8.2. Other Interest Rate Derivatives

Interest Rate Swaps

An interest rate swap is an agreement between two traders. A party A commits to payin
a party B the fixed interest rate r^* from the specified amount which can be assumed to b
unity. The party B is committed to pay to the party A a floating interest rate r. Assume tha
the short rate r follows a stochastic process of the form (7.1) and that payments are realize
continuously. We can reformulate the interest rate swap agreement as follows: the party A
holds a bond paying a coupon $r - r^*$. At the maturity T, i.e., the time of the end of the swa
agreement, the bond has the zero value. Derivation of a governing equation for the price
$P(t,r)$ of such a bond is very similar to the one of a zero-coupon bond. Using Itō's lemma
we obtain a stochastic differential equation for the bond price

$$dP = \mu_P(t,r)dt + \sigma_P(t,r)dw,$$

where

$$\mu_P = \frac{\partial P}{\partial t} + \mu\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2},\ \sigma_P = \sigma\frac{\partial P}{\partial r}.$$

We can construct a portfolio consisting of one bond with maturity T_1 and Δ bonds with maturity T_2. Its value is therefore equal to: $\pi = P(T_1) + \Delta P(T_2)$. A change in the value due to the change of bond prices equals $dP(T_1) + \Delta dP(T_1)$. In addition to this, there is a change $(1+\Delta)(r-r^*)dt$ due to coupon payments. In summary, we have

$$\begin{aligned} d\pi &= dP(T_1) + \Delta dP(T_1) + (1+\Delta)(r-r^*)dt \\ &= (\mu_P(T_1) + \Delta\mu_P(T_2) + (1+\Delta)(r-r^*))\,dt + (\sigma_P(T_1) + \Delta\sigma_P(T_2))\,dw. \end{aligned} \quad (8.23)$$

We make our portfolio risk-neutral by choosing Δ such that $-\frac{\sigma_P(T_1)}{\sigma_P(T_2)}$. Then

$$d\pi = \left(\mu_P(T_1) + -\frac{\sigma_P(T_1)}{\sigma_P(T_2)}\mu_P(T_2) + (1+\Delta)(r-r^*)\right)dt.$$

In order to avoid possibility of an arbitrage opportunity, the yield of the portfolio should be equal to the riskless interest rate r, i.e.

$$d\pi = r\pi dt = r\left(P(T_1) - \frac{\sigma_P(T_1)}{\sigma_P(T_2)}P(T_2)\right)dt.$$

Hence

$$\mu_P(T_1) + -\frac{\sigma_P(T_1)}{\sigma_P(T_2)}\mu_P(T_2) + (1+\Delta)(r-r^*) = r\left(P(T_1) - \frac{\sigma_P(T_1)}{\sigma_P(T_2)}P(T_2)\right)$$

from which we obtain

$$\frac{\mu_P(T_1) + (r-r^*) - rP(T_1)}{\sigma_P(T_1)} = \frac{\mu_P(T_2) + (r-r^*) - rP(T_2)}{\sigma_P(T_2)}.$$

It means that the above expressions are independ of the maturity T. Hence there is a function $\lambda = \lambda(r,t)$ such that

$$\frac{\mu_P(T) + (r-r^*) - rP(T)}{\sigma_P(T)} = \lambda$$

for all maturities T. Similarly as in the case of pricing zero-coupon bonds, this function is called a market price of risk. Finally, substituting μ_P and σ_P yields the partial differential equation for the bond price (and hence for the swap price) $P(t,r)$

$$\frac{\partial P}{\partial t} + (\mu - \lambda\sigma)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2} - rP + r - r^* = 0. \tag{8.24}$$

The terminal condition $P(T,r) = 0$ follows from the fact that the bond pays only the coupon and there is no such a payment at maturity (see e.g., Kwok [75]).

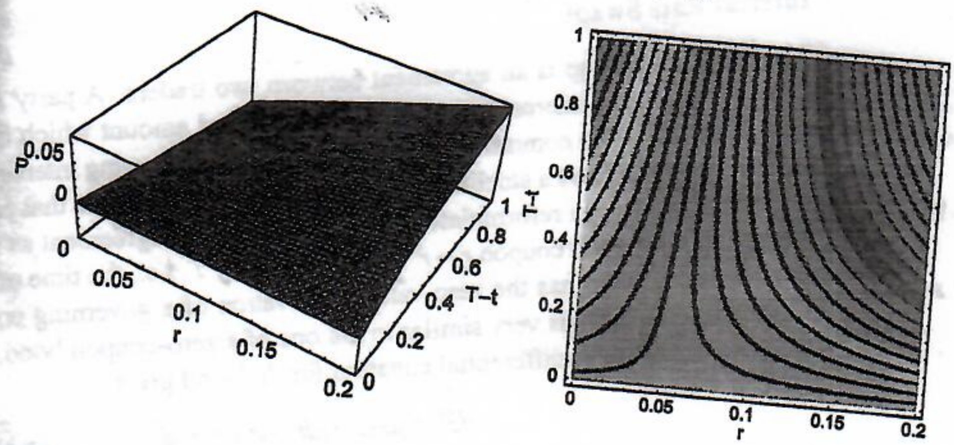

Figure 8.3. A 3D and contour plots of the function $P(r,t)$ describing a price of the interest rate swap.

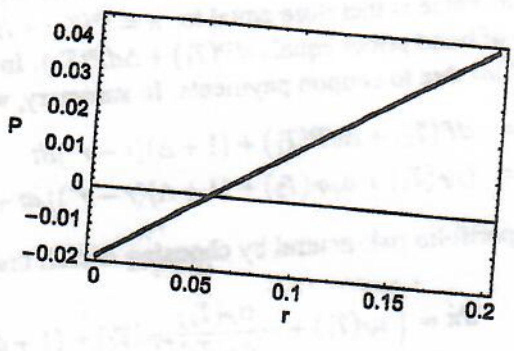

Figure 8.4. A graph of dependence of the interest rate swap price $P(r,0)$ on the floating interest rate $r > 0$.

Equation (8.24) can be solved numerically. For example, let us consider the short rate follows the stochastic differential equation for the CIR process, i.e.

$$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}dw.$$

means that $\mu(r,t) = \kappa(\theta - r)$ and $\sigma(r,t) = \sigma\sqrt{r}$. If we consider the market price of risk