Modelling volatility - ARCH and GARCH models

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Time series analysis
Stock prices

- Weekly stock prices (library quantmod)
- Continuous returns:

```r
library(quantmod)
library(astsa)
getSymbols("EBAY", from="2010-01-01", to="2014-12-31", auto.assign=TRUE)
EBAY <- to.weekly(EBAY)
returns <- diff(log(EBAY$EBAY.Adjusted))[-1] # omit NA at the beginning
chartSeries(returns, theme="white")
```

- At the beginning of the term we analyzed their autocorrelations in a HW
Returns

- Time evolution:
Returns

- Based on ACF, they look like a white noise:
Returns

- We model them as a white noise:

```r
11 model1 <- sarima(returns, 0, 0, 0, details = FALSE)
12 acf2(model1$fit$residuals)  # the same as acf2(vynosys) -> ok
13 acf2(model1$fit$residuals^2)  # -> PROBLEM
```

- residuals are just - up to a contant - the returns
- If the absolute value of a residual is small, usually follows a residual with a small absolute value
- Similarly, after a residual with a large absolute value, there is often another residual with a large absolute value - it can be positive or negative, so it cannot be seen on the ACF
- **Second powers will likely be correlated** (but this does not hold for a white noise)
Returns

- ACF of squared residuals:

  \[\text{Series: model1$fit$residuals}^2\]

  \[\text{LAG}\]

  \[\text{ACF}\]

  \[0.1\]

  \[0.3\]

  \[0.5\]

  \[-0.1\]

  \[-0.3\]

  \[-0.5\]

  \[5\]

  \[10\]

  \[15\]

  \[20\]

  \[25\]

  \[\rightarrow\]

  significant autocorrelation

- **QUESTION:**
  Which model can capture this property?
Returns

- Possible explanation: nonconstant variance
**ARCH and GARCH models**

- $u$ is not a white noise, but
  \[ u_t = \sqrt{\sigma_t^2} \eta_t, \]
  where $\eta$ is a white noise with unit variance, i.e.,
  \[ u_t \sim N(0, \sigma_t^2) \]

- **ARCH model** (autoregressive conditional heteroskedasticity) - equation for variance $\sigma_t^2$:
  \[ \sigma_t^2 = \omega + \alpha_1 u_{t-1}^2 + \cdots + \alpha_q u_{t-q}^2 \]

- **Constraints on parameters:**
  - variance has to be positive:
    \[ \omega > 0, \alpha_1, \ldots, \alpha_{q-1} \geq 0, \alpha_q > 0 \]
  - stationarity:
    \[ \alpha_1 + \cdots + \alpha_q < 1 \]
ARCH and GARCH models

- **Disadvantages of ARCH models:**
  - a small number of terms $u_{i−i}^2$ is often not sufficient - squares of residuals are still often correlated
  - for a larger number of terms, these are often not significant or the constraints on parameters are not satisfied

- **Generalization:** **GARCH models** - solve these problems
ARCH and GARCH models

- **GARCH(p,q) model** (generalized autoregressive conditional heteroskedasticity) - equation for variance $\sigma_t^2$:

$$\sigma_t^2 = \omega + \alpha_1 u_{t-1}^2 + \ldots + \alpha_q u_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \ldots + \beta_p \sigma_{t-p}^2$$

- **Constraints on parameters**:
  - variance has to be positive:
    $$\omega > 0, \alpha_1, \ldots, \alpha_{q-1} \geq 0, \alpha_q > 0$$
    $$\beta_1, \ldots, \beta_{p-1} \geq 0, \beta_p > 0$$
  - stationarity:
    $$\alpha_1 + \ldots + \alpha_q + (\beta_1 + \ldots + \beta_p) < 1$$

- A popular model is GARCH(1,1).
GARCH models in R

- Modelling YHOO returns - continued
- In R:
  - library fGarch
  - function garchFit, model is written for example like arma(1,1)+garch(1,1)
  - parameter trace=FALSE - we do not want the details about optimization process
- We have a model constant + noise; we try to model the noise by ARCH/GARCH models
ARCH(1)

- Estimation of ARCH(1) model:

```r
# arch(1) = garch(1,0)
model10 <- garchFit(~garch(1,0), data=returns, trace=FALSE)
stand.res <- model10@residuals/model10@sigma.t # standardized residuals
acf2(stand.res)
acf2(stand.res^2)
summary(model10)
```

- We check
  1. ACF of standardized residuals
  2. ACF of squared standardized residuals
  3. **summary** with tests about standardized residuals and their squares
GARCH models in R

- Useful values:
  - `@fitted` - fitted values
  - `@residuals` - residuals
  - `@h.t` - estimated variance
  - `@sigma.t` - estimated standard deviation

- Standardized residuals - residuals divided by their standard deviation should be a white noise

- Also their squares should be a white noise
ARCH(1)

- Residuals:
ARCH(1)

- Squares:
Tests about residuals

- Tests:

<table>
<thead>
<tr>
<th>Standardised Residuals Tests</th>
<th>Statistic</th>
<th>p-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jarque-Bera Test R Chi^2</td>
<td>29.18907</td>
<td>4.588523e-07</td>
</tr>
<tr>
<td>Shapiro-Wilk Test R W</td>
<td>0.9821336</td>
<td>0.002392983</td>
</tr>
<tr>
<td>Ljung-Box Test R Q(10)</td>
<td>12.29485</td>
<td>0.2658079</td>
</tr>
<tr>
<td>Ljung-Box Test R Q(15)</td>
<td>21.08263</td>
<td>0.1342099</td>
</tr>
<tr>
<td>Ljung-Box Test R Q(20)</td>
<td>28.34245</td>
<td>0.1015375</td>
</tr>
<tr>
<td>Ljung-Box Test R^2 Q(10)</td>
<td>27.55866</td>
<td>0.002123322</td>
</tr>
<tr>
<td>Ljung-Box Test R^2 Q(15)</td>
<td>44.23218</td>
<td>0.0001011277</td>
</tr>
<tr>
<td>Ljung-Box Test R^2 Q(20)</td>
<td>46.02944</td>
<td>0.0007985573</td>
</tr>
<tr>
<td>LM Arch Test R TR^2</td>
<td>25.98164</td>
<td>0.01079828</td>
</tr>
</tbody>
</table>

- We have: normality test, Ljung-Box for standardized residuals and their squares
- What is new: testing homoskedasticity for the residuals
**ARCH(2)**

- We try ARCH(2) - results of the tests:

<table>
<thead>
<tr>
<th>Test</th>
<th>Type</th>
<th>Statistic</th>
<th>p-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jarque-Bera Test</td>
<td>R</td>
<td>Chi^2</td>
<td>34.65834, 2.97e-08</td>
</tr>
<tr>
<td>Shapiro-Wilk Test</td>
<td>R</td>
<td>W</td>
<td>0.9814854, 0.001841151</td>
</tr>
<tr>
<td>Ljung-Box Test Q(10)</td>
<td>R</td>
<td>Q(10)</td>
<td>10.75724, 0.3767411</td>
</tr>
<tr>
<td>Ljung-Box Test Q(15)</td>
<td>R</td>
<td>Q(15)</td>
<td>19.54396, 0.1901334</td>
</tr>
<tr>
<td>Ljung-Box Test Q(20)</td>
<td>R</td>
<td>Q(20)</td>
<td>26.93409, 0.1371285</td>
</tr>
<tr>
<td>Ljung-Box Test Q(10)</td>
<td>R^2</td>
<td>Q(10)</td>
<td>14.11704, 0.1677196</td>
</tr>
<tr>
<td>Ljung-Box Test Q(15)</td>
<td>R^2</td>
<td>Q(15)</td>
<td>27.9337, 0.02198778</td>
</tr>
<tr>
<td>Ljung-Box Test Q(20)</td>
<td>R^2</td>
<td>Q(20)</td>
<td>30.13048, 0.06776678</td>
</tr>
<tr>
<td>LM Arch Test</td>
<td>R</td>
<td>TR^2</td>
<td>20.09535, 0.06530357</td>
</tr>
</tbody>
</table>
ARCH(3)

- ARCH(3) - results of the tests:

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</thead>
<tbody>
<tr>
<td>Jarque-Bera Test</td>
<td>33.88186</td>
<td>4.391851e-08</td>
</tr>
<tr>
<td>Shapiro-Wilk Test</td>
<td>0.9816512</td>
<td>0.001968254</td>
</tr>
<tr>
<td>Ljung-Box Test (Q(10))</td>
<td>10.70552</td>
<td>0.3809165</td>
</tr>
<tr>
<td>Ljung-Box Test (Q(15))</td>
<td>19.44674</td>
<td>0.1941989</td>
</tr>
<tr>
<td>Ljung-Box Test (Q(20))</td>
<td>26.77638</td>
<td>0.1416733</td>
</tr>
<tr>
<td>Ljung-Box Test (Q(10) ^ 2)</td>
<td>14.01256</td>
<td>0.1724194</td>
</tr>
<tr>
<td>Ljung-Box Test (Q(15) ^ 2)</td>
<td>27.74511</td>
<td>0.02322011</td>
</tr>
<tr>
<td>Ljung-Box Test (Q(20) ^ 2)</td>
<td>29.97015</td>
<td>0.07033882</td>
</tr>
<tr>
<td>LM Arch Test</td>
<td>20.00903</td>
<td>0.06691536</td>
</tr>
</tbody>
</table>
ARCH(4)

- ARCH(4) - results of the tests:

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<td>12.79082</td>
<td>0.001669201</td>
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<td>Shapiro-Wilk Test R W</td>
<td>0.9880808</td>
<td>0.0302226</td>
</tr>
<tr>
<td>Ljung-Box Test R Q(10)</td>
<td>10.54746</td>
<td>0.3938435</td>
</tr>
<tr>
<td>Ljung-Box Test R Q(15)</td>
<td>18.98255</td>
<td>0.2145258</td>
</tr>
<tr>
<td>Ljung-Box Test R Q(20)</td>
<td>26.02379</td>
<td>0.1650277</td>
</tr>
<tr>
<td>Ljung-Box Test R^2 Q(10)</td>
<td>14.41393</td>
<td>0.1549343</td>
</tr>
<tr>
<td>Ljung-Box Test R^2 Q(15)</td>
<td>21.97894</td>
<td>0.1083574</td>
</tr>
<tr>
<td>Ljung-Box Test R^2 Q(20)</td>
<td>23.88737</td>
<td>0.2473466</td>
</tr>
<tr>
<td>LM Arch Test R TR^2</td>
<td>18.14053</td>
<td>0.1114896</td>
</tr>
</tbody>
</table>

- No autocorrelation in residuals and their squares.
ARCH(4)

- ACF of squared residuals:

→ without significant correlation
But ARCH coefficients $\alpha_i$ are not significant:

| Error Analysis                      | Estimate | Std. Error | t value | Pr(>|t|) |
|-------------------------------------|----------|------------|---------|----------|
| mu       | 4.022e-03 | 2.433e-03 | 1.653   | 0.0983   |
| omega    | 1.170e-03 | 2.003e-04 | 5.842   | 5.16e-09 ***
| alpha1   | 3.839e-02 | 1.250e-01 | 0.307   | 0.7587   |
| alpha2   | 1.259e-01 | 8.058e-02 | 1.563   | 0.1181   |
| alpha3   | 1.000e-08 | 1.475e-01 | 0.000   | 1.0000   |
| alpha4   | 1.098e-01 | 8.052e-02 | 1.364   | 0.1726   |

Signif. codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
**GARCH(1,1)**

- We try GARCH(1,1)
- Tests:

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<td>( Q(10) )</td>
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<td>R^2</td>
<td>( Q(15) )</td>
</tr>
<tr>
<td>Ljung-Box Test</td>
<td>R^2</td>
<td>( Q(20) )</td>
</tr>
<tr>
<td>LM Arch Test</td>
<td>R</td>
<td>( \text{TR}^2 )</td>
</tr>
</tbody>
</table>
### GARCH(1,1)

- **Estimates:**

| Error Analysis: | Estimate | Std. Error | t value | Pr(>|t|) |
|-----------------|----------|------------|---------|----------|
| mu              | 3.262e-03| 2.351e-03  | 1.388   | 0.1652   |
| omega           | 2.428e-05| 3.596e-05  | 0.675   | 0.4994   |
| alpha1          | 4.334e-02| 2.401e-02  | 1.805   | 0.0711   |
| beta1           | 9.396e-01| 3.923e-02  | 23.954  | <2e-16   *** |

---

Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
Estimated standard deviation

- We obtain it using $\sigma_t$:
Estimated standard deviation

- Another access to the graphs - `plot(model11)`: 

```r
> plot(model11)

Make a plot selection (or 0 to exit):

1: Time Series
2: Conditional SD
3: Series with 2 Conditional SD Superimposed
4: ACF of Observations
5: ACF of Squared Observations
6: Cross Correlation
7: Residuals
8: Conditional SDs
9: Standardized Residuals
10: ACF of Standardized Residuals
11: ACF of Squared Standardized Residuals
12: Cross Correlation between r^2 and r
13: QQ-Plot of Standardized Residuals

Selection: |
Predictions

- We use the function `predict` with parameter `n.ahead` (number of observations)
Predictions

- Parameter $nx$ - we can change the number of observations from the data which are shown in the plot (here $nx=100$):

![Prediction with confidence intervals](image)
Predictions

- Predicted standard deviation: `plot(ts(predictions[3]))`
Predictions

- For a longer time (exercise: compute its limit):
Application: Value at risk (VaR)

- Value at risk (VaR) is basically a quantile
- Let $X$ be a portfolio value, then

$$\mathcal{P}(X \leq \text{VaR}) = \alpha,$$

for example for $\alpha = 0.05$
- A standard GARCH assumes normal distribution - we can compute quantiles
- Shortcomings:
  - normality assumptions
  - there are also better risk measures than VaR
Apication: Value at risk (VaR)

- **WHAT WE WILL DO:**
  - Start with $N$ observations of returns.
  - Estimate the GARCH model.
  - Make a prediction for standard deviation and using the prediction we construct VaR for returns for the following day.
  - Every day move the window with data (we have a new observation), estimate GARCH again and compute the new VaR.
Not required - for those interested

- [https://systematicinvestor.wordpress.com/2012/01/06/trading-using-garch-volatility-forecast/](https://systematicinvestor.wordpress.com/2012/01/06/trading-using-garch-volatility-forecast/)

- "... Now, let’s create a strategy that switches between mean-reversion and trend-following strategies based on GARCH(1,1) volatility forecast." + R code

- From the website:
Other models for volatility

- Threshold GARCH:
  - $u_t > 0$ - "good news", $u_t < 0$ - "bad news"
  - TARCH can model their different effect on volatility
  - leverage effect: bad news have a higher impact

- We do not model variance (as in ARCH/GARCH models), but
  - its logarithm → exponential GARCH
  - any power of standard deviation → power GARCH

- and others...