Cvičenia z PDR 2007/2008

Príklady na precvičenie - rovnica vedenia tepla na ohraničenom intervale

1. Nájdite riešenia nasledujúcich rovníc:

   (a)
$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x^2},\ \ x \in (0,1), t > 0 \\ u(x,0) &= x^2,\ \ x \in (0,1) \\ u(0,t) &= 0, u(1,t) = 0,\ \ t > 0. \end{aligned}$$

   (b)
$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x^2},\ \ x \in (0,1), t > 0 \\ u(x,0) &= x^2,\ \ x \in (0,1) \\ u(0,t) &= 0, u(1,t) = 1,\ \ t > 0. \end{aligned}$$

   (c)
$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x^2},\ \ x \in (0,1), t > 0 \\ u(x,0) &= x^2,\ \ x \in (0,1) \\ u(0,t) &= t, u(1,t) = 1,\ \ t > 0. \end{aligned}$$

   (d)
$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x^2},\ \ x \in (0,1), t > 0 \\ u(x,0) &= 3\sin(3\pi x)\cos(3\pi x),\ \ x \in (0,1) \\ u(0,t) &= 1, u(1,t) = 2,\ \ t > 0. \end{aligned}$$

   (e)
$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x^2},\ \ x \in (0,1), t > 0 \\ u(x,0) &= 2x,\ \ x \in (0,1) \\ u(0,t) &= \sin t, u(1,t) = 1,\ \ t > 0. \end{aligned}$$

(f)

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x^2}, \;\; x \in (0,1), t > 0 \\
u(x,0) &= x^2, \;\; x \in (0,1) \\
\frac{\partial u}{\partial x}(0,t) &= 0, \frac{\partial u}{\partial x}(1,t) = 0, \;\; t > 0.
\end{aligned}$$

(g)

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x^2} + x, \;\; x \in (0,1), t > 0 \\
u(x,0) &= x^2, \;\; x \in (0,1) \\
\frac{\partial u}{\partial x}(0,t) &= 0, \frac{\partial u}{\partial x}(1,t) = 0, \;\; t > 0.
\end{aligned}$$

(Postup riešenia nehomogénnej rovnice odvoďte rovnakým spôsobom ako v prípade okrajových podmienok $u(0,t) = 0, u(1,t) = 0$.)

2. Nech $u(x,t)$ je riešením rovnice

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x^2} + f(x,t), \;\; x \in (0,1), t > 0 \\
u(x,0) &= u_0(x), \;\; x \in (0,1) \\
\frac{\partial u}{\partial x}(0,t) &= 0, \frac{\partial u}{\partial x}(1,t) = 0, \;\; t > 0,
\end{aligned}$$

pričom $f(x,t) > 0$ pre všetky (x,t). (To znamená, že krajné body intervalu sú izolované a v každom čase a mieste dodávame teplo.) Dokážte, že funkcia $F(t) = \int_0^1 u(x,t)dx$, vyjadrujúca celkové teplo v čase t, je rastúca.