# On a conjecture of Gnot, Trenkler and Zmyślony :-)

Radoslav Harman, KAMŠ, FMFI UK
and, in the future, perhaps also some readers of the QED blog

July 22, 2009

## 1 Introduction

The aim of the paper is to show our progress towards proving the following conjecture from (1):

**Conjecture 1.** *Let* $\mathbf{A}$ *and* $\mathbf{V}$ *be nonnegative definite matrices. Define the function* $f : [0, \infty) \to \mathbb{R}$ *by*

$$f(\lambda) = \mathrm{tr}((\mathbf{A} - \lambda\mathbf{V})_+\mathbf{V}).$$

*Then* $f$ *is convex.*

Remark: For a symmetric matrix $\mathbf{H}$ of type $n \times n$ we define its positive part $\mathbf{H}_+$ as follows: If $u_1, ..., u_n$ is an orthonormal system of eigenvectors of $\mathbf{H}$ and $h_1, ..., h_n$ are corresponding eigenvalues of $\mathbf{H}$, then

$$\mathbf{H}_+ = \sum_{i=1}^{n} \max(h_i, 0) u_i u_i^T.$$

Notation: For a vector $a$, the symbol $\langle a \rangle$ denotes the diagonal matrix with the elements of $a$ on the diagonal.

## 2 What we know

**Lemma 2.** *If matrices* $\mathbf{A}$ *and* $\mathbf{V}$ *commute then the function* $f$ *is convex, nonincreasing and piecewise-linear.*

*Proof.* It is a standard result in the matrix theory that if nonnegative definite matrices $\mathbf{A}$ and $\mathbf{V}$ commute than they have simultaneous spectral decomposition, i.e., $\mathbf{A} = \mathbf{U}\langle a\rangle\mathbf{U}^T$, and $\mathbf{V} = \mathbf{U}\langle v\rangle\mathbf{U}^T$, where $\mathbf{U}$ is the matrix of eigenvectors of both $\mathbf{A}$ and $\mathbf{V}$, and the components of the vectors $a = (a_1, ..., a_n)'$, $v = (v_1, ..., v_n)'$ are eigenvalues of $\mathbf{A}$ and $\mathbf{V}$. Note also that if $\mathbf{U}$ is an orthogonal matrix and $\Lambda$ is a diagonal matrix, then $(\mathbf{U}\Lambda\mathbf{U}^T)_+ = \mathbf{U}\Lambda_+\mathbf{U}^T$.

Therefore for any $\lambda > 0$:

$$\begin{aligned} f(\lambda) &= tr((\mathbf{A} - \lambda\mathbf{V})_+\mathbf{V}) \\ &= tr((\mathbf{U}\langle a\rangle\mathbf{U}^T - \lambda\mathbf{U}\langle v\rangle\mathbf{U}^T)_+\mathbf{V}) \\ &= tr(\mathbf{U}(\langle a\rangle - \lambda\langle v\rangle)_+\mathbf{U}^T\mathbf{U}\langle v\rangle\mathbf{U}^T) \\ &= tr((\langle a\rangle - \lambda\langle v\rangle)_+\langle v\rangle) \\ &= \sum_{i=1}^{n} \max(a_i v_i - \lambda v_i^2, 0). \end{aligned}$$

The last term of the previous chains of equalities is obviously a sum of convex, nonincreasing and piecewise-linear functions of $\lambda$, i.e., a convex, nonincreasing and piecewise-linear function itself. □

# References

[1] Gnot S, Trenkler G, Zmyšlony R (1995): Nonnegative minimum biased quadratic estimation in the linear regression models, Journal of Multivariate Analysis 54, 1, 113-125