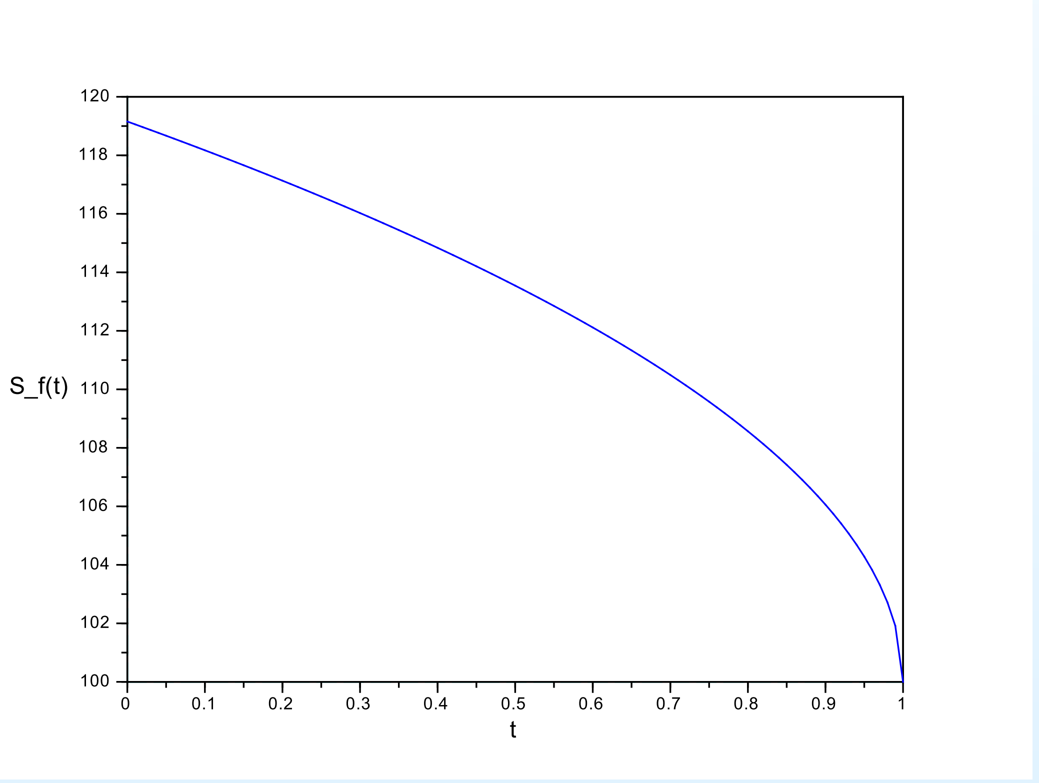
$$S_f(t) \geq E \max(1, r/q)$$

Research in financial mathematics - examples

- J. N. Dewyne, S. D. Howison, J. Rupf, P. Wilmott (1993): **Some mathematical results in the pricing of American options.** Euro. Journal on Applied Mathematics 4, 381-398

Asymptotics for a call option, if $r \leq q$ and $t \to T$:

$$S_f(t) \approx E(1 + 0.638\sigma\sqrt{T - t})$$



Research in financial mathematics - examples

- D. Ševčovič (2001): **Analysis of the free boundary for the pricing of an American Call option.** Euro. Journal on Applied Mathematics 12, 25-37.

  Nonlinear integral equation for $S_f(t)$ and its numerical solution

- S. P. Zhu (2006): **A new analytical approximation formula for the optimal exercise boundary of American put options.** International Journal of Theoretical and Applied Finance 9, 1141-1177.

  Closed-form (but complicated) expression for $S_f(t)$

Linear complementarity problem

- Denote by $V(S,t)$ the price of an American option and by $\bar{V}(S)$ its payoff
- We know that $V(S,t) \geq \bar{V}(S)$ must hold
- We show (for a call) that

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV \leq 0$$

  - if $S < S_f$: we have equality
  - if $S \geq S_f$: then $V(S,t) = S - E$ (because $S_f \geq E$); we substitute it into the left-hand side and use that $S_f \geq Er/q$
- We see that we cannot have both inequalities satisfied as strict inequalities

Linear complementarity problem

- Hence we have a linear complementarity problem:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV \leq 0$$

$$V(S,t) \geq \bar{V}(S)$$

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV\right)\left(V(S,t) - \bar{V}(S)\right) = 0$$

for $S \in (0,\infty), 0 < \tau < T$.

Linear complementarity problem

- The sequence of transformations from the earlier lectures: $V(S,t) \to Z(x,\tau) \to u(x,\tau)$
- Resulting problem for $u(x,\tau)$ in the case of a call:

$$
\begin{aligned}
\left(\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}\right)(u(x,\tau) - g(x,\tau)) &= 0, \\
\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} \geq 0, \quad u(x,\tau) - g(x,\tau) &\geq 0
\end{aligned}
$$

  for $x \in \mathbb{R}, 0 < \tau < T$, where
  - $g(x,\tau) = Ee^{\alpha x + \beta \tau} \max(0, e^x - 1)$ is transformed payoff (α, β in the earlier lectures)
  - $g(x,0)$ is the initial condition $u(x,0)$
- For a put we have instead: $g(x,\tau) = Ee^{\alpha x + \beta \tau} \max(0, 1 - e^x)$