Príklady na precvičenie
Rovnica vedenia tepla na ohraničenom intervale s nulovými o.p.

Nájdite riešenia nasledujúcich rovníc:

1.

$$\begin{aligned} \frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} &= 0 \quad \text{pre } x \in (0,1), t > 0 \\ u(x,0) &= 2\sin(\pi x)\cos(4\pi x) \quad \text{pre } x \in [0,1], \\ u(0,t) = u(1,t) &= 0 \quad \text{pre } t > 0. \end{aligned}$$

2.

$$\begin{aligned} \frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} &= 0 \quad \text{pre } x \in (0,1), t > 0 \\ u(x,0) &= \sin^3(5x) \quad \text{pre } x \in [0,1], \\ u(0,t) = u(1,t) &= 0 \quad \text{pre } t > 0. \end{aligned}$$

K odvodzovaniu goniometrických identít - poznámka na nasledujúcej strane.

3.

$$\begin{aligned} \frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} &= 1 \quad \text{pre } x \in (0,1), t > 0 \\ u(x,0) &= x(1-x) \quad \text{pre } x \in [0,1], \\ u(0,t) = u(1,t) &= 0 \quad \text{pre } t > 0. \end{aligned}$$

4.

$$\begin{aligned} \frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} &= t^2 \quad \text{pre } x \in (0,1), t > 0 \\ u(x,0) &= x(1-x) \quad \text{pre } x \in [0,1], \\ u(0,t) = u(1,t) &= 0 \quad \text{pre } t > 0. \end{aligned}$$

Odvodzovanie goniometrických identít

- jeden z možných postupov

Jediný vzťah, ktorý si treba pamätať, je Eulerova formula

$$e^{ix} = \cos(x) + i\ \sin(x). \tag{1}$$

Pre $-x$ z nej dostaneme

$$e^{-ix} = \cos(-x) + i\ \sin(-x) = \cos(x) - i\ \sin(x). \tag{2}$$

Z (1) a (2) teraz vieme vyjadriť sínus a kosínus:

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2},\ \ \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}. \tag{3}$$

Počítajme teraz napríklad $\sin(x)\cos(y)$. Dosadíme vyjadrenie pomocou exponenciál (3) a upravujeme:

$$\begin{aligned}
\sin(x)\cos(y) &= \frac{e^{ix} + e^{-ix}}{2}\ \frac{e^{iy} - e^{-iy}}{2i} \\
&= \frac{1}{4i}\left[e^{i(x+y)} + e^{i(x-y)} - e^{-i(x-y)} - e^{-i(x+y)}\right] \\
&= \frac{1}{4i}\left[\left(e^{i(x+y)} - e^{-i(x+y)}\right) + \left(e^{i(x-y)} - e^{-i(x-y)}\right)\right] \\
&= \frac{1}{2}\left[\frac{e^{i(x+y)} - e^{-i(x+y)}}{2i} + \frac{e^{i(x-y)} - e^{-i(x-y)}}{2i}\right] \\
&= \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]
\end{aligned}$$