Regresne modely, 4.10.2025

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Minule: GLS: $\hat{\beta} = (X^T W^{-1} X)^{-} X^T W^{-1} y$ ... rieši $\min_{\beta} \|y - X\beta\|_W$ $\rightarrow$ pre $y = X\beta + \varepsilon$, $Var(\varepsilon) = \sigma^2 W$

- feasible GLS: predpokladáme štruktúru na W → v nej odhadujeme koeficienty
  - napr. časové dáta: predp. štruktúru korelácií v ε
    - ↳ napr. AR(1): $\varepsilon_i = \underbrace{\varphi}_{\text{odhadujeme}} \varepsilon_{i-1} + w_i$, $w_i \sim N(0, \sigma^2)$, nez.
  - koef. odhadujeme ⇒ už to nie je pravé GLS (stráca to teor. vlastnosti)

Rko: $unem_i = \beta_0 + \beta_1 HDP_i + \beta_2 ko_i + \varepsilon_i$

$\rightarrow \hat{\varepsilon}$: $\hat{cor}(\hat{\varepsilon}_t, \hat{\varepsilon}_{t-1}) = 0{,}3$

⇒ ten istý model, ale: $\varepsilon_t = \varphi \varepsilon_{t-1} + w_t$ ⇒ $\hat{\varphi} = 0{,}6$, $\hat{\beta} = \begin{pmatrix} 101 \\ 0{,}07 \\ -0{,}05 \end{pmatrix}$

Vlastnosti odhadov:

$\underset{n\times 1}{y} = \underset{n\times k}{X}\underset{k\times 1}{\beta} + \underset{n\times 1}{\varepsilon}$, $E(\varepsilon) = 0_n$, $Var(\varepsilon) = \sigma^2 I_n$

náhodné

- predp.: $rank(X) = k$ ⇒ $\hat{\beta} = (X^TX)^{-1}X^Ty$

$E(\hat{\beta}) = \beta$ (DÚ)

$Var(\hat{\beta}) = (X^TX)^{-1}X^T \cdot \underbrace{Var(y)}_{\sigma^2 I_n} \cdot X(X^TX)^{-1} = \sigma^2 (X^TX)^{-1} = \sigma^2 M^{-1}$

M: inf. matica

vieme: $\hat{\beta}$ je BLUE pre β (Gauss-Markov)

$[A^{-}: AA^{-}A = A]$

pre $rank(X) \leq k$: $\widehat{l^T\beta} = l^T\hat{\beta} = l^T(X^TX)^{-}X^Ty$ pre $l \in \mathcal{C}(X^T) = \mathcal{C}(X^TX)$

$E(y) = X\beta$

$Var(y) = \sigma^2 I_n$

$E(l^T\hat{\beta}) = l^T(X^TX)^{-}X^T \cdot X\beta = z^T \underbrace{X^TX(X^TX)^{-}X^TX}_{X^TX}\beta = \underbrace{z^TX^TX}_{l^T}\beta = l^T\beta$, nevychýlený //

$l \in \mathcal{C}(X^TX) \Rightarrow \exists z: l = X^TXz$

$\Rightarrow l^T = z^TX^TX$

$Var(l^T\hat{\beta}) = \underbrace{l^T}_{z^TX^TX}(X^TX)^{-}X^T \cdot \sigma^2 I_n \cdot X[(X^TX)^{-}]^T l = \sigma^2 \underbrace{z^TX^TX}_{l^T}[(X^TX)^{-}]^T l = \sigma^2 \underbrace{\underset{1\times k}{l^T}\,\underset{k\times k}{[(X^TX)^{-}]^T}\,\underset{k\times 1}{l}}_{1\times 1} \overset{(\,)^T}{=}$

$c^T = c$ pre $c \in \mathbb{R}$

$= \sigma^2 l^T(X^TX)^{-}l$ //

best linear unbiased estimator

Gaussova-Markovova veta:

Nech $l \in \mathcal{C}(X^TX)$. Potom $l^T\beta$ je odhadnuteľná a $l^T\hat{\beta}$ je BLUE pre $l^T\beta$, kde $\hat{\beta}$ je ľub. riešenie (NR): $X^TX\beta = X^Ty$.

Teda $Var(T) \geq Var(l^T\hat{\beta})$ $\forall T$ - lin. nevych. odhad $l^T\beta$.

(Bez dôkazu.)

Predpoklad normality:

__Predpoklad normality:__

$y = X\beta + \varepsilon, \quad \varepsilon \sim N(0_n, \sigma^2 I_n)$

predp. : $\underline{X - \text{plná hodnosť}}$

a) $\hat\beta \sim N(\beta, \sigma^2 (X^T X)^{-1})$

b) $\frac{1}{\sigma^2} \| y - X\hat\beta \|^2 \sim \chi^2_{n-k}$

↑ rezíduum : $\hat\varepsilon_i = y_i - \hat y_i = y_i - x_i^T \hat\beta$

$\hat\varepsilon = y - X\hat\beta \Rightarrow \| y - X\hat\beta\|^2 = \|\hat\varepsilon\|^2 = \sum \hat\varepsilon_i^2 = RSS$: súčet štvorcov rezíduí

$s^2 = \frac{RSS}{n-k} \Rightarrow RSS = (n-k)s^2$

$\Rightarrow$ b): $\frac{(n-k)s^2}{\sigma^2} \sim \chi^2_{n-k}$

$\left[\chi^2_n = \sum_{i=1}^{n} X_i^2, \quad X_i \sim N(0,1), \text{ nez.} \right]$

c) $\frac{1}{\sigma^2} \| X(\hat\beta - \beta)\|^2 \sim \chi^2_k$

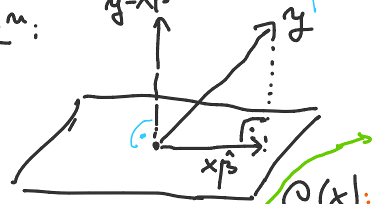
b) a c) sú nez.

__Intuícia__:
- nez., b) a c) žijú v ortogonálnych doplnkoch
- vnútri normy : N. rozdelenie
- $\|x\|^2 = \sum x_i^2$

$\Big\} \Rightarrow \chi^2$ (ešte vnútri musia byť normalizované)

- # stupňov voľnosti : koľko ich je ? – " v koľkorozmernom svete žijú "

$\mathbb{R}^n$:



$y - X\hat\beta = \hat\varepsilon$: v doplnku k $\mathcal{C}(X)$:
- dim = n − k
- $\Rightarrow \chi^2_{(n-k)}$
- dim = n − r

$\to y, \varepsilon \in \mathbb{R}^n$: n stupňov voľnosti (ľub. n-rozmerné vektory)

$\mathcal{C}(X)$: dim = k $\Rightarrow X\hat\beta$: k stupňov voľnosti $\Rightarrow \chi^2_k$

: dim = r

__Pre $rank(X) = r$__ $(r \le k)$:

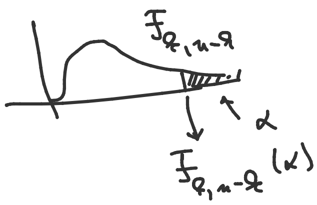
pre ľub. $\hat\beta$ ' riešenie (NR):

a) $\frac{1}{\sigma^2} \| y - X\hat\beta\|^2 \sim \chi^2_{n-r}$

b) $\frac{1}{\sigma^2} \| X(\hat\beta - \beta)\|^2 \sim \chi^2_r$

$\Big\}$ + sú nez.

$$\Rightarrow \frac{\| X(\hat\beta - \beta)\|^2 / r}{\| y - X\hat\beta\|^2 / (n-r)} \sim F_{r, n-r}$$



__Oblasť spoľahlivosti__ (pre $rank(X) = k$)

$\sigma \subseteq \mathbb{R}^k$: $P(\beta \in \sigma) = 1 - \alpha$

$$P\left(\frac{\| X(\hat\beta - \beta)\|^2 / k}{\| y - X\hat\beta\|^2 / (n-k)} < F_{k,n-k}(\alpha) \right) = 1 - \alpha$$

$\| y - X\hat\beta\|^2 / (n-k) = s^2$; $F_{k,n-k}(\alpha)$ ↑ krit. h.

$\left[\|z\| = z^T z \right]$

$\|$

$P(\| X(\hat\beta - \beta)\|^2 < k s^2 \cdot F)$

$$\parallel$$

$$P\left((\hat\beta-\beta)^T X^T X(\hat\beta-\beta) < q s^2 F\right)$$

$$\Rightarrow \sigma = \{\beta\in\mathbb{R}^q \mid (\hat\beta-\beta)^T \underbrace{X^T X}_{=M}(\hat\beta-\beta) < q S^2 F_{q,n-q}(\alpha)\}$$

Aký je to útvar? • $\beta\in\mathbb{R}^q$

• $(\hat\beta-\beta)^T M(\hat\beta-\beta) < c$

$M = Q\Lambda Q^T$: spekt. rozklad

$Q = [q_1 \dots q_q]$: vl. vekt.

$\Lambda = \begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_q\end{bmatrix}$: vl. č.

$$\Rightarrow \underbrace{(\hat\beta-\beta)^T Q}_{z^T}\Lambda \underbrace{Q^T(\hat\beta-\beta)}_{z\in\mathbb{R}^q} < c$$

$$(z_1 \dots z_q)\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_q\end{bmatrix}\begin{bmatrix}z_1\\ \vdots\\ z_q\end{bmatrix} < c$$

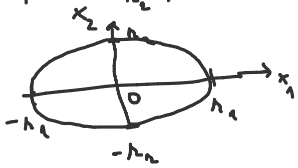
$$\sum_{i=1}^{q} \lambda_i z_i^2 < c$$

$$\sum_{i=1}^{q}\left(\frac{z_i}{\lambda_i^{-1/2}}\right)^2 < c$$

→ elipsoid pre z, nie pre β

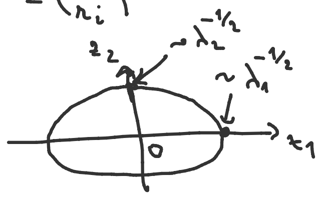
Elipsa

$$\left(\frac{x_1}{r_1}\right)^2 + \left(\frac{x_2}{r_2}\right) \le 1$$



Elipsoid

$$\sum\left(\frac{x_i}{r_i}\right)^2 \le 1$$



$z = Q^T(\hat\beta-\beta)$ $\quad / \cdot Q$ (ortog.) L

$Qz = \hat\beta - \beta$

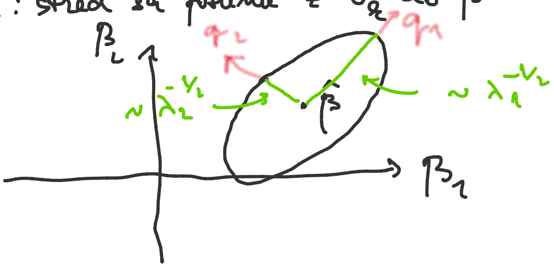
$\beta = \hat\beta - Qz$

Q - ortog. ⇒ Qz : otočenie (preklopenie) → hl. osi : v smere vl. vektorov $q_1 \dots$

↓

posunutie : stred sa posunie z 0_q do $\hat\beta$

⇒ σ :



⇒ σ: elipsoid

: stred $\hat\beta$

: hl. poloosi – v smeroch q_i

– dĺžky prop. $\lambda_i^{-1/2}$