Financial derivatives – exam
Winter term 2014/2015

Problem 1: *[max. 13 points]*
Determine whether the following assertions are true or false. Write your answers, without explanations.
Grading: correct answer +0.5 points, incorrect answer -0.5 points, no answer 0 points

1. The distribution of a Wiener process at a given time is normal.
2. The limiting distribution of the short rate in Cox-Ingersoll-Ross model is lognormal.
3. The limiting distribution of the short rate in Vasicek model is lognormal.
4. The expected value of a geometrical Brownian motion can be a decreasing function of time.
5. The variance of a geometrical Brownian motion can be a decreasing function of time.
6. If we modell a stock price by a geometrical Brownian motion, it means that we assume the daily returns to be independent random variables.
7. If we modell a stock price by a geometrical Brownian motion, it means that we assume the daily returns to be normally distributed.
8. Black-Scholes model assumes a geometrical Brownian motion as a model for the underlying stock price.
9. The Vasicek model allows the short rate to be less than zero with a positive probability.
10. In Vasicek model, the limit of term structure of interest rates, as maturity approaches infinity, is an increasing function of the short rate
11. Fokker-Planck equation is a partial differential equation for the density of a stochastic process at a given time.
12. Maximum likelihood estimates of the Vasicek model cannot be expressed in a closed form; they need to be computed numerically.
13. Black-Scholes price of a put option is a convex function of the underlying stock price.
14. Black-Scholes price of a put option is a decreasing function of the underlying stock volatility.
15. Black-Scholes price of a cash-or-nothing option is an decreasing function of the underlying stock volatility.
16. Leland model incorporates transaction costs into Black-Scholes framework.
17. Leland model allows us to model both bid and ask prices of an option.
18. The Leland bid and ask prices of any derivative can be computed by using an adjusted volatility in the Black-Scholes formula.
19. Price of an American derivative cannot lie below its payoff.
20. Price of an American put option can be computed as *max(payoff, price of the European put option with the same parmeters).*
21. If the stock price is lower than the early exercise boundary for a call option, it means that we should keep the option (instead of exercising it at the given moment).
22. SOR method for solving a system of linear equations is a generalization of the Gauss-Seidel method which converges for any system of equations.
23. When solving a system of linear equations, arising from the implicit scheme for numerically solving a heat equation, the SOR method converges for any positive value of the parameter omega.
24. Setting omega equal to 1.5 in the SOR method for solving a system of linear equations

always speeds up the convergence, compared to the Gauss-Seidel method.

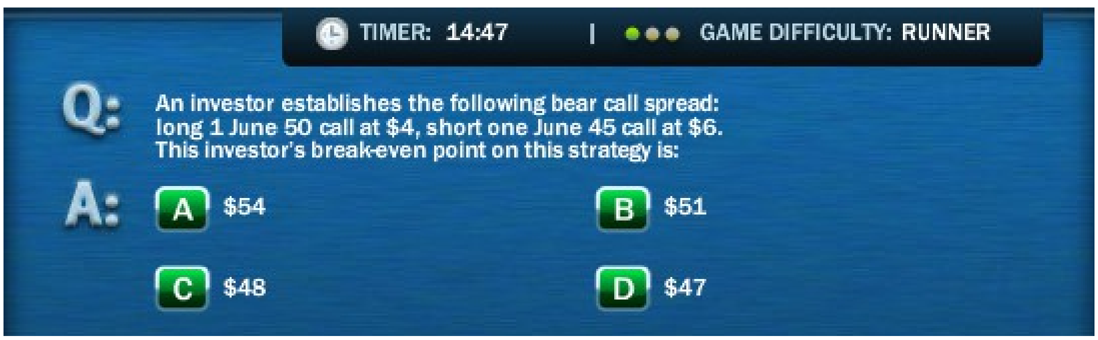
25. Projected SOR algorithm can be used to solve a linear complementarity problem.
26. Payoff of Asian options depends on average value of the underlying stock during the specified time period.

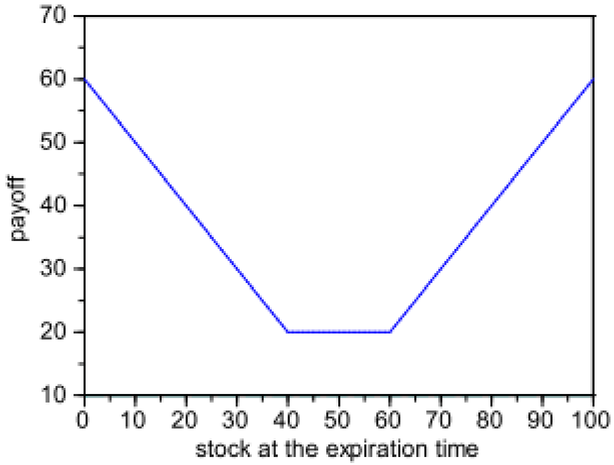
Problem 2: *[max. 4 points]*

Grading: 4 points for acorrect answer with a correct explanation (no credit for an answer without its correct justification, a partial credit is possible for relevant computations even if the final answer is not derived)

A break-even point is the stock price at the time of expiration, for which the strategy has a zero profit.

Solve the following problem from the OptionQuest CBOE online game (note that the numbers correspond to strike prices of the options, e.g. “50 call” means “call with strike price 50”):





Problem 3: *[max. 8 points]*
Consider the Black-Scholes framework, the underlying stock which does not pay dividends and a derivative with the payoff given on the left.

1. Write this payoff as a linear combination of call and put options.

2. Use the previous answer to find the Black-Scholes price of this derivative, if the stock volatility is 30 percent and its price today is 55 USD, interest rate is 1 percent and the expiration is in 1 year.

3. Derive the limit of the price of this derivative, as volatility of the underlying stock approaches infinity (for general values of the other parameters, i.e., *S, E, r, t, T*).

Grading:

- *2 points for a correct answer to question 1*
- *2 points for a correct answer to question 2*
- *4 points for a correct answer in question 3 with a correct explanation (no credit for an answer without its correct justification, a partial credit is possible for relevant computations even if the final answer is not derived)*

Problem 4: *[max. 10 points]*
Recall the barrier options from the lectures:

> **Barrier options - barrier and rebate**
>
> - Classification of barriers:
>   - down-and-out: if the stock price hits the barrier from above
>   - up-and-out: if the stock price hits the barrier from below
> - A typical example of a barrier:
>
> $$B(t) = bEe^{-\alpha(T-t)},$$
>
> where $0 < b \leq 1, \alpha \geq 0$ are constants

Solve the homework given in the lectures:

> **Up-and-out option: homework**
>
> - Write the mathematical formulation of pricing an up-and-out option: PDE (and its domain), boundary condition, terminal condition
> - Transform it to a PDE on a fixed domain

for an up-and-out call option with a barrier defined above and zero rebate.

Grading:

- *1 point for each of the following: PDE, domain for the PDE, boundary condition, terminal condition*
- *6 points for the transformation, partial credit can be awarded*

Problem 5: *[max. 5 points]*
Solve the following problem which comes from a preparation to an acturial exam:

You are given:

(i) $$\frac{dS(t)}{S(t)} = 0.3\,dt - \sigma\,dZ(t), \qquad t \geq 0,$$
where $Z(t)$ is a standard Brownian motion and σ is a positive constant.

(ii) There is a real number a such that

$$\frac{d[S(t)]^a}{[S(t)]^a} = -0.66\,dt + 0.6\,dZ(t), \quad t \geq 0.$$

Calculate σ.

(A) 0.16

(B) 0.20

(C) 0.27

(D) 0.60

(E) 1.60

<u>Show your work</u> (unlike on the original exam, which is a test), there is no credit for an answer without its correct derivation.

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1. *[From the fundamentals; max. 10 points]* Derive the explicit method for numerically solving the heat equation, which arises from solving the Black-Scholes PDE:

   Transformation to a heat equation

   - Transformation

   $$V(S,t) = e^{-\alpha x - \beta \tau} u(x,\tau),$$

   $$\alpha = \frac{r-q}{\sigma^2} - \frac{1}{2}, \beta = \frac{r+q}{2} + \frac{\sigma^2}{8} + \frac{(r-q)^2}{2\sigma^2}, \tau = T - t, x = \ln(S/E),$$

   transforms the Black-Scholes equation to the following heat equation:

   $$\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} = 0$$

   for $x \in \mathbb{R}, \tau \in [0,T]$

   - Initial condition: $u(x,0) = g(x)$

2. *[max. 10 points]* Give an example of an option strategy (combination of call and put options, explicitly give amounts of each and their strike prices) which has a limited loss and is profitable if there is a small change in stock price in either direction.

3. *[max. 10 points]* Consider the Black-Scholes PDE

   - Matematical formulation of the model:
     Find solution $V(S,t)$ to the partial differential equation (so called Black-Scholes PDE)

   $$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

   which holds for $S > 0, t \in [0,T)$.

   and solve the following exercise:

   EXERCISES:

   - Find the price of a derivative with payoff $V(S,T) = S^n$, where $n \in \mathbb{N}$.
     HINT: Look for the solution in the form $V(S,t) = A(t)S^n$