Financial derivatives
Revision of numerical methods

Review: numerical methods for a heat equation

Consider a heat equation

$$\frac{\partial u}{\partial t}(x,t) = a^2 \frac{\partial^2 u}{\partial x^2}(x,t) \;\; x \in [-L, L], t \in (0, T]$$

with initial condition $u(x, 0) = u_0(x)$ for $x \in [-L, L]$ and boundary conditions $u(-L, t) = \phi(t), u(L, t) = \psi(t)$ for $t \in (0, T]$.

Consider its discretization with time step k and space step h.

1. Write down the notation which you are going to use in your derivation of numerical schemes.
2. Derive the explicit and implicit scheme for the heat equation.
3. When using the explicit scheme, we cannot choose k, h arbitrarily. What condition has to be satified and why (shortly - what happens if it is not satisfied)?
4. Write the implicit scheme in a matrix form

$$Au^{(m)} = b,$$

   where $u^{(m)}$ is a vector with approximation of the solution on m-th time level (except for boundary points, where the values are determined by boundary conditions - so we do not compute them in our numerical scheme), A is a matrix (write its elements explicitly) and b is the right hand side vector.

Review: vectors and matrices, solving a system of linear equation

1. Define a vector norm (i.e. the properties of a general norm of a vector from $\mathbb{R}^n$).
2. Define a matrix norm (i.e. the properties of a general norm of an $n \times n$ matrix). Give an example of a matrix norm, example of a matrix (small one, so that the computation can be done by hand) and compute the given norm for the given matrix.
3. In general (for any pair of matrix and vector norms) the inequality $\|Ax\| \leq \|A\| \, \|x\|$ does not necessarily hold. Give an example of a pair of matrix and vector norms, for which this inequality holds. Where - in the analysis of numerical methods for solving linear systems - have we used this property?
4. Consider the iterative scheme for solving the system $Ax = b$ in the form $x^{(k+1)} = Tx^{(k)} + g$, where T is iteration matrix and g is a vector. What has to be satisfied by the matrix T, so that the iterative scheme is convergent?
5. Is it possible that a sequence of vectors in $\mathbb{R}^n$ (e.g. a sequence of iteration from the previous point) converges in some norm, but does not converge in another norm? Explain.

6. Derive the formulation of the Gauss-Seidel method

$$u_i^{p+1} = \frac{1}{A_{ii}} \left(b_i - \sum_{j<i} A_{ij} u_j^{p+1} - \sum_{j>i} A_{ij} u_j^p \right)$$

in the matrix form $x^{(k+1)} = Tx^{(k)} + \omega$. Explain your notation (in case you introduce any new matrices related to A).

7. Prove that the Gauss-Seidel method, applied to the system from implicit scheme for heat equation, is convergent. State the convergence criterion (without a proof) and show that it is satisfied by our matrix.

New in this course: system of linear equation continued (SOR methods), numerical solution of European and American option pricing

1. State a criterion for a convergence of the SOR method with arbitrary $\omega \in (0, 2)$ aplied to a system of linear equations. Show that it is satisfied for the matrix which arises in numerical solution of European option pricing.

2. Derive the optimal ω for the SOR method applied to a system of linear equations which arises in numerical solution of European option pricing.

3. Suppose that the American option prices are given by the table at the bottom of the webpage `http://www.iam.fmph.uniba.sk/institute/stehlikova/fd16/ex/ex07.html`. Find the interval of length 2, which contains the free boundary.

4. Shortly explain why we cannot price the Americal option like a maximum of the payoff and the corresponding European option price.

5. For which of these cases are the European and Americal option prices equal (for any S and t):

   - call option on a stock which does not pay dividends
   - put option on a stock which does not pay dividends
   - binary option on a stock which does not pay dividends
   - call option on a stock which pays continuous dividends
   - put option on a stock which pays continuous dividends
   - binary option on a stock which pays continuous dividends

6. Be able to explain what you did in the computer submission of the American option pricing.

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Sample Test 3

You can use a A4 cheat sheet. Calculators, mobile phones, notebooks, etc. are not permitted. Also no comunication during the test is allowed.

1. *(1 point)* Prove that the Gauss-Seidel method, applied to the system from implicit scheme for heat equation, is convergent. State the convergence criterion (without a proof) and show that it is satisfied by the matrix which arises in numerical solution of European option pricing.
2. *(1 point)* State a criterion for a convergence of the SOR method with arbitrary $\omega \in (0, 2)$ applied to a system of linear equations. Show that it is satisfied for the matrix which arises in numerical solution of European option pricing.
3. *(1 point)* When solving a system of linear equation using an iterative scheme, we would like to have $\|x^{(k)} - x^*\| \to 0$ where $x^{(k)}$ is k-th approximation and x^* is the exact solution. Is it possible that this sequence of iterations converges in some norm, but does not converge in another norm? Explain.
4. *(1 point)* Consider the PSOR algorithm for solving a discretized linear complementarity problem of the form $Ax \geq b$, $x \geq g$, $(Ax - b)_i(x - g)_i = 0$. Why the stopping criterion in the algorithm cannot be taken to be $\|Ax - b\| < \varepsilon$ (which is a good criterion in the SOR or Gauss-Seidel algorithm)?
5. *(1 point)* Consider a binary option on a stock which does not pay dividends. Show that the prices of the European and the American option of this kind are not equal
6. *(1 point)* Explain how did you compute the option prices in your program for American option pricing, which correspond to stock prices which were not among the grid points.