ARMA models
Part 1: Autoregressive models (AR)

Beáta Stehlíková

Time Series Analysis

Faculty of Mathematics, Physics and Informatics, Comenius University

ARMA models

- Terminology:
  - AR - autoregressive model
  - MA - moving average
  - ARMA - their combination
- Firstly: autoregressive process of first order - AR(1)
  - definition
  - stationarity, condition on parameters
  - calculation of moments and ACF
  - simulated data
  - practical example with real data
- Then:
  - autoregressive processes of higher order
  - how to choose a suitable order of an AR model for the data

I.

Autoregressive process of the first order - AR(1)

AR(1) - definition

- AR(1) process:

$$x_t = \delta + \alpha x_{t-1} + u_t,$$

where δ and α are constants and $\{u_t\}$ is a white noise

- Let for time $t = t_0$ we are given the value x_{t_0} :

$$\begin{aligned} x_{t_0+1} &= \delta + \alpha x_{t_0} + u_{t_0+1}, \\ x_{t_0+2} &= \delta + \alpha x_{t_0+1} + u_{t_0+2} = \\ &\delta(1+\alpha) + \alpha^2 x_{t_0} + (\alpha u_{t_0+1} + u_{t_0+2}), \\ x_{t_0+3} &= \ldots \end{aligned}$$

in general:

(1) $$x_{t_0+\tau} = \alpha^\tau x_{t_0} + \frac{1-\alpha^\tau}{1-\alpha}\delta + \sum_{j=0}^{\tau-1} \alpha^j u_{t_0+\tau-j}$$

AR(1) - stationarity

- From (1):

$$x_t = \alpha^{t-t_0} x_{t_0} + \frac{1-\alpha^{t-t_0}}{1-\alpha}\delta + \sum_{j=0}^{t-t_0-1} \alpha^j u_{t-j}$$

- Deterministic initial conditions: value of the process at time t_0 is $x_0 \rightarrow$ process
- Random initial conditions:
  - ◇ Process is generated for $t \in \mathbb{R} \rightarrow$ value x_{t_0} is random.
  - ◇ If $-1 < \alpha < 1$, then for $t_0 \rightarrow -\infty$ we obtain

$$x_t = \frac{1}{1-\alpha}\delta + \sum_{j=0}^{\infty} \alpha^j u_{t-j} \qquad (2)$$

  - ◇ Wold representation: $\psi_j = \alpha^j$ for $|\alpha| < 1 \rightarrow$ process is weakly stationary.

AR(1) - moments

- Recall the explicit expression of the process (2):

$$x_t = \frac{\delta}{1-\alpha} + \sum_{j=0}^{\infty} \alpha^j u_{t-j}$$

- Expected value:

$$\begin{aligned} E[x_t] &= E\left[\frac{\delta}{1-\alpha} + \sum_{j=0}^{\infty} \alpha^j u_{t-j}\right] \\ &= \frac{\delta}{1-\alpha} + \sum_{j=0}^{\infty} \alpha^j E[u_{t-j}] = \frac{\delta}{1-\alpha} \end{aligned}$$

  - ◇ $E[x_t] = 0$ iff $\delta = 0$
  - ◇ in general: $E[x_t] \neq \delta$, but they have the same sign (since $|\alpha| < 1$)

AR(1) - moments

- Variance:

$$\begin{aligned} Var[x_t] &= Var\left[\frac{\delta}{1-\alpha} + \sum_{j=0}^{\infty} \alpha^j u_{t-j}\right] \\ &= \sum_{j=0}^{\infty} Var[\alpha^j u_{t-j}] = \sum_{j=0}^{\infty} \alpha^{2j} Var[u_{t-j}] \\ &= \sigma^2 \sum_{j=0}^{\infty} \alpha^{2j} = \sigma^2 \frac{1}{1-\alpha^2} \end{aligned}$$

where

- ◇ we used that the dispersion of a sum of uncorrelated random variables is a sum of variances
- ◇ σ^2 is a variance of white noise $\{u_j\}$

AR(1) - moments

- Autocovariances (we use that že $Cov[u_k, u_l] = \sigma^2$ for $k = l$ and $Cov[u_k, u_l] = 0$ for $k \neq l$):

$$
\begin{aligned}
Cov[x_t, x_{t-s}] &= E\left[\left(\sum_{i=0}^{\infty} \alpha^i u_{t-i}\right)\left(\sum_{j=0}^{\infty} \alpha^j u_{t-s-j}\right)\right] \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \alpha^{i+j} E[u_{t-i}u_{t-s-j}] \\
&= \sigma^2 \sum_{j=0}^{\infty} \alpha^{s+2j} = \alpha^s \frac{\sigma^2}{1-\alpha^2}
\end{aligned}
$$

- Autocorrelations:

$$
Cor[x_t, x_{t-s}] = \frac{Cor[x_t, x_{t-s}]}{Var[x_t]Var[x_{t-s}]} = \alpha^s
$$

Example - simulated data

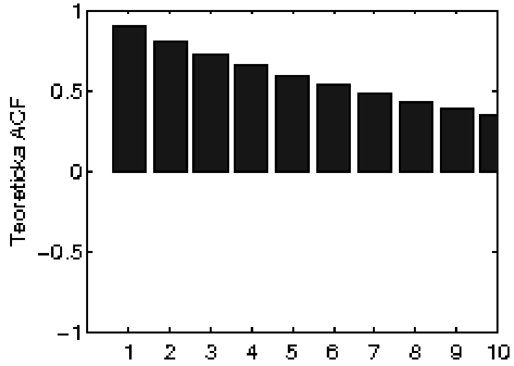
- AR(1) process

$$x_t = \delta + \alpha x_{t-1} + u_t,$$

  where the white noise u_t has a normal distribution, $\delta = 0$, $\sigma^2 = 1$
- We consider $\alpha = \{0.9, 0.6, -0.9\}$
- We present:
  - theoretical ACF
  - simulated trajectory
  - sample ACF estimated from the simulated data

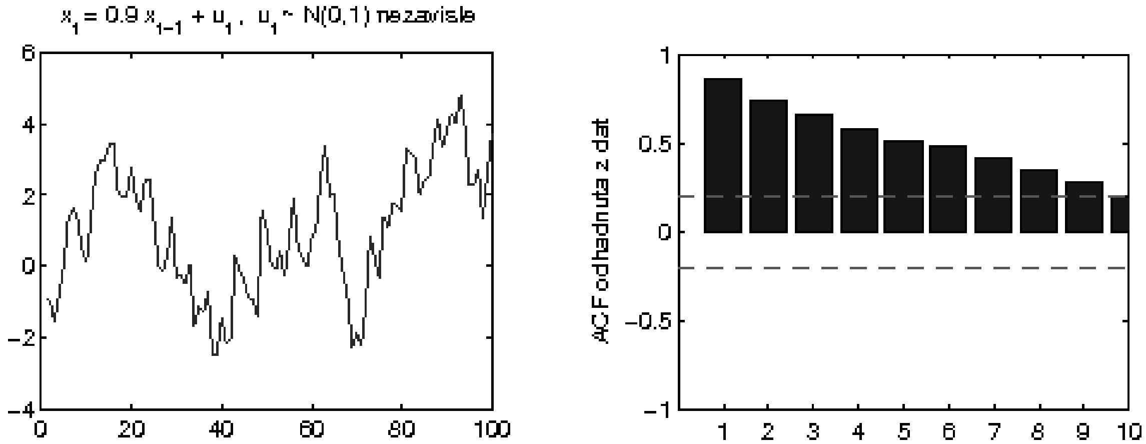
Example - simulated data, $\alpha = 0.9$

- Theoretical ACF:



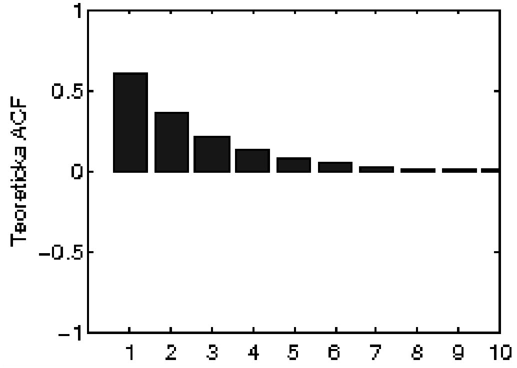
Revision question: What are its values equal to?

- Simulation of the process and its sample ACF:



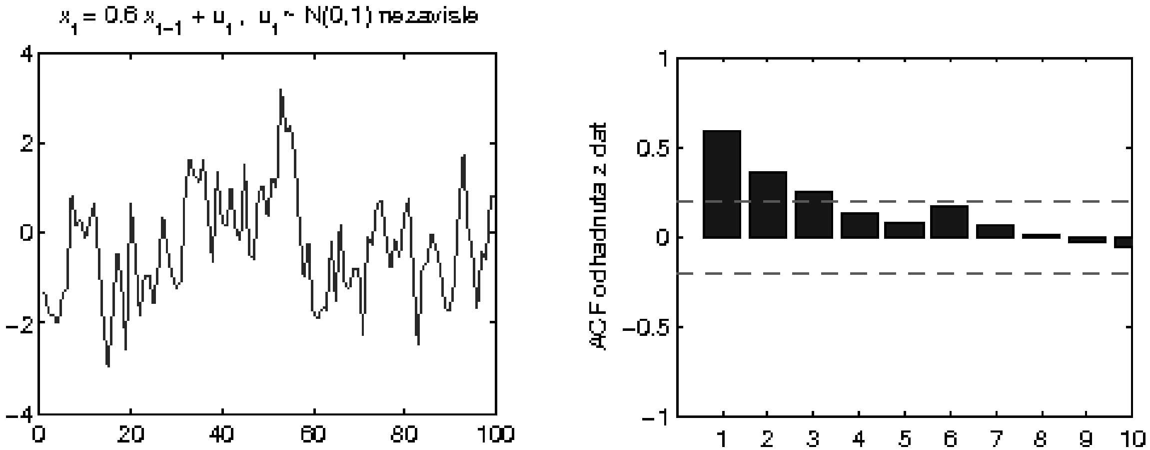
Example - simulated data, $\alpha = 0.6$

- Theoretical ACF:



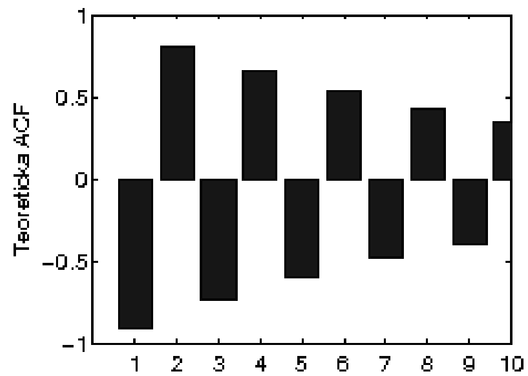
Revision question: What are its values equal to?

- Simulation of the process and its sample ACF:



Example - simulated data, $\alpha = -0.9$

- Theoretical ACF:



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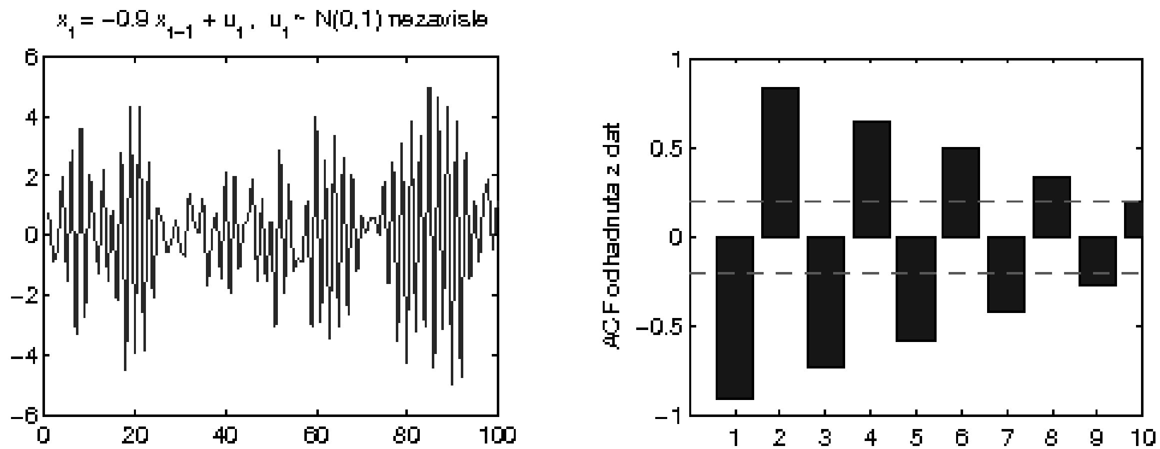
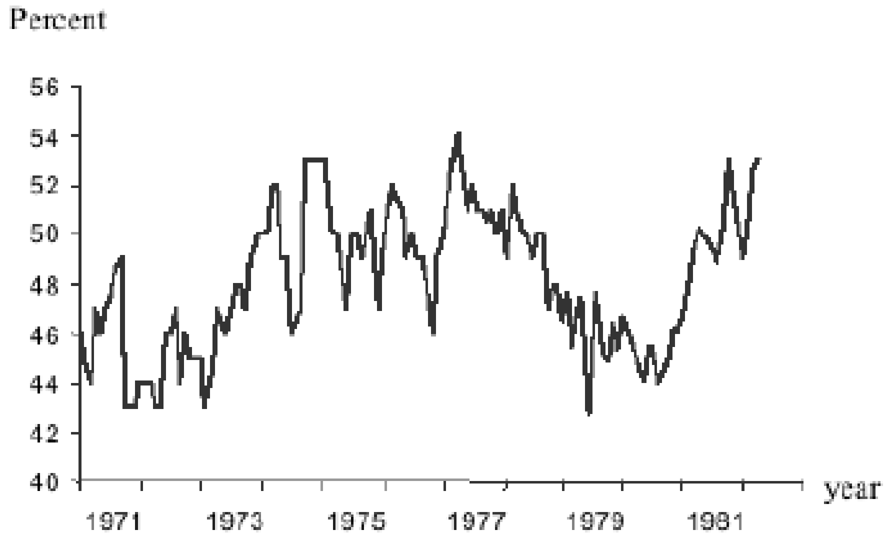


Example - real data

- G. Kirchgässner: **Causality Testing of the Popularity Function: An Empirical Investigation for the Federal Republic of Germany, 1971-1982**, Public Choice 45 (1985), p. 155-173.
- [Kirchgässner, Wolters], example 2.2

- Germany, January 1971 - April 1982
- CDU_t = popularity of the CDU/CSU



a) Popularity of the CDU/CSU, 1971 – 1982

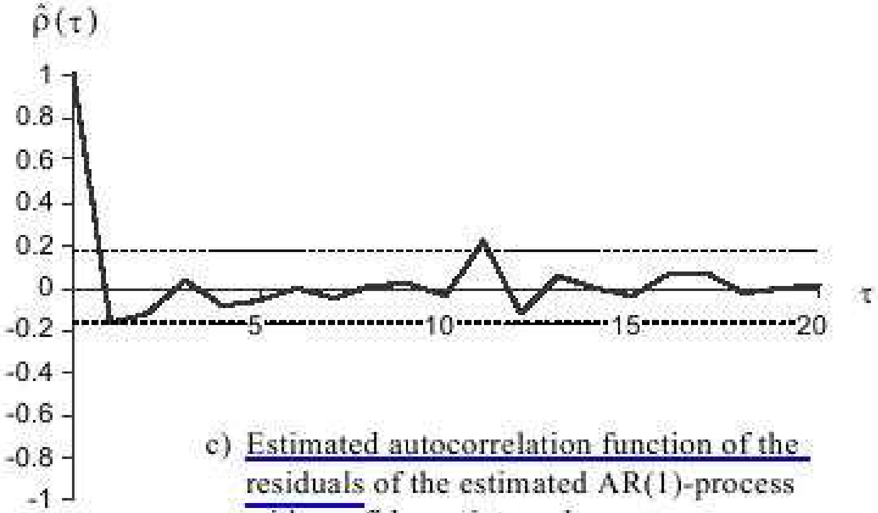
Example - real data

- Estimated AR(1) model:

$$CDU_t = \underset{(3.43)}{8.053} + \underset{(17.10)}{0.834}\, CDU_{t-1} + \hat{u}_t,$$

$$\bar{R}^2 = 0.683, \quad SE = 1.586, \quad Q(11) = 12.516 \ (p = 0.326).$$

The estimated t values are given in parentheses. The autocorrelogram, which is also given in *Figure 2.4*, does not indicate any higher-order process. Moreover, the Box-Ljung Q Statistic with 12 correlation coefficients (i.e. with 11 degrees of freedom) gives no reason to reject this model.



c) Estimated autocorrelation function of the residuals of the estimated AR(1)-process with confidence intervals

Example - real data

- Is the estimated model stationary?
- Residuals from the model should be a white noise:
  - On the graph with ACF there are intervals. What they are used for? Compute its bounds using the available data.
  - In the text authors mentioned ACF of the residuals and Ljung-Box Q statistics - what hypothesis are tested, how and what are the results?
- What is the expected value of the random variable CDU_t?

Predictions

- Process is stationary $\rightarrow$ it has a constant expected value
- It is also meaningful to compute conditional expected value
- In the previous example:
  - ◇ We have a stationary process as a model for popularity
  - ◇ We have found unconditional expected value of the process - it is constant
  - ◇ Conditional expected value - for example: *What is the expected popularity next month if its current value is 40 percent? What if the initial popularity is 35 percent?* - different answers

Predictions in an AR(1) model

- Intuition (more precisely in more complicated models, where it is not so obvious)
- For $x_t := CDU_t$ we have a model

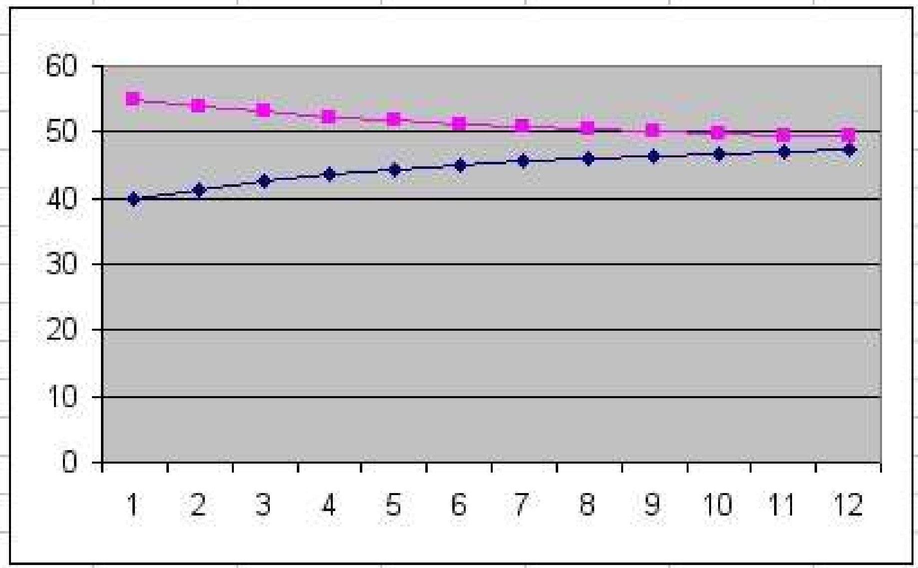
$$x_t = 8.053 + 0.834x_{t-1} + u_t$$

- White noise u_t will be replaced by its expected value (zero)
- For x_{t-1} we take
  - its realized value x_{t-1}, if it is available
  - prediction of the value x_{t-1}, if it has not been realized yet

Predictions in an AR(1) model

- For two different initial conditions:

	A	B
1	delta	8,053
2	alfa	0,834
3		
4	40	55
5	41,413	53,923
6	42,59144	53,02478
7	43,57426	52,27567
8	44,39394	51,65091
9	45,07754	51,12986
10	45,64767	50,6953
11	46,12316	50,33288
12	46,51971	50,03062
13	46,85044	49,77854
14	47,12627	49,5683
15	47,35631	49,39296
16		
17		
18		



- What is their common limit?

Motivation for more complicated models

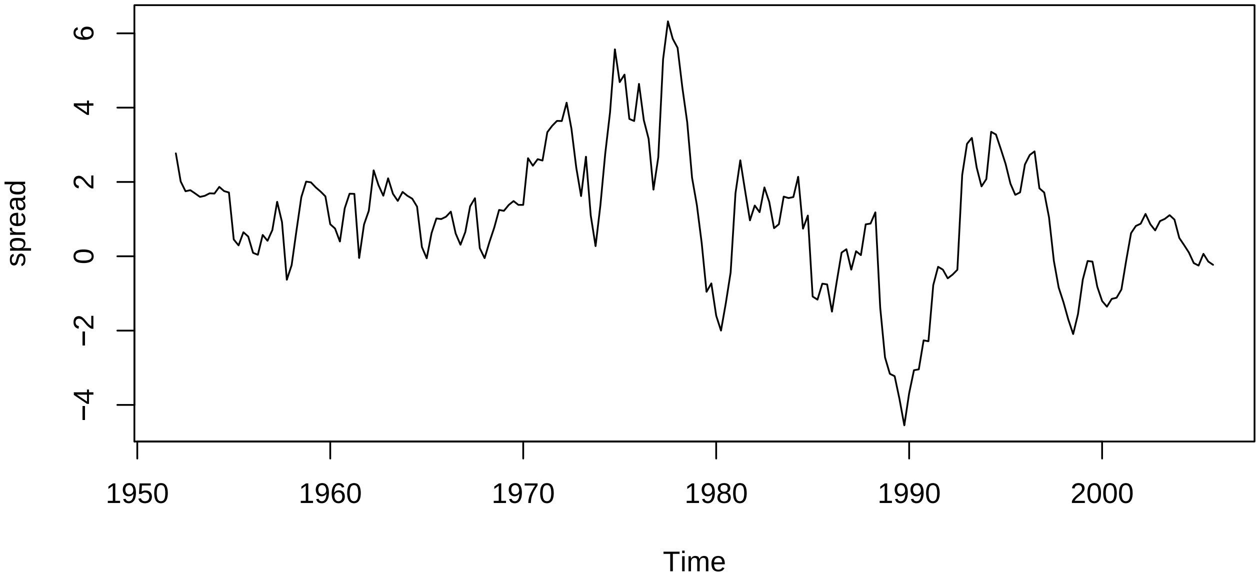
Mills, Markellos: **The Econometric Modelling of Financial Time Series**. Cambridge University Press, 2008

Dáta: http://www.lboro.ac.uk/departments/ec/cup/data.html

- Quarterly data, 1952Q1 - 2005Q4
- Variables:
  - short term interest rate (3 months))
  - long term interest rate (20 years)
- We will modell the difference between long term and short term rates

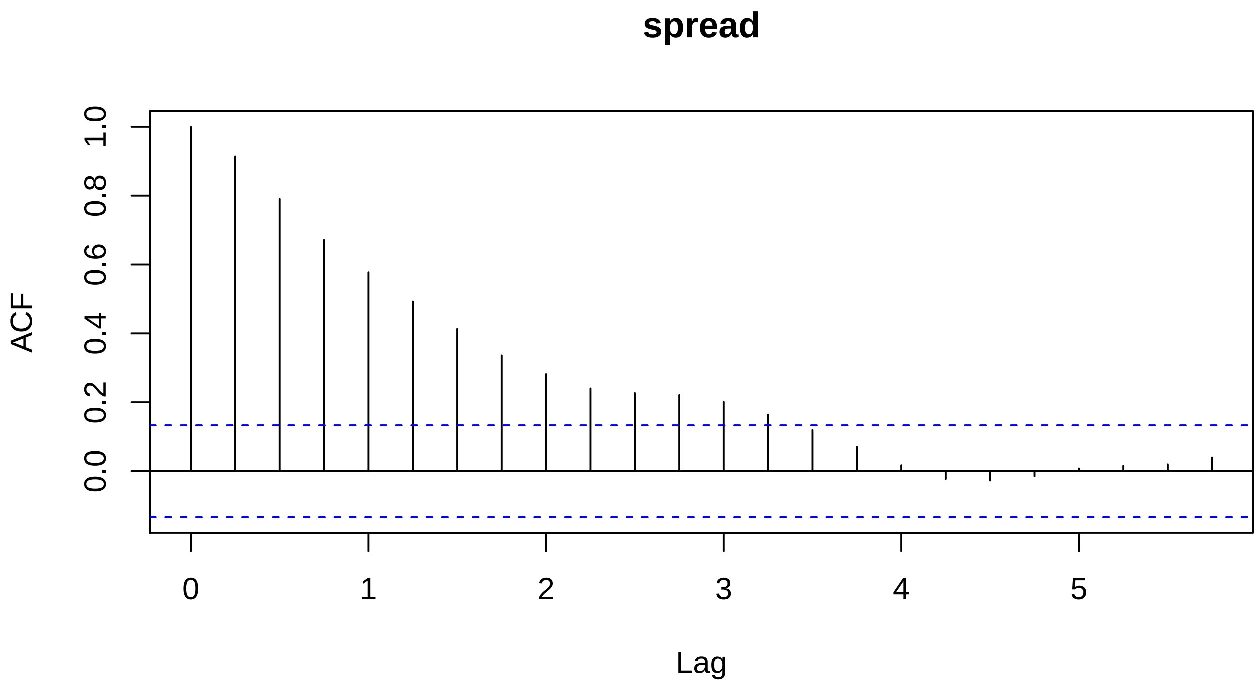
Motivation for more complicated models

- Behaviour of our time series:



Motivation for more complicated models

- Estimated ACF:



Motivation for more complicated models

- In R, we will use the package **astsa**: Applied Statistical Time Series Analysis
- We estimate an AR(1) model:

```
sarima(spread,1,0,0,details="FALSE")
```

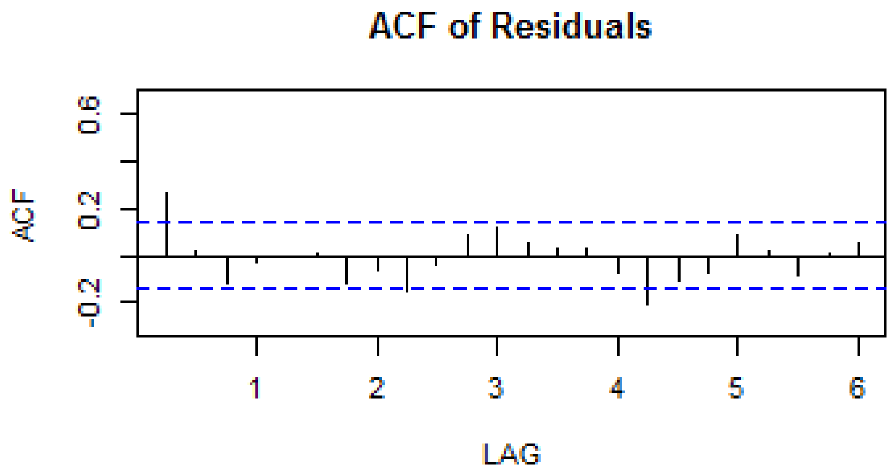
- For stationarity: the AR coefficient has to be less than 1 in absolute value
- **AR** in SARIMA relates to autoregressive terms
- SARIMA denotes more general models which we will study later

Motivation for more complicated models

```
> sarima(spread,1,0,0,details="FALSE")
$fit

Call:
stats::arima(x = xdata, order = c(p, d, q), seasonal = list(order = c(P, D,
    Q), period = S), xreg = xmean, include.mean = FALSE, optim.control = list(t$
    REPORT = 1, reltol = tol))

Coefficients:
         ar1   xmean
      0.9156  1.0473
s.e.  0.0266  0.5491

sigma^2 estimated as 0.5106:  log likelihood = -234.8,  aic = 475.61

$AIC
[1] 0.3463184

$AICc
[1] 0.3561018

$BIC
[1] -0.622429
```

Motivation for more complicated models

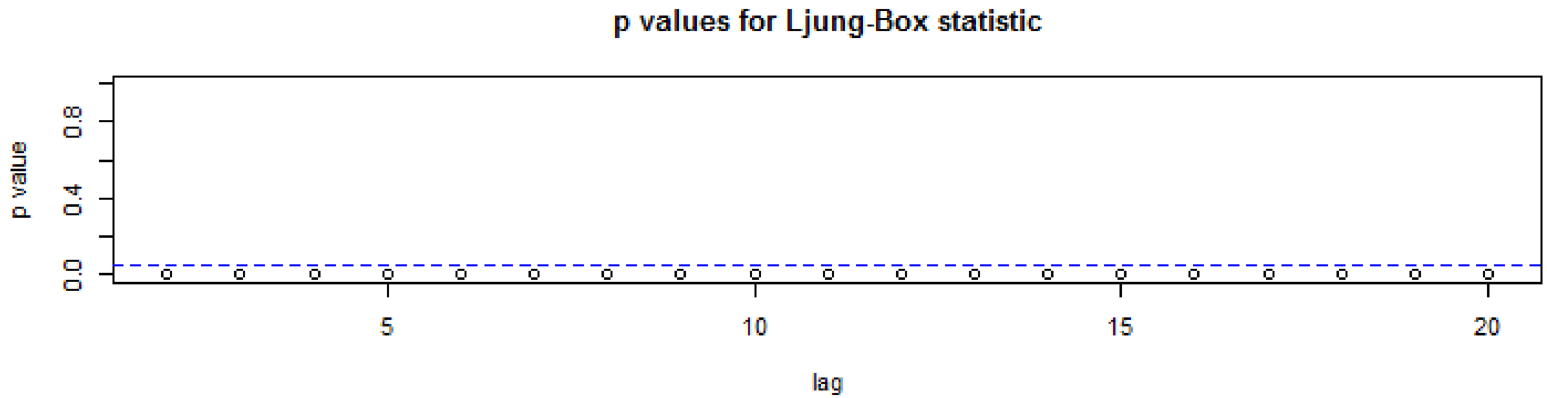
- Checking residuals - ACF:



- Revision:
  - ◇ What is the null hypothesis?
  - ◇ What are these intervals used for and how are they constructed?
  - ◇ What is the outcome?

Motivation for more complicated models

- Checking residuals - P values of Ljung-Box statistics:



- We have residuals from AR(1) model, the degress of freedom are decreased by 1
- Revision:
  - ◇ What is the null hypothesis? What is the result of the test?
  - ◇ How is the statistic computed and what is its distribution under null hpothesis?

II.

Autoregressive process of the second order - AR(2)

Previous example - modelling spread

- We found out that AR(1) model

$$x_t = \delta + \alpha x_{t-1} + u_t,$$

  is not suitable (residuals are not white noise)

- We try to use in addition to x_{t-1} also x_{t-2}:

$$x_t = \delta + \alpha_1 x_{t-1} + \alpha_2 x_{t-2} + u_t$$

- Such a process is called autoregressive process of second order
- In the same way autoregressive process of p-th order:

$$x_t = \delta + \alpha_1 x_{t-1} + \ldots + \alpha_p x_{t-p} + u_t$$

- Firstly we will study the AR(2) process

AR(2) - definition

- AR(2) process:

$$x_t = \delta + \alpha_1 x_{t-1} + \alpha_2 x_{t-2} + u_t$$

- Already without u_t it is more complicated than AR(1) - roots of the characteristic polynomial
- We try another approach (not substitution)
- Using lag operator:

$$\begin{aligned}(1 - \alpha_1 L - \alpha_2 L^2) x_t &= \delta + u_t \\ \alpha(L) x_t &= \delta + u_t\end{aligned}$$

- Wold representation and stacionarity:

$$x_t = \alpha^{-1}(L)\delta + \alpha^{-1}(L) u_t$$

$\rightarrow$ we need inverse operator $\alpha^{-1}(L)$

AR(2) - definition

- Inverse operator $\alpha^{-1}(L)$; we find it using a method of undetermined coefficients:

$$\alpha^{-1}(L) = \psi_0 + \psi_1 L + \psi_2 L^2 + \ldots$$

and

$$(3) \quad 1 = (1 - \alpha_1 L - \alpha_2 L^2)(\psi_0 + \psi_1 L + \psi_2 L^2 + \ldots)$$

- We compare coefficients in front of L^j on both sides of (3):

$$\psi_j - \alpha_1 \psi_{j-1} - \alpha_2 \psi_{j-2} = 0,$$

$$\psi_0 = 1, \; \psi_1 = \alpha_1$$

AR(2) - stationarity

- Stationarity conditions: To satisfy the condition $\sum \psi_j^2 < \infty$ the roots of the charakteristic equation
$$\lambda^2 - \alpha_1 \lambda - \alpha_2 = 0$$
need to be less than 1 in absolute value
- In other words: roots of the equation
$$\alpha(L) = 1 - \alpha_1 L - \alpha_2 L^2 = 0$$
have to be greater than 1 in absolute value, i.e. outside of the unit circle
- The same as for AR(1) before: roots of $\alpha(L) = 0$ are outside of unit circle

Example - modelling spread

Estimated AR(2) model:

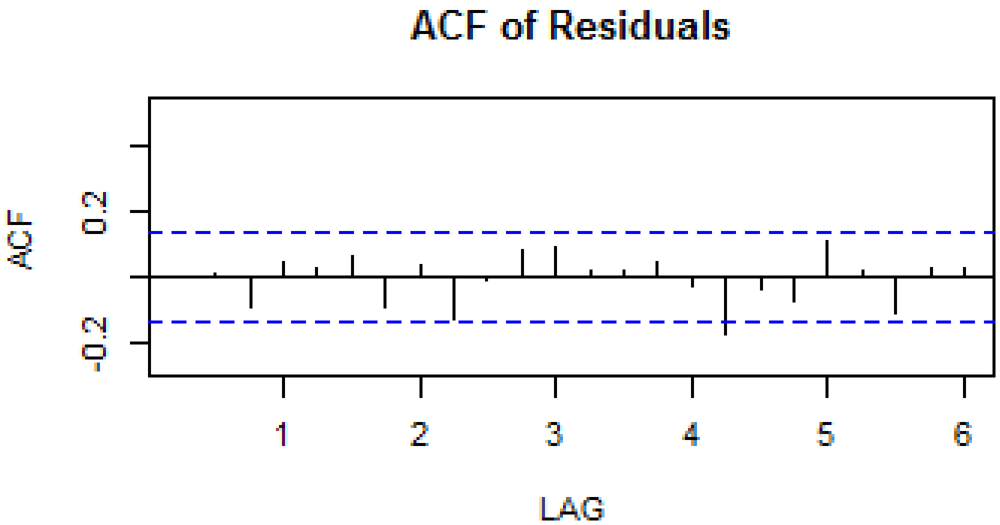
```
> sarima(spread,2,0,0,details="FALSE")
$fit

Call:
stats::arima(x = xdata, order = c(p, d, q), seasonal = list(order = c(P, D,
    Q), period = S), xreg = xmean, include.mean = FALSE, optim.control = list(t$
    REPORT = 1, reltol = tol))

Coefficients:
         ar1      ar2   xmean
      1.1809  -0.2886  1.0449
s.e.  0.0650   0.0651  0.4212

sigma^2 estimated as 0.4677:  log likelihood = -225.42,  aic = 458.84

$AIC
[1] 0.2678181

$AICc
[1] 0.277955

$BIC
[1] -0.6853031
```

Example - modelling spread

- Show that the estimate process is stationary.
- What we test about the residuals - state null hypotheses and explain the tests
- What is their result?

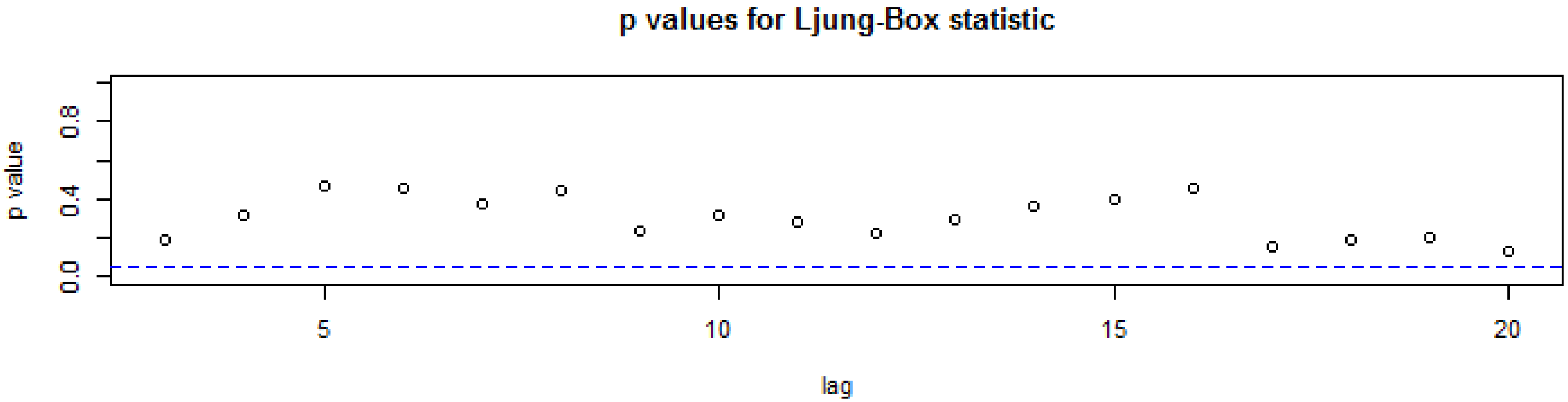
Example - modelling spread

- ACF:



Example - modelling spread

- P-values of Ljung-Box statistics



For residuals from AR(p) model the degrees of freedom are decreased by p.

AR(2) - moments

- Weakly stationary AR(2) process:

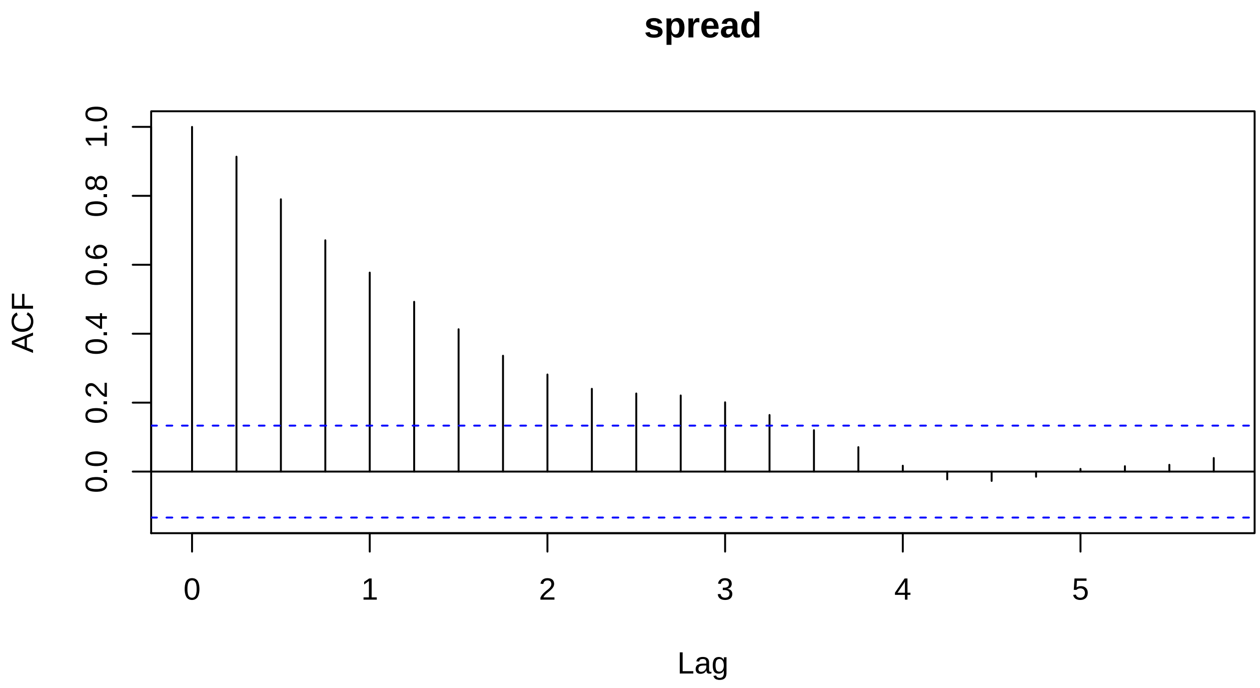
$$x_t = \delta + \alpha_1 x_{t-1} + \alpha_2 x_{t-2} + u_t$$

- Expected value:
  - denote $\mu = E[x_i]$; then

$$\begin{aligned} \mu &= \delta + \alpha_1 \mu + \alpha_2 \mu, \\ \mu &= \frac{\delta}{1 - \alpha_1 - \alpha_2} \end{aligned}$$

AR(2) - moments

- Autocovariances of AR(2) process - motivation :
  - recall - sample ACF for spread:



AR(2) - moments

- Autocovariances of AR(2) process - motivation:
  - sample ACF for spread was similar to AR(1) process
  - however, AR(1) was not a good model, but AR(2) was
  - what is the bahaviour of the ACF of AR(2) process?
  - can it be similar to ACF of AR(1)? (it seems so)
  - can it be "totally different"? (i.e. "this is certainly not AR(1), but it can be AR(2)")

AR(2) - moments

- Autocovariances - computation: we can assume zero expected value, i.e.

$$
\begin{aligned}
x_t &= \alpha_1 x_{t-1} + \alpha_2 x_{t-2} + u_t \quad \Big/ \quad \times x_{t-s},\ E[.] \\
E[x_{t-s}x_t] &= \alpha_1 E[x_{t-s}x_{t-1}] + \alpha_2 E[x_{t-s}x_{t-2}] + E[x_{t-s}u_t]
\end{aligned}
$$

- For $s = 0, 1, 2$ we obtain:

$$
\begin{aligned}
\gamma(0) &= \alpha_1\gamma(1) + \alpha_2\gamma(2) + \sigma^2 \\
\gamma(1) &= \alpha_1\gamma(0) + \alpha_2\gamma(1) \\
\gamma(2) &= \alpha_1\gamma(1) + \alpha_2\gamma(0)
\end{aligned}
$$

  - system of equations $\rightarrow \gamma(0) = Var[x_t]$, $\gamma(1)$, $\gamma(2)$

- For $s \geq 2$ - difference equation:

  (4) $\qquad \gamma(s) - \alpha_1\gamma(s-1) - \alpha_2\gamma(s-2) = 0,$

  with initial conditions from the previous point

AR(2) - moments

- Autocorrelations: we divide the difference equation (4) and its initial conditions by $\gamma(0)$:

$$\rho(s) - \alpha_1 \rho(s-1) - \alpha_2 \rho(s-2) = 0$$

$$\rho(0) = 1, \rho(1) = \frac{\alpha_1}{1-\alpha_2}$$

AR(2) - ACF - example 1

- Spread modelled by AR(2) process:

```
Coefficients:
         ar1      ar2   xmean
      1.1809  -0.2886  1.0449
s.e.  0.0650   0.0651  0.4212
```

- Difference equation for autocorrelations:

$$\rho(s) - 1.1809\rho(s-1) + 0.2886\rho(s-2) = 0$$

initial conditions: $\rho(0) = 1,\ \rho(1) = \frac{1.1809}{1+0.2886}$

AR(2) - ACF

- ACF is a solution to difference eqution

$$\rho(s) - \alpha_1 \rho(s-1) - \alpha_2 \rho(s-2) = 0$$

$\Rightarrow$ behaviour depends on roots of charakteristic equation

$$\lambda^2 - \alpha_1 \lambda - \alpha_2 = 0$$

- λ_1, λ_2 - real (and different): ACF has a form

$$\rho(s) = c_1 \lambda_1^s + c_2 \lambda_2^s$$

Stationarity: $|\lambda_{1,2}| < 1$

- λ_1, λ_2 - complex: ACF is a dumped combination of sine and cosine

$$\rho(s) = r^s (c_1 \cos(ks) + c_2 \sin(ks))$$

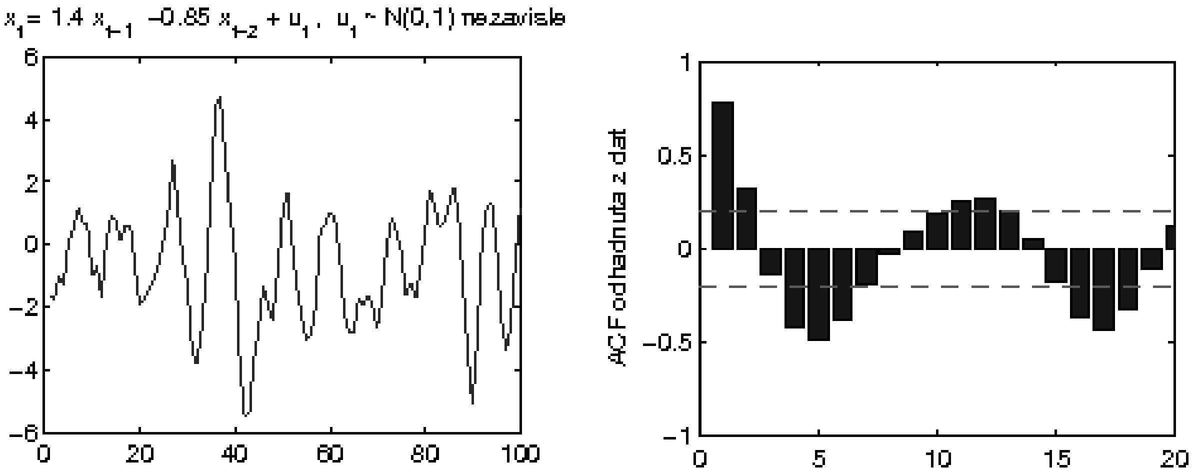
Stationarity: $r < 1$

AR(2) - ACF - example 2

- Process: $x_t = 1.4x_{t-1} - 0.85x_{t-2} + u_t$
  - ◇ correlations satisft the difference eqution
  $$\rho(t) - 1.4\rho(t-1) + 0.85\rho(t-2) = 0$$
  - ◇ and its solution
  $$\rho(t) = 0.922^t(c_1 \cos(0.709t) + c_2 \sin(0.709t))$$
  - ◇ c_1, c_2 from initial conditions $\rho(0), \rho(1)$
  - ◇ $\cos(kt), \sin(kt)$ $\rightarrow$ period $\frac{2\pi}{k}$
  in our case $\frac{2\pi}{k} = \frac{2\pi}{0.709} = 8.862 \approx 9$
  $\Rightarrow$ in data generated by this process we can expect this period

AR(2) - ACF - example

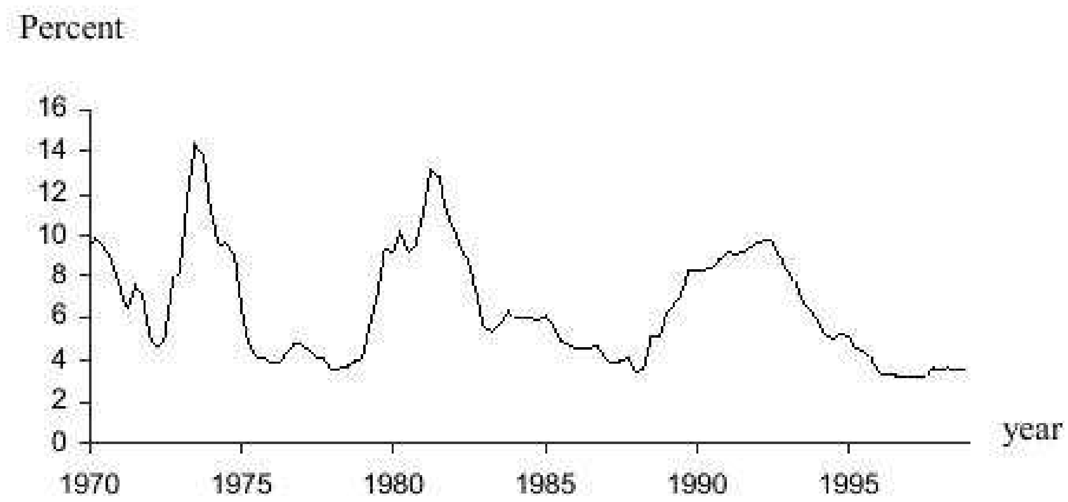
- Figure:
  - realization of the process
    $x_t = 1.4x_{t-1} - 0.85x_{t-2} + u_t$
  - sample ACF



AR(2) - real data

[Kirchgässner, Wolters], example 2.6

- 3-months interest rate, Germany, 1970q1-1998q4

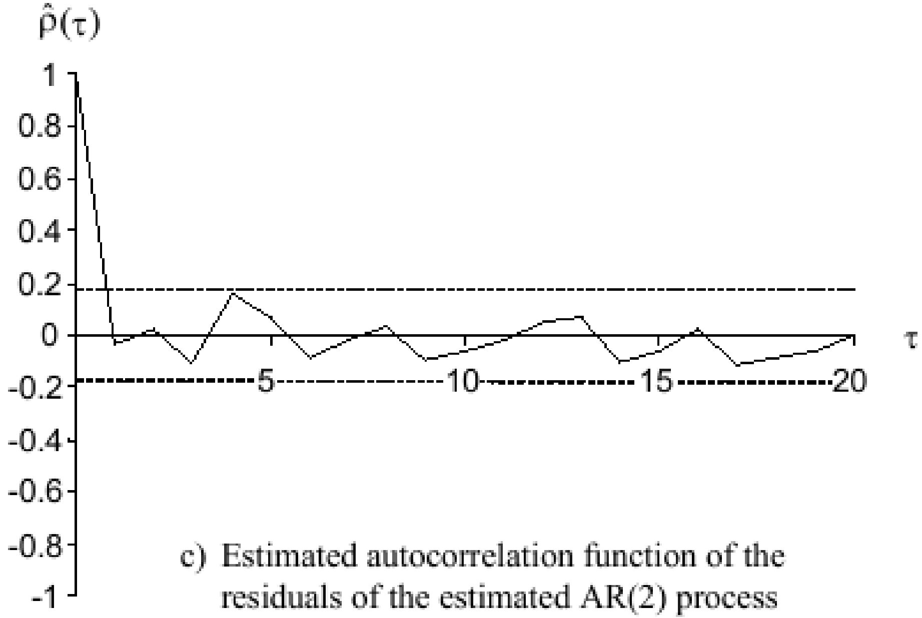


AR(2) - real data

- Estimated AR(2) model:

$$GSR_t = \underset{(2.82)}{0.577} + \underset{(17.49)}{1.407}\, GSR_{t-1} - \underset{(-6.16)}{0.498}\, GSR_{t-2} + \hat{u}_t,$$

$$\bar{R}^2 = 0.910, \quad SE = 0.812, \quad Q(6) = 6.431 \quad (p = 0.377)$$



c) Estimated autocorrelation function of the residuals of the estimated AR(2) process with confidence intervals

AR(2) - real data

- Questions about the model:
  - ◇ Is it stationary?
  - ◇ Check residuals - ACF, Q-statistics (what are the degrees of freedom?).
  - ◇ What is the expected value of the process?
  - ◇ What is the bahaviour of its ACF?
  - ◇ Explain the following assertion from the book (p.49) and compute the given values:
    "The two roots of the process are 0.70 +/- 0.06i, i.e. they indicate cycles ... the frequency $f = 0.079$ *corresponds to a period of 79.7 quarters and therefore of nearly 20 years."*

III.

Autoregressive process of p-th order - AR(p)

AR(p) - introduction

- We have seen AR(1) and AR(2) proceses, their ACF can be similar - how to distinguish them?
- In the same way we can define AR(p) process - what is its ACF?
- How to determine the correct order of a model for data?

- AR(p) process - we show:
  - stationarity: roots outside of the unit circle
  - ACF: given by a difference equation of p-th order
  - the first p autocorrelations (initial conditions for the difference equation): from the system of equations; useful computation, we will use it also later

AR(p) proces - stationarity

- AR(p) process:

  (5) $x_t = \delta + \alpha_1 x_{t-1} + \alpha_2 x_{t-2} + \ldots + \alpha_p x_{t-p} + u_t,$

  t. j. $\alpha(L)x_t = \delta + u_t$, where
  $\alpha(L) = 1 - \alpha_1 L - \ldots - \alpha_p L^p$

- Wold representation and stationarity:

  $$x_t = \alpha(L)^{-1}(\delta + u_t),$$

  inverse operator $\alpha(L)^{-1}$ in the form

  $$\alpha(L)^{-1} = 1 + \psi_1 L + \psi_2 L^2 + \ldots$$

- For coefficients ψ_j we obtain difference equation

  $$\psi_k - \alpha_1 \psi_{k-1} - \ldots - \alpha_p \psi_{k-p} = 0$$

  $\Rightarrow$ in order to $\sum \psi_j^2$ the roots of
  $\lambda^k - \alpha_1 \lambda^{k-1} - \ldots - \alpha_p = 0$ need to be inside the unit circle, i.e. roots of $\alpha(L) = 0$ have to be outside of the unit circle

AR(p) process - moments

- Expected value:
  we denote $\mu = E[x_t]$ and take expected value of both sides of (5):

$$\mu = \delta + \alpha_1 \mu + \ldots + \alpha_p \mu \;\Rightarrow\; \mu = \frac{\delta}{1 - \alpha_1 - \ldots - \alpha_p}$$

- Variance autocovariances - WLOG $\delta = 0$

$$\begin{aligned} x_t &= \alpha_1 x_{t-1} + \ldots + \alpha_p x_{t-p} + u_t \;\Big/\; \times x_{t-s},\ E[.] \\ \gamma(s) &= \alpha_1 \gamma(s-1) + \ldots \alpha_p \gamma(s-p) + E[u_t x_{t-s}] \end{aligned}$$

AR(p) process - moments

- Variance, autocovariances - continued:
  - $s = 0, 1, \ldots, p \rightarrow$ system of $p+1$ equations with unknowns $\gamma(0), \gamma(1), \ldots, \gamma(p)$:

$$\begin{aligned}
\gamma(0) &= \alpha_1\gamma(1) + \alpha_2\gamma(2) + \ldots + \alpha_p\gamma(p) + \sigma^2 \\
\gamma(1) &= \alpha_1\gamma(0) + \alpha_2\gamma(1) + \ldots + \alpha_p\gamma(p-1) \\
&\ldots \\
\gamma(p) &= \alpha_1\gamma(p-1) + \alpha_2\gamma(p-2) + \ldots + \alpha_p\gamma(0)
\end{aligned} \tag{6}$$

  - others from the difference eqution

$$\gamma(t) - \alpha_1\gamma(t-1) - \ldots - \alpha_p\gamma(t-p) = 0 \tag{7}$$

AR(p) process - moments

- ACF :
  - difference equation - we divide (7) by $\gamma(0)$:

$$\rho(t) - \alpha_1\rho(t-1) - \ldots - \alpha_p\rho(t-p) = 0$$

  - initial conditions - last p equations from (6) divided by $\gamma(0)$:

$$\begin{aligned} \rho(1) &= \alpha_1 + \alpha_2\rho(1) + \ldots + \alpha_p\rho(p-1) \\ \rho(2) &= \alpha_1\rho(1) + \alpha_2 + \ldots + \alpha_p\rho(p-2) \\ & \ldots \\ \rho(p) &= \alpha_1\rho(p-1) + \alpha_2\rho(p-2) + \ldots + \alpha_p \end{aligned} \quad (8)$$

    - called Yule-Wolker equations

AR(p) process - ACF - example 1

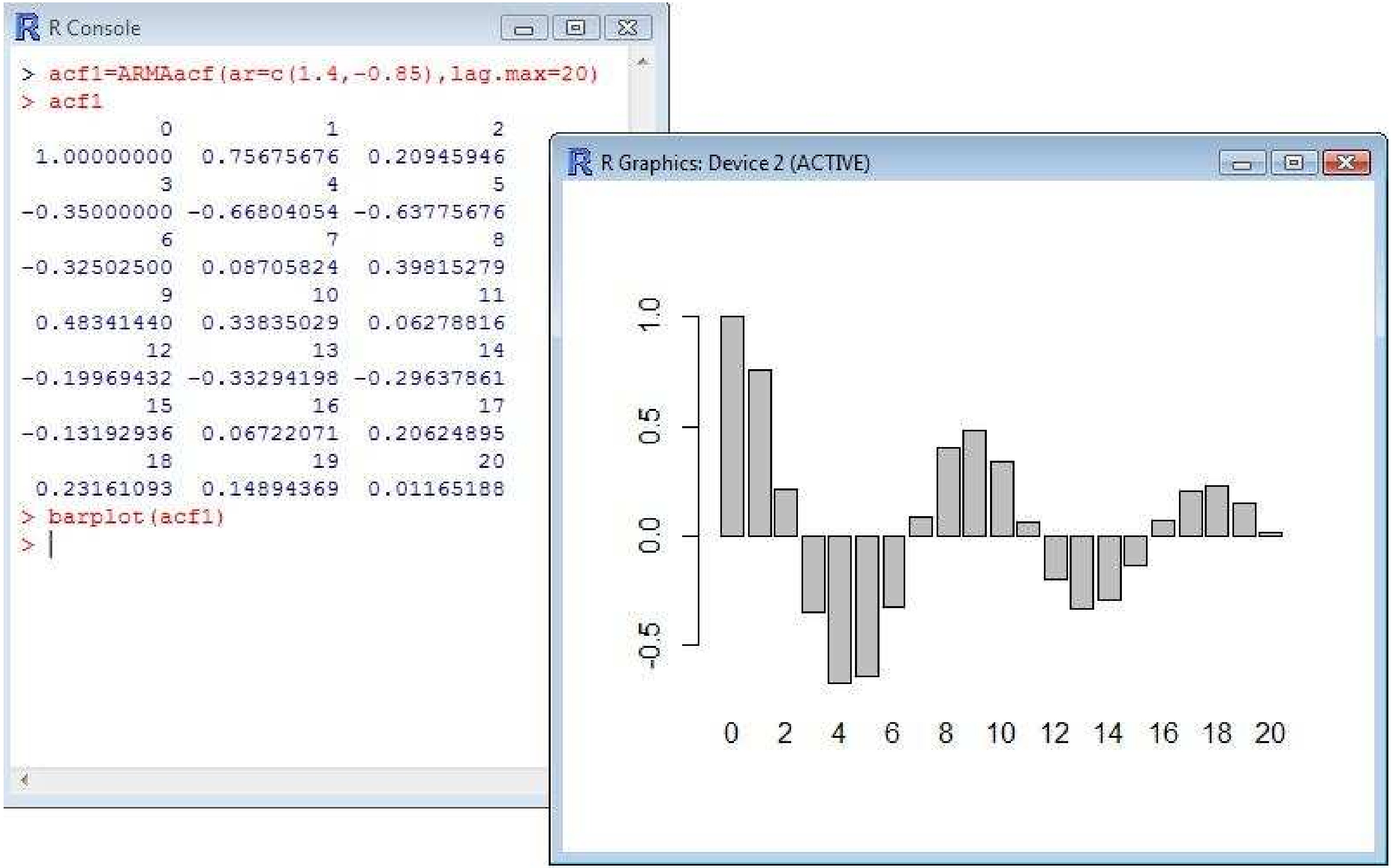
- ACF in R:
  - function **ARMAacf** from package **stats**
  - we computed ACF of the process

$$x_t = 1.4x_{t-1} - 0.85x_{t-2} + u_t$$

  - now in R:

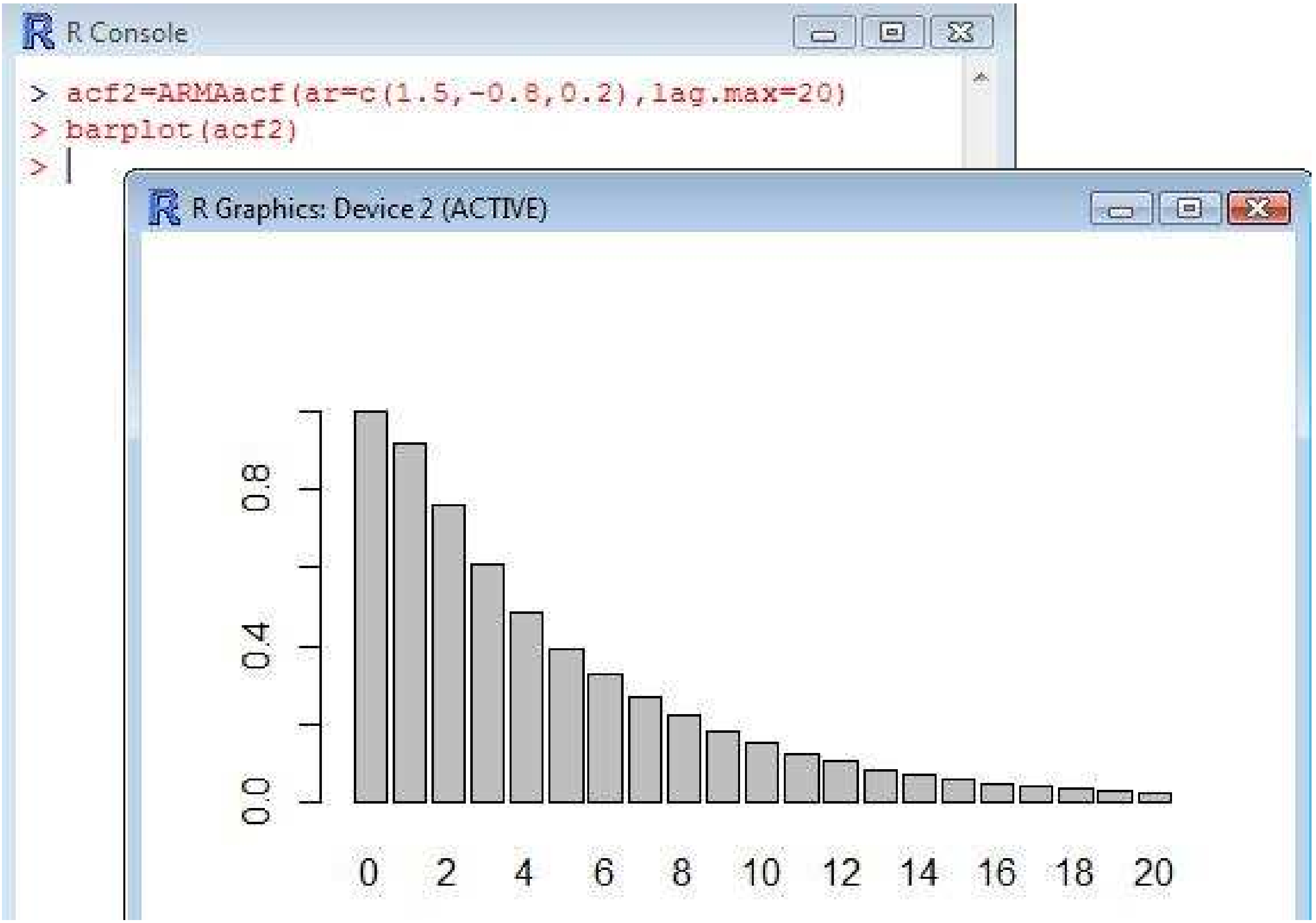
ARMAacf(ar=c(1.4,-0.85), lax.max=20)

AR(p) process - ACF - example 1



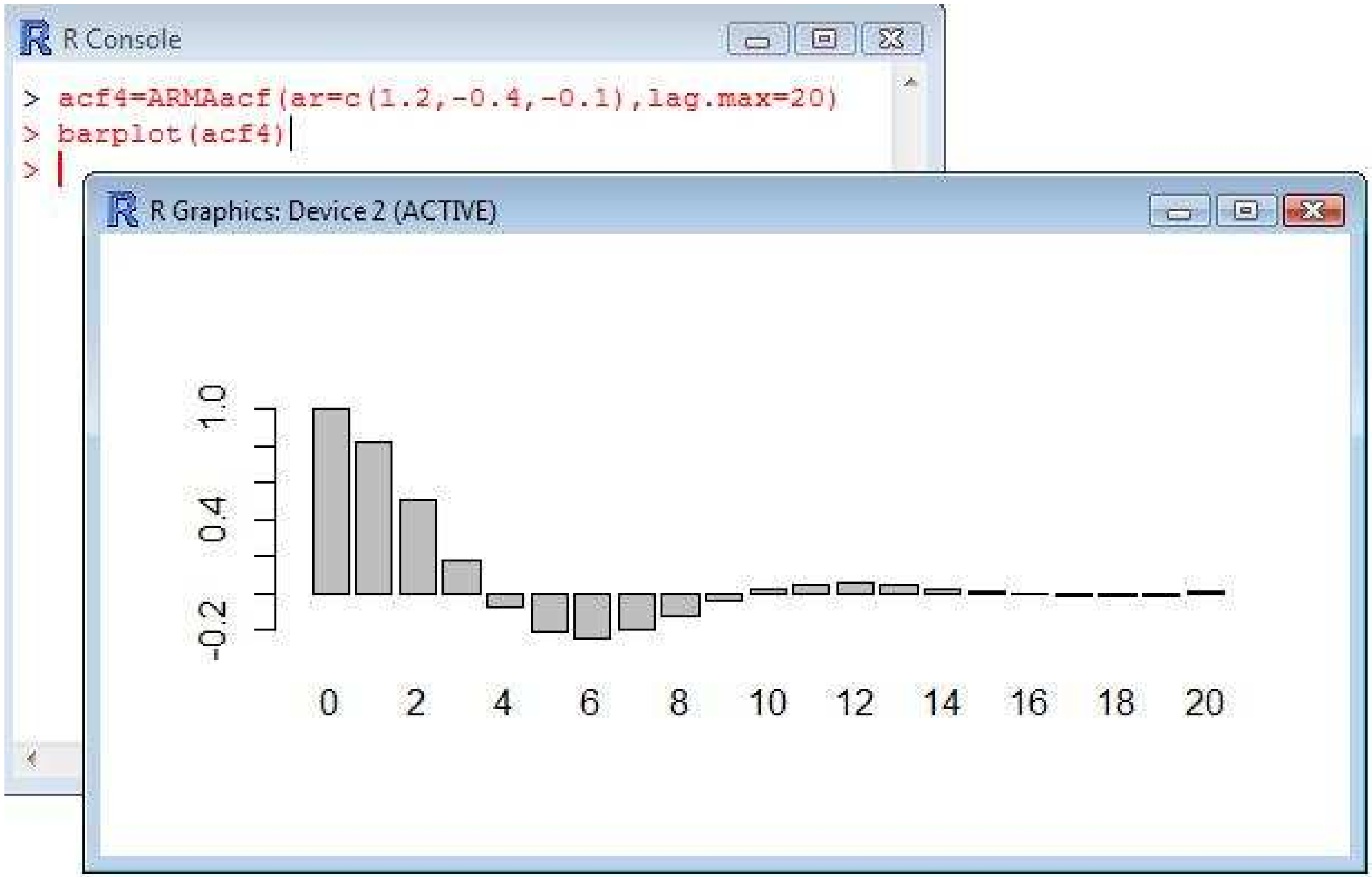
AR(p) process - ACF - example 2

- AR(3) process $x_t = 1.5\,x_{t-1} - 0.8\,x_{t-2} + 0.2\,x_{t-3} + u_t$



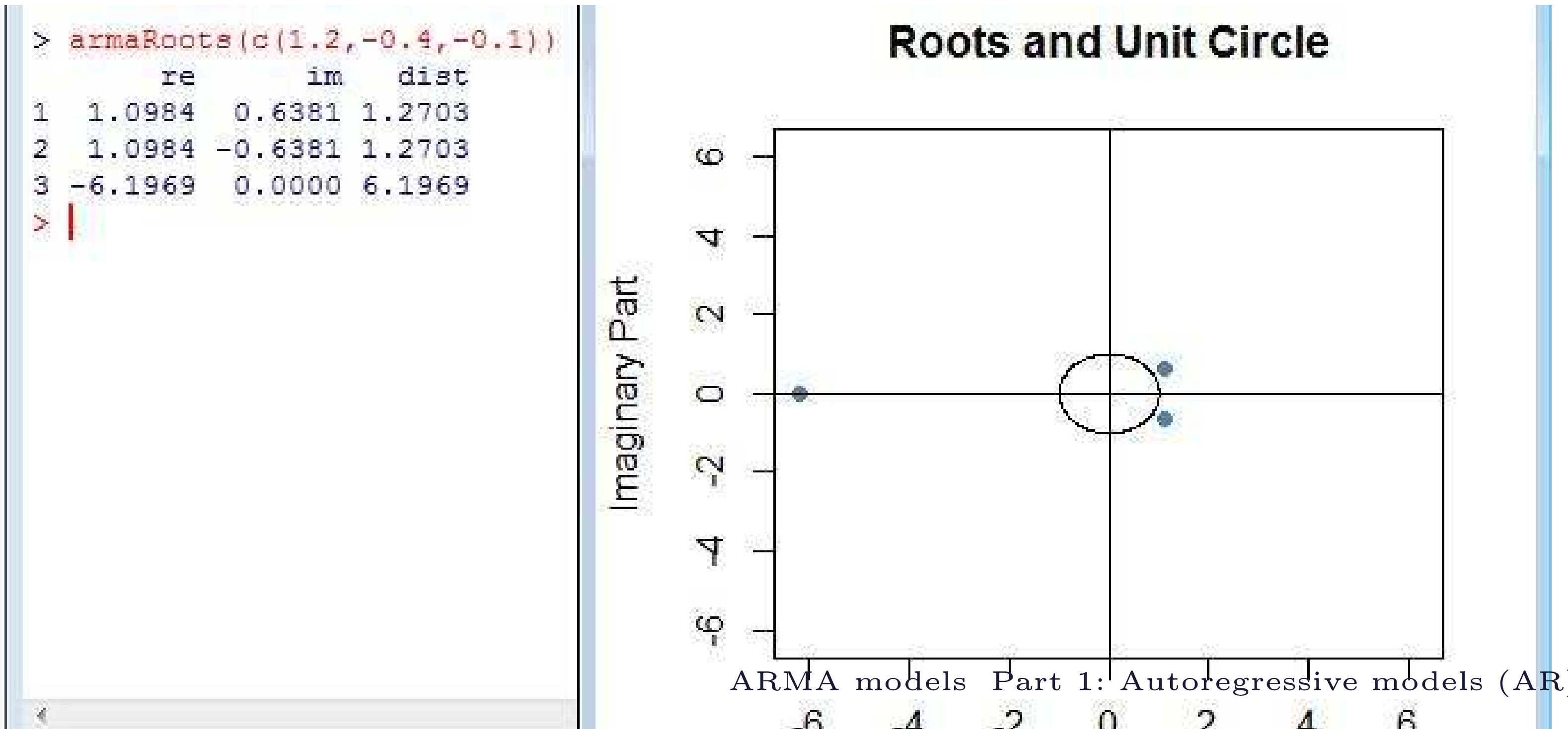
AR(p) process - ACF - example 3

- AR(3) process $x_t = 1.2\, x_{t-1} - 0.4\, x_{t-2} - 0.1\, x_{t-3} + u_t$
- We can expect complex roots.

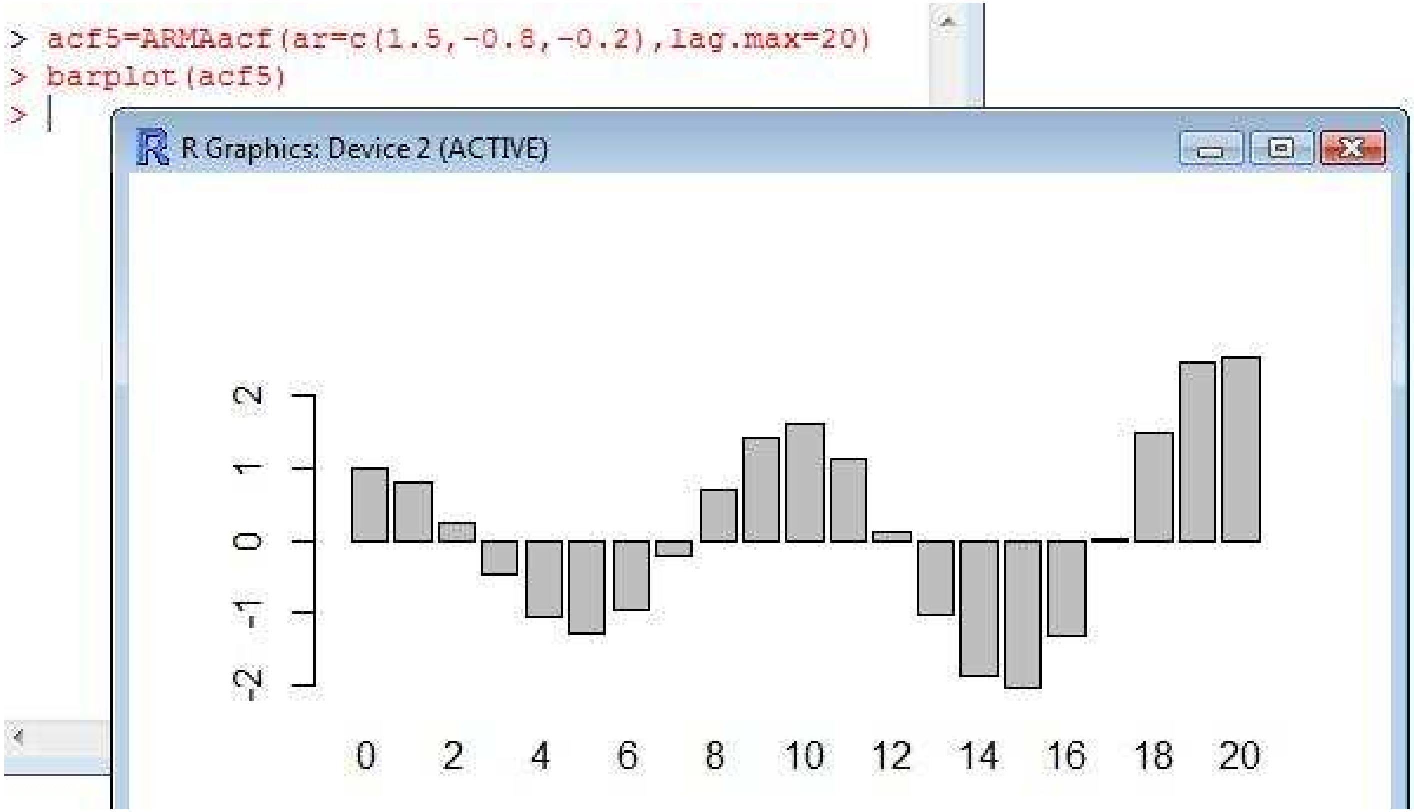


AR(p) process - ACF - example 3

- Roots v R:
  - ◇ function **armaRoots** from package **fArma**
  - ◇ returns values of the roots - they have to be outside of the unit circle
- EXERCSE: write down the polynomial, the roots of which we compute now

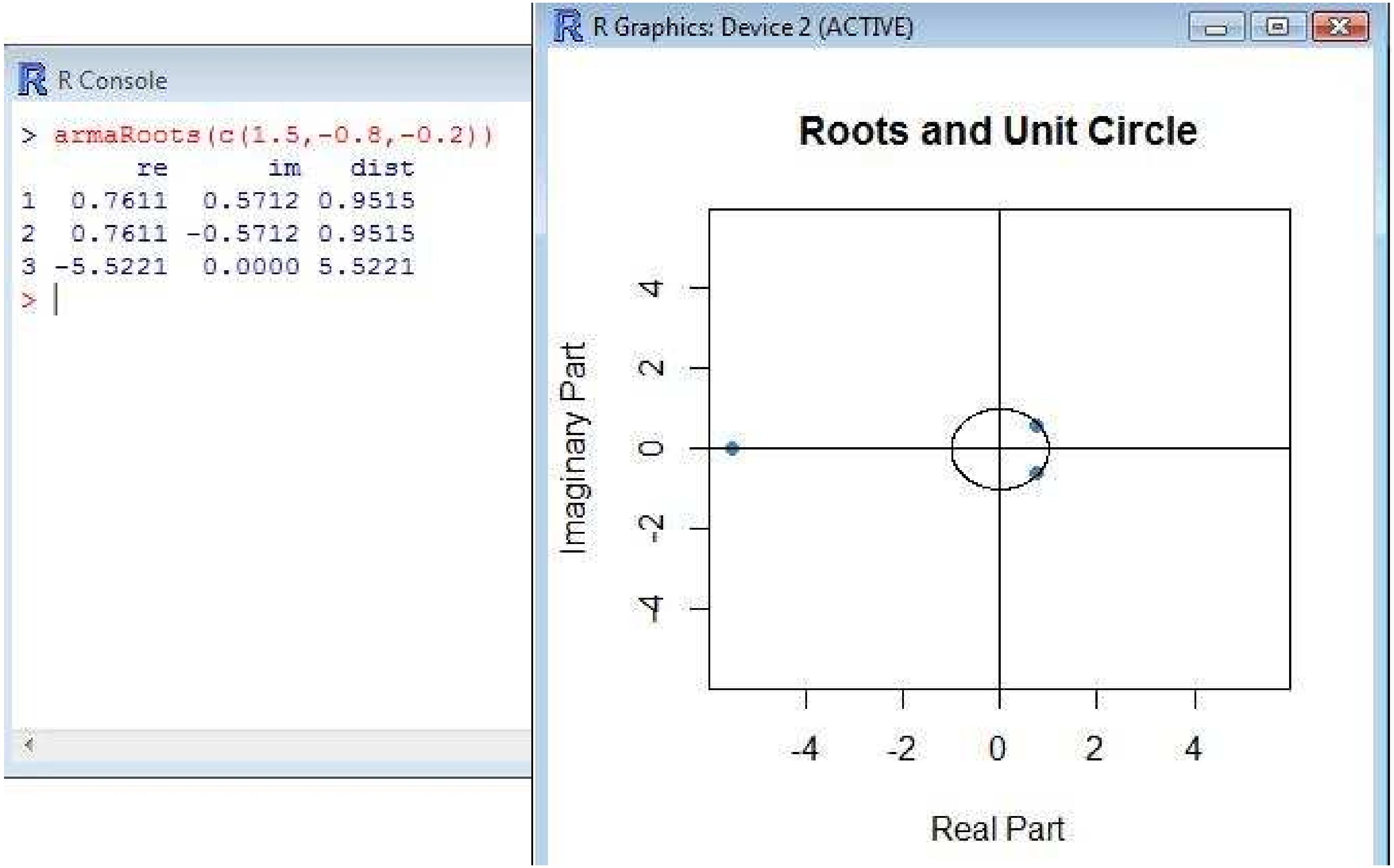


AR(p) process - ACF - example 4



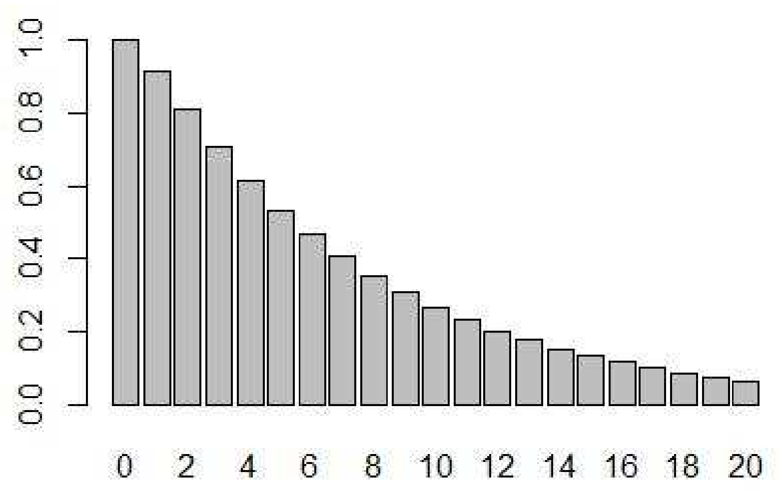
- How is it possible?
  - absolute value of ACF greater than 1
  - increasing

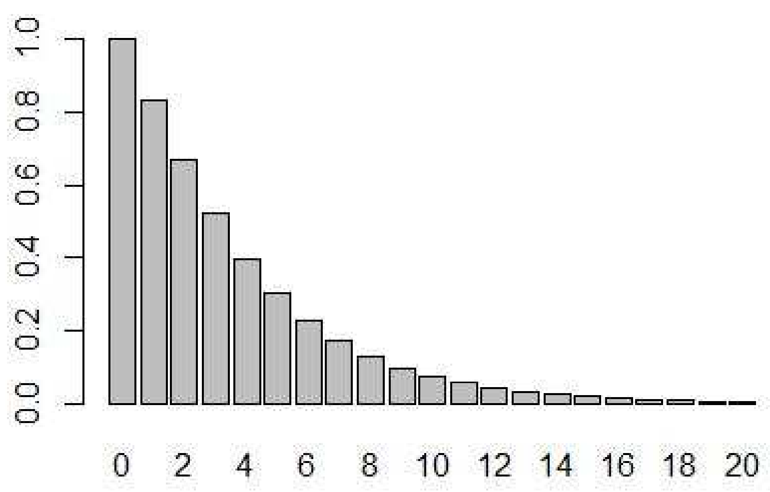
AR(p) process - ACF - example 4



- Process is not stationary → ACF calculation does not make sense

AR(p) process - ACF - example 5





- ACF for two processes: one is AR(2) and the other is AR(3)
- We cannot distinguish them
- Working with real data - moreover, we do not have exact values but estimates

IV.

Parctial autocorrelation function - determining the order of AR process

PACF - motivation

- consider some random process x_t with zero expected value and modell it using its k lagged values:

$$x_t = \beta_1 x_{t-1} + \beta_2 x_{t-2} + \ldots + \beta_k x_{t-k} + u_t$$

- Denote coefficients b Φ_{ki}, where k is the number of lags of x which we used and i is a coefficient at x_{t-i}
- So:

$$\begin{aligned} x_t &= \Phi_{11} x_{t-1} + u_t \\ x_t &= \Phi_{21} x_{t-1} + \Phi_{22} x_{t-2} + u_t \\ x_t &= \Phi_{31} x_{t-1} + \Phi_{32} x_{t-2} + \Phi_{33} x_{t-3} + u_t \\ &\ldots \\ x_t &= \Phi_{k1} x_{t-1} + \Phi_{k2} x_{t-2} + \Phi_{k3} x_{t-3} + \ldots + \Phi_{kk} x_{t-k} + u_t \end{aligned}$$

- If x is an AR(p) process, tthen $\Phi_{kk} = 0$ for $k > p$.

PACF - definition and computation

- Coefficient Φ_{kk} is called partial autocorrelation of order k
- Their sequence form the partial autocorrelation function (PACF)
- Computation: we start from

  $$x_t = \Phi_{k1} x_{t-1} + \Phi_{k2} x_{t-2} + \Phi_{k3} x_{t-3} + \ldots + \Phi_{kk} x_{t-k} + u_t$$

  and similarly as in the case of Yule-Wolker equations we get

$$\begin{aligned}
\rho(1) &= \Phi_{k1} + \Phi_{k2}\,\rho(1) + \ldots + \Phi_{kk}\,\rho(k-1) \\
\rho(2) &= \Phi_{k1}\,\rho(1) + \Phi_{k2} + \ldots + \Phi_{kk}\,\rho(k-2) \\
&\ldots \\
\rho(k) &= \Phi_{k1}\,\rho(k-1) + \Phi_{k2}\,\rho(k-2) + \ldots + \Phi_{kk}
\end{aligned}$$

PACF - definition and computation

- Matrix form:

$$\begin{bmatrix} 1 & \rho(1) & \dots & \rho(k-1) \\ \rho(1) & 1 & \dots & \rho(k-2) \\ & & \dots & \\ \rho(k-1) & \rho(k-2) & \dots & 1 \end{bmatrix} \begin{bmatrix} \Phi_{k1} \\ \Phi k2 \\ \dots \\ \Phi kk \end{bmatrix} = \begin{bmatrix} \rho(1) \\ \rho(2) \\ \dots \\ \rho(k) \end{bmatrix}$$

- We need only Φ_{kk}, we use Cramer rule:

$$\Phi_{kk} = \frac{\det \begin{pmatrix} 1 & \rho(1) & \dots & \rho(1) \\ \rho(1) & 1 & \dots & \rho(2) \\ & \dots & \dots & \\ \rho(k-1) & \rho(k-2) & \dots & \rho(k) \end{pmatrix}}{\det \begin{pmatrix} 1 & \rho(1) & \dots & \rho(k-1) \\ \rho(1) & 1 & \dots & \rho(k-2) \\ & \dots & \dots & \\ \rho(k-1) & \rho(k-2) & \dots & 1 \end{pmatrix}} \tag{9}$$

PACF - example: AR(1)

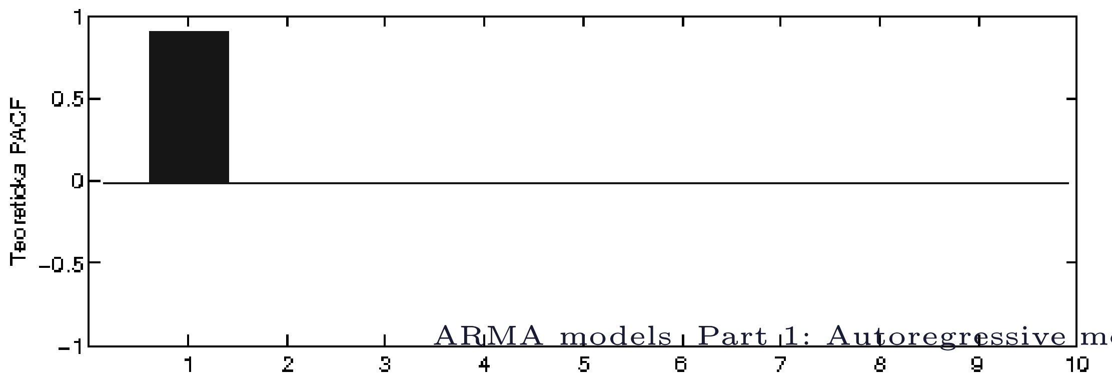
- We compute:

$$\Phi_{11} = \rho(1)$$

$$\Phi_{22} = \frac{\det\begin{pmatrix} 1 & \rho(1) \\ \rho(1) & \rho(2) \end{pmatrix}}{\det\begin{pmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{pmatrix}} = \frac{\rho(2) - \rho(1)^2}{1 - \rho(1)^2} = 0$$

$$\ldots$$

- From the definition of PACF - also the following $\Phi_{kk} = 0$
- For $\alpha = 0.9$:

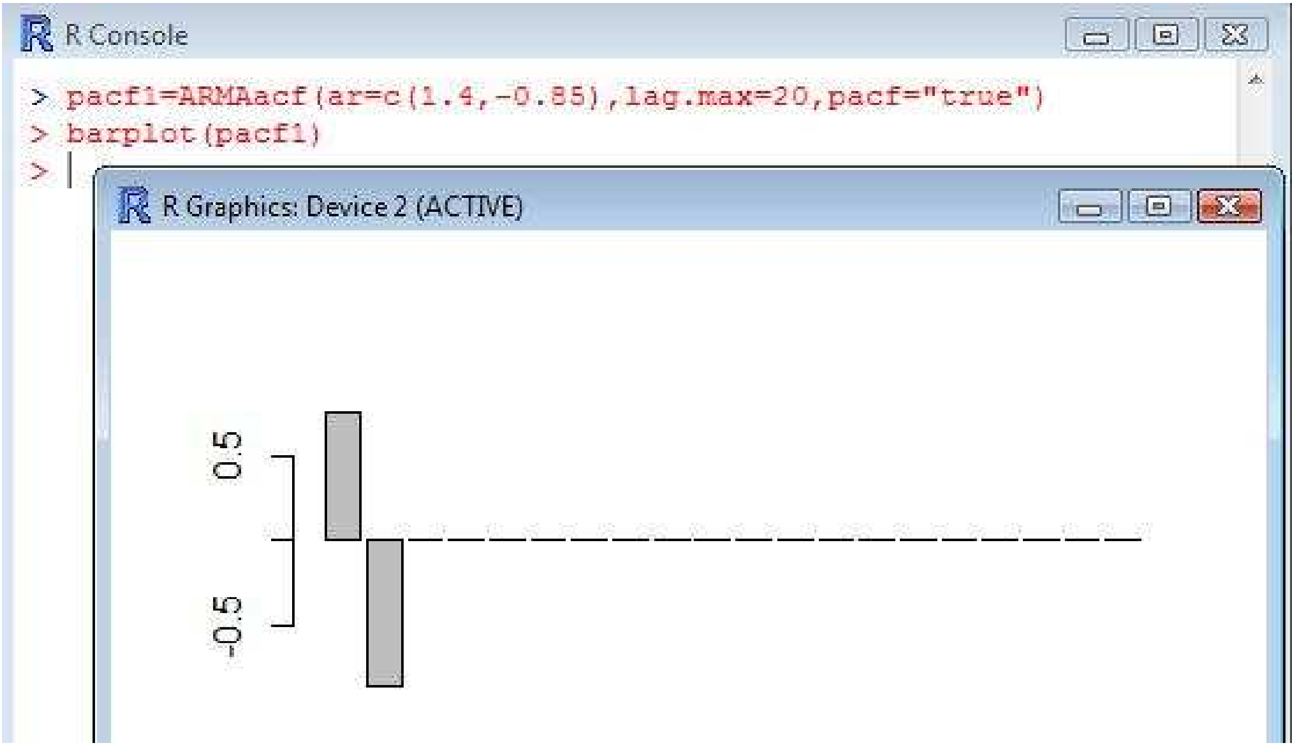


PACF - example 1

- PACF in R - again **ARMAacf** from package **stats**
- For $x_t = 1.4x_{t-1} - 0.85x_{t-2} + u_t$ we computed ACF, now PACF:

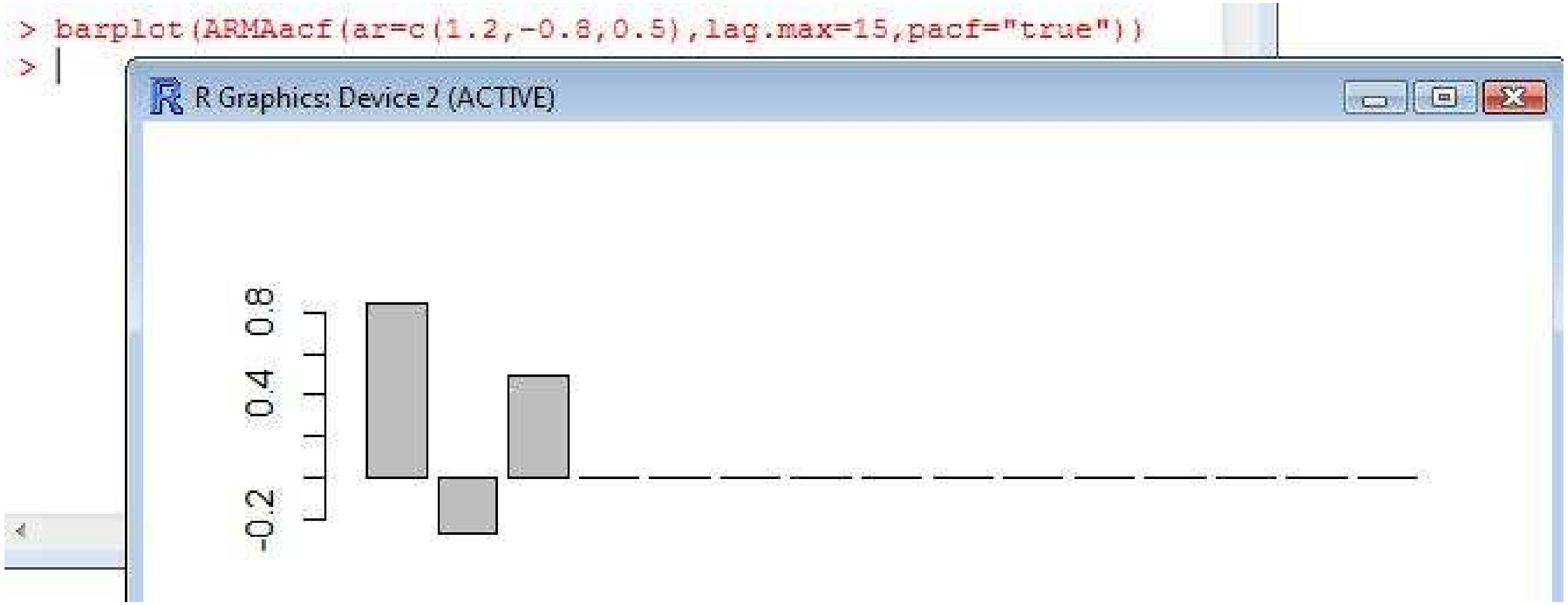
ARMAacf(ar=c(1.4,-0.85), lax.max=20, pacf="true")

PACF - example 1



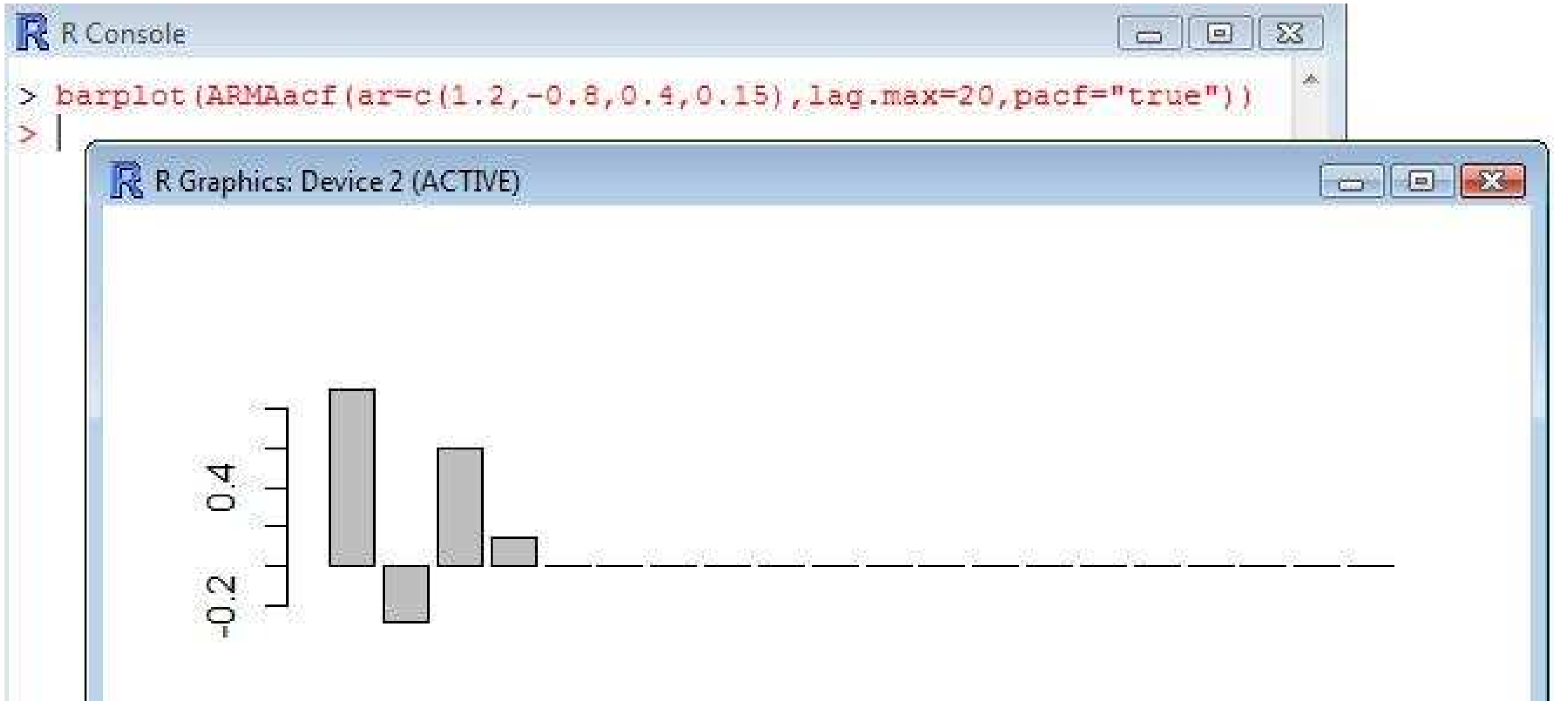
PACF - example 2

- AR(3) process $x_t = 1.2\, x_{t-1} - 0.8\, x_{t-2} + 0.5\, x_{t-3} + u_t$



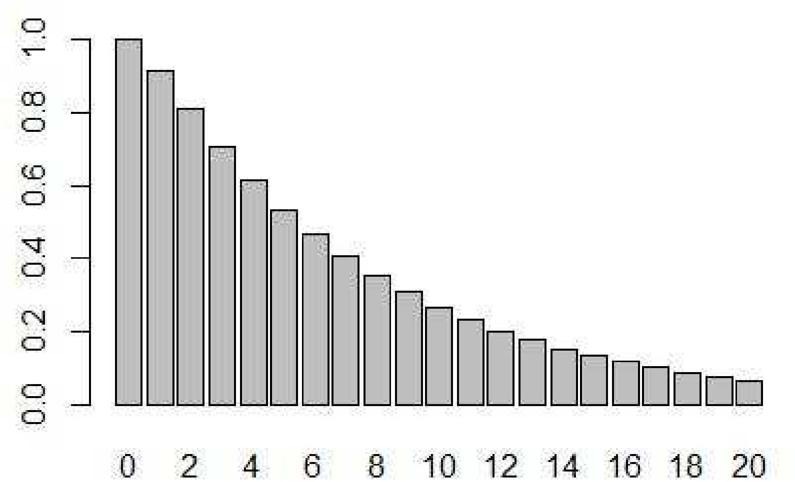
PACF - example 3

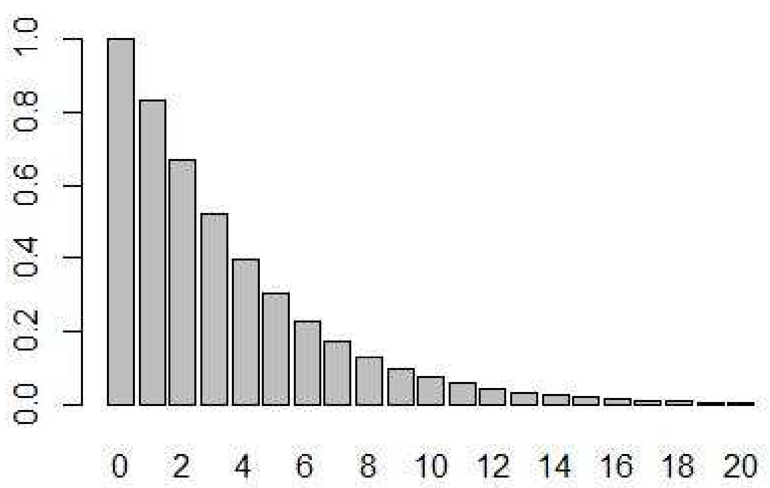
- AR(4) process
  $x_t = 1.2\, x_{t-1} - 0.8\, x_{t-2} + 0.4\, x_{t-3} + 0.15\, x_{t-4} + u_t$



PACF - example 4

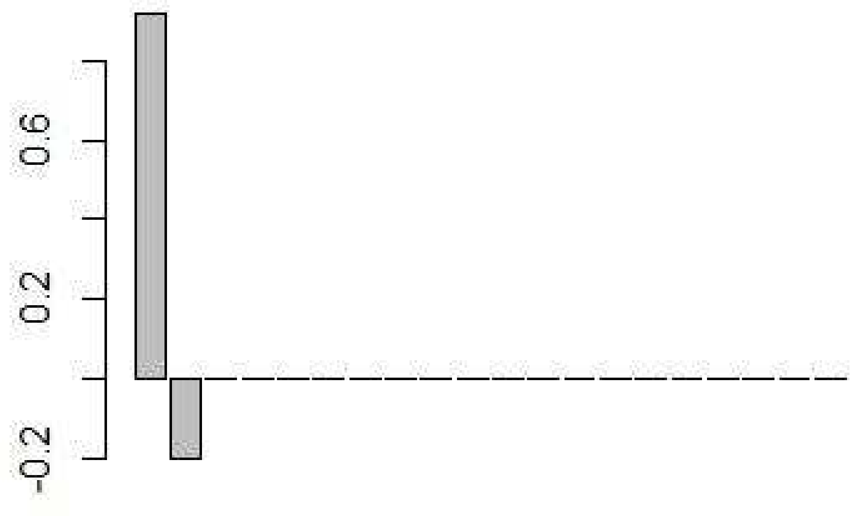
- Recall:
  ACF for two processes, one is AR(2) and the other one AR(3) , but we were not able to distinguish them:

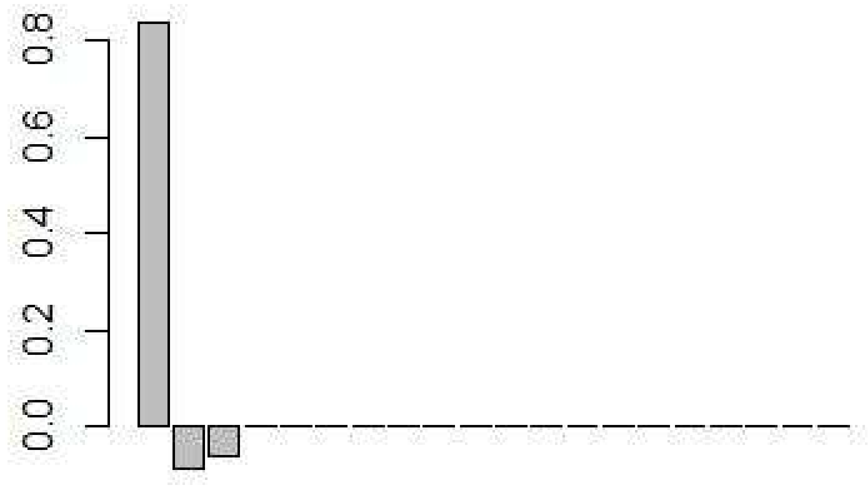




PACF - example 4

- PACF of these processes:





- Now it is clear that in the left we have AR(2) and in the right we have AR(3) process

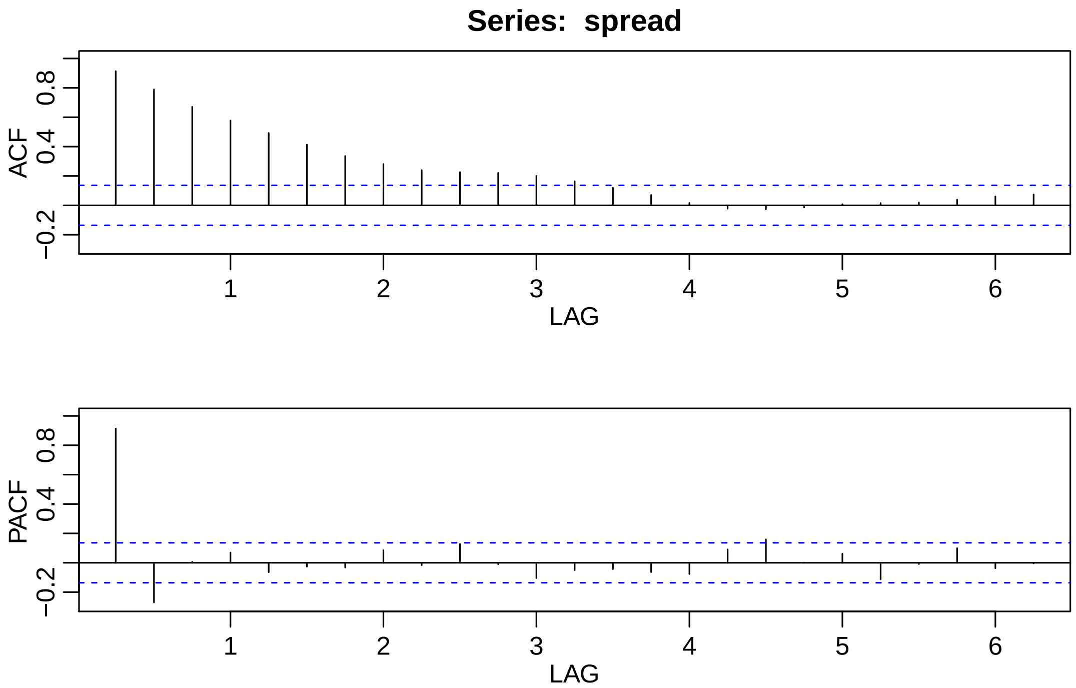
PACF - estimation from data

- Into (15) we set the consistent estimates of autocorrelations $\rightarrow$ consistent estimates of $\hat{\Phi}_{kk}$
- For AR(p) process we have $\Phi_{kk} = 0$ for $k > p$, for these k asymptotically

$$Var[\hat{\Phi}_{kk}] \approx \frac{1}{T}$$

PACF estimation - example 1

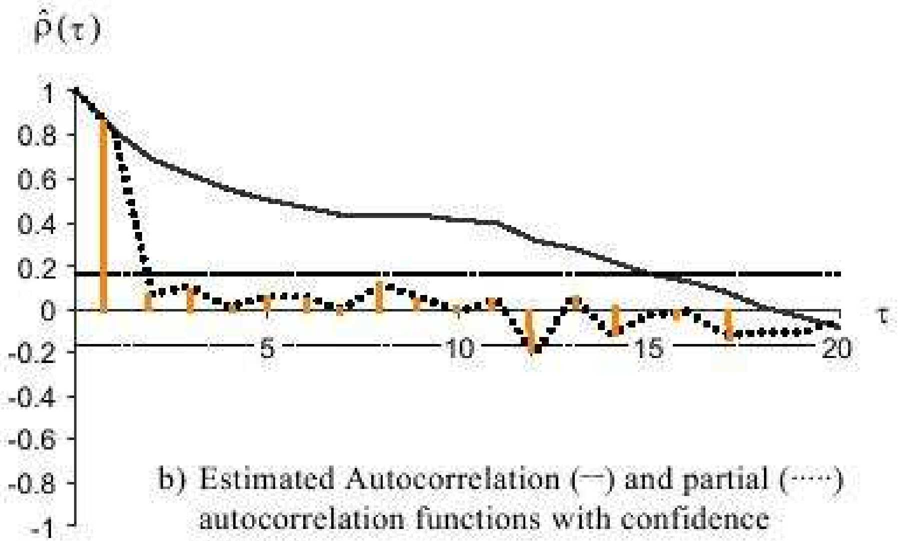
- We modelled spread; using function acf2(spread) we get ACF and PACF:



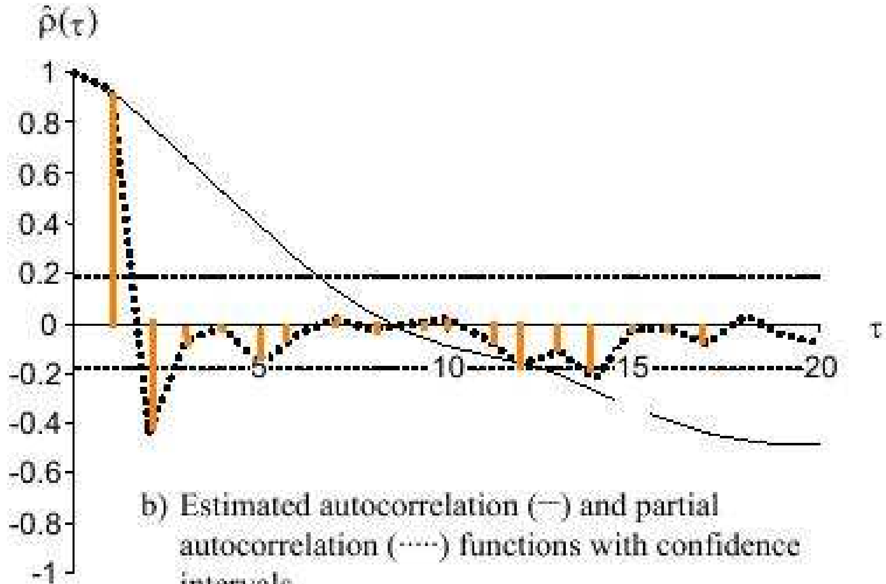
- We see that it suggest estimating AR(2) process (which we did)

PACF estimation - example 2

- Previous real data examples:
  - ◇ popularity (left) - AR(1)
  - ◇ interest rates (right) - AR(2)



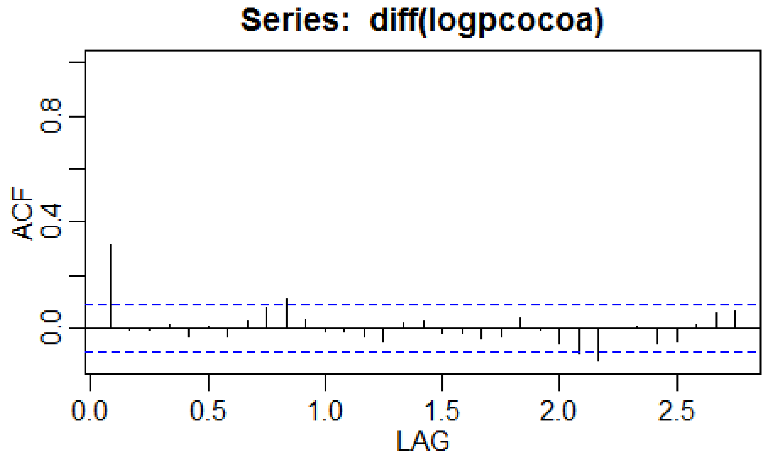
b) Estimated Autocorrelation (—) and partial (·····) autocorrelation functions with confidence intervals



b) Estimated autocorrelation (—) and partial autocorrelation (·····) functions with confidence intervals

Next lecture

- Data: **pcocoa** - cocoa prices; ACF for differences of lagarithms:



- Following lecture: models with this property