ARMA models
Part 2: Moving average models (MA)

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V.

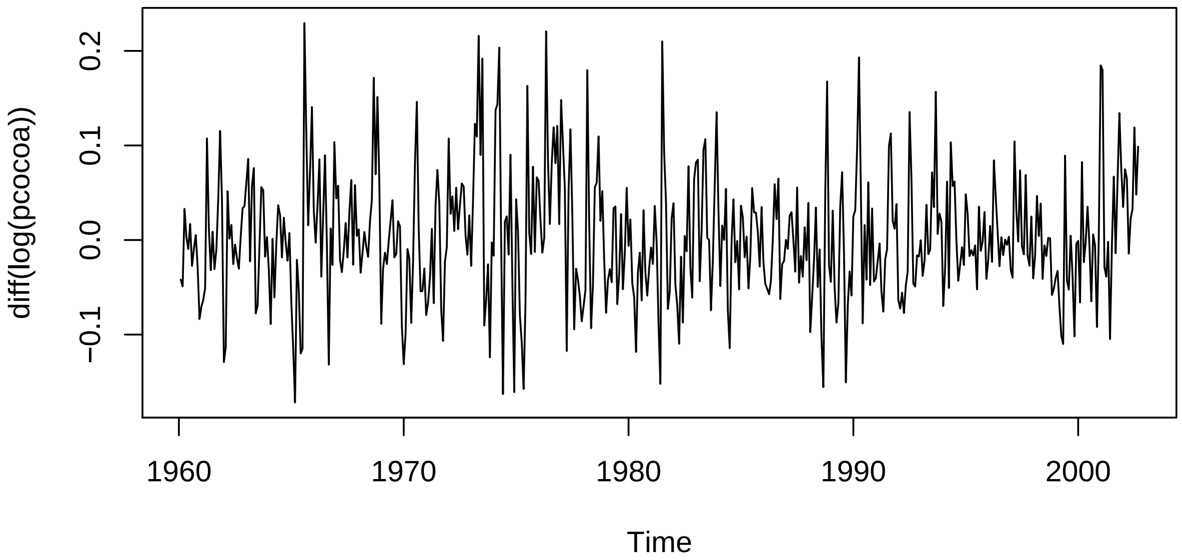
Moving average process of the first order - MA(1)

Data from previous lecture

Ben Vogelvang: **Econometrics. Theory and Applications with EViews.** Pearson Education Limited, 2005.

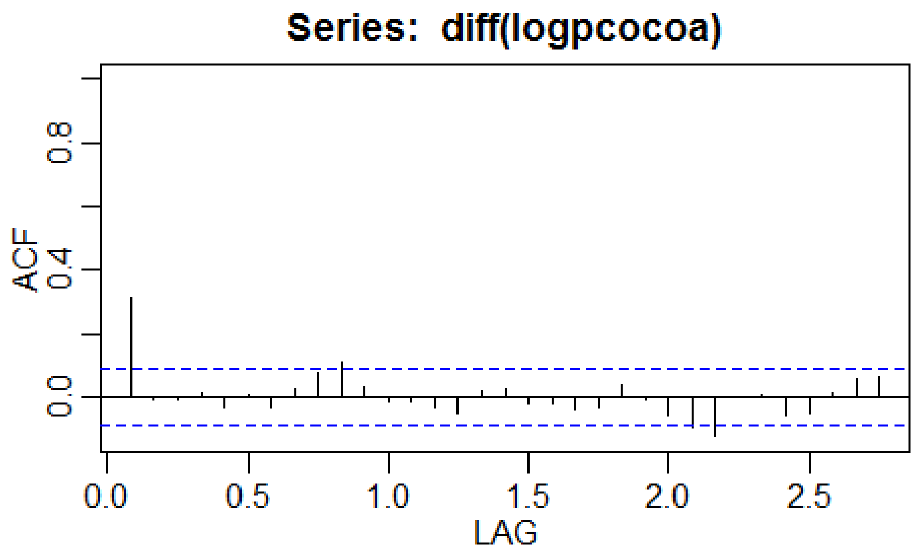
Chapter 14.7. - The Box-Jenkins Approach in Practice

- Monthy data, January 1960 - September 2002
- $pcocoa_t$ - cocoa prices, we take logarithms and because of stationarity we will work with differences



Data from previous lecture

- Estimated ACF:



- One significantly nonzero autocorrelation and the remaining ones are nearly zero

Example from the first lecture

- Let u_t be a white noise, we define

$$x_t = u_t + u_{t-1}$$

- We computed:

$$E[x_t] = 0, \quad Var[x_t] = 2\sigma^2$$

$$Cov[x_t, x_{t+\tau}] = \begin{cases} \sigma^2 & \text{pre } \tau = 1 \\ 0 & \text{pre } \tau = 2, 3, \ldots \end{cases}$$

$$Cor[x_t, x_{t+\tau}] = \begin{cases} 1/2 & \text{pre } \tau = 1 \\ 0 & \text{pre } \tau = 2, 3, \ldots \end{cases}$$

- ACF is zero for $\tau = 2, 3, \ldots$ - exactly the property which we need

Generalization - MA(1) process

- Let u_t be a white noise, then

  $$x_t = \mu + u_t - \beta u_{t-1}$$

  is called a moving average process of the first order - MA(1)
- Wold representation: $x_t = \mu + \sum_{j=0}^{\infty} \psi_j u_{t-j}$
  MA(1) process: $\psi_0 = 1, \psi_1 = -\beta, \psi_j = 0$ for $j = 2, 3, \dots$
- Moments and ACF:

$$E[x_t] = \mu, \;\; Var[x_t] = (1 + \beta^2)\sigma^2$$

$$Cov[x_t, x_{t+\tau}] = \begin{cases} -\beta\sigma^2 & \text{for } \tau = 1 \\ 0 & \text{for } \tau = 2, 3, \dots \end{cases}$$

$$Cor[x_t, x_{t+\tau}] = \begin{cases} -\frac{\beta}{1+\beta^2} & \text{for } \tau = 1 \\ 0 & \text{for } \tau = 2, 3, \dots \end{cases}$$

MA(1) process - examples

1. Let u_t be a white noise with distribution $N(0,4)$, we define

$$x_t = u_t + \frac{1}{2}u_{t-1}$$

Then: $E[x_t] = 0, \;\; Var[x_t] = (1 + (1/2)^2) \times 4 = 5$

$$Cor[x_t, x_{t+\tau}] = \begin{cases} \frac{1/2}{1+1/4} = 2/5 & \text{for } \tau = 1 \\ 0 & \text{for } \tau = 2, 3, \ldots \end{cases}$$

2. Let u_t be a white noise with distribution $N(0,1)$, we define

$$y_t = u_t + 2u_{t-1}$$

Then: $E[y_t] = 0, \;\; Var[y_t] = (1 + 4) \times 1 = 5$

$$Cor[y_t, y_{t+\tau}] = \begin{cases} \frac{2}{1+4} = 2/5 & \text{for } \tau = 1 \\ 0 & \text{for } \tau = 2, 3, \ldots \end{cases}$$

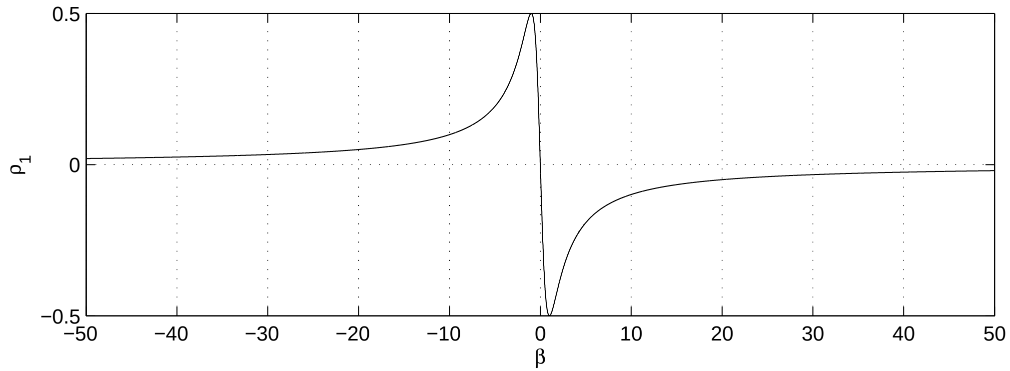
MA(1) process - generalization

- Let us take an MA(1) process, i.e. ACF of the form

$$Cor[x_t, x_{t+\tau}] = \begin{cases} -\frac{\beta}{1+\beta^2} & \text{for } \tau = 1 \\ 0 & \text{for } \tau = 2, 3, \ldots \end{cases}$$

- Suppose now that we are given the value $\rho_1 = \rho(1)$ and we want to find the coefficient β, i.e.

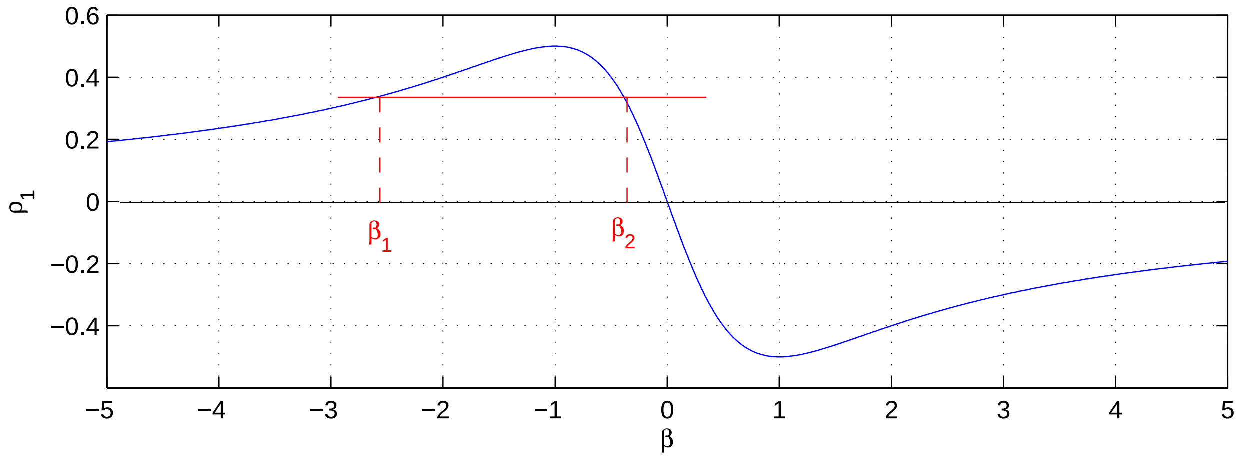
$$\rho_1 = -\frac{\beta}{1+\beta^2} \Rightarrow \beta = ?$$



MA(1) process - generalization

- We have therefore the equation:

  $\rho_1 = -\frac{\beta}{1+\beta^2} \Rightarrow \beta^2 + \frac{1}{\rho_1}\beta + 1 = 0$



  $\rightarrow$ two solutions β_1, β_2, they satisfy $\beta_1\beta_2 = 1$.

- Processes

  $$x_t = \mu + u_t - \beta u_{t-1}, \quad x_t = \mu + u_t - \frac{1}{\beta}u_{t-1}$$

  have the same ACF

- If we want a unique parametrization, we need an additional condition.

Invertibility of a process

- We will try to write the process in AR(∞) form:

$$x_t = \hat{\mu} + u_t + \psi_1 x_{t-1} + \psi_2 x_{t-2} + \psi_3 x_{t-3} + \ldots$$

  - if is it possible, the process is called invertible

- For MA(1) process:

$$\begin{aligned} x_t &= \mu + (1 - \beta L) u_t \\ (1 - \beta L)^{-1} x_t &= (1 - \beta L)^{-1} \mu + u_t \end{aligned}$$

  $(1 - \beta L)^{-1}$ exists for $|\beta| < 1$, then

$$(1 + \beta L + \beta^2 L^2 + \ldots) x_t = \mu / (1 - \beta) + u_t$$

$$x_t + \beta x_{t-1} + \beta^2 x_{t-2} + \ldots = \mu / (1 - \beta) + u_t$$

MA(1) - invertibility

- We obtained invertibility condition for MA(1) process: $|\beta| < 1$
- Another way how to express it:
  - we have a process $x_t = \mu + (1 - \beta L)u_t$
  - root of the polynomial $1 - \beta L$ is $1/\beta$
  - invertibility condition means that root of $1 - \beta L = 0$ has to be in absolute value greater than 1, i.e. outside of the unit circle

MA(1) - computation of PACF

- Recall the general formula:

$$\Phi_{kk} = \frac{\det\begin{pmatrix} 1 & \rho(1) & \dots & \rho(1) \\ \rho(1) & 1 & \dots & \rho(2) \\ & \dots & \dots & \\ \rho(k-1) & \rho(k-2) & \dots & \rho(k) \end{pmatrix}}{\det\begin{pmatrix} 1 & \rho(1) & \dots & \rho(k-1) \\ \rho(1) & 1 & \dots & \rho(k-2) \\ & \dots & \dots & \\ \rho(k-1) & \rho(k-2) & \dots & 1 \end{pmatrix}} \tag{1}$$

- For MA(1) we have $\rho(k) = 0$ for $k = 2, 3, \dots$

MA(1) - computation of PACF

- PACF is not zero after some lags (as it holds for AR):

$$\Phi_{11} = \rho(1)$$

$$\Phi_{22} = \frac{\det\begin{pmatrix} 1 & \rho(1) \\ \rho(1) & \rho(2) \end{pmatrix}}{\det\begin{pmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{pmatrix}} = \frac{\det\begin{pmatrix} 1 & \rho(1) \\ \rho(1) & 0 \end{pmatrix}}{\det\begin{pmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{pmatrix}} = \frac{-\rho(1)^2}{1-\rho(1)^2}$$

$$\Phi_{33} = \frac{\det\begin{pmatrix} 1 & \rho(1) & \rho(1) \\ \rho(1) & 1 & \rho(2) \\ \rho(2) & \rho(1) & \rho(3) \end{pmatrix}}{\det\begin{pmatrix} 1 & \rho(1) & \rho(2) \\ \rho(1) & 1 & \rho(1) \\ \rho(2) & \rho(1) & 1 \end{pmatrix}} = \frac{\det\begin{pmatrix} 1 & \rho(1) & \rho(1) \\ \rho(1) & 1 & 0 \\ 0 & \rho(1) & 0 \end{pmatrix}}{\det\begin{pmatrix} 1 & \rho(1) & 0 \\ \rho(1) & 1 & \rho(1) \\ 0 & \rho(1) & 1 \end{pmatrix}} = \frac{\rho(1)^3}{1-2\rho(1)^2}$$

$$\Phi_4 = \frac{-\rho(1)^4}{(1-\rho(1)^2)^2-\rho(1)^2}$$

$$\dots$$

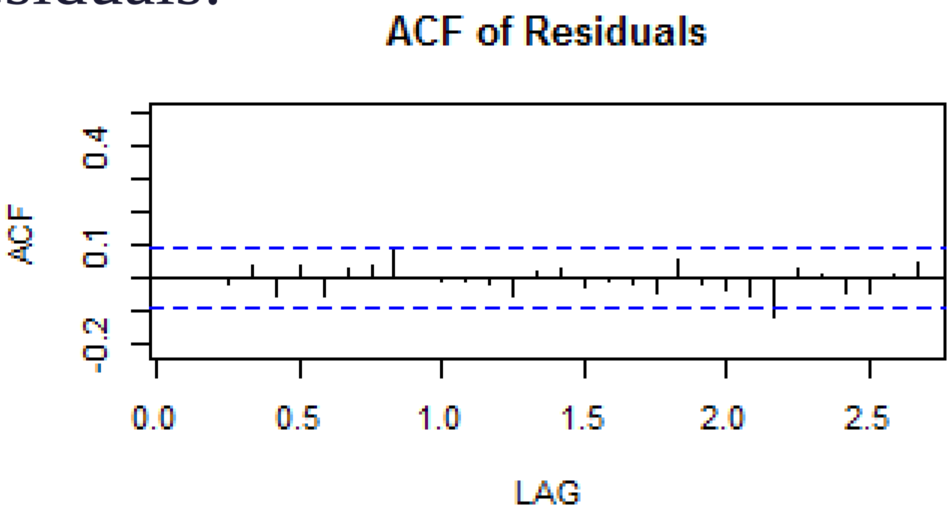
Real data - cocoa prices

- Data from the beginning of the lecture
- MA(1) model for differences of logarithms (variable y in the output from R):

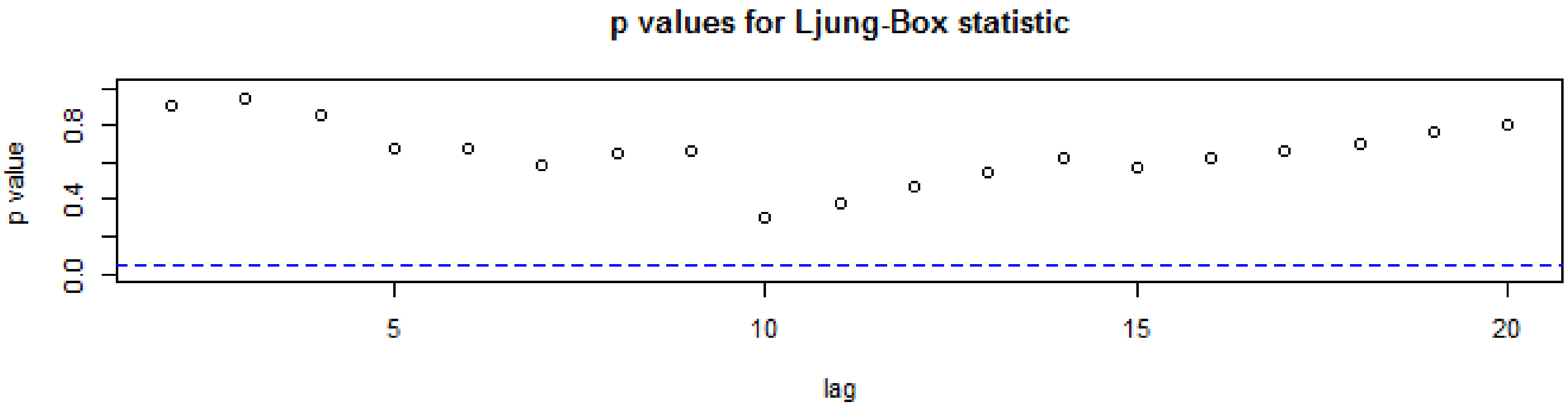
```
> sarima(y,0,0,1,details=FALSE)
$fit

Call:
stats::arima(x = xdata, order = c(p, d, q), seasonal = list(order = c(P, D,
    Q), period = S), xreg = xmean, include.mean = FALSE, optim.control = lis$
    REPORT = 1, reltol = tol))

Coefficients:
         ma1   xmean
      0.3520  0.0024
s.e.  0.0402  0.0037

sigma^2 estimated as 0.003897:  log likelihood = 693.61,  aic = -1381.21
```

Real data - cocoa prices

- ACF of residuals:



- Ljung-Box statistics:



- Model is OK.

VI.

Moving average process of order q - MA(q)

MA(q) process - definition and properties

- Let u_t be a white noise, then

$$x_t = \mu + u_t - \beta_1 u_{t-1} - \beta_2 u_{t-2} - \ldots - \beta_q u_{t-q}$$

  is called a moving average proces of q-th order - MA(q)

- Wold representation: $x_t = \mu + \sum_{j=0}^{\infty} \psi_j u_{t-j}$
  MA(q) process: $\psi_0 = 1, \psi_1 = -\beta_1, \ldots \psi_q = -\beta_q, \psi_j = 0$ for $j > q \to$ MA(q) proces is always stationary

- Moments, ACF, PACF:

$$E[x_t] = \mu, \quad Var[x_t] = (1 + \beta_1^2 + \ldots \beta_q^2)\sigma^2$$

$$Cov[x_t, x_{t+\tau}] = 0 \text{ for } \tau = q+1, q+2, \ldots$$

$$\Rightarrow Cor[x_t, x_{t+\tau}] = 0 \text{ for } \tau = q+1, q+2, \ldots$$

MA(q) process - definition and properties

- Computation of the first q autocorrelations (we can assume $\mu = 0$):

$$
\begin{aligned}
Cov[x_t, x_{t+\tau}] &= E[(u_t - \beta_1 u_{t-1} - \ldots - \beta_q u_{t-q}) \times \\
&\quad (u_{t+\tau} - \beta_1 u_{t+\tau-1} - \ldots - \beta_q u_{t+\tau-q})]
\end{aligned}
$$

$$
\begin{aligned}
= \; & E[u_t(u_{t+\tau} - \beta_1 u_{t+\tau-1} - \ldots - \beta_q u_{t+\tau-q})] \\
& -\beta_1 E[u_{t-1}(u_{t+\tau} - \beta_1 u_{t+\tau-1} - \ldots - \beta_q u_{t+\tau-q})] \\
& \ldots \\
& -\beta_q E[u_{t-q}(u_{t+\tau} - \beta_1 u_{t+\tau-1} - \ldots - \beta_q u_{t+\tau-q})]
\end{aligned}
$$

MA(q) process - definition and properties

- Continued:

$$
\begin{aligned}
\tau = 1 \quad &\Rightarrow \quad \gamma(1) = (-\beta_1 + \beta_1\beta_2 + \ldots + \beta_{q-1}\beta_q)\sigma^2 \\
\tau = 2 \quad &\Rightarrow \quad \gamma(2) = (-\beta_2 + \beta_1\beta_3 + \ldots + \beta_{q-2}\beta_q)\sigma^2 \\
&\quad\ \ \ldots \\
\tau = q \quad &\Rightarrow \quad \gamma(q) = (-\beta_q)\sigma^2
\end{aligned}
$$

- PACF - substitution of ACF into (1)

MA(q) process - definition and properties

- Invertibility:

$$x_t = \mu + u_t - \beta_1 u_{t-1} - \beta_2 u_{t-2} - \ldots - \beta_q u_{t-q}$$
$$x_t = \mu + (1 - \beta_1 L - \ldots - \beta_q L^q) u_t$$

- Existence of $(1 - \beta_1 L - \ldots - \beta_q L^q)^{-1}$ - roots of $1 - \beta_1 L - \ldots - \beta_q L^q = 0$ have to be outside of the unit circle