XV. Pricing exotic derivatives

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Financial derivatives

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Exotic options

- Path-dependent options - payoff depends not only on the value of the underlying asset at the expiration time, but also on its evolution (path) before the expiration time
- Lowers the risk coming from sudden changes in prices
- Extra credit 2013, the price rose during the last day before the expiration of the options:

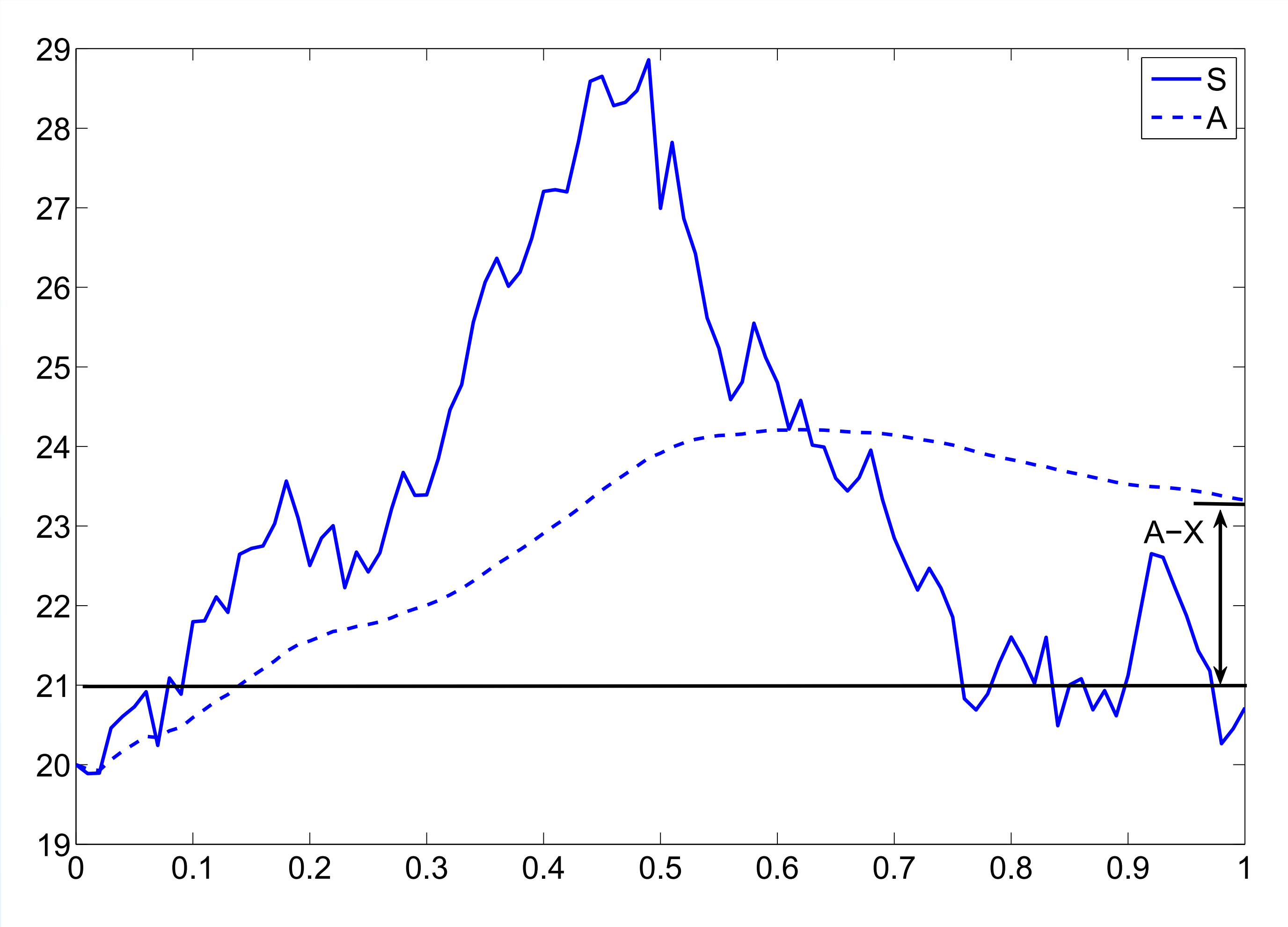
http://finance.google.com/

Asian options

- Payoff depends on historical average of stock prices
- Classification of Asian options:
  - based on averaging - arithmetic or geometric average
  - based on position of the average in the payoff - can take the role of the stock price or the exercise price
- Average:
  - arithmetic:
    discrete case $A_t = \frac{1}{n}\sum_{i=1}^{n} S_{t_i}$
    continuous case $A_t = \frac{1}{t}\int_0^t S_\tau d\tau$
  - geometric:
    discrete case $\ln A_t = \frac{1}{n}\sum_{i=1}^{n} \ln S_{t_i}$
    continuous case $\ln A_t = \frac{1}{t}\int_0^t \ln S_\tau d\tau$

Asian options

Stock price and its average (dashed line):



Asian options

- Position of the average in the payoff
  - the average price A enters the payoff taking the role of the stock price - the option is called average rate call, resp. put :

$$V(S, A, T) = \max(A - E, 0) \quad \text{for a call}$$
$$V(S, A, T) = \max(E - A, 0) \quad \text{for a put}$$

  - the average price A enters the payoff taking the role of the exercise price - the option is called average strike call, resp. put :

$$V(S, A, T) = \max(S - A, 0) \quad \text{for a call}$$
$$V(S, A, T) = \max(A - S, 0) \quad \text{for a put}$$

- So we have, for example:
  - *Asian arithmetically averaged average rate call option*,
  - *Asian geometrically averaged average strike put option*

Differential of the averaged price

- We will use continuous time
- Arithmetic average:

$$\frac{dA}{dt} = -\frac{1}{t^2}\int_0^t S_\tau d\tau + \frac{1}{t}S_t = \frac{S_t - A_t}{t}$$

- Geometric average:

$$\frac{dA}{dt} = A_t\left[-\frac{1}{t^2}\int_0^t \ln S_\tau d\tau + \frac{1}{t}\ln S_t\right] = A_t\,\frac{\ln S_t - \ln A_t}{t}$$

- In both cases:

$$dA = A\,f\left(\frac{S}{A}, t\right)dt,$$

where $f(x,t) = (x-1)/t$, resp. $f(x,t) = (\ln x)/t$

PDE for the Asian option price

- Geometric Brownian motion for the stock price $dS = \mu S dt + \sigma S dw$, stock pays continuous dividends with rate D
- Option price $V = V(S, A, t)$:

$$dV = \frac{\partial V}{\partial S} dS + \left(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial A} A f\left(\frac{S}{A}, t\right) \right) dt$$

- As in the case of the Black-Scholeso model:
  - portfolio: option + stocks
  - elimination the random part of the SDE for the portfolio value
  - yield of a riskless portfolio has to be equal to r (riskless instantaneous interest rate)

PDE for the Asian option price

- Resulting PDE for the Asian option price $V(S, A, t)$:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r - D) S \frac{\partial V}{\partial S} + A f\left(\frac{S}{A}, t\right) \frac{\partial V}{\partial A} - rV = 0$$

for $S \in (0, \infty)$, $A \in (0, \infty)$, $t \in (0, T)$

- Terminal condition depends on the option, e.g.,

$$V(S, A, T) = \max(S - A, 0)$$

for $S \in (0, \infty)$, $A \in (0, \infty)$

- Three variables, but only one derivative of the second order $\rightarrow$ the PDE is not in a suitable form for finding a numerical scheme $\rightarrow$ for average strike option we perform a transformation

Transformation for average strike options

- Transformation:

$$V(S, A, t) = AW(x, t), \quad x = \frac{S}{A}$$

- PDE for the function $W(x, t)$:

$$\frac{\partial W}{\partial t} + \frac{\sigma^2}{2} x^2 \frac{\partial^2 W}{\partial x^2} + (r - D) x \frac{\partial W}{\partial x} + f(x, t) \left(W - x \frac{\partial W}{\partial x} \right) - rW = 0$$

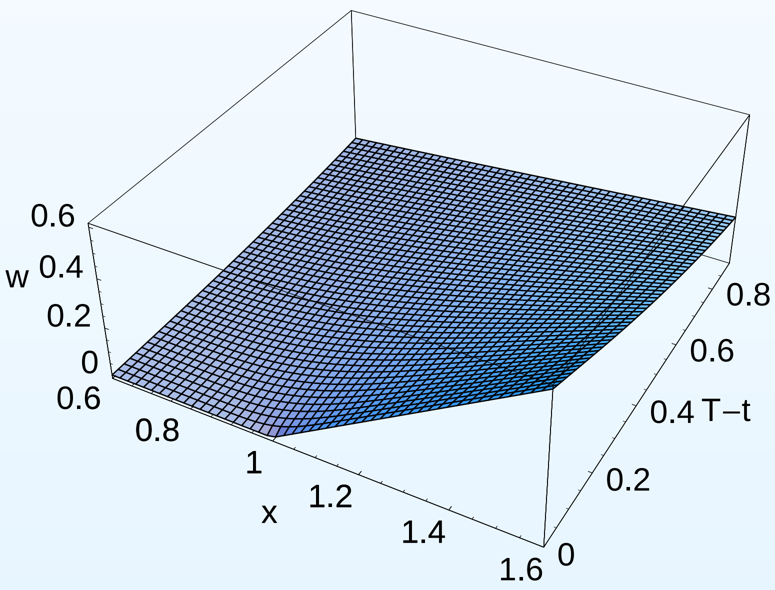
  for $x \in (0, \infty)$, $t \in (0, T)$

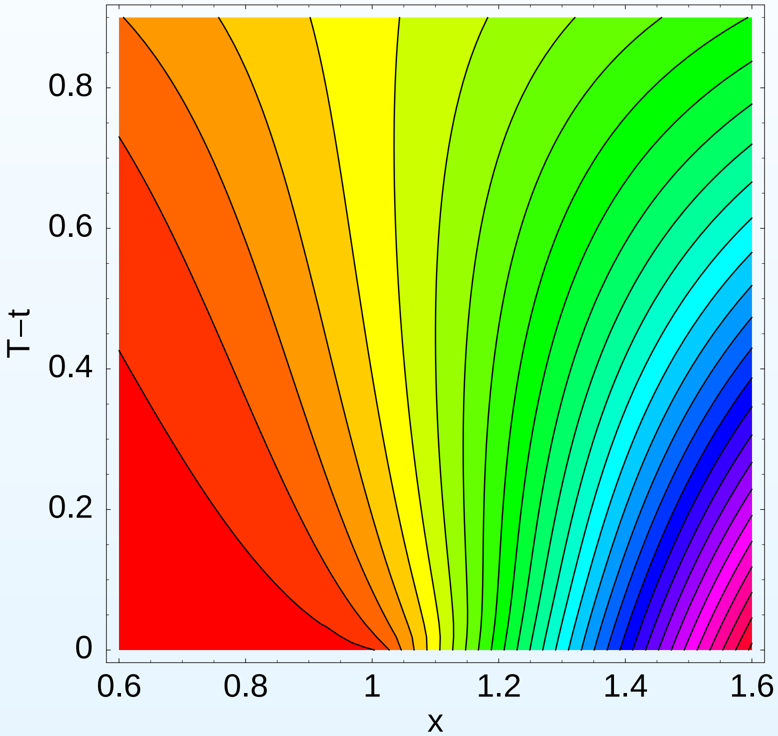
- Terminal condition for $x \in (0, \infty)$:

$$W^{call}(x, T) = \max(x - 1, 0), \quad W^{put}(x, T) = \max(1 - x, 0)$$

Average strike option - example

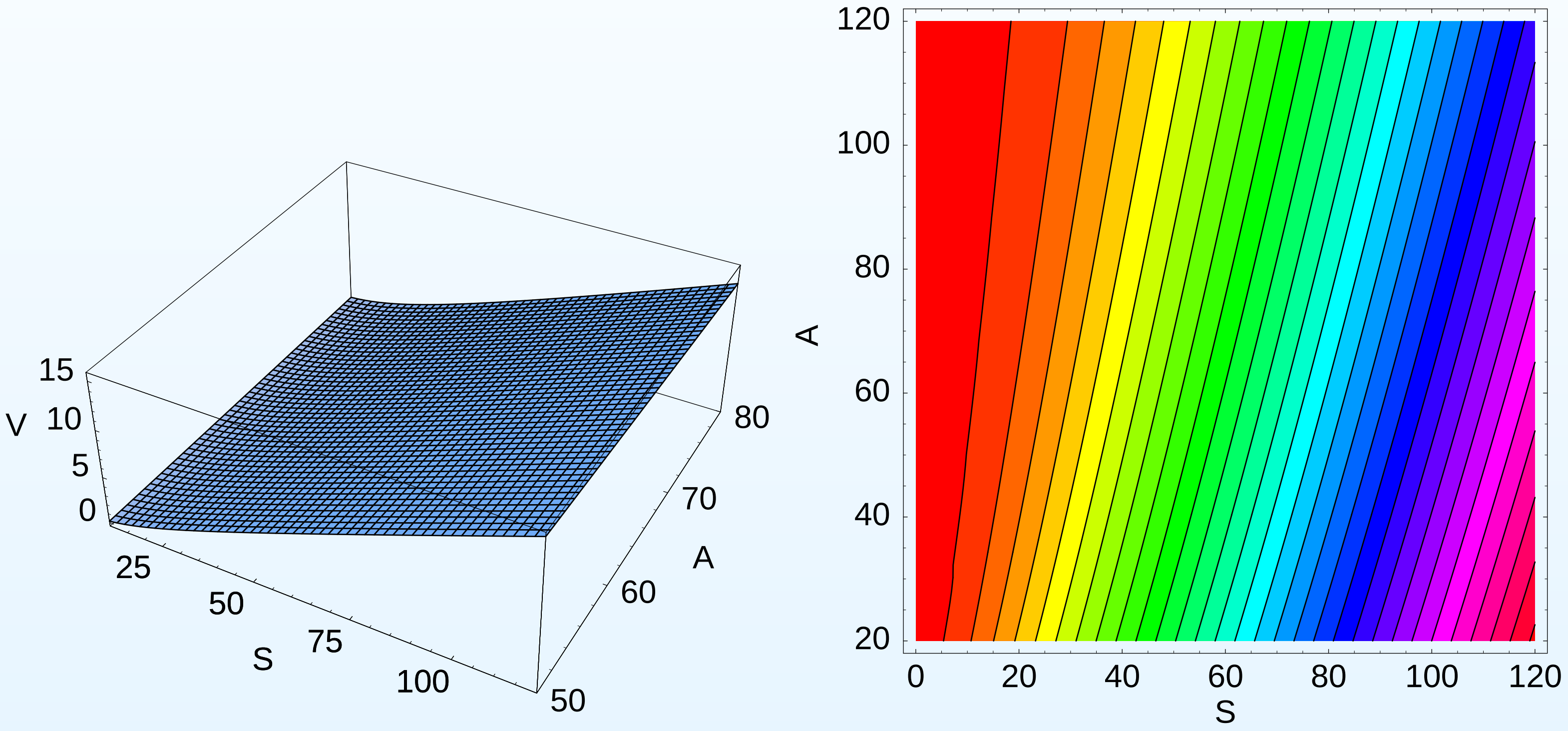
- Auxiliary function $W(x, t)$:





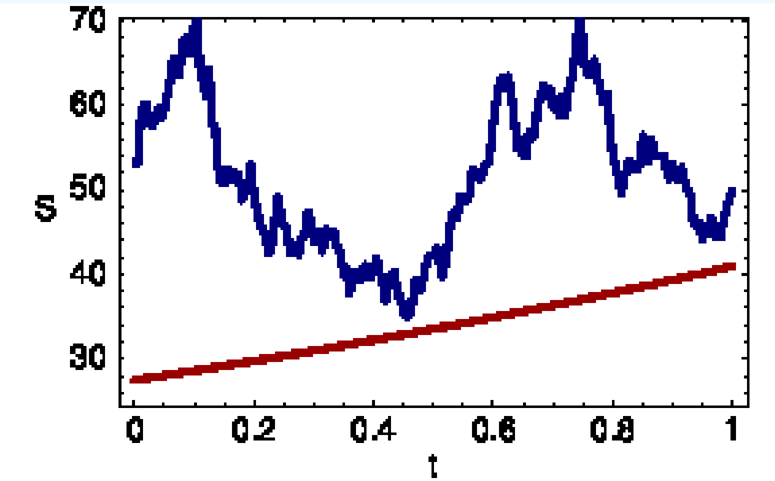
Average strike option - example

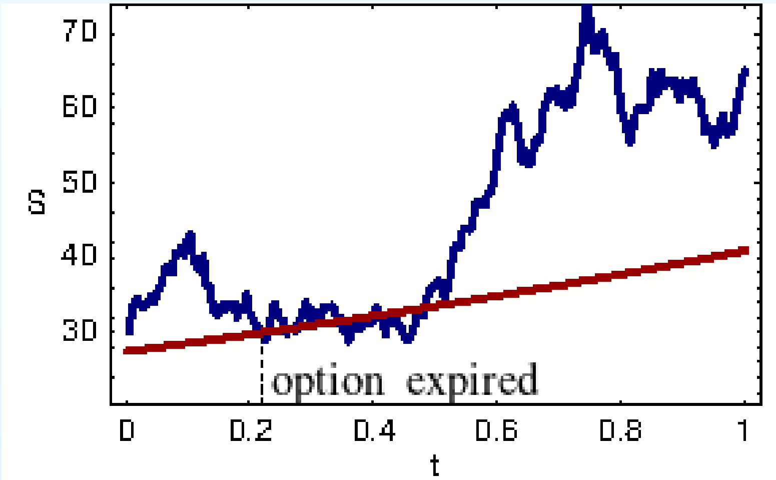
- Option price $V(S, A, t)$ for selected t:



Barrier options

- Similar to classical call and put options
- The difference: if at some time of the life of the options the stock price hits the given barrier, then:
  - the option is no longer valid
  - the option holder receives rebate from the writer
- Example: stock price (blue), barrier (brown)





Barrier options - barrier and rebate

- Classification of barriers:
  - down-and-out: if the stock price hits the barrier from above
  - up-and-out: if the stock price hits the barrier from below
- A typical example of a barrier:

$$B(t) = bEe^{-\alpha(T-t)},$$

where $0 < b \leq 1, \alpha \geq 0$ are constants

- Example of a rebate:

$$R(t) = E\left(1 - e^{-\beta(T-t)}\right),$$

where $\beta \geq 0$ is a constant - satisfies $R(T) = 0$

PDE for a down-and-out option

- Option is valid in the domain $S > B(t)$ - here, the Black-Scholes PDE holds:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV = 0$$

  for $S \in (B(t), \infty), t \in (0, T)$.

- On the boudary, i.e., for $S = B(t)$ - the option is cancelled and its value equals the rebate:

$$V(B(t), t) = R(t)$$

  for $t \in (0, T)$.

- Terminal condition for $S \in (B(t), \infty), t = T$ depends on the option type:

$$V^{call}(S, T) = \max(0, S - E), \;\; V^{put}(S, T) = \max(0, E - S)$$

PDE for a down-and-out option

- Transformation to a fixed domain $x \in (0, \infty)$:

$$V(S,t) = W(x,t), \quad x = \ln\left(\frac{S}{B(t)}\right),$$

- PDE for the function W:

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 W}{\partial x^2} + \left(r - D - \frac{\sigma^2}{2} - \alpha\right)\frac{\partial W}{\partial x} - rW = 0$$

  for $x \in (0,\infty), t \in (0,T)$.

- Boundary condition: $W(0,t) = R(t)$ pre $t \in (0,T)$.
- Terminal condition:

$$V^{call}(x,T) = E\max(0, be^x - 1)$$
$$V^{put}(x,T) = E\max(0, 1 - be^x)$$

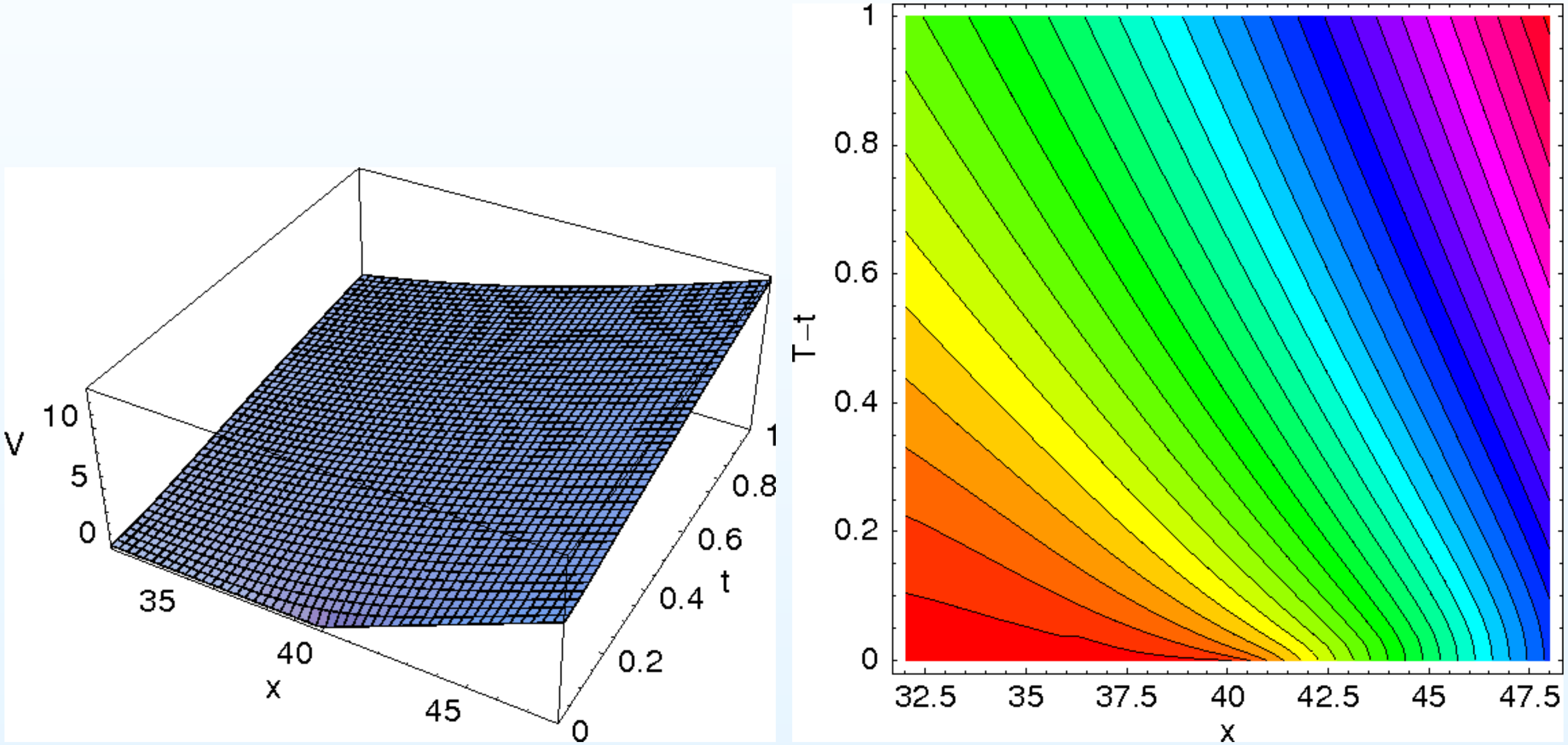
  for $x \in (0, \infty)$.

Up-and-out option: homework

- Write the mathematical formulation of pricing an up-and-out option: PDE (and its domain), boundary condition, terminal condition
- Transform it to a PDE on a fixed domain

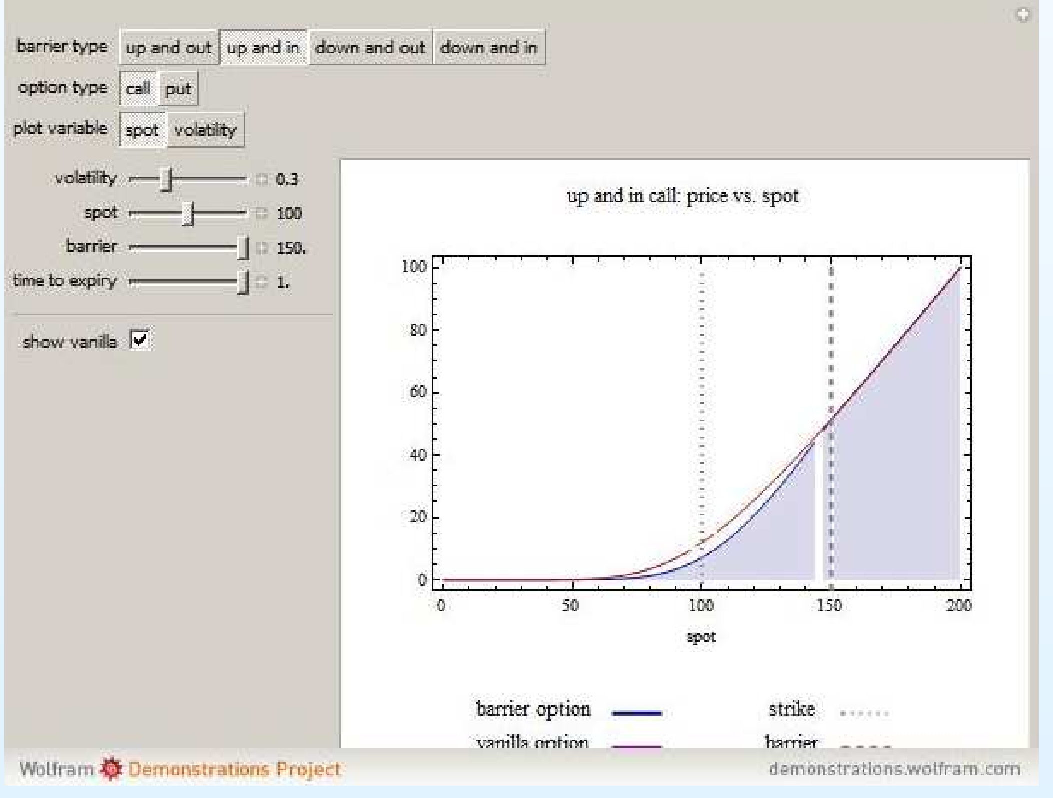
Barrier options: example

- Price of a barrier option:



Barrier options: interactive

- Web page:
  http://demonstrations.wolfram.com/BarrierOptionPricingWithinTheBlackScholesModel/
- Requires the player, available at:
  http://demonstrations.wolfram.com/download-cdf-player.html



Basket options, options on indices, etc.

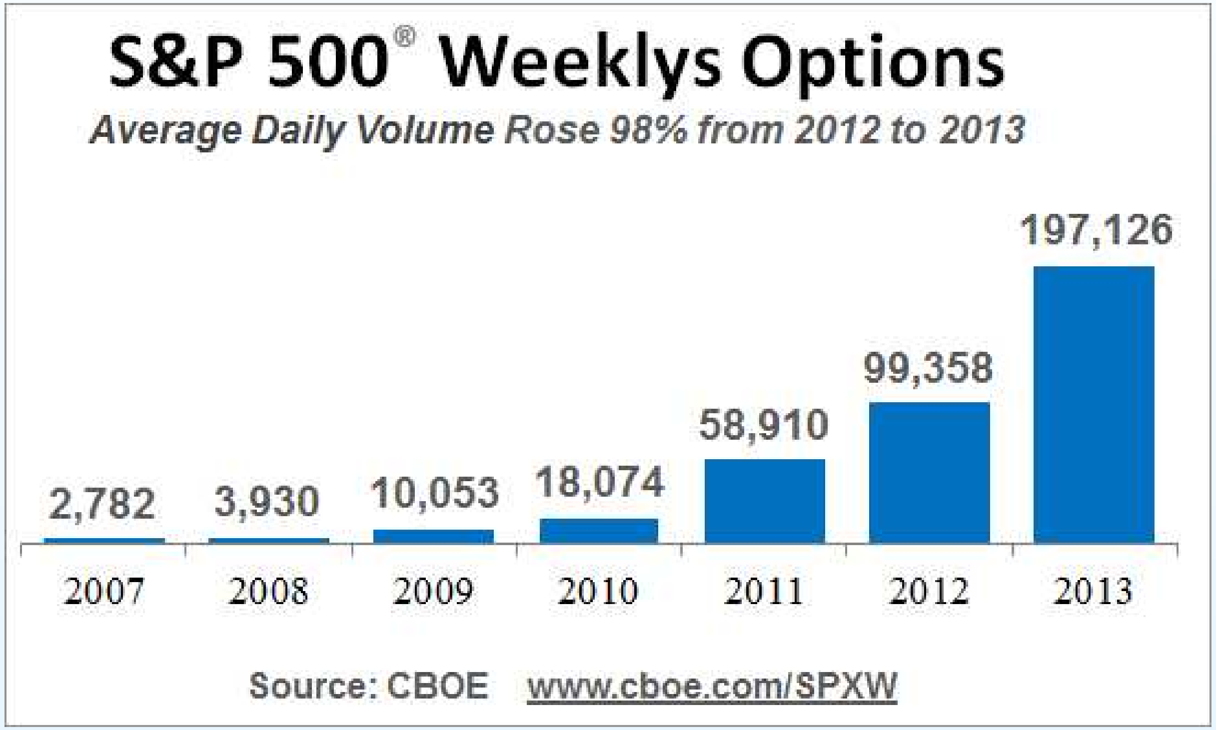
- Payoff of the option depends on the value of several assets or on the value of an index
- EXAMPLE 1: spread options - payoff depends on a difference between values of two assets at the expiration time, e.g.,

$$V(S_1, S_2, T) = \max((S_1 - S_2) - E, 0)$$

  - useful for example for commodities (prices of an input and an output)
- EXAMPLE 2: options on indices - for example S&P 500, NYSE, ...
  If each stock follows a GBM, we obtain n-dimensional Black-Scholes equation ($n =$ number of stock in the index)

Basket options, options on indices, etc.

- Trading S&P 500 options:



http://www.cboe.com/

Lookback options

- Lookback options - payoff depends on the maximal price of the underlying asset during the given period

$$M = M_{T_0}^{T} = \max(S_t, t \in [T_0, T]),$$

  where $T \geq 0$

- For example: maximum M instead of the stock price on the payoff:

$$V^{call}(S, M, T) = \max(0, M - E)$$
$$V^{put}(S, M, T) = \max(0, E - M)$$

Spread options: Margrabe formula

- Recall: spread options

$$V(S_1, S_2, T) = \max((S_1 - S_2) - E, 0)$$

- Suppose that the stocks do not pay dividends and that

$$\begin{aligned} dS_1 &= \mu_1 S_1 dt + \sigma_1 S_1 dw_1 \\ dS_2 &= \mu_2 S_2 dt + \sigma_2 S_2 dw_2 \end{aligned}$$

where $\mathbb{E}[dw_1 dw_2] = \rho dt$

- For the case of $E = 0$ there is an explicit formula for the option price - co called Margrabe formula
- We derive the PDE for the option price and find its solution

Spread options: Margrabe formula

- Similarly as in the derivation of the Black-Scholes model
- Portfolio:
  - one option V
  - $-\Delta_1$ stocks S_1
  - $-\Delta_2$ stocks S_2

  Portfolio value: $P = V - \Delta_1 S_1 - \Delta_2 S_2$
- Change in the pofolio value $P = dV - \Delta_1 dS_1 - \Delta_2 dS_2$, where
  - dS_1, dS_2 are in the assumptions
  - dV is given by the multidimensional Itō lemma (since $V = V(S_1, S_2, t)$)
- We eliminate randomness (terms dw_1, dw_2) - by setting $\Delta_1 = \frac{\partial V}{\partial S_1}$, $\Delta_2 = \frac{\partial V}{\partial S_2}$
- Yield of a riskless portfolio has to be r

Spread options: Margrabe formula

- The resulting PDE:

$$\frac{\partial V}{\partial t} + rS_1 \frac{\partial V}{\partial S_1} + rS_2 \frac{\partial V}{\partial S_2} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2}$$

$$+\rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 S_2} - rV = 0$$

  with terminal condition

$$V(S_1, S_2, T) = \max(S_1 - S_2, 0)$$

- Transformation:

$$V(S_1, S_2, t) = S_2 f(x, t), \quad x = \frac{S_1}{S_2}$$

Spread options: Margrabe formula

- PDE for the function $f(x,t)$:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\tilde{\sigma}^2 x^2 \frac{\partial^2 f}{\partial x^2} = 0,$$

kde $\tilde{\sigma}^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$

- Terminal condition $f(x,T) = \max(x-1, 0)$
- This is the Black-Scholes PDE for a call, where
  - the variable x corresponds to the stock price S
  - exercise price $E = 1$
  - interest rate is zero
- Hence, the solution is: $f(x,t) = xN(d_1) - N(d_2)$, where $d_1 = \frac{\log x + \frac{\tilde{\sigma}^2}{2}\tau}{\tilde{\sigma}\sqrt{\tau}}$, $d_2 = d_1 - \tilde{\sigma}\sqrt{\tau}$

Spread options: Margrabe formula

- Solution in the original variables (i.e., the spread option price):

$$V(S_1, S_2, t) = S_1 N(d_1) - S_2 N(d_2),$$

where

$$d_1 = \frac{\log \frac{S_1}{S_2} + \frac{\tilde{\sigma}^2}{2}\tau}{\tilde{\sigma}\sqrt{\tau}}, d_2 = d_1 - \tilde{\sigma}\sqrt{\tau}$$

- this is known as Margrabe formula

- HOMEWORK: Derive the spread option price, if the stocks pay continuous dividends with rates q_1, q_2.