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Diferenčné a diferenciálne rovnice

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Vysokoškolský učebný text pre potreby zabezpečenia výučby predmetov Diferenčné a diferenciálne rovnice vyučovanom na študijnom programe Ekonomická a finančná matematika.

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Diferenčné a diferenciálne rovnice

Úvod

V tomto predmete nám ide o matematický aparát pre opis dlhodobého časového vývoja viac-menej izolovaného systému v dôsledku vnútorných mechanizmov v ňom prebiehajúcich. Abstrahujeme od materiálnej podstaty systému, ktorá môže byť fyzikálna (pohyb planéty v dôsledku gravitačného zákona), biologická (frekvencia genotypov v dôsledku mechanizmu selekcie), ekonomická (vývoj cien komodity v dôsledku trhových mechanizmov), alebo iná.

Matematický model takéhoto systému budeme nazývať *dynamickým systémom* (ďalej DS). V závislosti od toho, či jeho stav budeme sledovať v oddelených časových okamihoch (sekunda, deň, rok, generácia), alebo nepretržite, budeme mu pridávať prívlastok diskrétny, resp. spojitý. V tomto predmete sa obmedzíme na konečnorozmerné DS, t.j. také, ktorých stav je opísateľný konečným počtom premenných.

Stavom DS nazývame takú informáciu o ňom, ktorej znalosť v nejakom časovom okamihu spolu so znalosťou vnútorných mechanizmov vývoja systému umožňuje predpovedať jeho vývoj v budúcnosti pri absencii vonkajších vplyvov.

I. Diskrétne dynamické systémy a diferenčné rovnice

1. Jednorozmerný, lineárny a afinný dynamický systém

1.1. Príklady

Populácia (s neprekrývajúcimi sa generáciami) sa rozmnožuje v "taktoch", pričom jedinec má v priemere $b > 0$ (birth rate) potomkov. Aký je časový vývoj populácie?

Označme $x(t)$ počet jedincov v t-tom časovom období.

Bilancia:

$$x(t+1) = bx(t).$$

Hodnoty $x(t)$ tvoria geometrickú postupnosť, teda platí

$$x(t) = b^{t-t_0}x(t_0).$$

Asymptotické správanie:

Ak $b > 1$:	populácia rastie nad všetky medze
$b = 1$:	zostáva konštantná
$b < 1$:	vyhynie,

čo je *Malthusov zákon.*

Na rovnaký matematický model vedie nasledovná úloha: Vklad v banke je úročený sadzbou i percent za úrokové obdobie. Ak sme na začiatku t_0-tého obdobia vložili čiastku $x(t_0)$, aká bude hodnota vkladu v období t?

Ak označíme $x(t)$ výšku vkladu v t-tom období, platí

$$x(t+1) = (1 + i/100)x(t),$$

preto

$$x(t) = (1 + i/100)^{t-t_0}x(t_0).$$

1.2. Lineárny jednorozmerný systém (LDS)

Zovšeobecnením príkladu 1.1 dostaneme lineárny dynamický systém (LDS). Ide o systém, ktorého stav je opísaný reálnym číslom x a ktorého hodnota v okamihu $t+1$ je určená jeho hodnotou v čase t vzťahom

$$x(t+1) = ax(t), \tag{1.2}$$

kde $a \in \mathbb{R}$. Za priestor stavov môžeme vziať $X = \mathbb{R}$; (ale aj $X = \{0\}$). Ak $a \geq 0$, môžeme za stavový priestor vziať aj $X = [0, \infty)$, $X = (-\infty, 0]$; ak $a > 0$, aj $X = (0, \infty)$, $X = (-\infty, 0)$ (prečo?).

Predpis (1.2) môžeme prepísať do tvaru

$$x(t+1) - x(t) = (a-1)x(t).$$

Preto sa klasicky nazýva tiež diferenčná rovnica.

Trajektóriou dynamického systému (1.2) cez bod x_0 (kratšie trajektóriou bodu x_0) nazývame postuponosť $x(t) = x(t, x_0),\ t = 0, 1, 2, ...$, ktorá spĺňa (1.2) a $x(0) = x_0$. Platí

$$x(t, x_0) = a^t x_0. \tag{1.3}$$

Rovnovážnym stavom (pevným bodom) DS (1.2) nazývame stav $\hat{x}$ taký, že $x(t_0) = \hat{x} = x(t)$ pre $t \geq t_0$; zrejme je to práve vtedy, ak $\hat{x} = a\hat{x}$, preto mu hovoríme aj pevný bod. Pre $a \neq 1$ má LDS (1.2) zrejme jediný pevný bod 0. Tento je

- asymptoticky stabilný pre $|a| < 1$ (t.j. každá postupnosť $x(t)$ generovaná vzťahom (2) k nemu konverguje pre $t \to \infty$),
- nestabilný v zrejmom zmysle pre $|a| > 1$

Vyššie uvedeným pojmom dáme presný zmysel v Kapitole 2 v abstraktnom kontexte.

Ak $a = 1$, sú všetky body pevné. Ak $a = -1$, trajektória $x(t, x_0)$ ľubovoľného bodu $x_0 \neq 0$ pozostáva z dvoch bodov $\pm x_0$, presnejšie $x(t, x_0) = (-1)^t x_0$. Je teda *periodickou trajektóriou* s minimálnou periódou 2.

1.3. Lineárna nehomogénna diferenčná rovnica a afinný jednorozmerný LDS

Lineárnou nehomogénnou diferenčnou rovnicou nazývame rovnicu

$$x(t+1) = ax(t) + b(t). \tag{1.4}$$

Indukciou si môžeme overiť, že pre postupnosti, vyhovujúce DR (1.4) platí

$$x(t) = a^{t-t_0} x(t_0) + \sum_{s=t_0}^{t-1} a^{t-1-s} b(s). \tag{1.5}$$

Analógia formuly (1.5) sa nám objaví u lineárnych diferenciálnych rovníc ako *formula variácie konštánt.*

Ak $b(t) \equiv b$ nezávisí od t, dostávame z (1.3) *afinný dynamický systém*

$$x(t+1) = ax(t) + b. \tag{1.6}$$

Ak $a \neq 1$, (1.4) má jediný pevný bod

$$\hat{x} = \frac{b}{1-a}. \tag{1.7}$$

Odchýlka od tohoto bodu $\tilde{x}(t) = x(t) - \hat{x}$ vyhovuje vzťahu

$$\tilde{x}(t+1) = x(t+1) - \hat{x} = ax(t) + b - a\hat{x} - b = a(x(t) - \hat{x}) = a\tilde{x}(t). \tag{1.8}$$

Teda $\tilde{x}$ vyhovuje LDR (1.2) a z toho vyplývajú zrejmé dôsledky pre stabilitu rovnovážneho stavu (1.7). Všimnime si úlohu, ktorú hrá znamienko a. Ak $a > 0$, trajektorie su monotónne, ak $a < 0$, trajektórie oscilujú okolo pevného bodu x.

Ak $a \neq 1$, možno z (1.4) trajektóriu $x(t) = x(t, x_0)$ bodu x_0 vypočítať v tvare

$$x(t) = a^t x_0 + b\frac{a^t - 1}{a - 1}. \tag{1.9}$$

Ak $a = 1$ a $b \neq 0$, systém nemá pevný bod a platí

$$x(t, x_0) = x_0 + tb.$$

1.4. Lineárny pavučinový (cobweb) model

modeluje dynamiku ceny produktu na čiastkovom trhu s jedným produktom v konkurenčnom prostredí.

Označíme

$$\begin{aligned} q^d(t) &\ldots \text{ dopyt po produkte} \qquad (d - \textit{demand}) \\ q^s(t) &\ldots \text{ ponuka produktu} \qquad (s - \textit{supply}) \\ p(t) &\ldots \text{ cena produktu} \qquad (p - \textit{price}) \end{aligned}$$

Predpokladáme, že

- dopyt sa riadi okamžitou cenou, kým ponuka cenou v predchádzajúcom okamihu:

$$q^d(t) = D(p(t)), \quad q^s(t) = S(p(t-1)) \tag{1.10}$$

- dopyt klesá s cenou a ponuka s ňou stúpa, obidvoje lineárne:

$$\begin{aligned} D(p) &= a - bp \\ S(p) &= -c + dp \end{aligned} \qquad a, b, d > 0, c \geq 0 \tag{1.11}$$

- trh je v každom okamihu v rovnováhe,

$$q^d(t) = q^s(t). \tag{1.12}$$

Ak do (1.12) dosadíme (1.10) a (1.11) dostaneme

$$a - bp(t) = -c + dp(t-1),$$

a po posune $t \mapsto t+1$

$$p(t+1) = \frac{1}{b}(a+c) - \frac{d}{b}p(t), \tag{1.13}$$

teda správanie p sa modeluje afinným DS.

Prirodzeným stavovým priestorom by bolo $X = [0, \infty)$, to však sa zobrazením (1.11) vo všeobecnosti nezobrazí do seba; nezostáva nám iné ako vziať za stavový priestor $X = \mathbb{R}$. Skutočnosť, že postupnosť $p(t)$ generovaná DS (1.13) môže prejsť z kladných do záporných hodnôt nasvedčuje, že model je nedokonalý.

Pevný bod

$$\hat{p} = \frac{a+c}{b+d}$$

je podľa (1.3) asymptoticky stabilný, ak $d < b$ (teda ak ponuka reaguje opatrnejšie, než dopyt, alebo v ekonomickej terminológii dopyt je *elastickejší* než ponuka), nestabilný, ak $d > b$. Keďže znamienko koeficientu pri $p(t)$ v (1.3) je záporné, trajektórie oscilujú okolo bodu $\hat{p}$.

1.5 Úlohy zloženého úrokovania

Ak v (1.4) intepretujeme $x(t)$ ako výšku vkladu s úrokovou mierou $100(a-1)$ na obdobie, môžeme $b(t)$ chápať ako výšku vkladu v čase t. Aparát lineárnej diferenčnej rovnice nám umožňuje systematicky formulovať a riešiť rozličné úlohy zloženého úrokovania. Uvedieme dva príklady:

Súčasná hodnota kupónového dlhopisu.
Majiteľ kupónového dlhopisu dostane od jeho vypisovateľa v určených termínoch do doby jeho splatnosti vyplatené čiastky nazývané *kupónmi* a v termíne jeho *splatnosti* jeho *nominálnu hodnotu* spolu s kupónom. *Súčasnú hodnotu* dlhopisu v danom časovom okamihu definujeme ako výšku vkladu s referenčným úrokom, ktorá by umožnila vyberať v budúcnosti čiastky rovné výnosom z dlhopisu a nič viac.

Ak označíme $x(t)$ súčasnú hodnotu dlhopisu v čase t, $d(t)$ výber z referenčného vkladu v čase t, a termín jeho splatnosti T, platí

$$x(t+1) = ax(t) - d(t+1), \tag{1.14}$$

a

$$x(T) = 0, \tag{1.15}$$

kde $a = 1 + i/100$ a i je referenčný úrok. Ďalej platí

$$d(t) = \begin{cases} 0 & \text{pre } t = 0 \\ c & \text{pre } t = 1, ..., T-1 \\ n + c & \text{pre } t = T \end{cases}, \tag{1.16}$$

kde c je výška kupónu a n je nominálna hodnota dlhopisu.

Z (1.14) a (1.15) využitím (1.5) dostávame pre súčasnú hodnotu $x(0)$ rovnicu

$$0 = a^T x(0) - \sum_{t=0}^{T-1} a^{T-t-1} d(t+1),$$

alebo ekvivalentne

$$x(0) = \sum_{s=1}^{T} a^{-s} d(s).$$

Dosadením za $d(t)$ z (1.16) a použitím vzorca pre súčet geometrického radu dostaneme

$$x(0) = a^{-T}\left[(n+c) + a\frac{a^{T-1}-1}{a-1}c\right].$$

Ako sa zabezpečiť na starobu
V súčasnosti je na ročné živobytie treba čiastku z, ročná inflácia, resp. ročná úroková miera sa predpokladá vo výške $100(a-1)\%$, resp. $100(b-1)\%$. Aladár pôjde o 30 rokov do penzie. Koľko musí dovtedy našetriť, aby z našetreného mohol financovať svoje živobytie na ďalších 25 rokov?

Ak pre $30 \le t \le 55$ označíme $x(t)$ výšku Aladárových úspor, platí

$$x(t+1) = bx(t) - a^t z,$$
$$x(55) = 0$$

Podobne ako pre súčasnú hodnotu dlhopisu dostaneme pre $x(30)$ rovnicu a jej riešenie.

2. Jednorozmerný nelineárny dynamický systém

2.1. Pevné body, periodické body a ich stabilita

Ide o DS, ktorého postupnosti $\{x(t)\}_t$ majú hodnoty v $X \subset \mathbb{R}$ a sú generované všeobecným rekurentným vzťahom

$$x(t+1) = f(x(t)) \tag{2.1}$$

Podľa stupňa hladkosti f hovoríme o C^r-dynamickom systéme, $r \ge 0$.

Pevné body $\hat{x}$ sú riešeniami nelineárnej rovnice

$$\hat{x} = f(\hat{x})$$

Pevný bod zobrazenia $f^r = \underbrace{f \circ f \circ \cdots \circ f}_{r\ razy}$ sa nazýva periodickým bodom periódy r; ak r je jeho minimálna perióda, nazýva sa r-cyklom.

Dôležitú úlohu hrajú pevné body a cykly, ktoré predstavujú ustálené stavy modelovaného systému. Pevný bod $\hat{x}$ nazývame:

- stabilným, ak z $x_0 \to \hat{x}$ vyplýva $x(t, x_0) \to \hat{x}$ rovnomerne v t, alebo ekvivalentne $\forall \varepsilon > 0\ \exists \delta > 0$ také, že ak $|x_0 - \hat{x}| < \delta$ potom $|x(t, x_0) - \hat{x}| < \varepsilon$ pre každé $t \ge 0$
- asymptoticky stabilným, ak je stabilný a $\exists \varepsilon_0 > 0$ také, že

$$|x_0 - \hat{x}| < \varepsilon_0 \Rightarrow x(t, x_0) \to \hat{x} \text{ pre } t \to \infty$$

- nestabilným, ak nie je stabilný.

Veta. *Nech f je C^1 a nech $\hat{x}$ je pevný bod f. Potom $\hat{x}$ je*
- *asymptotický stabilný, ak $|f'(\hat{x})| < 1$*
- *nestabilný, ak $|f'(\hat{x})| > 1$.*

Dôkaz.

1. Nech $|f'(\hat{x})| < 1$. Potom existuje ε také, že $|f'(x)| < \alpha < 1$ pre $|x - \hat{x}| < \varepsilon$. Ak $|x - \hat{x}| < \varepsilon$, potom podľa vety o strednej hodnote

$$|f(x) - \hat{x}| = |f(x) - f(\hat{x})| = |f'(\vartheta)||x - \hat{x}| \le \alpha|x - \hat{x}| \tag{2.2}$$

kde ϑ je medzi x a $\hat{x}$; ak $|x(0) - \hat{x}| < \varepsilon$, potom z (2.2) dostávame indukciou

$$|x(t) - \hat{x}| < \varepsilon, \quad |x(t) - \hat{x}| \le \alpha^t |x(0) - \hat{x}|$$

2. Ak $|f'(\hat{x})| > 1$, podobne $\exists \varepsilon > 0$ a $\alpha > 1$ také, že $|f'(x)| > \alpha > 1$ pre $|x - \hat{x}| \le \varepsilon$. Ak $|x - \hat{x}| < \varepsilon$, platí z podobných dôvodov ako v 1.

$$|f(x) - \hat{x}| \ge \alpha(x - \hat{x}),$$

preto $|x(t) - \hat{x}| \ge \alpha^t |x(0) - \hat{x}|$, pokiaľ $|x(s) - \hat{x}| \le \varepsilon$ pre $0 \le s \le t$. Nech by x_0 bolo ľubovoľne blízke $\hat{x}$, pre $t \ge \frac{\log\varepsilon - \log|x(0) - \hat{x}|}{\log\alpha}$, platí $|x(t, x_0) - \hat{x}| \ge \varepsilon$.

Kým $|f'(\hat{x})|$ rozhoduje o stabilite bodu $\hat{x}$, znamienko $f'(\hat{x})$ zasa rozhodne o tom, či sa v okolí bodu $\hat{x}$ trajektórie správajú monotónne (ak $f'(\hat{x}) > 0$) alebo oscilatoricky (ak $f'(\hat{x}) < 0$).

2.2 Logistický model populačnej dynamiky

Realistickejšie než v Malthusovom modeli 1.1 je predpokladať, že s veľkosťou populácie klesá rýchlosť rozmnožovania (napr. vzrastom úmrtnosti vzhľadom na obmedzenosť potravinových zdrojov). Zjednodušene predpokladáme, že rýchlosť rastu populácie klesá lineárne s jej veľkosťou:

$$r(x) = q(K - x).$$

Potom správanie populácie môžeme opísať vzťahom

$$x(t+1) = f(x(t)), \tag{2.3}$$

kde

$$f(x) = r(x)x = q(K - x)x.$$

Transformáciou $x = Ky$, $qK = \mu$ môžeme (2.3) normalizovať do tvaru

$$y(t+1) = \mu y(1 - y).$$

Budeme teda uvažovať vzťah

$$x(t+1) = f_\mu(x(t)), \tag{2.4}$$

kde

$$f_\mu(x) = \mu x(1 - x)$$

Ak $0 \le \mu \le 4$, (2.4) generuje DS na stavovom priestore $X = [0, 1]$ (potom totiž

$$\max f = f(1/2) = \mu/4 \le 1). \tag{2.5}$$

Skúmajme, ako vyzerajú pevné body a ich stabilita pre rôzne hodnoty μ.

Pevný bod $\hat{x}$ je riešením rovnice

$$\hat{x} = \mu\hat{x}(1-\hat{x}).$$

Jedným riešením je $\hat{x}_0 = 0$. Pre $\mu \geq 1$ je ďalším riešením $\hat{x}_1(\mu) = 1 - 1/\mu$.

Stabilita: $f'(0) = \mu$, preto $\hat{x}_0$ je asymptoticky stabilný pre $\mu \in [0,1)$ a nestabilný pre $\mu > 1$. Ďalej platí $f'(1-1/\mu) = 2-\mu$, preto $x_1(\mu)$ je asymptoticky stabilný pre $1 < \mu < 3$. Pre $\mu > 3$ sú obidva pevné body nestabilné. Čo sa deje s trajektóriami? Ukážeme, že pre $\mu = 3$ sa od bodu $x_1(3) = 2/3$ oddelia 2 periodické body. Platí

$$\begin{aligned} f_\mu^2(x) &= \mu\,[\mu x(1-x)]\,[1-\mu x(1-x)] \\ &= -\mu^3x^4 + 2\mu^3x^3 - (\mu^2+\mu^3)x^2 + \mu^2 x \end{aligned}$$

Nenulové pevné body f_μ^2 sú riešením rovnice

$$F(x,\mu) = \mu^3x^3 - 2\mu^3x^2 + (\mu^2+\mu^3)x - \mu^2 + 1 = 0. \tag{2.6}$$

Platí $F(2/3,3) = 0$ a $\frac{\partial F}{\partial x}(2/3,3) = 0$, preto nemôžeme v okolí bodu $(2/3,3)$ použiť vetu o implicitnej funkcii na to, aby sme z rovnice (2.6) vyjadrili x jednoznačne ako funkciu μ.

Keďže vieme, že jedným z koreňov rovnice (2.6) je aj pevný bod $x = 1 - 1/\mu$ zobrazenia f, môžeme polynóm na ľavej strane (2.6) vydeliť koreňovým činiteľom $x = 1 - 1/\mu$. Dostaneme polynóm

$$\mu^3x^2 - (\mu^3+\mu^2)x + \mu^2 + \mu.$$

Tento polynóm môžeme ďalej vydeliť μ a dostaneme

$$G(x,\mu) = \mu^2x^2 - (\mu^2+\mu)x + \mu + 1.$$

Bod x je 2-cyklus (teda pevným bodom f_μ^2 a nie pevným bodom f_μ) práve vtedy, ak $G(x,\mu) = 0$. Keďže $\partial G/\partial x(\frac{2}{3},3) = 0$, stále ešte nemôžeme z rovnice $G = 0$ vyjadriť x jednoznačne ako funkciu μ. Platí však $\partial G/\partial\mu(\frac{2}{3},3) = -1 \neq 0$, preto podľa vety o implicitnej funkcii existuje funkcia φ na okolí bodu $2/3$ taká, že lokálne v okolí bodu $(2/3,3)$ je $G(x,\mu) = 0$ práve vtedy, ak $\mu = \varphi(x)$. Derivovanie vzťahu

$$G(x,\varphi(x)) = 0$$

dostávame

$$\varphi'\left(\frac{2}{3}\right) = -\frac{\partial G/\partial x(\frac{2}{3},3)}{\partial G/\partial\mu(\frac{2}{3},3)} = 0; \tag{2.7}$$

ďalším derivovaním (pri zohľadnení (2.6)) dostaneme

$$\varphi''\left(\frac{2}{3}\right) = -\frac{\partial^2 G/\partial x^2(\frac{2}{3},3)}{\partial G/\partial\mu(\frac{2}{3},3)} = 18.$$

Lokálne v okolí bodu $(\frac{2}{3}, 3)$ teda body 2-cyklu ležia na krivke

$$\mu = \varphi(x) = 3 + 9\left(x - \frac{2}{3}\right)^2 + o\left(x - \frac{2}{3}\right)^2.$$

Je to krivka tvaru paraboly, otvorená smerom rastu μ. Znamená to, že cyklus sa vyskytuje pre $\mu > 3$.

Aby sme mohli rozhodnúť o stabilite cyklu pre μ blízke 3, potrebujeme zistiť hodnoty funkcie $g(x) = (f_\mu^2)'(x)|_{\mu=\varphi(x)}$ pre x blízke 2/3. Platí $g(2/3) = 1, g'(2/3) = 0, g''(2/3) < 0$, preto $g(x) < 1$ pre x blízke 2/3. To značí, že pre μ blizke 3 je cyklus asymptoticky stabilný. Detaily prenechávame čitateľovi ako cvičenie.

Logistický model (2.4), (2.5) bol v posledných troch desaťročiach predmetom intenzívneho skúmania. Dokázalo sa, že existuje nekonečná postupnosť $\mu_1 = 3 < \mu_2 < \mu_3 \cdots < 4$ taká, že pre $\mu = \mu_k$ sa zo stabilného k-cyklu odštepí stabilný $(k+1)$-cyklus, pričom k-cyklus stabilitu stratí. Postupnosť $\{\mu_k\}$ nemá 4 ako limitu; pre hodnoty μ dosť blízke 4 sa vyskytuje 3-cyklus. Dá sa dokázať, že ak pre niektoré $0 < \mu < 4$ existuje 3-cyklus, musí pre toto μ existovať k-cyklus pre ľubovoľné k. Presnejšie, možno existenciu cyklov vyčítať z tzv. Šarkovského postupnosti usporiadania prirodzených čísel

$$\begin{aligned} 3 < 5 < 7 < \cdots < 2 \cdot 3 < 2 \cdot 5 < \cdots < \cdots < 2^k \cdot 3, 2^k \cdot 5 < \dots \\ < \cdots < 2^3 < 2^2 < 2 < 1 \end{aligned} \tag{2.8}$$

takto: ak sa pre nejaké $0 < \mu < 4$ vyskytuje k-cyklus, potom sa vyskytujú všetky l-cykly také, že $l > k$ v usporiadaní (28).

Ďalej platí, že ak sa vyskytne 3-cyklus, potom sa systém správa "chaoticky". Voľne povedané to značí, že trajektórie ktoré vychádzajú z ľubovoľne blízkych bodov sa môžu správať celkom odlišne.

2.3 Nelineárny pavučinový model

O "pavučinovom" modeli 1.4 niet nijakého dôvodu predpokladať, že funkcie S, D sú lineárne. Prirodzené je však očakávať, že S je rastúca, kým D je klesajúca. Vyplýva to aj zo základnej mikroekonomickej teórie.

Predpokladajme teda, že S, D sú C^1 a $D(p) \geq 0$ pre $p \geq 0$ $S'(p) > 0$, $D'(p) < 0$ pre $p \geq 0$. Potom zrejme S je rastúca, D je klesajúca a má C^1 inverznú funkciu D^{-1}, pre ktorú platí

$$(D^{-1})'(q) = \frac{1}{D'(D^{-1}(q))}.$$

Ak existuje $\hat{p} \geq 0$ také, že $D(\hat{p}) = S(\hat{p})$ (čo budeme v ďalšom predpokladať), potom je jediné (prečo?). Rovnako ako v (1.4) odvodíme z podmienky rovnováhy (1.10) pre správanie jednorozmerného dynamického systému

$$p(t+1) = D^{-1} \circ S(p(t)), \tag{2.9}$$

ktoré má jediný rovnovážny bod $\hat{p}$. Platí

$$(D^{-1} \circ S)'(\hat{p}) = \frac{1}{D'(\hat{p})} \cdot S'(\hat{p}).$$

Podľa Vety 2.1 je $\hat{p}$ asymptotický stabilný ak $S'(\hat{p}) < D'(\hat{p})$ (čo značí, že spotrebiteľ reaguje na zmenu ceny "elastickejšie", ako dodávateľ) a nestabilný, ak $S'(\hat{p}) > |D'(\hat{p})|$. Lokálne trajektórie vždy oscilujú okolo bodu $\hat{p}$, pretože $S'(\hat{p})/D'(\hat{p}) < 0$.

Zaujímavé dynamické správanie pavučinového modelu môžeme dostať, ak S nie je monotónna funkcia. Ekonomicky to nie je celkom za vlasy pritiahnuté - možno si napríklad predstaviť, že dodávateľ po prekročení určitej cenovej hladiny (a tým mieru zisku) má viac než o zvýšenie svojich príjmov záujem napr. o voľný čas, aby mohol príjmy užiť. Predpokladajme napríklad, že $D(p)$ je lineárna ako v 1.9,

$$D(p) = a - bp,$$

ale $S(p)$ je kvadratická,

$$S(p) = c + dp - ep^2,$$

a teda pre veľké $p > 0$ je klesajúca. Samozrejme, že tento model má opodstatnenie len na takom intervale kladných hodnôt p, v ktorých je $D(p) > 0$ aj $S(p) > 0$.

Ukážeme, že pri vhodne zvolených kladných hodnotách a, b, c, d, e možno transformáciou

$$p = A - Bx \tag{2.10}$$

dynamický systém (2.7),

$$p(t+1) = \frac{1}{b}\left[a - c - dp(t) + ep^2(t)\right] \tag{2.11}$$

pretransformovať na (2.4) tak, aby v intervale $x \in (0,1)$ bolo $p > 0$, $D(p) > 0$ aj $S(p) > 0$.

Skutočne, ak zvolíme $a = 3 + 2\mu$, $b = 1$, $c = 1$, $d = 3\mu$ a $e = \mu$, potom pre $A = 2$, $B = 1$ dosadením do (2.10) z (2.9) dostaneme

$$2 - x(t+1) = 3 + 2\mu - 1 = 3\mu(2 - x(t)) + \mu(2 - x(t))^2,$$

teda

$$x(t+1) = \mu x(t)(1 - x(t)).$$

Ďalej, ak $x \in [0,1]$, potom $p \in [1,2]$, $D(p) \geq 1$ a $S(p) \geq 1 + 3\mu p - \mu p^2 = 1 + \mu(3p - p^2) \geq 3$.

To značí, že pre uvedené hodnoty a, b, c, d má pavučinový model rovnako komplikovanú dynamiku, ako logistický model z 2.2.

3. Viacrozmerné lineárne DS

3.1. Príklad. Fibonacciho postupnosť. (Leonardo Pisano, 13. str.)

Sledujme populáciu králikov, ktoré počnúc od 2. mesiaca života splodia každý mesiac pár králikov. Ako sa bude asymptoticky vyvíjať ich populácia?

Odhliadnime od zrejmých nedostatkov modelu (úmrtnosť, obmedzenosť zdrojov) a označme $y(t)$ počet králičích párov v t-tom mesiaci. Potom platí

$$y(t+1) = y(t) + y(t-1).$$

Ak začneme s jedným párom, potom $y(0) = y(1) = 1$.

Úlohu môžeme preformulovať: označíme $x_1(t) = y(t-1)$, $x_2(t) = y(t)$, $x(t) = (x_1(t), x_2(t))$. Potom platí

$$\begin{aligned} x_1(t+1) &= x_2(t) \\ x_2(t+1) &= x_1(t) + x_2(t), \end{aligned}$$

teda

$$x(t+1) = Ax(t), \quad x(0) = (1,1)$$

kde

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

3.2 Explicitný výpočet trajektórií lineárneho dynamického systému

$$x(t+1) = Ax(t) \tag{3.1}$$

$x \in \mathbb{R}^n$, $A - n \times n -$ matica.

Zrejme platí

$$x(t) = A^t x(0).$$

Ide teda o to, počítať mocniny matice tak, aby bola vidno ich asymptotika. Ak sú vlastné hodnoty matice A rozličné, označme ich $\lambda_1, \dots, \lambda_n$, a označme $v_1, \dots, v_n$ ich vlastné vektory. Potom platí

$$Av_i = \lambda_i v_i, \ i = 1, ..., n,$$

t. j.

$$AV = V\Lambda \qquad \text{kde } V = (v_1, \dots, v_n)$$
$$\Lambda = \text{diag } (\lambda_1, \dots, \lambda_n)$$

a V je regulárne. Z toho dostávame

$$A = V\Lambda V^{-1}$$

a

$$A^t = V\Lambda^t V^{-1} = V \begin{pmatrix} \lambda_1^t & & \\ & \ddots & \\ & & \lambda_n^t \end{pmatrix} V^{-1} \tag{3.2}$$

Všimnime si, že sa nám môže stať, že λ_i sú komplexné.

Pre viacnásobné korene môžeme postup modifikovať, v tomto texte to však nebudeme robiť.

Užitočné je chápať to ešte inak. Ak pretransformujeme vektor x na vektor $\tilde{x}$ transformáciou

$$x(t) = V\tilde{x}(t) \tag{3.3}$$

potom platí

$$V\tilde{x}(t+1) = AV\tilde{x}(t)$$

a teda časový vývoj $\tilde{x}(t)$ sa riadi LDS

$$\tilde{x}(t+1) = V^{-1}AV\tilde{x}(t) = \Lambda\tilde{x}(t); \tag{3.4}$$

LDS (3.4) sa rozpadá na n nezávislých jednorozmerných LDS

$$\tilde{x}_i(t+1) = \lambda_i\tilde{x}_i(t) \tag{3.5}$$

(ak λ_i sú imaginárne, môžu byť niektoré LDS imaginárne (pozri cvičenie!)).

Z (1.3) vyplýva, že každú trajektóriu systému (3.4) resp. (3.5) možno napísať v tvare

$$\tilde{x}_i(t) = c_i\lambda_i^t,$$

kde $c_i = x_i(0)$, alebo ekvivalentne

$$\tilde{x}(t) = \sum_{i=1}^{n} c_i e_i \lambda_i^t = \Lambda^t c \tag{3.6}$$

kde e_i je i-ty jednotkový súradnicový stĺpcový vektor a $c = (c_1, \ldots, c_n)^T$. Z (3.3), (3.6) vyplýva, že každú trajektóriu (3.1) možno vyjadriť v tvare

$$x(t) = V\Lambda^t c,$$

alebo

$$x(t) = \sum_{i=1}^{n} c_i v_i \lambda_i^t, \tag{3.7}$$

kde v_i sú vlastné vektory vlastných hodnôt λ_i a c_i sú vhodné konštanty. Ak poznáme $x(0)$, môžeme vektor c koeficientov určiť z rovnice

$$Vc = x(0), \tag{3.8}$$

alebo ekvivalentne zo sústavy rovníc

$$\sum_{i=1}^{n} c_i v_i = x(0), \tag{3.9}$$

pretože matica V je regulárna.

Spolu s imaginárnou vlastnou hodnotou λ matice A je jej vlastnou hodnotou aj $\bar{\lambda}$. Rovnako sú komplexne združené ich vlastné vektory $v, \bar{v}$. Napriek tomu, že v pravej strane formuly (3.7) sa vtedy vyskytujú komplexné čísla, jej hodnota

$x(t)$ samozrejme musí byť reálna. Vo vyjadrení (3.7) pre $x(t)$ sa vyskytnú členy $cv\lambda^t + d\bar{v}\bar{\lambda}^t$. Pretože $x(0)$ je reálne, platí

$$cv + d\bar{v} = x(0) = \overline{x(0)} = \bar{c}\bar{v} + \bar{d}v,$$

a teda

$$(c - \bar{d})v + (\bar{c} - d)\bar{v} = 0.$$

Keďže $v, \bar{v}$ sú vlastnými vektormi rozličných vlastných hodnôt, sú lineárne nezávislé, z čoho vyplýva $c = \bar{d}$. Vo vyjadrení (3.7) pre $x(t)$ budú teda členy s $\lambda, \bar{\lambda}$ mať tvar

$$x(t) = ... + cv\lambda^t + \overline{cv\lambda^t}... = ...2Re(cv\lambda^t), ..$$

Ak λ vyjadríme v polárnom tvare $\lambda = \rho(\cos\varphi + i\sin\varphi)$, $\rho = |\lambda|$, $\varphi = \arg\lambda$, potom platí $\varphi \neq 0$ a

$$\begin{aligned}\lambda^t &= \rho^t(\cos t\varphi + i\sin t\varphi),\\ \bar{\lambda}^t &= \rho^t(\cos t\varphi - i\sin t\varphi),\end{aligned}$$

a teda

$$\begin{aligned}x(t) &= cv\lambda^t + \bar{c}\bar{v}\overline{\lambda}^t =\\ &= 2Recv\lambda^t = \rho^t(a\cos t\varphi + b\sin t\varphi)\end{aligned}$$

pre vhodné vektory a, b.

Príklad. Compute the solution of LDE (3.1) with the matrix

$$A = \begin{pmatrix} 1 & 1\\ -1 & 1\end{pmatrix},$$

i. e.

$$\begin{aligned}x_1(t+1) &= x_1(t) + x_2(t)\\ x_2(t+1) &= -x_1(t) + x_2(t)\end{aligned}$$

satisfying $x_1(0) = 1$,$x_2(0) = 0$.

The eigenvalues of A are $\lambda = 1 + i$, $\overline{\lambda} = 1 - i$ and $v = (1, i)$, $\overline{v} = (1, -i)$ are the corresponding eigenvectors.

Hence $x_1(t)$, $x_2(t)$ satisfy

$$\begin{aligned}x_1(t) &= 2Re\{(\gamma + i\delta)(1+i)^t\}\\ x_2(t) &= 2Re\{(\gamma + i\delta)i(1+i)^t\}\\ &= 2Re\{-\delta + i\gamma)(1+i)^t\},\end{aligned}$$

where γ, δ are determined by the condition for $x(0)$. Moivre's formula enables us to compute $x(t)$ without dealing with complex numbers. We have $\lambda = 1 + i = \sqrt{2}(\cos(t\frac{\pi}{4}) + i\sin(t\frac{\pi}{4}))$ which yields $\lambda^t = (\sqrt{2})^t(\cos(t\frac{\pi}{4}) + i\sin(t\frac{\pi}{4}))$, so

$$\begin{aligned}x_1(t) &= 2(\sqrt{2})^t(\gamma\cos(t\frac{\pi}{4}) - \delta\sin(t\frac{\pi}{4}))\\ x_2(t) &= 2(\sqrt{2})^t[-\delta\cos(t\frac{\pi}{4}) - \gamma\sin(t\frac{\pi}{4})]\end{aligned}$$

Since $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, the values of γ, δ are solutions of the equations $2\gamma = 1$, $\delta = 0$ so we have $\gamma = \frac{1}{2}$, $\delta = 2$. Thus, the solution is

$$\begin{aligned}x_1(t) &= (\sqrt{2})^t\cos(t\frac{\pi}{4})\\ x_2(t) &= (\sqrt{2})^t\sin(t\frac{\pi}{4})\end{aligned}$$

3.3. Príklad. Diskrétny model dynamiky populácie s vekovou štruktúrou.

Pri presnejších modeloch populačnej dynamiky dlhšie žijúcich populácií s prekrývajúcimi sa generáciami treba brať do úvahy, že jedinci sú schopní mať potomstvo iba v určitom časovom intervale.

V zjednodušenom modeli ľudskej populácie predpokladajme, že potomstvo môžu mať jedinci vo veku od 20 do 60 rokov, pričom nerobíme rozdiel medzi mužmi a ženami. Stav populácie opíšeme vektorom $x = (x_1, x_2, x_3)$, kde

x_1 je počet jedincov s vekom v intervale $[0, 20]$

x_2 je počet jedincov s vekom v intervale $[20, 60]$

x_3 je počet jedincov s vekom v intervale nad 60 rokov.

Označíme m_k úmrtnosť k-tej vekovej skupiny pre $k = 1, 2, 3$ a b pôrodnosť druhej vekovej skupiny.

Ak zjednodušene predpokladáme, že vo vekových skupinách je populácia vzhľadom na vek rovnomerne rozložená a za jednotku času vezmeme rok, dostaneme nasledovný systém:

$$\begin{aligned} x_1(t+1) &= \left(1 - m_1 - \frac{1}{20}\right) x_1(t) + b x_2(t) \\ x_2(t+1) &= \frac{1}{20} x_1(t) + \left(1 - m_2 - \frac{1}{40}\right) x_2(t) \\ x_3(t+1) &= \frac{1}{40} x_2(t) + (1 - m_3) x_3(t). \end{aligned} \tag{3.10}$$

Zo štatistickej ročenky Slovenska za rok 1991 sa dajú parametre systému odhadnúť takto: $m_1 = 98 \times 10^{-4}$, $m_2 = 4.13 \times 10^{-3}$, $m_3 = 5.5 \times 10^{-2}$, $b = 2.87 \times 10^{-2}$. Pre tieto hodnoty bude teda systém (3.10) mať tvar

$$x(t+1) = Ax(t),$$

kde

$$A = \begin{pmatrix} 0.94 & 0.029 & 0 \\ 0.05 & 0.971 & 0 \\ 0 & 0.025 & 0.945 \end{pmatrix}.$$

Matica A má zrejme vlastnú hodnotu $\lambda_3 = 0.945$ s vlastným vektorom $v_3 = (0, 0, 1)$. Ďalšie dve vlastné hodnoty sú vlastnými hodnotami matice

$$\begin{pmatrix} 0.94 & 0.029 \\ 0.05 & 0.971 \end{pmatrix}.$$

Jej charakteristická rovnica je

$$\lambda^2 - 1.91\lambda + 0.91 = 0$$

a jej koreňmi sú $\lambda_1 = 0.997$ a $\lambda_2 = 0.914$. Vlastnými vektormi v_k vlastných hodnôt λ_k sú $v_1 = (0.42,\ 0.84,\ 0.4)$ a $v_2 = (0.66,\ -0.58,\ 0.47)$. Podľa (3.56) možno každú trajektóriu napísať v tvare

$$x(t) = c_1 v_1 \lambda_1^t + c_2 v_2 \lambda_2^t + c_3 v_3 \lambda_3^t.$$

Keďže $|\lambda_i| < 1,\ i = 1,2,3$, platí $x(t) \to 0$ pre $t \to 0$. To značí, že pri týchto hodnotách pôrodnosti a úmrtnosti by populácia vyhynula. Ďalej si všimnime, že

$$x(t) = \lambda_1^t \left[c_1 v_1 + c_2 v_2 (\lambda_2/\lambda_1)^t + c_3 v_3 (\lambda_3/\lambda_1)^t\right].$$

Keďže $\lambda_2 < \lambda_1$, $\lambda_3 < \lambda_1$, druhý a tretí člen v zátvorke konvergujú k nule pri $t \to \infty$. Vekový profil populácie sa teda bude asymptoticky približovať vektoru v_1.

3.4 Výpočet riešení lineárnej diferenčnej rovnice druhého a vyšších rádov.

Všeobecná lineárna diferenčná rovnica 2. rádu má tvar

$$y(t+1) = a_1 y(t) + a_0 y(t-1),$$

alebo ekvivalentne

$$y(t+2) = a_1 y(t+1) + a_0 y(t).$$

Je špeciálnym prípadom DR n-tého rádu

$$y(t+n) = a_{n-1} y(t+n-1) + ... + a_0 y(t). \tag{3.11}$$

Substitúciami

$$y(t) = x_1(t), ..., y(t+n-1) = x_n(t)$$

môžeme rovnicu pretransformovať na n-rozmerný lineárny DS

$$x(t+1) = Ax(t)$$

so "sprievodnou maticou"

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 & 1 \\ a_0 & a_1 & a_2 & \dots & a_{n-1} \end{pmatrix},$$

o ktorej je známe, že jej charakteristický polynóm je

$$\lambda^n - a_{n-1}\lambda^{n-1} - ... - a_1\lambda - a_0, \tag{3.12}$$

Tento polynóm môžeme dostať priamo z rovnice (3.11) zámenou λ^k za $y(t+k)$. Jeho trajektórie môžeme počítať postupom z odseku 3.2. Pretože nám stačí počítať iba prvú zložku vektora x, nemusíme hľadať vlastné vektory. Vieme totiž že ak sú korene polynómu (3.12) navzájom rôzne, platí

$$y(t) = x_1(t) = p_1\lambda_1^t + \cdots + p_n\lambda_n^t.$$

Koeficienty $p_1, ..., p_n$ môžeme určiť, ak máme dané napr. x_0, alebo ekvivalentne $y(0) = y_0, ..., y(n-1) = y_{n-1}$, a to zo systému n nezávislých lineárnych rovníc

$$p_1\lambda_1^j + \cdots + p_n\lambda_n^j = y_j,\ j = 1, \cdots, n. \tag{3.13}$$

Ak má charakteristický polynóm dvojicu komplexne združených koreňov $\lambda \neq \bar{\lambda}$, potom koeficienty pri nich budú komplexné, $p, \bar{p}$. V prípade $n = 2$ bude $p_1 = p,\ p_2 = \bar{p}$ a pre určenie p dostaneme rovnice

$$2\text{Re } p(= p + \bar{p}) = y_0$$

$$2\text{Re}(p\lambda)(= \lambda p + \overline{\lambda p}) = y_1.$$

3.5. Riešenie úlohy 3.1

Vlastné hodnoty sú riešením rovnice $\lambda^2 - \lambda - 1 = 0$, teda

$$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

Podľa 3.4 možno riešenie $y(t)$ vyjadriť v tvare

$$y(t) = p\lambda_1^t + q\lambda_2^t,$$

kde p, q sú konštanty, nezávislé od t. Z rovníc $y(0) = y(1) = 1$ dostávame pre p, q systém rovníc

$$\begin{aligned} p + q &= 1 \\ \lambda_1 p + \lambda_2 q &= 1. \end{aligned}$$

Ich riešením je

$$p = \frac{1 - \lambda_2}{\lambda_1 - \lambda_2}, \qquad q = \frac{\lambda_1 - 1}{\lambda_1 - \lambda_2}$$

Pre $t \mapsto \infty$ platí

$$\frac{y(t+1)}{y(t)} = \frac{p\lambda_1^{t+1} + q\lambda_2^{t+1}}{p\lambda_1^t + q\lambda_2^t} =$$

$$= \lambda_1 \frac{p + q\left(\frac{\lambda_2}{\lambda_1}\right)^t \lambda_2}{p + q(\frac{\lambda_2}{\lambda_1})^t} \to \lambda_1, \qquad \text{pre } t \to \infty,$$

teda pomer dvoch nasledujúcich členov konverguje k zlatému rezu pre $t \to \infty$.

3.6. Príklad. Pravdepodobnosť hráčovho bankrotu. Predpokladajme, že hráč hrá hazardnú hru, v ktorej s pravdepodobnosťou p vyhrá 1 korunu a s pravdepodobnosťou $1 - p$ ju prehrá. Jeho počiatočná hotovosť je n_0 korún, ak vyhrá $N > 0$ korún, prestane hrať. Aká je pravdepodobnosť jeho bankrotu (t.j. vyprázdnenia jeho hotovosti)?

Označme $P(n)$ pravdepodobnosť bankrotu, ak hráč má hotovosť n. Keďže po jednej hre sa s pravdepodobnosťou p dostane do stavu $n+1$ a s pravdepodobnosťou $1 - p$ do stavu $n - 1$, platí

$$P(n) = pP(n+1) + (1-p)P(n-1),$$

z čoho pre P dostaneme diferenčnú rovnicu

$$P(n+1) = \frac{1}{p}P(n) - \frac{1-p}{p}P(n-1).$$

Ďalej platí $P(0) = 1$ (bankrot je istý) a $P(N) = 0$ (hráč končí hru a preto s istotou nezbankrotuje).

Všimnime si, že v tejto diferenčnej rovnici premenná n (zodpovedajúca doterajšej premennej t) nemá charakter času. Ďalej si všimnime, že ide o "okrajovú" úlohu, v ktorej sú hodnoty P dané na koncoch intervalu.

3.7 Pevný bod a asymptotické správanie trajektórií

Ak $A - I$ je regulárna matica, potom LDS

$$x(t+1) = Ax(t) \tag{3.14}$$

má jediný pevný bod 0. Jeho stabilitné vlastnosti definujeme rovnako ako v jednorozmernom prípade s tým, že $|\cdot|$ chápeme ako normu v $\mathbb{R}^n$. Platí

Veta.
a) *Ak sú absolútne hodnoty všetkých vlastných hodnôt matice* A < 1*, potom je* 0 *asymptoticky stabilný.*
b) *ak sú absolútne hodnoty všetkých vlastných hodnôt* ≤ 1 *a všetky vlastné hodnoty s* $|\cdot| = 1$ *sú jednoduché, potom je* 0 *stabilný.*
c) *ak je* $|\cdot|$ *aspoň jednej vlastnej hodnoty* > 1*, potom je* 0 *nestabilný pevný bod.*

Pozn. Všimnime si, že $A - I$ je regulárna práve vtedy, ak A nemá vlastnú hodnotu 1.

Dôkaz vety. Urobíme ho pre prípad navzájom rozličných vlastných hodnôt. V takom prípade transformácia $x = V\tilde{x}$ z 3.2 prevedie DS (3.14) na DS (3.4), V, Λ ako v 3.2. Platí

$$|x| \leq |V||\tilde{x}|, \tag{3.15}$$

$$|\tilde{x}| \leq |V^{-1}||x|. \tag{3.16}$$

Z (3.5) vyplýva

$$|\tilde{x}(t)| \leq \overline{\lambda}^t |\tilde{x}(0)|, \text{ kde } \overline{\lambda} = \max |\lambda_i|,$$

z čoho podĺa (3.15),(3.16) dostávame

$$|x(t)| \leq \bar{\lambda}^t |V||V^{-1}||x(0)|.$$

Z toho vyplýva a), b).

Ak $|\lambda_i| > 1$ pre nejaké i, potom pre $\tilde{x}(0) = \delta e_i$ (e_i = i-ty jednotkový vektor) platí $\tilde{x}(t) = \delta\lambda_i^t e_i$, $|\tilde{x}(t)| = \lambda_i^t \delta$. Trajektória $x(t) = V\tilde{x}(t)$ spĺňa $|x(0)| \leq |V||\tilde{x}(0)| = |V|\delta$ podĺa (3.15), avšak podĺa (3.16) $|x(t)| \geq |V^{-1}|^{-1}|\tilde{x}(t)| = |V^{-1}|^{-1}|\lambda_i|^t\delta \geq |V^{-1}|^{-1}|V|^{-1}|\lambda_i|^t|x(0)|$. To značí, že 0 je nestabilný. □

3.8 Viacrozmerný afinný dynamický systém a formula variácie konštánt

Analogicky ako v jednorozmernom prípade nazývame afinným dynamický systém tvaru

$$x(t+1) = Ax(t) + b, \tag{3.17}$$

kde $x, b \in \mathbb{R}^n$ a A je $n \times n$ matica. Ak je matica $A - I$ regulárna, potom (3.9) má jediný pevný bod, a to

$$\hat{x} = (I - A)^{-1}b.$$

Ľahko si overíme, že správanie odchýľky od pevného bodu $y(t) = x(t) - \hat{x}$ sa riadi lineárnym dynamickým systémom

$$y(t+1) = Ay(t).$$

Z toho ihneď vyplýva, že o stabilite pevného bodu $\hat{x}$ rozhodujú vlastné hodnoty matice A podľa Vety 3.5.

Pre lineárnu nehomogénnu diferenčnú rovnicu (nejde o dynamický systém, lebo b závisí od t)

$$x(t+1) = Ax(t) + b(t).$$

odvodíme analogicky ako v jednorozmernom prípade indukciou vzťah

$$x(t) = A^{t-t_0}x(t_0) + \sum_{s=t_0}^{t-1} A^{t-1-s}b(s).$$

Analogická formula pre lineárne nehomogénne diferenciálne rovnice sa nazýva *formulou variácie konštánt.*

3.9 Stabilita Samuelsonovho modelu národného hospodárstva

V modeli sa stav hospodárstva opisuje v diskrétnych časových okamihoch $t = \ldots, 0, 1, \ldots$ nasledovnými premennými:
- celkovými príjmami $Y(t)$
- prostriedkami vynaloženými na spotrebu $C(t)$
- investíciami $I(t)$ a
- vládnymi výdavkami $G(t)$

Predpokladá sa, že:
- vládne výdavky sú konštantné v čase - $G(t) \equiv G$
- rozpočet je vyrovnaný, t.j.

$$Y(t) = C(t) + I(t) + G \tag{3.18}$$

- spotreba je priamo úmerná príjmom v predchádzajúcom období,

$$C(t) = \gamma Y(t-1), \quad (0 < \gamma < 1) \tag{3.19}$$

- investície sú úmerné zmene spotreby,

$$I(t) = \alpha(C(t) - C(t-1))) \quad (\alpha > 0); \tag{3.20}$$

Konštanta γ sa nazýva hraničným sklonom (*propensity*) k spotrebe, α je tzv. *akcelerátor.*

Dosadením z (3.19) a (3.20) do (3.18) dostaneme

$$\begin{aligned} Y(t) &= \gamma Y(t-1) + \alpha(C(t) - C(t-1)) + G \\ &= \gamma Y(t-1) + \alpha\gamma\left[Y(t-1) - Y(t-2)\right] + G \end{aligned} \tag{3.21}$$

t.j. $Y(t)$ spĺňa afinnú diferenčnú rovnicu

$$Y(t) = \gamma(\alpha+1)Y(t-1) - \alpha\gamma Y(t-2) + G \tag{3.22}$$

ktorá má pevný bod

$$\hat{Y} = \frac{G}{1-\gamma}.$$

Vyšetríme jeho stabilitu.

Charakteristická rovnica rovnice (3.22),

$$\lambda^2 - \gamma(\alpha+1)\lambda + \alpha\gamma = 0$$

má korene

$$\lambda_{1,2} = \frac{1}{2}\left[(\alpha+1)\gamma \pm \sqrt{\gamma^2(1+\alpha)^2 - 4\alpha\gamma}\right].$$

Ak odhliadneme od hraničného prípadu dvojnásobného koreňa, treba rozlíšiť dve možnosti: 1. Ak

$$\gamma > \frac{4\alpha}{(1+\alpha)^2} \tag{3.23}$$

potom λ_1, λ_2 sú reálne a platí

$$\begin{aligned} \lambda_1 &= \frac{1}{2}(\alpha+1)\gamma + \sqrt{\gamma^2(1+\alpha^2) - 4\alpha\gamma} \\ &> \frac{1}{2}(\alpha+1)\gamma - \sqrt{\gamma^2(1+\alpha)^2 - 4\alpha\gamma} = \lambda_2 > 0. \end{aligned}$$

Všimnime si, že krivka $\gamma = \frac{4\alpha}{(1+\alpha)^2}$ prechádza bodom $\alpha = 0$, $\gamma = 0$, rastie pri $\alpha < 1$ a klesá pri $\alpha > 1$; maximálna hodnota $\hat{\gamma} = 1$ sa dosahuje pri $\alpha = 1$.
Keďže $|\lambda_2| < |\lambda_1|$, rovnovážny bod $\hat{Y}$ bude stabilný, ak $|\lambda_1| < 1$ a nestabilný ak $|\lambda_1| > 1$. Aby bolo $|\lambda_1| < 1$, musí platiť

$$(\alpha+1)\gamma < 2 \tag{3.24}$$

a teda nerovnosť $|\lambda_1| < 1$ bude ekvivalentná nerovnosti

$$\sqrt{\gamma^2(1+\alpha)^2 - 4\alpha\gamma} < 2 - (\alpha+1)\gamma$$

resp.

$$\gamma^2(1+\alpha)^2 - 4\alpha\gamma < [2 - (\alpha+1)\gamma]^2,$$

ktorá je splnená práve vtedy, ak $\gamma < 1$.

Z (3.23) a (3.24) dostávame $(\alpha+1)\frac{4\alpha}{(1+\alpha)^2} < 2$, a teda aj $\alpha < 1$

Ak teda platí (3.23) a

$$\alpha < 1 \text{ a } \gamma < 1, \tag{3.25}$$

vtedy pevný bod $\hat{Y}$ je stabilný, ak jedna z podmienok (3.25) nie je splnená a (3.23) splnené je, $\hat{Y}$ je nestabilný.
2. Ak

$$\gamma < \frac{4\alpha}{(1+\alpha)^2}, \tag{3.26}$$

potom sú λ_1 a $\lambda_2 = \bar{\lambda_1}$ komplexne združené a platí

$$|\lambda_1| = \sqrt{\alpha\gamma},$$

stacionárny bod $\hat{Y}$ je teda asymptoticky stabilný, ak $\alpha\gamma < 1$ a nestabilný, ak $\alpha\gamma > 1$.

Ďalej, z 3.7 vyplýva, že trajektórie budú oscilovať okolo rovnovážneho bodu.

4. Nelineárne viacrozmerné DS

4.1 Rovnovážne stavy a ich stabilita

Uvažujeme nelineárny DS, daný rekurentným vzťahom

$$x(t+1) = f(x(t)), \tag{4.1}$$

kde $x \in X \subset \mathbb{R}^n$ a $f : X \to X$ je $C^r,\ r \geq 0$.

Rovnovážny stav $\hat{x}$ DS (4.1) je riešením rovnice (vektorovej)

$$f(\hat{x}) = \hat{x},$$

čo je vlastne systém n rovníc a n neznámych

$$f_i(x_1, \ldots, x_n) = x_i, \qquad i = 1, \ldots, n.$$

Na rozdiel od afinného DS nemožno vo všeobecnosti tento systém rovníc explicitne riešiť a často nie je iná možnosť ako hľadať riešenia numericky. Predpokladajme však, že sme taký bod našli a zaujíma nás jeho stabilita. Platí

Veta. *Nech f je C^1 a nech $\hat{x}$ je pevným bodom zobrazenia f. Potom $\hat{x}$ je*
- *asymptoticky stabilným, ak sú absolútne hodnoty všetkých vlastných hodnôt operátora $Df(\hat{x})$ menšie ako* 1
- *nestabilný, ak má niektorá z vlastných hodnôt operátora $Df(\hat{x})$ absolútnu hodnotu väčšiu ako* 1.

Poznámka. Voľne povedané, veta hovorí, že o stabilite pevného bodu rozhoduje v nekritických prípadoch linearizácia zobrazenia v ňom.

Urobíme dôkaz stability pre prípad rozličných vlastných hodnôt. Dôkaz nestability je vo viacrozmernom prípade náročnejší a je mimo rámca tohoto textu.

Označme $y = x - \hat{x}$. Platí

$$\hat{x} + y(t+1) = f(\hat{x} + y(t)) = f(\hat{x}) + Df(\hat{x})y(t) + \omega(y(t))y(t), \tag{4.2}$$

teda

$$y(t+1) = (A + \omega(y))y,$$

kde

$$A = Df(\hat{x})$$

a

$$\lim_{y \to 0} \omega(y) = 0.$$

Ak sú vlastné hodnoty matice A navzájom rozličné, platí podľa (3.2)

$$AV = V\Lambda$$

kde V je regulárna a $\Lambda = \ \text{diag}\ \{\lambda_1, \ldots, \lambda_n\}$. Položme $y(t) = Vz(t)$. Z (4.2) dostaneme

$$z(t+1) = [\Lambda + V^{-1}\omega(Vz(t))V]z(t),$$

a teda

$$|z(t+1)| \le (\alpha + |V^{-1}||V||\omega(Vz(t))|)|z(t)|, \tag{4.3}$$

kde

$$\alpha = \max\{|\lambda_1|, \dots, |\lambda_n|\} < 1.$$

Pretože $\lim_{y\to 0}\omega(y) = 0$, existuje $\delta > 0$ také, že ak $|z| < \delta$, potom $|\omega(Vz)| < \frac{1}{2}(1-\alpha)|V||V^{-1}|^{-1}$. Zvoľme $\varepsilon > 0$ a položme $\delta^* < \min\{\epsilon/|V|, \delta\}$. Ak $|z(0)| < \delta^*$, potom zo (4.3) dostávame

$$|z(1)| \le (\alpha + ||V||V^{-1}||\omega(V(z(0))|)|z(0)| \le \frac{1}{2}(\alpha+1)z(0) < z(0) < \delta^*,$$

pretože $\frac{1}{2}(\alpha+1) < 1$.

Indukciou vzhľadom na t dostaneme

$$|z(t)| \le |z(0)| < \delta^*,$$

čo prechodom k premennej y dáva

$$\begin{aligned} |y(t)| &= |Vz(t)| \le |V||z(t)| \le |V||z(0)| \\ &= |V||V^{-1}y(0)| \le |V||V^{-1}||y(0)| \le |V||V^{-1}|\delta^* \end{aligned}$$

čo značí, že $\hat{x}$ je stabilný, a

$$|y(t)| \le |V||z(t)| \le |V||[\frac{1}{2}(\alpha+1)]^t|z(0)| \le |V||V^{-1}||[\frac{1}{2}(\alpha+1)]^t|y(0)| \to \text{ pre } t \ge 0,$$

teda $\hat{x}$ je aj asymptoticky stabilný.

4.2 Model krátkodobých fluktuácií výmenných kurzov

V diplomových prácach K. Boďovej (2004), J. Szolgayovej (2006) a T. Bokesa (2007) sa vyšetruje diferenčná rovnica

$$x(t+1) - x(t) = a(x(t) - x(t-1)) - b|x(t)|x(t)$$

s $a,\ b > 0$. Modeluje sa ňou vplyv trhu na krátkodobé správanie výmenného kurzu vybranej cudzej meny v domácej mene. Model je rovnako použiteľný na modelovanie cien iných špekulatívnych aktív, ako sú akcie, nehnuteľnosti, atď. Rozumieme nimi aktíva, ktoré sa kupujú a predávajú v rozličných časových okamihoch s cieľom zisku. Ide o "heterogeneous agents model", zohľadňujúci heterogenitu aktérov na trhu.

V rovnici x označuje krátkodobú odchýľku od rovnovážneho výmenného kurzu, určeného ekonomickými fundamentami (ako napr. paritou kúpnej sily, či úrokovým diferenciálom). Prvý člen na pravej strane predstavuje zmenu kurzu v dôsledku zmeny dopytu *chartistov*, ktorí predpokladajú že smer pohybu kurzu v predchádzajúcom časovom intervale bude pokračovať aj v budúcnosti. Na druhej strane *fundamentalisti*, vplyv ktorých predstavuje druhý člen predpokladajú, že odchýlený kurz sa v budúcnosti bude vracať do rovnovážnej polohy; faktorom $|x(t)|$ sa vyjadruje, že aktérov s fundamentalistickým správavním pribúda s veľkosťou odchýlky. Týmto model zohľadňuje to, že rovnovážny výmenný kurz nie je jednoznačne definovaný.

Zaujíma nás v prvom rade stabilita rovnovážneho kurzu $x = 0$. Charakteristickou rovnicou linearizácie v bode $x = 0$ (**3.4**) je

$$\lambda^2 - (1 + a)\lambda - a = 0$$

s koreňmi 1, a. Podľa vety z **4.1** je teda pre $a > 1$ rovnovážny kurz nestabilný. Pre $a < 1$ veta o stabilite nerozhodne, pretože charakteristická rovnica má okrem toho aj koreň 1.

Použitím jemnejších metód (vety o centrálnej variete) sa dá dokázať, že rovnovážny bod je pre $a < 1$ asymptoticky stabilný. Simulácie priebehu riešenia naznačujú, že pre $a > 1$ dochádza k trvalým fluktuáciám kurzu. Záver teda je, že ak akcia chartistov, charakterizovaná parametrom a prekročí prahovú hodnotu 1, rovnovážny kurz stráca stabilitu a prichádza k trvalým fluktuáciám výmenného kurzu.

4.3. Abstraktný dynamický systém

Nech $X \subset \mathbb{R}^n$. (C^r-diskrétnym) dynamickým systémom, $r \geq 0$ nazývame postupnosť C^r zobrazení $\varphi_t : X \to X \quad t = 0, 1, \ldots$ takých, že
1. φ_0 je identické zobrazenie
2. $\varphi_t \circ \varphi_s = \varphi_{t+s} \qquad$ pre $t, s = 0, 1, 2 \ldots$

C^r - zobrazenie $f : X \to X$ zrejme generuje C^r - dynamický systém predpisom

$$\varphi_t(x) = f^t(x) \quad (= \underbrace{f \circ \cdots \circ f}_{t\ razy}(x)) \quad \text{ pre } x \in X,\ t = 1, 2, \ldots$$

Naopak, každý dynamický systém je takto generovaný zobrazením φ_1. Ak $x(t), t = 0, 1, \ldots,$ je postupnosť, generovaná rekurentným vzťahom

$$x(t + 1) = f(x(t)), \qquad t = 0, 1, 2 \ldots,$$

a φ_t je dynamický systém, generovaný zobrazením f, potom zrejme platí

$$x(t) = \varphi_t(x(0)).$$

Hodnoty x zrejme môžeme v súlade s Úvodom nazvať stavmi DS.

1. Differential equation as a "limit" of a discrete dynamical system

1.1 Example. Ideally stirred vessel

A vessel of volume V is filled by a liquid I. Starting at $t = 0$ liquid II flows into the vessel with volumetric flow rate q. The liquid in the vessel is stirred very efficiently. We idealize the stirring by assuming that at any time instant the concentration of liquid II is the same at any point of the vessel. We would like to determine this concentration as a function of t.

Modelling the situation in the spirit of Chapter I we follow the concentration in discrete time instants $t_j = j\Delta t$ and write down the material balance of the instant t and the instant $t + \Delta t$:

$$Vx(t + \Delta t) = Vx(t) + q\Delta t - qx(t)\Delta t \tag{1.1}$$

or

$$x(t + \Delta t) = \left[1 - \frac{q}{V}\Delta t\right] x(t) + \frac{q}{V}\Delta t \tag{1.2}$$

The equation (1.2) generates an affine DDS; we can e.g. compute its stationary state and stability of the latter.

The model (1.2) is far from being flawless. It implicitly assumes that $x(t)$ remains constant in the interval $[t, t + \Delta t]$, then jumps to $x(t + \Delta t)$ given by (1.2).

To remedy the model we rewrite (1.1) as

$$\frac{x(t + \Delta t) - x(t)}{\Delta t} = \frac{q}{V}(1 - x(t)).$$

Assuming now that $x(t)$ is differentiable we pass to the limit for $\Delta t \to 0$ to obtain

$$\frac{dx(t)}{dt} = \frac{q}{V}(1 - x(t)). \tag{1.3}$$

We have obtained a new object which will be called *differential equation*. By analogy to the recurrence relation (1.2) we expect that once we fix $x(t_0)$, it should determine a unique function $x(t)$ (to be called *solution* of(1.3)) for all $t \geq t_0$. Unlike in the time discrete case this is by no means obvious and a theory is needed to establish it. In fact, it is not true in general unless certain assumptions are imposed on the equation.

1.2 Example. Population dynamics

Consider a population with overlapping generations. Unlike in the case of non-overlapping generations (I. 1.1) we follow the population continuously and expect its size to vary nearly continuously in time. Therefore, to assume that the population size $x(t)$ varies differentiably with time is not a gross distortion.

If we denote by $r\Delta t$ the number of offsprings an individual has during a time interval of length Δt, and by $x(t)$ the size of the population at time t, we can approximately write

$$x(t + \Delta t) = x(t) + r\Delta t x(t) = (1 + r\Delta t)x(t)$$

or

$$\frac{x(t+\Delta t)-x(t)}{\Delta t}=rx(t).$$

Passing to the limit as in (1.1) we obtain the differential equation

$$\dot{x}(t)=rx(t).$$

2. One-dimensional autonomous equation

2.1. Linear equation

This is the equation

$$\dot{x}=ax \tag{2.1}$$

By direct inspection it can be checked that for each $c\in\mathbb{R}$ the function

$$x(t)=ce^{at} \tag{2.2}$$

solves equation (2.1). The constant c can be uniquely computed from the condition that $x(t)$ *passes through the point* (t_0,x_0), i.e., that $x(t_0)=x_0$. Indeed, substituting into this condition from (2.2) we have

$$ce^{at_0}=x_0,$$

or,

$$c=x_0e^{-at_0}.$$

Substituting into (2.2) we obtain

$$x(t)=x_0e^{a(t-t_0)} \tag{2.3}$$

Note that (2.3) gives a solution not only for $t\geq t_0$ but for all $t\in\mathbb{R}$.

The formula (2.3) represents the *Malthus law* for populations with overlapping generations.

We do not know so far that (2.3) is the *only* solution passing through (t_0,x_0). To prove the latter, let $x(t)$ be *any* solution of (2.1) through (t_0,x_0). Consider the function

$$z(t)=e^{-at}(x(t)-x_0e^{a(t-t_0)}).$$

. We have

$$\begin{aligned}\frac{d}{dt}z(t)&=-ae^{-at}(x(t)-x_0e^{a(t-t_0)})\\&+e^{-at}(\dot{x}(t)-ax_0e^{a(t-t_0)})\\&=e^{-at}(\dot{x}(t)-ax(t))=0.\end{aligned}$$

Hence, $z(t)$ is constant. Since, in addition, $z(t_0)=0$, we have $z(t)\equiv 0$, which completes the proof.

Note that $\hat{x}=0$ is a *stationary* solution, i. e. constant solution of (2.1); if $a\neq 0$, it is the only one.We define stability, asymptotic stability and instability of a stationary solution almost literally as it was defined for discrete time dynamical systems in I.2.1. The only difference is that we allow t to run through all positive reals instead of integers.Then, obviously, $\hat{x}=0$ is stable if $a\leq 0$, asymptotically stable is $a<0$ and unstable if $a>0$.

2.2 Affine equation

We consider the equation

$$\dot{x} = ax + b \tag{2.4}$$

1. If $a = 0$, then the unique solution passing through the point (t_0, x_0) is

$$x(t) = x_0 + b(t - t_0)$$

2. If $a \neq 0$ then (2.4) has a stationary (equilibrium) - i.e. constant solution $\hat{x}(t) \equiv \hat{x}$. It has to satisfy $\dot{\hat{x}}(t) \equiv 0$, i.e.

$$a\hat{x} + b = 0,$$

or,

$$\hat{x} = -\frac{b}{a}$$

For the deviation $y(t)$ of a solution $x(t)$ from $\hat{x}$, $y(t) = x(t) - \hat{x} = x(t) + \frac{b}{a}$ we obtain,

$$\dot{y}(t) = \dot{x}(t) = ax(t) + b = a\left(y(t) - \frac{b}{a}\right) + b = ay(t),$$

Consequently, $y(t)$ is a solution of the LDE (2.1). Its solutions are

$$y(t) = y(t_0)e^{a(t-t_0)}.$$

Hence, the function

$$x(t) = -\frac{b}{a} + \left[x(t_0) + \frac{b}{a}\right] e^{a(t-t_0)}$$

is a solution of (2.4), passing through the point (t_0, x_0).

Note that (1.3) is an affine equation (2.4) with $a = -q/V$, $b = q/V$. Therefore, if initially V is filled by liquid I, i. e. $x(0) = 0$, we have

$$x(t) = 1 - e^{-qt/V}.$$

Obviously, in the case $a \neq 0$, a decides the stability properties of the stationary solution $\hat{x}$ in the same way as in 2.1

2.3. General autonomous equation

We consider the equation

$$\dot{x} = f(x) \qquad f \in C^1(U, \mathbb{R}),$$

$U \in \mathbb{R}$ open, and we are looking for a solution $x(t)$ passing through (t_0, x_0).

If $f(x_0) = 0$ then such a solution is obviously $x(t) \equiv x_0$. We do not know so far that this solution is unique.

Let now $f(x_0) \neq 0$; without loss of generality suppose $f(x_0) > 0$. Then, $f(x) \neq 0$ for $x \in V$, V a neighborhood of x_0. If $x(t)$ is a solution through (t_0, x_0), then $x(t) \in V$ for $t \in I$, where I is a neighborhood of t_0. Therefore, $\dot{x}(t) = f(x(t)) > 0$

on I and, consequently, $x(t)$ has an inverse function $t = \tau(x)$ on I. For $t \in I$ we have

$$\frac{d\tau}{dx} = \frac{1}{f(x)},$$

hence

$$t - t_0 = \tau(x(t)) - \tau(x(t_0)) = F(x(t)), \tag{2.5}$$

where

$$F(x) = \int_{x_0}^{x} \frac{1}{f(y)} dy.$$

For $x \in V$, F has an inverse; from (2.5) we obtain

$$x(t) = F^{-1}(t - t_0). \tag{2.6}$$

Theorem. *Let $f \in C^1(U, \mathbb{R})$, U an open subset of $\mathbb{R}$. Then, for each $t_0 \in \mathbb{R}$, $x_0 \in U$ there exists a unique solution (i.e. a function $x(t)$ such that $\dot{x}(t) = f(x(t))$) through the point (t_0, x_0) (i.e. such that $x(t_0) = x_0$) on some open interval I containing t_0. One has:*

(i) $x(t) \equiv x_0$ *if* $f(x_0) = 0$

(ii) $\dot{x}(t) > 0$ *for all* $t \in I$ *if* $f(x_0) > 0$

(iii) $\dot{x}(t) < 0$ *for all* $t \in I$ *if* $f(x_0) < 0$.

(iv) If $x_1 \in U$ is such that $f(x_1) = 0$, $f(x) > 0$ for $x_0 \le x < x_1$ then $x(t) \to x_1$ for $t \to \infty$.

In cases (ii) and (iii) (2.6) holds for all t in the interval of existence of the solution.

Remark. The interval on which we consider the solution $x(t)$ is not uniquely defined. Therefore the solution is unique only after we have fixed this interval. However, there is a maximal interval of existence of a solution. As we will see later, it may be bounded even if $U = \mathbb{R}$.

Proof. We postpone the proof of uniqueness to a later chapter. It will be proved in the context of the general existence and uniqueness theorem.

Conclusion (i) follows immediately by integration and uniqueness.

(ii) Assume $f(x_0) > 0$. We prove that $\dot{x}(t) > 0$ as long as $x(t)$ exists. Assume the contrary, i. e., there exists a t_1 such that $\dot{x}(t_1) = 0$. Then, we have

$$f(x(t_1)) = 0.$$

Hence, $x(t) \equiv x(t_1)$ is the unique solution through $(t_1, x(t_1))$. This contradicts $\dot{x}(t_0) > 0$.

Case (iii) can be turned to the Case (ii) by the transformation $t \mapsto -t$.

(iv) By uniqueness of the constant solution x_1 we have

$$x(t) < x_1. \tag{2.7}$$

Since $\dot{x}(t)$ is increasing it has a limit $x^* \le x_1$. Assume $x^* < x_1$. Then $f(x) > 0$ for $x \in [x_0, x^*]$ and, consequently, $f(x) > \delta > 0$ for $x \in [x_0, x^*]$. We have $\dot{x}(t) = f(x(t)) \ge \delta$, hence $x(t) \ge x_0 + \delta(t - t_0) \to \infty$ which contradicts (2.7).

Let now V be the maximal interval containing x_0 such that $f(x) > 0$ for $x \in V$. By continuity, V is open. Define $x(t)$ by (2.6) for $t \in F(V)$. By the formula of the derivative of an inverse function we have

$$\dot{x}(t) = \frac{1}{F'(x(t))} = f(x(t)),$$

hence $x(t)$ is a solution of (2.6) on $I = F(V)$. □

Remark.

(1) The formula (2.6) gives a solution through (t_0, x_0) even if f is merely continuous. However, as we will see later it may not be the only one, if f is not differentiable.

(2) Conclusions analogous to (iv) for $t \to= -\infty$ and/or $x_1 < x_0$ can be obtained from (iv) by the transformation $t \mapsto -t$, $x \mapsto -x$; the details are left to the reader.

Corollary. *Let* $f(\hat{x}) = 0$. *Then,* $\hat{x}$ *is asymptotically stable if* $f'(\hat{x}) < 0$ *and unstable if* $f'(\hat{x}) > 0$.

.

In case $f'(\hat{x}) = 0$, $\hat{x}$ can be asymptotically stable, stable but not asymptotically stable, or unstable. We leave it to the reader to find examples for all three cases.

2.4. Example. Logistic equation

As in example I. 2.2 we assume that the rate of proliferation of a population (with overlapping generations) decreases linearly with population size,

$$\dot{x} = r_0(K - x)x.$$

From Theorem 2.2 it follows that this equation has exactly two equilibria, namely 0 (unstable) and K (stable). Physically interesting nonnegative solutions $x(t)$ are increasing while $0 < x(t) < K$ and decreasing while $x(t) > K$. We can compute the solutions explicitly by the method discussed in 2.3. Indeed if $x(t)$ is a nonconstant solutions passing through (t_0, x_0), we have

$$\int_{t_0}^{t} ds = \frac{1}{r_0} \int_{x_0}^{x(t)} \frac{1}{(K - x)x} dx,$$

hence

$$r_0(t - t_0) = \frac{1}{K} \left(\log \frac{x(t)}{K - x(t)} + \log \frac{K - x_0}{x_0} \right)$$

which is equivalent to

$$x(t) = \frac{K x_0}{x_0 + e^{-K r_0 (t - t_0)}(K - x_0)}. \tag{2.8}$$

From (2.8) it follows that every positive solution tends to K for $t \to \infty$.

2.5. Example. The Solow model of economic growth

Consider an economy with aggregated product. The production function Y (measuring product amount) is a function of the amount of invested capital K and labor L, $Y = f(K, L)$, $K, L > 0$, $f \in C^1$.

Assumptions

1. $\partial f/\partial K, \partial f/\partial L > 0,\ \partial^2 f/\partial K^2, \partial^2 f/\partial L^2 < 0$
2. f is doubly homogeneous,

$$f(\gamma K, \gamma L) = \gamma f(K, L).$$

(find such a function!)

Then, we have

$$f(K, L) = L\varphi(k),$$

where

$$k = K/L,\ \varphi(k) = f(k, 1).$$

We have

$$\varphi'(k) > 0,\ \varphi''(k) < 0. \tag{2.9}$$

Note that, because of (2.9), φ' is bounded below and decreasing, hence converges for $k \to \infty$.

Solow's assumptions are that a constant part $0 < s < 1$ of the capital is invested and that labor proliferates by the Malthus law. Mathematically, this can be expressed by

$$\dot{K} = sY,\ \dot{L} = \lambda L,\ \lambda > 0.$$

Consequently,

$$\dot{k} = (K/L)^{\cdot} = \dot{K}/L - KL^{\cdot}/L^2 = sY/L - \lambda K/L = g(k), \tag{2.10}$$

where

$$g(k) = s\varphi(k) - \lambda k.$$

Assume $\varphi(0) = 0,\ s\varphi'(0) > \lambda,\ s\varphi'(\infty) < \lambda$. Then we have $g(0) = 0$ and

$$g'(0) = s\varphi'(0) - \lambda > 0, \tag{2.11}$$

hence $g(k) > 0$ for $k > 0$ sufficiently small.

On the other hand, there is a k^* such that for $k > k^*$ one has $s\varphi'(k) \le \lambda - \delta$ for some $\delta > 0$, hence

$$g'(k) = s\varphi'(k) - \lambda \le -\delta < 0. \tag{2.12}$$

Integrating (2.12) we obtain

$$g(k) \le g(k^*) - \delta(k - k^*) \tag{2.13}$$

for $k > k^*$ which implies $g(k) \to -\infty$ for $k \to \infty$.

From (2.10) it follows that there is a $\hat{k} > 0$ such that $g(\hat{k}) = 0$.

We have

$$g''(k) = s\varphi''(k) < 0,$$

hence g' is decreasing. From (2.11) and (2.13) it follows that $g(k)$ has a positive maximum at a point $k_1 > 0$. We have $g'(k_1) = 0$ and, since g' is decreasing, $g'(k) < 0$ for $k > k_1$. Obviously $\hat{k} > k_1$, $\hat{k}$ is unique and

$$g'(\hat{k}) < 0, g(k) > 0 \text{ for } 0 < k < \hat{k} \text{ and } g(k) < 0 \text{ for } k > \hat{k}. \tag{2.14}$$

Hence (2.10) has two equilibria 0 and $\hat{k}$, 0 being unstable by (2.11) and $\hat{k}$ being asymptotically stable. In fact, by Theorem 2.3 (iv) and 2.3, Remark 2, $\hat{k}$ is globally asymptotically stable in $(0, \infty)$, i.e. every solution $k(t)$ with $k(0) > 0$ satisfies $k(t) \to \hat{k}$ for $t \to \infty$.

2.6. Equations with separable variables

Those are differential equation of the form

$$\frac{dx}{dt} = g(t)f(x) \tag{2.15}$$

with g, f continuous $I \to \mathbb{R}$, $J \to \mathbb{R}$ respectively, I, J open subsets of $\mathbb{R}$.

Let $t_0 \in I$, $x_0 \in J$. If $f(x_0) = 0$ then a solution through (t_0, x_0) is $x(t) \equiv x_0$.

Let now $f(x_0) \neq 0$; without loss of generality assume $f(x_0) > 0$. Then, $f(x) > 0$ on some open interval $V \ni x_0$. If $x(t) \in V$ for $t \in U$, from (2.15) we obtain

$$\int_{x_0}^{x(t)} \frac{1}{f(x)} dx = \int_{t_0}^{t} \frac{1}{f(x(s))} \frac{dx}{ds} ds = \int_{t_0}^{t} g(s) ds \tag{2.16}$$

Integrating (2.16) we obtain

$$x(t) = F^{-1} \left(\int_{t_0}^{t} g(s) ds \right). \tag{2.17}$$

for $t \in U$, where F is as in 2.3. Differentiating (2.17) one can readily verify that it provides a solution of (2.15) through (t_0, x_0). Again, with f merely continuous it may not be the only such one.

A precise analysis of solutions in the spirit of Theorem 2.3 is more complicated and will not be given here.

3. Higher dimensional linear equations

3.1 Linear autonomous (homogeneous) equation

We consider the equation

$$\dot{x} = Ax, \tag{3.1}$$

where $x \in \mathbb{R}^n$, $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ - i.e. A is an $n \times n$- matrix. In components (3.1) writes as

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + \cdots + a_{1n}x_n \\ &\dots\dots\dots\dots \\ \dot{x}_n &= a_{n1}x_1 + \cdots + a_{nn}x_n \end{aligned} \tag{3.2}$$

For this reason we call the equation also a system of linear autonomous equations.

We assume (as usual) that the eigenvalues of A are mutually distinct. We proceed as in (I.3.2) and by a suitable linear transformation decompose (3.1) into a system of mutually independent one-dimensional linear equation which we know to solve:

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues and $v_1, \ldots, v_n$ the corresponding eigenvectors. Denote

$$\begin{aligned} V &= (v_1, \ldots, v_n) \\ \Lambda &= \text{ diag } \{\lambda_1, \ldots, \lambda_n\}. \end{aligned}$$

The transformation $x = V\tilde{x}$ brings (3.1) to the equation

$$\dot{\tilde{x}} = V^{-1}AV\tilde{x} = \Lambda\tilde{x}.$$

If $x(t_0)$ is a solution of (3.1) satisfying the condition $x(t_0) = x^0$ then $\tilde{x}(t) = V^{-1}x(t)$ satisfies the condition $\tilde{x}(t_0) = \tilde{x}^0 = V^{-1}x^0$ and its components $\tilde{x}_1(t), \ldots, \tilde{x}_n(t)$ satisfy the equations

$$\dot{\tilde{x}}_j = \lambda_j \tilde{x}_j, \qquad j = 1, \ldots, n. \tag{3.3}$$

For λ_j real we know by 2.1, 2.3 that the unique solution of the j- th equation of (3.3) satisfying the condition $\tilde{x}_j(t_0) = \tilde{x}_j^0$ is

$$\tilde{x}_j(t) = e^{\lambda_j(t-t_0)}\tilde{x}_j^0. \tag{3.4}$$

To deal with imaginary λ_j we introduce the *Euler formula*:

$$e^{i\gamma} = \cos\gamma + i\sin\gamma \text{ for } \gamma \in \mathbb{R}.$$

Note that $\gamma \mapsto e^{i\gamma}$ maps the real line to the unit circle in the complex plane, points 2π apart being mapped to the same point.

Adopting Euler's formula we readily check that (3.4) remains valid for complex λ_j as well. Indeed, if $\lambda_j = a + i\beta$ we have

$$\begin{aligned} \frac{d}{dt}\left(e^{\lambda_j(t-t_0)}\right) &= \frac{d}{dt}\left[e^{\alpha(t-t_0)}(\cos\beta(t-t_0) + i\sin\beta(t-t_0))\right] \\ &= \alpha e^{\alpha(t-t_0)}(\cos\beta(t-t_0) + i\sin\beta(t-t_0)) \\ &\quad + e^{\alpha(t-t_0)}(-\beta\sin\beta(t-t_0) + i\beta\cos\beta(t-t_0)) \\ &= e^{\alpha(t-t_0)}\left[\alpha\cos\beta(t-t_0) - \beta\sin\beta(t-t_0)\right. \\ &\quad \left. + i(\alpha\sin\beta(t-t_0) + \beta\cos\beta(t-t_0))\right] = \\ &= (\alpha + i\beta)e^{\alpha(t-t_0)}\left[\cos\beta(t-t_0) + i\sin\beta(t-t_0)\right] \\ &= \lambda_j e^{\lambda_j(t-t_0)}. \end{aligned}$$

Thus we have

Theorem. *Assume that the spectrum of A consists of mutually distinct eigenvalues. Then, the unique solution $x(t)$ of the equation (3.1) satisfying the condition $x(t_0) = x^0$ is*

$$x(t) = V \; diag \left(e^{\lambda_1(t-t_0)}, \dots e^{\lambda_n(t-t_0)} \right) V^{-1} x^0, \tag{3.5}$$

where $V = (v_1, \dots, v_n)$, λ_j are the eigenvalues of A and v_j are their respective eigenvectors.

Remark.

1. It is possible to compute the solution of (3.1) in a closed form also in the case of A having multiple eigenvalues.
2. In practice, to compute a particular solution with a given initial value, one proceeds as follows: From (3.5) it follows that

$$x(t) = c_1 v_1 e^{\lambda_1(t-t_0)} + \dots + c_n v_n e^{\lambda_n(t-t_0)} \tag{3.6}$$

$c = (c_1, \dots, c_n)^T$, $c = V^{-1} x_0$. That is, $(c_1, \dots, c_n)$ is the solution of the linear system of independent linear equations

$$Vc = x_0 \tag{3.7}$$

or, in components,

$$\begin{aligned} v_{11} c_1 + \dots + v_{1n} c_n &= x_{10} \\ v_{n1} c_1 + \dots + v_{nn} c_n &= x_{n0} \end{aligned} \tag{3.8}$$

with $v_j = (v_{1j}, \dots, v_{nj})$, $x_0 = (x_{10}, \dots, x_{n0})$. In this way, to compute a particular solution one can reduce the computation of the inverse matrix V^{-1} to the solution of the system of equation (3.8) (the computation of V^{-1} requires the solution of n such systems).

3. Computing real solutions one can entirely avoid dealing with complex numbers. Indeed, if λ is an imaginary eigenvalue, with eigenvector v then $\bar{\lambda}$ is an eigenvalue as well with eigenvector $\bar{v}$. Therefore, the sum (3.6) contains terms

$$cve^{\lambda(t-t_0)} + d\bar{v}e^{\bar{\lambda}(t-t_0)}.$$

Since we look for real solutions we have $x(t) = \bar{x}(t)$; as in I.3.2 we obtain $d = \bar{c}$. If we denote $\lambda = \alpha + i\beta$, $v = z + iw$, $c = a + ib$, for $d = \bar{c}$ we have

$$cve^{\lambda(t-t_0)} + \bar{c}\bar{v}e^{\bar{\lambda}(t-t_0)} = 2Re(cve^{\lambda(t-t_0)}) =$$
$$2e^{\alpha(t-t_0)} \left\{ a \left[z \cos\beta(t-t_0) - w \sin\beta(t-t_0) \right] - b \left[w \cos\beta(t-t_0) + z \sin\beta(t-t_0) \right] \right\}.$$

Rearranging all the terms of (3.6) with imaginary λ_j we can completely avoid dealing with complex numbers. Let us note that for fixed λ complex the two terms in equation (3.8) containing a, b turn out very simple, namely

$$az - bw.$$

Example. Compute the solution of LDE (3.1) with the matrix

$$A = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$$

following $x_1(0) = 1$, $x_2(0) = 0$.

The eigenvalues of A are $\lambda = -1 + i$, $\overline{\lambda} = -1 - i$ and $v = (1, -i)$, $\overline{v} = (1, i)$ are the corresponding eigenvectors.

We will search for the solution in the form $x(t) = 2Re[c \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(-1+i)t}]$ where $c = \gamma + i\delta$ is a complex constant to be computed from the initial conditions. Carrying out the multiplication in the right-hand side we obtain

$$x(t) = 2Re[(\gamma + i\delta) \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-t}(\cos t + i \sin t)] = 2e^{-t} \begin{pmatrix} \gamma \cos t & -\delta \sin t \\ -\delta \cos t & -\gamma \sin t \end{pmatrix},$$

from the conditions for $x(0)$ we obtain the equations for γ, δ: $2\gamma = 1$, $-2\delta = 0$
So, the solution is $x_1(t) = e^{-t} \cos t$, $x_2(t) = e^{-t} \sin t$

3.2 Stability of the zero solution of a LDE

Every linear equation has at least one stationary (time constant) solution. The stability concepts are defined literally as in the time discrete case of Chapter I. The only difference is that, in the definition, t can take all nonnegative values rather than nonnegative integers.

Theorem. *The zero solution of (3.1) is*
(i) asymptotically stable, if the real parts of all eigenvalues of A are negative,
(ii) stable, if the real parts of all eigenvalues of A are non-positive and the eigenvalues with vanishing real parts are simple,
(iii) unstable if some eigenvalue has a positive real part.

Proof. As in the case of Theorem I.3.7, we prove conclusions (i) and (iii) for A having distinct eigenvalues,

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A and let $v_1, \ldots, v_n$ be respectively their eigenvectors. If $\bar{\lambda} = \max\{\Re\lambda_j : j = 1, \ldots, n\} < 0$, we have

$$|e^{\lambda_j t}| = e^{\Re\lambda_j t} |\cos \Im\lambda_j t + i \sin \Im\lambda_j t| \leq e^{\bar{\lambda} t}.$$

By (3.5), we have

$$|x(t)| \leq |V| e^{\bar{\lambda} t} |V^{-1}| |x(0)| \to 0 \text{ for } t \to \infty$$

and

$$|x(t)| \leq \varepsilon \text{ provided } |x(0)| \leq |V|^{-1} |V^{-1}|^{-1} \varepsilon.$$

This proves (i).

To prove (iii) let j be such that $\Re\lambda_j > 0$. Choose $x(0) = \delta v_j$. Then, $V^{-1}x(0) = e_j$ (the j-th unit coordinate vector) and, by (3.5),

$$\begin{aligned} |x(t)| &= \delta |V e_j e^{\lambda_j t}| = \delta |v_j| |e^{\lambda_j t}| \\ &= \delta |v_j| e^{\Re\lambda_j t} (\cos \Im\lambda_j t + \sin \Im\lambda_j t). \end{aligned}$$

No matter how small δ is, for $t = 2\pi/\Im\lambda_j$ we have

$$|x(t)| \geq \delta|v_j|e^{\Re\lambda_j t} \to \infty \text{ for } t \to \infty$$

which proves (iii).

Remark. Why real part for DE while absolute value for discrete DS? Consider a DE

$$\dot{x} = \lambda x.$$

If we approximate the derivative by finite difference, we obtain

$$\frac{x(t + \Delta t) - x(t)}{\Delta t} = \lambda x(t);$$

i.e. the sequence $x(t)$ is generated by the recurrent relation

$$x(t + \Delta t) = (1 + \lambda\Delta t)x(t).$$

This gives

$$x(1) = (1 + \lambda\Delta t)^{\frac{1}{\Delta t}} x(0) \approx e^{\lambda} x(0).$$

Note that $|e^{\lambda}| < 1$ precisely if $Re\lambda < 0$.

4.STRUCTURE OF SOLUTIONS OF A LINEAR DIFFERENTIAL EQUATION

4.1 The space of solutions

Consider the differential equation

$$\dot{x} = A(t)x \tag{4.1}$$

for $x \in \mathbb{R}^n$, $A \in C(\mathbb{R}, L(\mathbb{R}^n, \mathbb{R}^n))$ (i.e. A is an $n \times n$-matrix continuously depending on t). For the moment we *assume* that for each $t_0 \in \mathbb{R}, x^0 \in \mathbb{R}^n$ there is a unique solution through (t_0, x^0) and this solution is global (i.e. it exists on all $\mathbb{R}$). We know already that this assumption holds true in case A is constant; we show in the next section that its validity extends to all linear equations.

We prove the following

Theorem. *The solutions of (4.1) form an n-dimensional linear space and the map $x(\cdot) \mapsto x(t)$ ($x(\cdot)$ a solution of (4.1)) is a linear isomorphism for each t.*

In a more classical language Theorem 4.1 claims that a linear combination of solutions is a solution again and that there are precisely n linearly independent solutions. Moreover, solutions are linearly independent if and only if so are their values at any $t_0 \in \mathbb{R}$.

Proof. Take $t_0 \in \mathbb{R}$, $x^0 \in \mathbb{R}^n$. Consider the map $x^0 \mapsto x(\cdot, x^0)$, where $x(\cdot, x^0)$ is the solution of (4.1) through (t_0, x_0). This map is obviously linear (by uniqueness), surjective (by existence) an injective (by uniqueness). This proves the Theorem (why?) □

4.2 Fundamental matrix

If $\varphi_1(t), \ldots, \varphi_m(t)$ are solutions of (4.1), i.e. if

$$\dot{\varphi}_i(t) = A(t)\varphi_i(t), \qquad i = 1, \ldots, m \tag{4.2}$$

for all $t \in \mathbb{R}$, we can equivalently write

$$\dot{\Phi}(t) = A(t)\Phi(t), \tag{4.3}$$

where $\Phi(t)$ is the matrix with columns $\varphi_i(t)$ (we understand $\dot{\Phi}$ as the matrix with $\dot{\varphi}_i$ as columns). If $m = n$ and $\varphi_1(t), \ldots \varphi_n(t)$ are linearly independent then $\Phi(t)$ is called *fundamental matrix of solutions* (henceforth abbreviated as FM). In a more classical terminology, $\varphi_1(t), \ldots, \varphi_n(t)$ are called *fundamental system of solutions.* One has

Theorem.

(i) $\Phi(t)$ is a fundamental matrix if and only it solves (4.3) and is regular for each t.
(ii) If $\Phi(t)$ is a FM then every solution $x(t)$ can be written as $x(t) = \Phi(t)c$ for a suitable constant vector c.
(iii) If $\Phi(t)$ is FM and C is regular then $\Phi(t)C$ is a FM.
(iv) If $\Phi(t), \bar{\Phi}(t)$ are FM then there exists a regular matrix C such that $\bar{\Phi}(t) = \Phi(t)C$.

The proof is immediate and is left as an exercise. □

Of particular importance is the FM $\Phi(t, \tau)$ (called *transfer matrix*) defined by the normalization condition

$$\Phi(\tau, \tau) = I.$$

By Theorem (iii), $\Phi(t, \tau)$ can be obtained from any FM $\Phi(t)$ by

$$\Phi(t, \tau) = \Phi(t)\Phi^{-1}(\tau)$$

By uniqueness of solutions, we have obviously

$$\Phi(\tau, t) = [\Phi(t, \tau)]^{-1}$$

and

$$\Phi(t, s)\Phi(s, \tau) = \Phi(t, \tau) \tag{4.4}$$

(prove as exercise).

4.3. Autonomous equation

We show that for the autonomous equation we have

$$\Phi(t + s, \tau + s) = \Phi(t, \tau) \qquad \forall s \in \mathbb{R} \tag{4.5}$$

Indeed, let $\Psi(t) = \Phi(t + s, \tau + s)$. We have

$$\dot{\Psi}(t) = \dot{\Phi}(t + s, \tau + s) = A\Phi(t + s, \tau + s) = A\Psi(t)$$

and

$$\Psi(\tau) = \Phi(\tau + s, \tau + s) = I$$

which proves (4.5).

The equality (4.5) means that $\Phi(t, \tau)$ is a function of $t - \tau$ only; we write

$$\Phi(t, \tau) = e^{A(t-\tau)}.$$

Then, e^{At} indeed has properties of the exponential: one has $\frac{d}{dt}e^{At} = Ae^{At}$, $e^{A0} = I$, and, by (4.4),

$$e^{A(t+s)} = e^{At}e^{As} \qquad \forall s, t \in \mathbb{R}.$$

4.4. The linear nonhomogeneous and the affine equation

We consider the equation

$$\dot{x} = A(t)x + b(t) \tag{4.6}$$

with A as in 4.1, $b \in C(\mathbb{R}, \mathbb{R}^n)$. We have

Theorem. *The solution $x(t)$ of (4.6) through the point (t_0, x^0) is given by the variation of constants formula*

$$x(t) = \Phi(t, t_0)x^0 + \int_{t_0}^{t} \Phi(t, s)b(s)ds \tag{4.7}$$

Proof. Although the formula can be proved simply by verifying the properties of $x(t)$ given by (4.6), we prefer to derive it in a similar way as in the one-dimensional case.

Let $x(t)$ be the solution of (4.6) through (t_0, x^0). Multiplying (4.6) by $\Phi(t_0, t)$ we obtain

$$\Phi(t_0, t)\dot{x}(t) - \Phi(t_0, t)A(t)x(t) = \Phi(t_0, t)b(t) \tag{4.8}$$

Differentiating

$$\Phi(t_0, t)\Phi(t, t_0) = I$$

with respect to t we obtain

$$\Phi(t_0, t)A(t)\Phi(t, t_0) + \frac{d}{dt}\Phi(t_0, t)\Phi(t, t_0) = 0,$$

or, multiplying by $\Phi(t_0, t)$ from the right,

$$-\frac{d}{dt}\Phi(t_0, t) = \Phi(t_0, t)A(t) \tag{4.9}$$

Substituting (4.9) into (4.8) we obtain

$$\frac{d}{dt}(\Phi(t_0, t)x(t)) = \Phi(t_0, t)b(t).$$

Integrating from t_0 to t we obtain

$$\Phi(t_0, t)x(t) - x^0 = \int_{t_0}^{t} \Phi(t_0, s)b(s)ds \tag{4.10}$$

The formula (4.7) is obtained by multiplying (4.10) by $\Phi(t, t_0)$ from the left. $\square$

Remarks. 1. Since $\Phi(t, s) = \Phi(t)\Phi^{-1}(s)$ for any FM Φ, we can write (4.7) as

$$x(t) = \Phi(t)\left[\Phi^{-1}(t_0)x^0 + \int_{t_0}^{t} \Phi^{-1}(s)b(s)ds\right] = \Phi(t)c(t) \tag{4.11}$$

with

$$c(t) = \left[\Phi^{-1}(t_0)x^0 + \int_{t_0}^{t} \Phi^{-1}(s)b(s)ds\right].$$

The formula (4.11) is an explanation of the name *variation of constants.*

2. Formula (4.7) also shows that every solution of (4.7) is obtained as a sum of one particular solution of the nonhomogeneous equation plus an appropriate solution of the homogeneous one. This is an obvious consequence of the fact that the difference of two solutions of a linear nonhomogeneous equation is a solution of the corresponding linear homogeneous equation. Indeed, if $x(t), \tilde{x}(t)$ are two solutions of (4.6) then $y(t) = x(t) - \tilde{x}(t)$ satisfies

$$\dot{y}(t) = \dot{x}(t) - \dot{\tilde{x}}(t) = A(t)(x(t) - \tilde{x}(t)) = A(t)y(t).$$

3. Formula (4.7) (or, (4.10)) enables us to compute the solutions of a linear nonhomogeneous equation in case we know the solution of the corresponding homogeneous equation. The latter is the case if e. g. A is constant. Then, (4.7) reads

$$\begin{aligned} x(t) &= e^{A(t-t_0)}x^0 + \int_{t_0}^{t} e^{A(t-s)}b(s)ds \\ &= e^{A(t-t_0)}\left(x^0 + \int_{t_0}^{t} e^{A(t_0-s)}b(s)ds\right). \end{aligned}$$

4. In general, the most effective procedure to compute a solution by the variation of constants method is as follows:

Take *any* fundamental matrix $\Phi(t)$. From Remark 1 we know that $\Phi(t)c(t)$ is a solution of the nonhomogeneous equation for a suitable $c(t)$. To find $c(t)$, we substitute into the differential equation (4.6) to obtain

$$\frac{d}{dt}[\Phi(t)c(t)] = A(t)\Phi(t)c(t) + b(t).$$

Differentiating the left-hand side we obtain

$$A(t)\Phi(t)c(t) + \Phi(t)\dot{c}(t) = A(t)\Phi(t)c(t) + b(t),$$

hence

$$\Phi(t)\dot{c}(t) = b(t).$$

If we denote $w(t)$ the solution of this t-dependent system of linear algebraic equations and integrate, we obtain

$$c(t) = \int w(t)dt,$$

where $\int w(t)$ is *any* primitive of $w(t)$. An arbitrary solution of the nonhomogeneous equation is then given by the formula

$$x(t) = \Phi(t)\left(c + \int w(t)dt\right)$$

with a suitable c which can, e. g., be determined by initial conditions.

Example. Find the solution of equations

$$\dot{x}_1 = x_1 - 2x_2 + e^t$$
$$\dot{x}_2 = 3x_1 - 4x_2 + 1,$$

satisfying $x_1(0) = 1$,$x_2(0) = 2$.

The eigenvalues of the matrix of the homogeneous equation are $\lambda_1 = -1$, $\lambda_2 = -2$ and $v_1 = (1, 1)$, $v_2 = (2, 3)$ are the corresponding eigenvectors.

The matrix

$$\Phi(t) = \begin{pmatrix} e^{-t} & 2e^{-2t} \\ e^{-t} & 3e^{-2t} \end{pmatrix}$$

is the fundamental matrix of the homogeneous equation.

We can find the solution of the nonhomogeneous equation as $x(t) = \Phi(t)c(t)$, where $c(t)$ is a solution of the equation

$$\Phi(t)\dot{c}(t) = \begin{pmatrix} e^t \\ 1 \end{pmatrix},$$

i. e.

$$e^{-t}\dot{c}_1(t) + 2e^{-2t}\dot{c}_2(t) = e^t$$
$$e^{-t}\dot{c}_1(t) + 3e^{-2t}\dot{c}_2(t) = 1$$

Solving this system of linear equations with unknowns $\dot{c_1}(t)$, $\dot{c_2}(t)$ we obtain the formulas

$$\dot{c}_1(t) = -2e^t + 3e^{2t}$$
$$\dot{c}_2(t) = -e^{3t} + e^{2t},$$

integrating them with respect to t we obtain

$$c_1(t) = -2e^t + \frac{3}{2}e^{2t} + d_1$$
$$c_2(t) = -\frac{1}{3}e^{3t} + \frac{1}{2}e^{2t} + d_2$$

where d_1, d_2 are integration constants to be still determined. Substituting for $c(t)$ and $\Phi(t)$ we have formulas for $x(t)$:

$$x_1(t) = (-2e^{-t}+\frac{3}{2}e^{2t}+d_1)e^{-t}+2(-\frac{1}{3}e^{3t}+\frac{1}{2}e^{2t}+d_2)e^{-2t} = d_1e^{-t}+2d_2e^{-2t}-1+\frac{5}{6}e^t$$

$$x_2(t) = (-2e^{-t}+\frac{3}{2}e^{2t}+d_1)e^{-t}+3(-\frac{1}{3}e^{3t}+\frac{1}{2}e^{2t}+d_2)e^{-2t} = d_1e^{-t}+3d_2e^{-2t}-\frac{1}{2}+\frac{1}{2}e^t$$

Conditions for $x(0)$ give equations for the integration constants d_1, d_2:

$$d_1 + 2d_2 - 1 + \frac{5}{6} = 1$$
$$d_1 + 3d_2 - \frac{1}{2} + \frac{1}{2} = 2$$

the solutions of which are $d_1 = -\frac{1}{2}$, $d_2 = \frac{5}{6}$

In conclusion, the solution is:

$$x_1(t) = \frac{5}{6}e^t - 1 - \frac{1}{2}e^{-t} - \frac{10}{6}e^{-2t}$$
$$x_2(t) = \frac{1}{2}e^t - 1 + \frac{1}{2}e^{-t} + \frac{5}{3}e^{-2t}.$$

In the special case $A(t) \equiv \text{const} = A$, $b(t) \equiv \text{const} = b$ we shall call (4.6) an *affine equation*. If A is regular then (4.6) has a unique stationary solution

$$x(t) \equiv \hat{x} = -A^{-1}b.$$

By Remark 2, the deviation $y(t) = x(t) - \hat{x}$ from $\hat{x}$ satisfies the linear homogeneous equation $\dot{y} = Ay$. Consequently, stability properties of the stationary solution $\hat{x}$ are determined by the real parts of the eigenvalues of the matrix A according to Theorem 2.3.

4.5 Higher order linear equations

We consider the equation

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_0(t)y = 0, \tag{4.12}$$

(homogeneous), or

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_0(t)y = h(t) \tag{4.13}$$

(nonhomogeneous), where $y \in \mathbb{R}$, $a_0, \ldots, a_{n-1}, h \in C(\mathbb{R}, \mathbb{R})$. We say that the function $y(t)$ is a solution of 4.12 resp. 4.13 on the interval I if we have

$$y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_0(t)y(t) = 0$$

resp.

$$y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_0(t)y(t) = h(t)$$

for all $t \in I$.

By introducing the variables $x_1 = y,\ x_2 = \dot{y}, \ldots, x_n = y^{(n-1)}$ we transform the equation (4.12) resp. (4.13) into the equation

$$\dot{x} = A(t)x \tag{4.14}$$

resp.

$$\dot{x} = A(t)x + b(t) \tag{4.15}$$

in $\mathbb{R}^n$, where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$A(t) = \begin{pmatrix} 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 \\ -a_0(t) & -a_1(t) & \ldots & -a_{n-1}(t) \end{pmatrix}, \quad b(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ h(t) \end{pmatrix}.$$

If $A(t) \equiv A$, $b(t) \equiv b$ constant, we will call (4.15) *affine*. Note that $y(t) = x_1(t)$. It follows from Sections 4.1, 4.4 that:

- the solutions of (4.12) form an n-dimensional linear space,
- for every $t_0, y^0 = (y_0^0, \dots, y_{n-1}^0)$ there is a unique solution of (4.12) or (4.13) such that
$$y(t_0) = y_0^0,\ \dot{y}(t_0) = y_1^0, \dots, y^{(n-1)}(t_0) = y_{n-1}^0.$$
- the unique solution of (4.13) through (t_0, y^0) is given by
$$y(t) = \sum_{i=1}^{n} \varphi_{1i}(t, t_0) y_{i-1}^0 + \int_{t_0}^{t} \varphi_{1n}(t, s) h(s) ds \tag{4.16}$$
where $\varphi_{ij}(t, s)$ is the ij-th entry of $\Phi(t, s)$,
- since the rows of the fundamental matrix of the system (4.14) are derivatives of its first row, in case the coefficient $a_i(t)$ are constant and the roots of the characteristic equation $\lambda_1, \dots \lambda_n$ are mutually distinct, a fundamental matrix $\Phi(t)$ of solutions of the corresponding system (4.14) can be easily obtained by
$$\Phi(t) = \begin{pmatrix} e^{\lambda_1 t} & \dots & e^{\lambda_n t} \\ \lambda_1 e^{\lambda_1 t} & \dots & \lambda_n e^{\lambda_n t} \\ \dots & \dots & \dots \\ \lambda_1^{n-1} e^{\lambda_1 t} & \dots & \lambda_n^{n-1} e^{\lambda_n t} \end{pmatrix}.$$
- since A is a companion matrix, in case the coefficients a_i are constant, its characteristic polynomial is $\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0$.
- it can be readily checked that if λ is a root of the characteristic polynomial $\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0$ of multiplicity k then the equation (4.12) with constant coefficients $a_1, \dots, a_n$ has solutions $t^j e^{\lambda t}, j = 0, \dots, k-1$; it can be proved that if we let λ run through all roots these solutions constitute a fundamental system.

Remark. In some particular cases, e.g. in the case of an autonomous equation with simple eigenvalues $\lambda_1, \dots, \lambda_n$ of its characteristic polynomial and $h(t) = e^{\beta t}$ $\beta \neq \lambda_i$ an exponential, the formula allows us to compute the solution in a shortened procedure. Indeed, in such a case $\varphi_{1i}(t, s) = \varphi_{1i}(t - s) = \sum_{j=1}^{n} d_{ij} e^{\lambda_j (t-s)}$, hence

$$y(t) = \sum_{i,j=1}^{n} d_{ij} e^{\lambda_j (t - t_0)} y_{i-1}^0 + \sum_{j=1}^{n} d_{nj} e^{\lambda_j t} \int_{t_0}^{t} e^{(-\lambda_j + \beta)s} ds,$$
$$\int_{t_0}^{t} e^{(-\lambda_j + \beta)s} ds = \frac{1}{\beta - \lambda_j} \left[e^{(\beta - \lambda_j)t} - e^{(\beta - \lambda_j)t_0} \right]$$

Therefore,
$$y(t) = \sum_{j=1}^{n} c_j e^{\lambda_j t} + c e^{\beta t}, \tag{4.17}$$

where
$$c_j = e^{-\lambda_j t_0} \left(\sum_{i=1}^{n} d_{ij} y_{i-1}^0 - \frac{d_{nj}}{\beta - \lambda_j} e^{\beta t_0} \right)$$
$$c = \sum_{j=1}^{n} \frac{d_{nj}}{\beta - \lambda_j}$$

The solution can be computed directly by substituting (4.17) into the equation and the initial conditions. We show this in the case $n = 2$.

We rewrite (4.17) as

$$y(t) = y_H(t) + y_{NH}(t)$$

where

$$\begin{aligned} y_H(t) &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ y_{NH}(t) &= c e^{\beta t}. \end{aligned}$$

Since $y_H(t)$ is a solution of the homogeneous equation

$$\ddot{y}_H + a_1 \dot{y}_H + a_0 y_H = 0,$$

we have

$$\begin{aligned} \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) &= \ddot{y}_{NH}(t) + a_1 \dot{y}_{NH}(t) + a_0 y_{NH}(t) \\ &= \beta^2 c e^{\beta t} + a_1 \beta c e^{\beta t} + a_0 c e^{\beta t}. \end{aligned}$$

In order that $y(t)$ is a solution of the equation

$$\ddot{y} + a_1 \dot{y} + a_0 y = e^{\beta t}$$

we therefore must have

$$(\beta^2 + a_1 \beta + a_0) c e^{\beta t} = e^{\beta t}$$

which gives

$$c = \frac{1}{\beta^2 + a_1 \beta + a_0}.$$

In a similar way the solution can be computed also in other cases: e.g., if $\beta = \lambda_j$ for some j, one takes $y_{NH}(t) = c t e^{\beta t}$, if $h(t) = a \cos \beta t + b \sin \beta t$, one takes $y_{NH}(t) = c \cos \beta t + d \sin \beta t$, etc.

The general rule is that if the roots of the characteristic polynomial are mutually distinct and $h(t) = t^\beta e^{\alpha t}$ with β a positive integer then $y_{NH}(t)$ can be sought in the form $p(t) e^{\alpha t}$ where $p(t)$ is a polynomial of degree β if α is not equal to any of the roots of the characteristic polynomial and of degree $\beta + 1$ otherwise. This rule extends also to the case of α complex which means that if $\alpha = \mu + i\nu$ (or, $h(t) = t^\beta [a \cos \nu t + b \sin \nu t]$) then $y_{NH}(t)$ can be sought in the form $e^{\mu t}[p_1(t) \cos \nu t + p_2(t) \sin \nu t]$, the degrees of the polynomial being determined in the same way as the degree of p in the case of α real.

Note that if $h(t) = a h_1(t) + b h_2(t)$ and $y_i(t)$ are solutions of the equations

$$y^{(n)} + a_{n-1}(t) y^{(n-1)} + \cdots + a_0(t) y = h_i(t),$$

$i = 1, 2$ then $a y_1(t) + b y_2(t)$ is a solution of (4.13). This linearity property allows us to obtain solutions of nonhomogeneous equations with right hand sides being linear combinations of the above special ones.

Example. Find the solution $y(t)$ of the equation:

$$\ddot{y} - 3\dot{y} + 2y = e^{-t}$$

satisfying $y(0) = 0$, $\dot{y}(0) = 0$.

The characteristic polynomial $\lambda^2 - 3\lambda + 2$ has roots 1,2, both different from the exponent of the right-hand side of the equation. Hence, we can find the solution in the form: $y(t) = c_1 e^t + c_2 e^{2t} + de^{-t}$

If we differentiate this formula and substitute its derivatives for $y(t)$, $\dot{y}(t)$, $\ddot{y}(t)$ into the equation, terms with e^t and e^{2t} will cancel out and we obtain an equation for d:

$$d(1 + 3 + 2) = 1 \qquad \text{which gives } d = \frac{1}{6}.$$

Substituting into the conditions for $y(0)$, $\dot{y}(0)$ we obtain equations for c_1, c_2:

$$c_1 + c_2 + \frac{1}{6} = 0,\ c_1 + 2c_2 - \frac{1}{6} = 0,$$

which gives $c_1 = -\frac{1}{3}$, $c_2 = \frac{1}{6}$. Hence, we obtain the solution

$$y(t) = \frac{1}{6}e^t - \frac{1}{3}e^{2t} + \frac{1}{6}e^{-t}$$

In case we replace the right-hand side e^{-t} by the function $\cos t$ the exponential expression of which is $\cos t = \frac{e^{it}+e^{-it}}{2}$. Because i is equal to none of the roots 1 or 2, we can find the solution in the same way as in the case e^t, in the form $y(t) = c_1 e^t + c_2 e^{2t} + d_1 \cos t + d_2 \sin t$ where d_1, d_2 can be determined by substituting this formula and its derivatives into the differential equation and c_1, c_2 can be computed from the conditions for $y(0)$ and $\dot{y}(0)$.

If in the nonhomogenous equation was, e. g., ae^t, the exponent would be equal to one of the roots of the characteristic polynomial. Hence, the solution would be given by the formula: $y(t) = c_1 e^t + c_2 e^{2t} + dte^t$

Example. Find the solution $y(t)$ of the equation:

$$\ddot{y} - 2\dot{y} + 5y = e^t \sin 2t$$

satisfying $y(1) = 0$, $\dot{y}(1) = 1$.

The roots of the characteristic polynomial $\lambda^2 - 2\lambda + 5$ are $1 \pm 2i$. Because $e^t \sin 2t = e^t \frac{e^{2it}+e^{-2it}}{2i} = \frac{e^{(1+2i)t}+e^{(1-2i)t}}{2i}$ and the coefficients $1 \pm 2i$ are equal to the roots of the characteristic polynomial, we consider the solution in the form:

$$y(t) = Re[(c + dt)e^{(1+2i)(t-1)}],$$

where c, d are complex constants, or directly in the real version

$$y(t) = (c_1 + d_1 t)e^{t-1} \cos 2(t-1) + (c_2 + d_2 t)e^{t-1} \sin 2(t-1).$$

Equivalently, we can replace the argument $t - 1$ by t. However, since the initial conditions are prescribed in point the $t = 1$, computing with an argument $t - 1$ is simpler.

4.6 The harmonic oscillator

Assume that we have a point of mass m attached to a spring the other end of which is fixed. If we move the point from its equilibrium position, the *restoring* force of the spring will push it back. For small deviations (to be denoted by y) the restoring force F can be assumed to depend linearly on y and point to the direction opposite to y, i.e.

$$F(y) = -ky$$

for some $k > 0$. Applying the second Newton's law we obtain for y the differential equation

$$\ddot{y} + \omega_0^2 y = 0, \tag{4.18}$$

where $\omega_0 = (k/m)^{1/2}$.

The characteristic equation

$$\lambda^2 + \omega_0^2 = 0$$

has two conjugate imaginary roots $\pm i\omega_0$. By **4.5** the two real independent solutions are $\cos\omega_0 t$ and $\sin\omega_0 t$. Therefore, every solution can be written in the form

$$y(t) = c\cos\omega_0 t + d\sin\omega_0 t$$

for suitable c, d. Note that we have

$$y(t) = A\cos(\omega_0 t - \varphi) \tag{4.19}$$

where the *amplitude* $A = (c^2+d^2)^{1/2}$ and the *phase shift* φ is defined by $\cos\varphi = c/A$, $\sin\varphi = d/A$.

From the expression (4.19) it follows that every solution exhibits periodic oscillations the frequency of which increases with increasing restoring force and decreasing mass m.

4.7. Example. Forced oscillations of an electric network. Resonance

Consider an electric network involving self-induction (L), resistance (R) and capacity (C) which is forced by an external source with time-dependent voltage $u(t)$. The equation for the current in the network is

$$RI + L\dot{I} + \frac{1}{C}\int I - u = 0.$$

Differentiating, we obtain

$$\ddot{I} + \frac{R}{L}\dot{I} + \frac{1}{LC}I = \dot{u}/L.$$

If we denote $I = y$, $b = R/(2L)$ and $\omega_0 = (LC)^{-1/2}$, we obtain for y the equation

$$\ddot{y} + 2b\dot{y} + \omega_0^2 y = \dot{u}/L. \tag{4.20}$$

Note that in the absence of resistance and external forcing equation (4.20) coincides with the equation (4.18), and therefore will exhibit periodic oscillations.

We assume that the forcing $u(t)$ is sinusoidal, with frequency ω,

$$u(t) = a \sin \omega t,$$

then

$$\dot{u}(t) = a\omega \cos \omega t;$$

because of linearity we can without loss of generality take $a\omega/L = 1$. We assume that the damping b is nonzero but small, in particular, that $0 < b < \omega_0$. Then, the characteristic equation

$$\lambda^2 + 2b\lambda + \omega_0^2 = 0$$

has a pair of complex conjugate roots

$$\lambda_{1,2} = -b \pm i\Omega,$$

where $\Omega = (\omega_0^2 - b^2)^{1/2}$.

Then, any solution of (4.20) can be expressed in the form

$$\begin{aligned} y(t) =& e^{-bt}(\gamma \cos \Omega t + \delta \sin \Omega t) \\ &+ c \cos \omega t + d \sin \omega t \end{aligned} \tag{4.21}$$

with suitable γ, δ, c, d. Note that the solution of the homogeneous equation e^{-bt} $(\gamma \cos \Omega t + \delta \sin \Omega t)$, representing "free" oscillations of the network without forcing, decays to zero with $t \to \infty$. Let us now compute the constants c, d, appearing in the terms representing "forced" oscillations.

Differentiating

$$y(t) = c \cos \omega t + d \sin \omega t \tag{4.22}$$

with respect to time we obtain

$$\dot{y} = -c\omega \sin \omega t + d\omega \cos \omega t \tag{4.23}$$

$$\ddot{y} = -c\omega^2 \cos \omega t - d\omega^2 \sin \omega t. \tag{4.24}$$

Substituting (4.22) - (4.24) into (4.20) we obtain

$$(-c\omega^2 + 2bd\omega + \omega_0^2 c) \cos \omega t + (-d^2 - 2bc\omega + \omega_0^2 d) \sin \omega t = \cos \omega t$$

hence

$$\begin{aligned} c(\omega_0^2 - \omega^2) + 2b\omega d = 1 \\ -2b\omega c + (\omega_0^2 - \omega^2)d = 0. \end{aligned}$$

Solving this system of linear equation for c, d we obtain

$$\begin{aligned} c &= \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2}, \\ d &= -\frac{2b\omega}{(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2}. \end{aligned}$$

Hence, we have

$$y(t) = A\cos(\omega t - \psi)$$

where the amplitude A is given by

$$A = \left[(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2\right]^{-1/2},$$

and the phase shift ψ is up to an integer multiple of 2π given by the equalities

$$\cos\psi = (\omega_0^2 - \omega^2)\left[(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2\right]^{-1/2}$$
$$\sin\psi = 2b\omega\left[(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2\right]^{-1/2}.$$

Note that if b is small, the amplitude may reach very large values for the frequency of forcing ω equal to the internal frequency ω_0 of the network. This phenomenon is called *resonance* and makes wireless transmission of electromagnetic oscillations possible. It may have unwanted effects as well, e.g. "humming"of car bodies at certain revolution frequencies of a car engine etc.

5. General theory of differential equations

5.1. The need of theory

Besides one-dimensional equations with separated variables, linear homogeneous equations (autonomous in higher dimensions) and linear nonhomogeneous equations (with homogeneous part autonomous in higher dimensions) only rather special classes of equations allow their solutions to be expressed in a closed form, i.e., roughly speaking, as a formula consisting of a finite number of terms containing constants, elementary functions and their primitives. The vast majority of them is listed in a classical book by E. Kamke from the twenties. No serious effort is made nowadays to find additional equations which are "integrable" in this sense. There is a good reason for it: in the beginning of the 19th century Liouville proved that an equation as simple as $\dot{x} = x^2 + t$ is not integrable.

Since we are not able to compute the solutions explicitly, the question arises whether our expectations of existence and uniqueness of a solution passing through a given point is true. The following examples show that this is not the case unless certain assumptions are imposed on the equation.

(i) *Non-existence:*
Consider the equation

$$\dot{x} = f(x)$$

where $f(x) = 1$ for $x \leq 0$, $f(x) = 0$ for $x > 0$. Obviously, there is no solution through the point $(t_0, 0)$ for any t_0 (why?). Note that f is not continuous.

(ii) *Non-uniqueness:*
The equation

$$\dot{x} = x^{1/3}$$

has at least two solutions through the point $(t^0, 0)$ namely $x(t) \equiv 0$ and

$$x(t) = \left[\frac{2}{3}(t - t_0)\right]^{3/2}.$$

Note that f is continuous but has no derivative at $x = 0$.

(iii) *Non-extendability:*
The solution through (t_0, x^0) with $x^0 \neq 0$ of the equation

$$\dot{x} = x^2$$

is

$$x(t) = \frac{1}{\frac{1}{x_0} - t + t_0}$$

Obviously, $x(t) \to \infty$ for $t \to \frac{1}{x_0} + t_0$ and, therefore, is not defined on all $\mathbb{R}$.

Note that the right-hand side of the equation has superlinear growth for $x \to \pm\infty$.

5.2. Basic theorem on existence, uniqueness and extendability of solutions

Theorem. *Let $U \subset \mathbb{R} \times \mathbb{R}^n$ be open, $f \in C^1(U, \mathbb{R}^n)$. Then, for each $(t_0, x^0) \in U$ there exists a unique solution of the differential equation*

$$\dot{x} = f(t, x). \tag{5.1}$$

through (t_0, x^0). The solution can be extended to the boundary of U in the following sense: for each compact $K \subset U$ the solution can be extended to contain a point in $U \setminus K$.

Remarks.

1. Existence should be understood in the following way: there exists an open interval $I \ni t_0$ and a function $x \in I : I \to U$ satisfying (5.1) and $x(t_0) = x^0$.
2. Uniqueness should be understood in the following way: whenever we fix an interval $I \subset \mathbb{R}$ there exists at most one function $x : I \to \mathbb{R}^n$ satisfying (5.1) and $x(t_0) = x^0$.
3. It follows from Remark 2 that if $I \subset J$, x is a solution on I and $\hat{x}$ a solution on J then $x(t) = \hat{x}(t)$ for $t \in I$. This circumstance allows us to skip the interval when talking about existence. Moreover, it allows us to define the maximal interval of existence $I_{\max}$ by

$$I_{\max} = \bigcup \left\{ (t_1, , t_2) : \text{ solution through } (t_0, x^0) \text{ exists on } (t_1, t_2) \right\}.$$

4. If $U = \mathbb{R} \times \mathbb{R}^n$, then extendability to the boundary means that the solution is either unbounded for $t > t_0$ ($t < t_0$) or can be extended to an interval containing $[t_0, \infty)$ ($(-\infty, t_0]$, respectively).
5. If continuity of f is assumed instead of continuous differentiability, existence is preserved but uniqueness is not. Uniqueness can be proved under a somewhat less restrictive assumption than continuous differentiability, though, namely "local Lipschitz continuity in x". We will not pursue this issue in this text, however.

There are essentially two ways to prove the theorem. We will discuss them in the following two sections.

5.3. The method of Cauchy-Peano

The methods reflects the idea of the differential equation being a "limit" of a discrete dynamical system (Section 1).

Consider the differential equation (5.1) with U, f as in **5.2**. Looking for a solution through a point $t_0, x^0 \in U$, we take a $\tau > 0$, N a positive integer and denote $\Delta t = \tau/N$. We approximate $\dot{x} = dx/dt$ in (5.1) by the finite difference $(x(t+\Delta t) - x(t))/\Delta t$ to obtain the recurrence formula

$$x(t_{i+1}) = x(t_i) + f(t_i, x(t_i))\Delta t \tag{5.2}$$

for $i = 0, 1, \ldots, N-1,\ t_i = t_0 + i\Delta t$. With $x(t_0) = x^0$ fixed it defines uniquely the values $x(t_i),\ i = 0, \ldots, N$. We now define a function (the *Euler polygon* $x_N(t)$ on $[t_0, t_0 + \tau]$ in such a way that its graph consists of segments joining the points $(t_i, x(t_i))$, i.e.

$$\begin{aligned} x_N(t) =& x_N(t_i) + \frac{t - t_i}{t_{i+1} - t_i}\left[x_N(t_{i+1}) - x_N(t_i)\right] \\ =& x_N(t_i) + (t - t_i) f(t_i, x_N(t_i)) \text{ for } t_i \le t \le t_{i+1}. \end{aligned} \tag{5.3}$$

Approximating the derivative by the backward difference $(x(t) - x(t - \Delta t))/\Delta t$ we can extend x_N to $[t_0 - \tau, t_0]$.

The existence theorem of Cauchy-Peano says that if f is continuous and τ is chosen sufficiently small then the sequence x_N is well defined and one can extract from it a uniformly convergent subsequence whose limit is a solution of (5.1) through (t_0, x^0). It can be proved that if f is C^1 then $x_N(t)$ converges and its limit is the unique solution of (5.1) through (t_0, x^0).

The advantages of the methods are

- it yields existence if f is merely continuous (then, one can prove merely existence but not uniqueness),
- it suggests a certain simple method to compute solutions approximately.

The disadvantage is that the proof is technically rather complicated.

5.4. Picard's approximations

This method employs the natural iteration procedure: guess a solution, substitute it to one side of the equation, then compute the other side of equation to obtain a new, hopefully better guess.

Note that $x(t)$ is a solution of (5.1) if and only if

$$x(t) = x^0 + \int_{t_0}^{t} f(s, x(s))ds \tag{5.4}$$

A function $x(t)$ satisfies (5.4) if and only if it is a fixed point of the operator $\mathcal{T}$ defined by

$$(\mathcal{T}x)(t) = x^0 + \int_{t_0}^{t} f(s, x(s))ds$$

We prove that on a suitable chosen functional metric space $\mathcal{T}$ is a contraction and, thus, has a unique fixed point.

Given $(t_0, x^0) \in U$, there exists a $d > 0$ such that the rectangle $Q = \{(t,x) : |t - t_0| < d,\ |x - x^0| < d\}$ is contained in U. Denote

$$K = \sup_{(t,x)\in Q} |f(t,x)|$$
$$L = \sup_{(t,x)\in Q} |Df(t,x)|,$$

the set of continuous functions on the interval $[t_0 - \delta,\ t_0 + \delta]$ with values in the set $\{x : |x - x_0| \leq \varepsilon\}$, δ, ε to be chosen later.

$\mathcal{M}$ is a closed subset of the Banach space $C([t_0 - \delta,\ t_0 + \delta], \mathbb{R}^n)$ and, therefore, is a complete metric space; we denote by $\|\cdot\|$ the norm on $\mathcal{M}$ induced by the norm in
$C([t_0 - \delta,\ t_0 + \delta], \mathbb{R}^n)$, i.e.

$$\|x\| = \sup_{t\in[t_o-\delta, t_0+\delta]} |x(t)|.$$

We need the following

Lemma. *Let $Q \in \mathbb{R}^m$ be convex and let $f \in C^1(Q, \mathbb{R}^n)$, $L = \sup|Df|$. Then, we have*

$$|f(x) - f(\bar{x})| \leq L|x - \bar{x}|$$

for all $x, \bar{x} \in Q$.

Proof. By the Leibniz-Newton formula

$$|f(x) - f(\bar{x})| = |\int_0^1 Df(\bar{x} + \vartheta(x - \bar{x}))(x - \bar{x})d\vartheta| \leq$$
$$\leq |\int_0^1 |Df(\bar{x} + \vartheta(x - \bar{x}))d\vartheta||x - \bar{x}| \leq L|x - \bar{x}|$$

since $f(\bar{x} + \vartheta(x - \bar{x})) \in Q$ and, therefore $|f(\bar{x} + \vartheta(x - \bar{x})| \leq L$ for all $0 \leq \vartheta \leq 1$.

Let $x \in \mathcal{M}$. For $|t - t_0| \leq \delta$, we have

$$|(\mathcal{T}x)(t) - x^0| \leqq |\int_{t_0}^t f(s, x(s))ds| \leq |\int_{t_0}^t |f(s, x(s)|ds|$$
$$\leq K|t - t_0| \leq K\delta.$$

Hence, $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ provided $\delta \leq \varepsilon/K$. Further, by the Lemma, for $x, \bar{x} \in \mathcal{M}$, and $|t - t_0| \leq \delta$ we have

$$|(\mathcal{T}x)(t) - (\mathcal{T}\bar{x})(t)| \leqq \int_{t_0}^t |f(s, x(s)) - f(s, \bar{x}(s))|ds|$$
$$\leq L|\int_{t_0}^t |x(s) - \bar{x}(s)|ds \leq \alpha\|x - \bar{x}\|,$$

where $\alpha = L\delta$. Thus, $\mathcal{T}$ is a contraction provided $\delta \leq 1/L$.

It follows that once $\delta \le \min\{\varepsilon/K, 1/L\}$, $\mathcal{T}$ is a contraction and, by the Banach fixed point theorem, has a unique fixed point $\hat{x}$. Obviously $\hat{x}$ is the unique solution through x^0 on the interval $[t_0 - \delta, t_0 + \delta]$.

To prove the second part of the theorem suppose that x is a solution of (5.1) on an interval I, $\{(t, x(t)) : t \in I\} \subset K \subset U$ with K compact and that the solution cannot be extended to the right.

Denote $t^* = \sup I$.

For $t, \tau \in I$ we have

$$|x(t) - x(\tau)| \le |\int_t^\tau |f(t, x(t))| ds| \le M|t - \tau|$$

where $M = \sup_{(t,x)\in K} |f(t, x)| < \infty$. It follows that $x(t)$ satisfies the Bolzano-Cauchy condition for $t \to t^*$ and, therefore, has a limit $x^* \in K$.

By the first part of the theorem, there exists a solution $\tilde{x}(t)$ through (t^*, x^*). By uniqueness, this solution coincides with the solution $x(t)$ for $t < t^*$ and, therefore extends is for $t > t^*$. This contradicts the assumption of I to be maximal. A similar contradiction is obtained for the lower limit of I. The two contradictions complete the proof of the theorem. □

Remark. 1. We assumed f to be C^1 by simplicity. From the proof it is easy to see that it is sufficient that f is continuous and its partial derivative with respect to x is continuous.

5.6. Approximate computation of solutions

Theorem 5.2 on existence and uniqueness makes it legitimate to compute a particular solution satisfying a given initial condition by various approximation schemes.

One can obtain an estimate by employing the Euler polygon of 5.3. As an example, consider the problem

$$\dot{x} = x^2 + t, \qquad x(0) = 1 \tag{5.5}$$

from 5.1 and estimate $x(0.2)$ by employing (5.4) with $t_0 = 0$, $\tau = 0.2$, $N = 4$, $\Delta t = 0.05$. One has

$$\begin{aligned} x_4(0.05) &= 1 + 0.05(1 + 0) = 1.05 \\ x_4(0.1) &= 1.05 + 0.05((1.05)^2 + 0.05) = 1.108 \\ x_4(0.15) &= 1.108 + 0.05((1.108)^2 + 0.1) = 1.174 \\ x_4(0.2) &= 1.174 + 0.05((1.174)^2 + 0.15) = 1.250 \end{aligned}$$

At this moment, we do not know how good the estimate 1.250 for $x(0.2)$ is. In order to improve it it is natural to try to use (5.4) with a smaller step size Δt. However, due to round-off errors this idea has its limits. Therefore, more sophisticated numerical method have to be introduced. We leave this topic as well as the problem of estimation of accuracy, of the results to the subject "Numerical methods".

In the case of one-dimensional equations, another way to estimate solutions is to employ inequalities. To this end we prove the following

Theorem. *Let $f \in C^1(U, \mathbb{R})$, $U \subset \mathbb{R}^2$ open. Let $x(t)$, $t \in [t_0, \tau]$ be differentiable and satisfy*

$$\dot{x}(t) \le f(t, x(t))$$

for $t \in (t_0, \tau)$. Let $y(t)$, $t \in [t_0, \tau]$, be a solution of the equation

$$\dot{y} = f(t, y)$$

such that $y(t_0) = x(t_0)$. Then, $x(\tau) \le y(\tau)$.

Proof. Denote

$$z(t) = x(t) - y(t).$$

Using the Leibnitz-Newton formula, we obtain

$$\frac{dz}{dt}(t) = \frac{d}{dt}(x(t) - y(t)) \le f(t, x(t)) - f(t, y(t)) = a(t)z(t), \tag{5.6}$$

where

$$a(t) = \int_0^1 \partial f / \partial x(t, y(t) + \vartheta(x(t) - y(t)))d\vartheta,$$

is continuous. Denote $Z(t) = e^{-\int_{t_0}^t a(s)ds} z(t)$. We have

$$\frac{dZ(t)}{dt} = \frac{d}{dt}\left[e^{-\int_{t_0}^t a(s)ds} z(t)\right] = e^{-\int_{t_0}^t a(s)ds}\left[\dot{z}(t) - a(t)z(t)\right] \le 0,$$

hence

$$Z(\tau) = Z(0) + \int_{t_0}^{\tau} \frac{dZ}{ds}(s)ds \le 0 + 0 = 0$$

and, consequently $z(\tau) \le 0$ as well.

Obviously, the Theorem holds with $\le$ replaced by $\ge$. We demonstrate the use of the Theorem on the Problem (5.5).

We have

$$\frac{dx}{dt}(t) \ge x^2(t),$$

hence $x(0.2) \ge y(0.2)$, where $y(t)$ is the solution of the problem

$$dy/dt = y^2 \qquad y(0) = 1.$$

We have

$$y(t) = (1 - t)^{-1},$$

hence

$$x(0.2) \ge (1 - 0.2)^{-1} = 1.25.$$

On the other hand for $t \in [0, 0.2]$, we have

$$\dot{x}(t) \le x^2(t) + 0.2,$$

hence

$$x(0.2) \le z(t)$$

where $z(t)$ is the solution of

$$\dot{z} = z^2 + 0.2, \qquad z(0) = 1.$$

We have

$$z(t) = (0.2)^{1/2} \tan((0.2)^{1/2} t + \arctan(0.2)^{-1/2}),$$

hence

$$x(0.2) \le (0.2)^{1/2} \tan((0.2)^{3/2} + \arctan(0.2)^{-1/2}) = 1.301.$$

We see that by the Euler polygon we have estimated the solution with accuracy 10^{-1}.

5.7 Linear equations

Consider the equation

$$\dot{x} = A(t)x$$

with $A : \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ continuous. By the Remark in 5.2, through each point (t_0, x^0) there exists a unique solution. We show that each solution can be extended to $\mathbb{R}$. By Theorem 5.2 it suffices to prove that, if $x(t)$ is a solution on some interval I which is bounded from above then $x(t)$ is bounded on I.

Let $t^* = \sup I$; take some $t_0 \in I$ and denote $x(t_0) = x^0$. Further, denote

$$M = 2 \sup_{t \in I} |A(t)|.$$

If $|\cdot|$ is the Euclidean norm, we have

$$\begin{aligned} \frac{d}{dt}|x(t)|^2 = \frac{d}{dt} < x(t), x(t) > = < x(t), A(t)x(t) > + \\ + < A(t)x(t), x(t) > = 2 < x(t), A(t)x(t) > \\ \leq |A(t)||x(t)|^2 \leq M|x(t)|^2, \end{aligned}$$

By the Theorem from 5.6 we have

$$|x(t)|^2 \leq (x^0)^2 e^{M(t-t_0)} \leq (x^0)^2 e^{M(t^*-t_0)}.$$

Hence, x is bounded on I for $t \to t^*$, a cotradiction. A similar argument works for the left hand limit of I.

6. General autonomous differential equations.

6.1. Stability of stationary solutions of autonomous differential equations

Consider the differential equation

$$\dot{x} = f(x) \tag{6.1}$$

with $f \in C^1(U, \mathbb{R}^n, U \subset \mathbb{R}^n,\ U$ open.

Theorem. *Let $f \in C^1,\ f(\hat{x}) = 0$. Then, the stationary solution $x(t) \equiv \hat{x}$ of the differential equation* (6.1) *is*

- *asymptotically stable if the real parts of all eigenvalues of the matrix $Df(\hat{x})$ are negative*
- *unstable, if the real part of at least one eigenvalue of $Df(\hat{x})$ is positive.*

Proof. We prove asymptotic stability for the case of mutually distinct eigenvalues.

Denote $A = Df(\hat{x})$, then

$$f(x) = (A + \omega(x - \hat{x}))(x - \hat{x})$$

where

$$\lim_{x \to 0} \omega(x) = 0.$$

Let λ_i be the eigenvalues, v_i their eigenvectors. Then, as in I.4.1 we find that $x(t)$ is a solution of (6.1) if and only if $y(t) = V^{-1}(x(t) - \hat{x})$ satisfies the differential equation

$$\dot{y} = (\Lambda + V^{-1}\omega(Vy))y \tag{6.2}$$

where $V = (v_1, \ldots . v_n)$.

We have (note that y may be imaginary)

$$\begin{aligned}
\frac{d}{dt}|y(t)|^2 &= \frac{d}{dt} < y(t), \bar{y}(t) > \\
&=< \bar{y}(t), \Lambda y(t) + V^{-1}\omega(Vy(t)) > + < y(t), \overline{\Lambda y(t) + V^{-1}\omega(Vy(t))} > \\
&= 2\sum_{i=1}^{n} \Re\lambda_i |y_i|^2 + 2\Re[< y(t), V^{-1}\omega(Vy(t)) >] \\
&\leq 2\lambda_0 |y(t)|^2 + 2|V^{-1}||y(t)||\omega(y(t))|
\end{aligned}$$

where $\lambda_0 = \max_i \text{Re } \lambda_i < 0$ (note that for λ_i imaginary we have $\lambda_j = \bar{\lambda}_i, y_j(t) = \bar{y}_i(t)$ for some j, hence

$$\begin{aligned}
< \bar{y}_i, \lambda_i y_i > + < \bar{y}_j, \lambda_j y_j > &=< \bar{y}_i, \lambda_i y_i > + < y_i, \bar{\lambda}_i \bar{y}_i > \\
&= 2(\lambda_i + \bar{\lambda}_i) < \bar{y}_i, y_i > \\
&= 4\Re\lambda_i |y_i|^2 \\
&= 2[\Re\lambda_i |y_i|^2 + \Re\lambda_j |y_j|^2]).
\end{aligned}$$

Let δ be chosen so small that $|V^{-1}||\omega(Vy)| < -\frac{1}{2}\lambda_0|y|$. Then, if $y(t)$ is a solution of (6.2) and $|y(t)| \leq \delta$, we have

$$\frac{d}{dt}|y(t)|^2 \leq \lambda_0 |y(t)|^2. \tag{6.3}$$

Similarly as in 5.3, from (6.3) we conclude

$$\frac{d}{dt}[|y(t)|^2 e^{-\lambda_0 t}] \leq 0,$$

hence

$$|y(t)| \leq |y(0)|e^{\lambda_0 t/2} \text{ for } t \geq 0. \tag{6.4}$$

For $x(t) = Vy(t)$ this means

$$|x(t) - \hat{x}| = |V||y(t)| = |V||y(0)|e^{\lambda_0 t/2} \leq |V||V^{-1}||x(0) - \hat{x}|e^{\lambda_0 t/2}.$$

Hence, $|x(t) - \hat{x}| \leq |V||V^{-1}||x(0) - \hat{x}|$ and $x(t) \to \hat{x}$ for $t \to \infty$. This proves that $\hat{x}$ is asymptotically stable.

The proof of instability is more involved and will be omitted.

6.2. Example. Continuous dynamics of the supply and demand cobweb model

In the linear (I.1.4) and nonlinear (I.2.3) cobweb model we assumed that suppliers and consumers make their decisions in discrete time instants to achieve immediate equilibrium. Now, we assume that the decisions are made continuously with the intention to move towards equilibrium.

The underlying microeconomic theory postulates perfect competition: the product is supplied by a large number of firms for which price is exogeneous, i. e. cannot be affected by their individual actions. The equilibrium value $q = S(p)$ results from the rational choice of the profit maximizing quantity of production.

Unlike in the discrete model, we do not assume that equilibrium can be achieved immediately. Rather, we assume that firms move gradually towards production equilibrium by changing the quantity of their production and market mechanisms move price gradually towards the demand equilibrium.

We denote price resp. quantity by p resp. q and suppose that equilibrium demand resp. supply is given by $q = D(p)$, $q = S(p)$ where $D : (0, \infty) \mapsto (0, \infty)$ and $S : [p_0, \infty) \mapsto [0, \infty)$ are C^1 and satisfy $D'(p) < 0$, $S'(p) > 0$. Further, suppose that there is and equilibrium price $\hat{p} \in [p_0, \infty)$ for which supply meets demand, i.e. $S(\hat{p}) = D(\hat{p}) (= \hat{q})$. Note that $\hat{p}, \hat{q}$ are uniquely determined (why?). We assume that the rate of movement towards equilibrium is proportional to the deviation from the latter. Then, we have

$$\begin{aligned} \dot{p} &= \beta(D(p) - q) \\ \dot{q} &= \alpha(S(p) - q). \end{aligned}$$

with $\alpha, \beta > 0$. By our assumption, this system of equations has a unique positive stationary solution $p \equiv \hat{p}$, $q \equiv \hat{q}$. To determine its stability we employ Theorem 6.1. If $x = (p, q)$, $\hat{x} = (\hat{p}, \hat{q})$ and if $f(x)$ is the vector of the right-hand sides, we have

$$Df(\hat{x}) = \begin{pmatrix} \beta D'(p) & -\beta \\ \alpha S'(p) & -\alpha \end{pmatrix}.$$

The eigenvalues λ_1, λ_2 of $Df(\hat{x})$ satisfy $\lambda_1 + \lambda_2 = -\alpha\beta S'(\hat{p}) < 0$, $\lambda_1\lambda_2 = \alpha\beta\,[S'(\hat{p}) - D'(\hat{p})] > 0$. Hence, their real parts are negative which means that $\hat{x} = (\hat{p}, \hat{q})$ is asymptotically stable. Note that the result is different to the discrete time model I.1.4 in which equilibrium may be unstable.

A similar conclusion can be obtained if it is assumed that the supplier sets price while the consumer chooses quantity or both agents decide on both price and quantity.

6.3. Example. The IS-LM model

The dynamic version of the famous IS-LM model of the Keynes theory of equilibrium in the money and capital markets leads to the same mathematical model as 6.2. Denote:

- $i(t)$ interest rate
- $Y(t)$ gross national product
- $L(Y, i)$ aggregated liquidity demand
- $S(Y)$ savings

- $I(i)$ investment
- M money supply
- P price level.

It is assumed that $\partial L/\partial Y > 0$, $\partial L/\partial i < 0$, $S' > 0$, $Y' < 0$ and

$$L(Y,i) = M/P \qquad \text{(money market equilibrium)}$$
$$S(Y) = I(i) \qquad \text{(capital market equilibrium)}.$$

The dynamic model assumes that interest changes so as to move the money market towards equilibrium while income changes so as to move towards equilibrium of the capital market. That is, we have

$$\begin{aligned} \frac{dY}{dt} &= \alpha(I(i) - S(Y)) \\ \frac{di}{dt} &= \beta(L(Y,i) - M/P) \end{aligned}$$

with $\alpha, \beta > 0$ or, more generally,

$$\begin{aligned} \frac{dY}{dt} &= H_1(I(i) - S(Y)) \\ \frac{di}{dt} &= H_2(L(Y,i) - M/P) \end{aligned} \tag{6.5}$$

with $H_1(0) = H_2(0) = 0$, $H_1'(0) > 0$, $H_2'(0) > 0$.

Let $(\hat{Y}, \hat{\imath})$ be the IS-LM equilibrium, which is a stationary point of (6.5). To determine its stability we compute the differential of the vector of the right-hand sides of (6.5) at $(\hat{Y}, \hat{\imath})$ which is

$$\begin{pmatrix} -H_1'(0)S'(\hat{Y}) & H_1'(0)I'(\hat{Y}) \\ -H_1'(0)\frac{\partial L}{\partial Y}(\hat{Y},\hat{\imath}) & H_2'(0)\frac{\partial L}{\partial i}(\hat{Y},\hat{\imath}) \end{pmatrix}.$$

The arguments dropped, we have $\lambda_1 + \lambda_2 = H_1'S' + H_2'\partial L/\partial i < 0$ and $\lambda_1\lambda_2 = H_1'H_2'(-S'\partial L/\partial i + I'\partial L/\partial i) > 0$ which means asymptotic stability as in 6.2.

6.4. Trajectories of autonomous differential equations

By a trajectory of a solution $x(t)$ of a differential equation

$$\dot{x} = f(x) \qquad x \in U \in \mathbb{R}^n \tag{6.6}$$

we understand the set

$$\{x(t) : t \in \text{ interval of existence of } x(\cdot)\}.$$

We can visualize a trajectory as the
- natural projection of the graph $(t, x(t))$ of the solution $x(t)$ into the x-space
- a parametrically defined "curve" $x = x(t)$ in the x-space.

Why is "curve" in quotation marks? The answer can be found in the following

Theorem. *Let f be C^1. Then, there is a unique trajectory of the equation* (6.6) *through every $x^0 \in U$ which is*
- *the point x^0 if $f(x^0) = 0$*
- *a simple differentiable curve if $f(x^0) \neq 0$; this curve is closed if and only if the solution through x^0 is periodic.*

Remark. We say that $C \subset \mathbb{R}^n$ is a (C^1) differentiable curve if there exists a C^1 map $\varphi : I \to \mathbb{R}^n$, I an interval of the real line, such that $C = \varphi(I)$ and $\varphi'(t) \neq 0$ for all $t \neq I$ (and, thus φ has local differentiable inverse in a neighborhood of t).

For the proof we will need the following

Lemma. *If $f \in C^1$ $\tau \in \mathbb{R}$ and $x(t)$ is a solution, then also $\tilde{x}(t) = x(t - \tau)$ is a solution.*

Proof. We have
$$\frac{d}{dt}\tilde{x}(t) = \frac{d}{dt}x(t-\tau) = f(x(t-\tau)) = f(\tilde{x}(t)).$$
□

Note that the proof does not go through if the right-hand side of the equation depends on t.

Proof of Theorem. Let $x(t)$, $\bar{x}(t)$ be two solutions passing through x^0 then, there are t_1, t_2 such that $x(t_1) = \bar{x}(t_2) = x^0$. Denote $y(t) = \bar{x}(t - t_1 + t_2)$. By the Lemma, $y(t)$ is a solution of (6.6). Further, we have
$$y(t_1) = \bar{x}(t_1 - t_1 + t_2) = \bar{x}(t_2) = x(t_1).$$
Therefore, by uniqueness (Theorem 5.2)
$$\bar{x}(t - t_1 + t_2) = y(t) = x(t)$$
for all t. This, however, means that $x(t)$ and $\bar{x}(t)$ define the same trajectory by a different parametrization (shifted by $t_1 - t_2$).

For the sequel we note that from the above argument it follows that all solutions having the same trajectory are time shifts of each other.

Let $f(x^0) = 0$. Then, a solution $x(t)$ passing through x^0 for some t is stationary, i.e. $x(t) \equiv x^0$. This means that its trajectory is x^0.

Let now
$$f(x^0) \neq 0 \tag{6.7}$$
and let $x(t)$ be a trajectory passing through x^0 for some $t = t_0$. Then, we can take $x(t)$ as the parametrization of its trajectory required in the definition of a differentiable curve provided we prove that $\dot{x}(t) \neq 0$ for all t.

Assume the contrary, i.e. that there is a $t^\star$ such that $\dot{x}(t^*) = 0$. Since $x(t)$ solves (6.7) we have $f(x(t^*)) = \dot{x}(t^*) = 0$. By uniqueness of solutions, $x(t) \equiv x(t^*)$, hence $\dot{x}(t_0) = 0$. This contradicts (6.7).

It is obvious that the trajectory of a non stationary periodic solution is a closed curve. To prove the converse, suppose that $x(t)$ is a closed curve. Then, there exist t_0, τ such that $x(t_0 + \tau) = x(t_0)$. Denote $y(t) = x(t + \tau)$. By the Lemma, $y(t)$ is a solution. Further, we have $y(t_0) = x(t_0 + \tau) = x(t_0)$. Hence, by uniqueness we have
$$x(t + \tau) = y(t) = x(t)$$
for all t, which means that $x(t)$ is periodic.

7. Two-dimensional autonomous equations

7.1 Trajectories of two-dimensional equations

Consider a two dimensional equation

$$\dot{x} = f(x)$$

or, equivalently,

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned} \tag{7.1}$$

with $f \in C^1(U, \mathbb{R}^2)$. Let $x(t)$ be a trajectory such that $\dot{x}_1(t) = f_1(x_1(t), x_2(t)) \neq 0$ for $t \in I$. Then, the part $\{x(t) : t \in I\}$ of its trajectory satisfies

$$dx_2/dx_1 = \frac{dx_2/dt}{dx_1/dt} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}. \tag{7.2}$$

This circumstance allows us to obtain the nonconstant trajectories as curves which at any of their points satisfy either (7.2) or

$$dx_1/dx_2 = f_1(x_1, x_2)/f_2(x_1, x_2) \tag{7.3}$$

(note that if $x(t)$ is a nonconstant trajectory, then, by Theorem 6.1, at any t we have either $\dot{x}_1(t) \neq 0$ or $\dot{x}_2(t) \neq 0$). To investigate the solutions of (7.2), (7.3) we can employ methods for the analysis of one-dimensional equations (like separation of variables, etc.)

Example. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 \end{aligned} \tag{7.4}$$

There $f_1(x_1, x_2) = x_2$, $f_2(x_1, x_2) = -x_1$. First, we see that $f_1(0, 0) = f_2(0, 0) = 0$, so the origin is a point trajectory. Since $f_1(x_1, x_2) \neq 0$ for $x_2 \neq 0$, the parts of trajectories in the open halfplanes $x_2 > 0$ and $x_2 < 0$ are graphs of solutions of the differential equation

$$dx_2/dx_1 = -x_1/x_2.$$

By the separation method (Section **2.6**) one obtains for the solution the formula

$$x_2 = \pm\sqrt{C - {x_1}^2} \tag{7.5}$$

with $C > 0$, the $\pm$ sign referring the halfplane $\pm x_2 > 0$, respectively.

Similarly, one can obtain trajectories in the halfplanes $\pm x_1 > 0$ as graphs of solutions of the differential equation

$$dx_1/dx_2 = -x_2/x_1$$

which are

$$x_1 = \pm\sqrt{C - {x_2}^2}, \; C > 0. \tag{7.6}$$

Note that the formulas (7.5) and (7.6) are consistent in the overlap of their validity halfplanes. They can be synthesized into the implicit formula

$${x_1}^2 + {x_2}^2 = C, \quad C \geq 0.$$

Consequently, for $C > 0$ the trajectories are concentric circles with the origin as center; for $C = 0$ we obtain the point trajectory $(0, 0)$. Altogether, the trajectories fill the plane.

7.2. Classification of two-dimensional linear equations

We consider the equation

$$\dot{x} = Ax \tag{7.7}$$

where $x \in \mathbb{R}^2$. We will draw the "phase portraits" of the linear (7.7), i.e. the partition of $\mathbb{R}^2$ into the trajectories of (7.7).

We assume that A is regular and has distinct eigenvalues.

Then, we have the following possibilities:

- the eigenvalues of A are both real and of the same sign - the case of *node*
- the eigenvalues of A are real with different signs (*saddle*)
- the eigenvalues of A are complex conjugate - (*focus or center*)

7.3. Node

Let the eigenvalues λ_1, λ_2 of A satisfy $0 < \lambda_1 < \lambda_2$ and denote v_1, v_2 their eigenvectors, $V = (v_1, v_2)$. Then, the transformation $x = Vy$ transforms (7.4) into the system of equations (cf. 3.1)

$$\begin{aligned} \dot{y}_1 &= \lambda_1 y_1 \\ \dot{y}_2 &= \lambda_2 y_2. \end{aligned} \tag{7.8}$$

Note that if $y(t)$ is a solution of (7.8) and $y_i(t) = 0$ for some t then $y_i(t) = 0$ for all $t, i = 1, 2$. This means that we have four types of solutions:

(i) $y_1(t) \equiv y_2(t) \equiv 0$

(ii) $y_1(t) \equiv 0, \quad y_2(t) \neq 0$ for all t

(iii) $y_1(t) \neq 0, \quad y_2(t) \equiv 0$ for all t

(iv) $y_1 \neq 0, \quad y_2(t) \neq 0$ for all t.

The trajectory of the solution of type (i) is the origin. The solutions of type (ii) have two different trajectories: the $y_2 > 0$ and the $y_2 < 0$ parts of the $y_1 -$ axis. The case (iii) is obtained by interchanging y_1 and y_2.

In case (iv) we can parametrize the trajectory of the solution by y_1. That is, from $y_1 = y_1(t)$ we express t as a function of y_1. Then, we have

$$\frac{d}{dy_1} y_2(t(y_1)) = \frac{dy_2}{dt}(t(y_1))\frac{dt}{dy_1} = \frac{\lambda_2}{\lambda_1}\frac{y_2}{y_1}.$$

In other words the trajectory of the solution is a part of the graph of a solution of the differential equation

$$\frac{dy_2}{dy_1} = \frac{\lambda_2}{\lambda_1}\frac{y_2}{y_1}. \tag{7.9}$$

Solving this equation by the method of separation of variables we obtain

$$y_2 = c|y_1|^{\lambda_2/\lambda_1} \qquad \text{with } c \neq 0. \tag{7.10}$$

We have

$$\frac{dy_2}{dy_1} = c\frac{\lambda_2}{\lambda_1} y_1^{\lambda_2/\lambda_1 - 1}.$$

Since $\lambda_2/\lambda_1 - 1 > 0$, for $y_1 \to 0$ we have $dy_2/dy_1 \to 0$. Hence, the trajectories are tangent to the axis y_1 for $y_1 \to 0$.

Note that for t increasing all solutions except the zero solution travel away from the origin. This is why the origin is called *unstable node* in this case. The case $\lambda_2 < \lambda_1 < 0$ (*stable node*) can be transformed into the case of unstable node by time reverse $t \mapsto -t$. In this case all trajectories tend to the origin for $t \to \infty$. Also, note that all trajectories except of the two ones of type (ii) are tangent to the y_1-axis at the origin.

Transforming the equation to the original x-coordinates we find that, at the stationary point, two of the trajectories are tangent to the eigenvector of the eigenvalue with larger modulus while all the remaining ones are tangent to the other eigenvector.

Example. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_1 - 3x_2 \end{aligned}$$

Here

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$$

has eigenvalues -2, -1 with corresponding eigenvectors $(1,-2)$, $(1,-1)$. Hence, the origin is a stable node, the trajectories being the origin, the halflines spanned by the eigenvectors and halfparabolas tangent to the eigenvector $(1,-1)$ at the origin (Fig. 1)

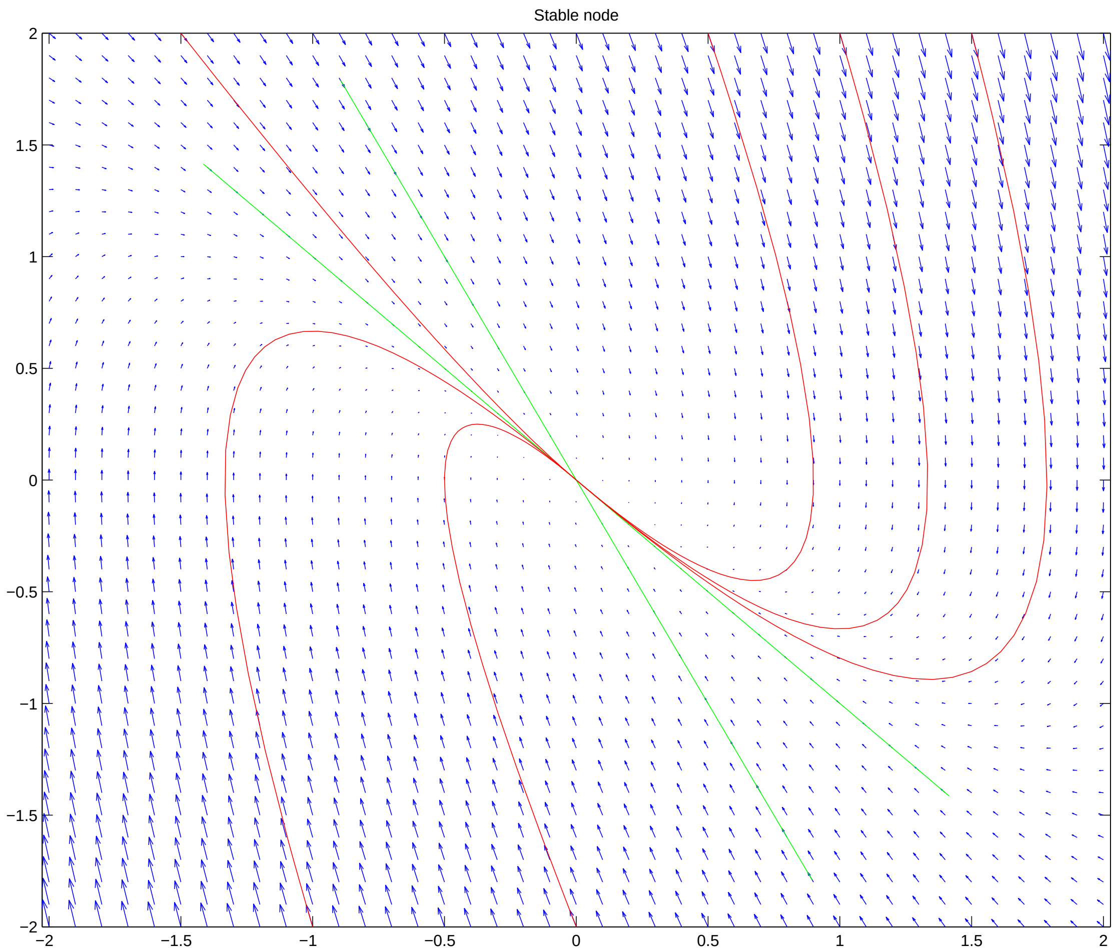

Fig. 1. Stable node

7.4. Saddle

Let $\lambda_1 > 0 > \lambda_2$. Transforming the equation as in **7.3**, we end up with the system of equations (7.8). Again, there are four types of solutions. The difference is that, while the solutions of type (ii) tend to 0 for $t \to -\infty$, solution of type (iii) tend to 0 for $t \to \infty$.

Solutions of type (iv) have trajectories which are parts of the curves (7.10). Since now $\lambda_2/\lambda_1 < 0$, the trajectories resemble hyperbolas $y_1 y_2 = \text{ const }$ in shape. Since the trajectories of the solutions of type (ii) and (iii) separate trajectories of different behavior, they are called *separatrices*. In the original x-variables the separatrices are halflines generated by the eigenvectors at the stationary point.

Example. Consider the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = 4x_1.$$

Here

$$A = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix},$$

its eigenvalues being ± 2 with corresponding eigenvalues $(1, \mp 2)$. Hence, the origin is a saddle, the trajectories being the origin, the halflines spanned by the eigenvectors as separatrices and hyperbolas filling the quadrants separated by the separatrices (Fig. 2)

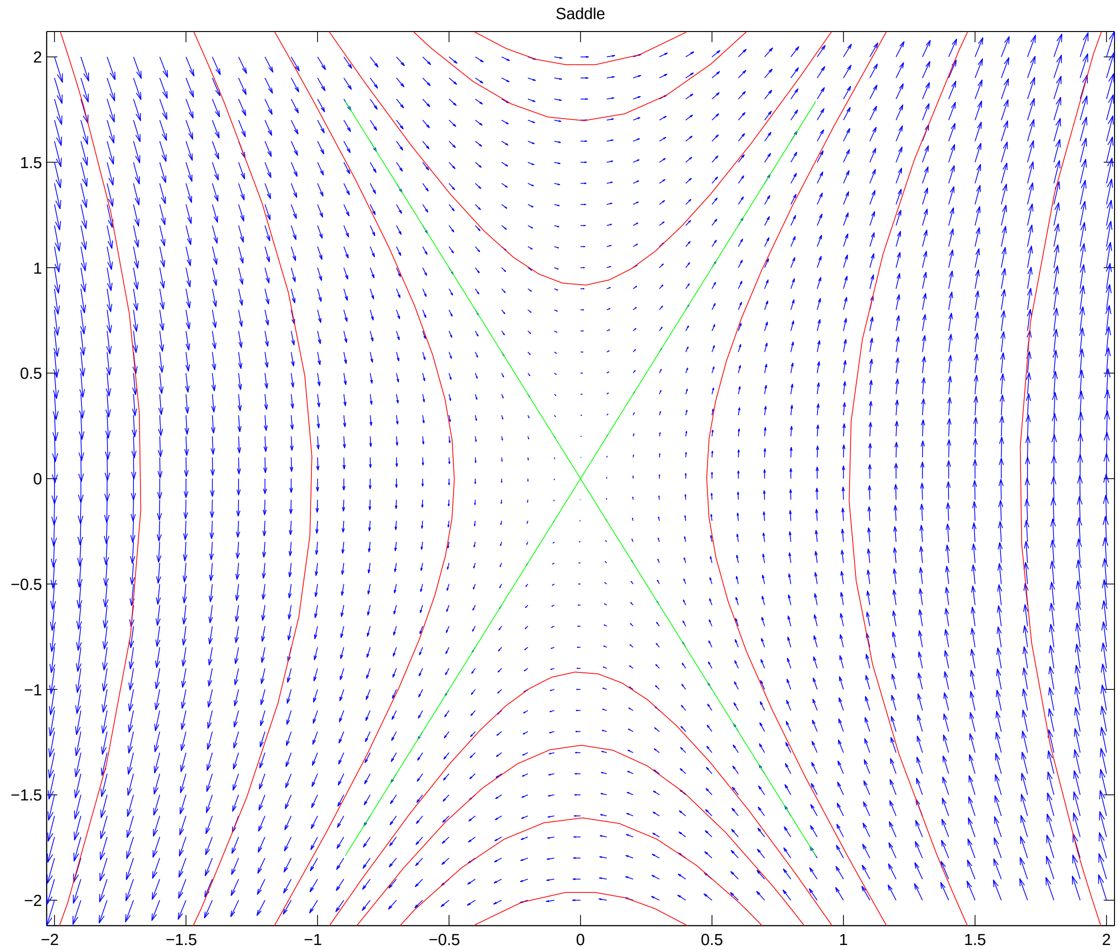

Fig. 2. Saddle

7.5. Focus and center

In this case there is a λ such that $\lambda_1 = \lambda,\ \lambda_2 = \bar{\lambda}$. The eigenvectors of $\lambda, \bar{\lambda}$ are complex conjugate as well and from the equations (7.5) it is sufficient to consider one of the equations

$$\dot{y} = \lambda y. \tag{7.11}$$

The equation (7.11) represents in fact a system of equations for the real and imaginary parts of y.

Let $\lambda = \alpha + i\beta, \quad y = y_1 + iy_2$.

We represent y in the polar form $y = re^{i\vartheta}$, where $r = (y_1^2 + y_2^2)^{1/2}$ and $\vartheta =$ arg y. Then, (7.11) reads

$$\frac{d}{dt}(y_1 + iy_2) = (\alpha + i\beta)(y_i + iy_2),$$

or, after separating the real and imaginary part,

$$\begin{aligned} \dot{y}_1 &= \alpha y_1 - \beta y_2 \\ \dot{y}_2 &= \beta y_1 + \alpha y_2 \end{aligned} \tag{7.12}$$

(this system of equations is called the *real canonical form* of (7.11). Representing (y_1, y_2) in polar coordinates, $y_1 = r\cos\vartheta$, $y_2 = r\sin\vartheta$, we transcribe (7.12) to

$$\begin{aligned} \dot{r}\cos\vartheta - r\sin\vartheta\dot{\vartheta} &= \alpha r\cos\vartheta - \beta r\sin\vartheta \\ \dot{r}\sin\vartheta + r\cos\vartheta\dot{\vartheta} &= \beta r\cos\vartheta + \alpha r\sin\vartheta. \end{aligned} \tag{7.13}$$

Solving (7.13) for $\dot{r}, \dot{\vartheta}$ we obtain

$$\begin{aligned} \dot{r} &= \frac{1}{r}\left[(\alpha r\cos\vartheta - \beta r\sin\vartheta)r\cos\vartheta + (\beta r\cos\vartheta + \alpha r\sin\vartheta)r\sin\vartheta\right] \\ &= \alpha r \\ \dot{\vartheta} &= \frac{1}{r}\left[\cos\vartheta(\beta r\cos\vartheta + \alpha r\sin\vartheta) - \sin\vartheta(\alpha r\cos\vartheta - \beta r\sin\vartheta)\right] \\ &= \beta(\cos^2\vartheta + \sin^2\vartheta) = \beta. \end{aligned}$$

Hence, the solutions of (7.11) or, equivalently, (7.12) are represented in polar form by the solutions of

$$\begin{aligned} \dot{r} &= \alpha r \\ \dot{\vartheta} &= \beta, \end{aligned} \tag{7.14}$$

which are

$$\begin{aligned} r(t) &= r(0)e^{\alpha t} \\ \vartheta(t) &= \vartheta(0) + \beta t. \end{aligned}$$

It follows that, if $\alpha \neq 0$, the nonzero trajectories are spirals winding inside with t increasing for $\alpha < 0$ (*stable focus*) and winding outside for $\alpha > 0$ (*unstable focus*). They rotate clockwise for $\beta < 0$ and counterclockwise for $\beta > 0$ (Fig. 3).

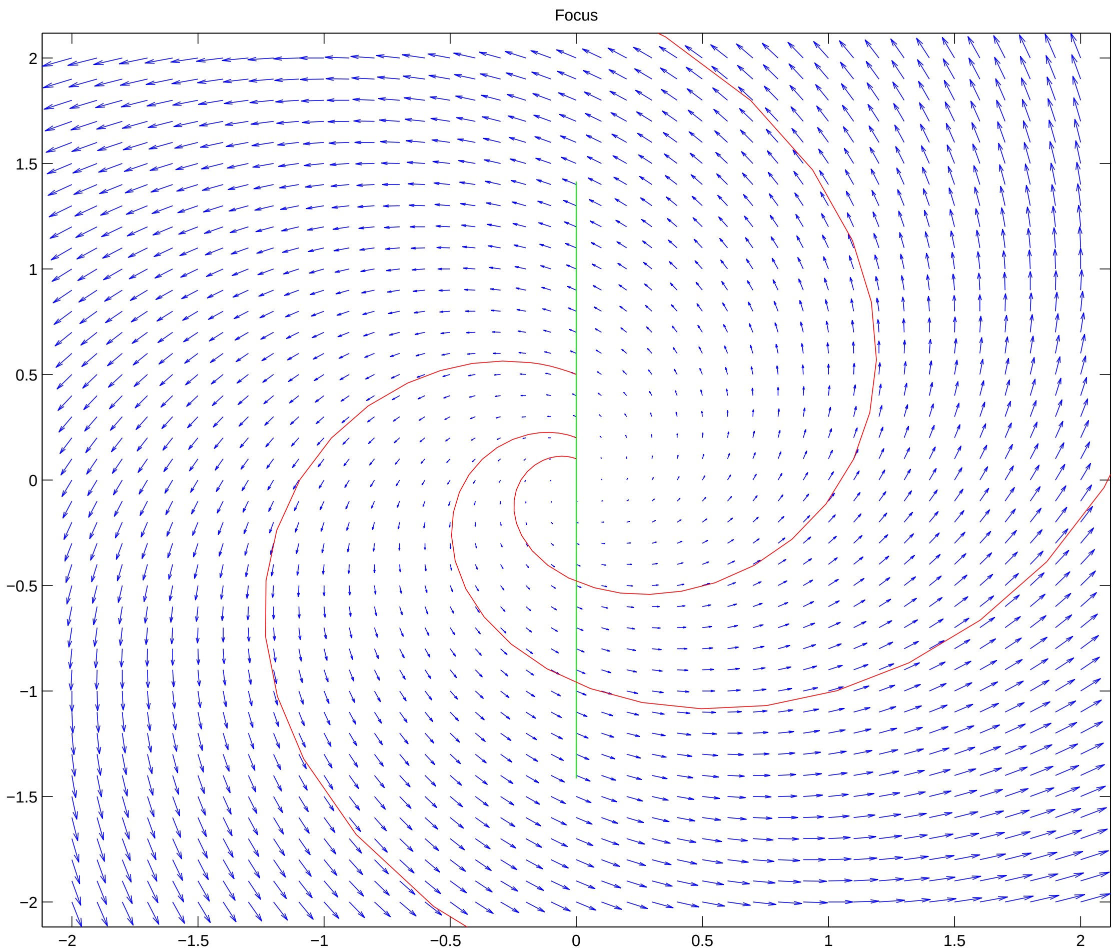

Fig. 3. Unstable focus

If $\alpha = 0$ (i.e., $\lambda, \bar{\lambda}$ are pure imaginary) then the trajectories are circles (the case of *center*). As in the case of focus movement goes clockwise for $\beta < 0$ and counterclockwise for $\beta > 0$ (Fig. 4).

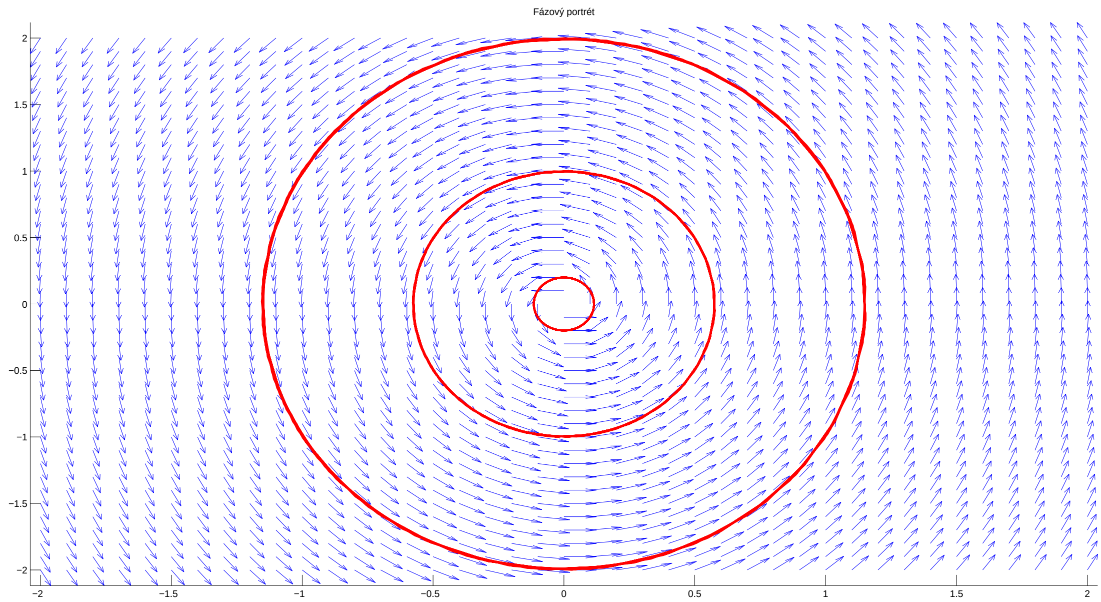

Fig. 4. Center

Note that (7.14) may be obtained more simply by working with the complex representation of solutions: if $y(t) = r(t)e^{i\vartheta(t)}$ is a nonzero solution, substituting into (7.11) we obtain

$$\dot{r}(t)e^{i\vartheta(t)} + ir(t)\dot{\vartheta}(t)e^{i\vartheta(t)} = \lambda r(t)e^{i\vartheta(t)}$$

Since $e^{i\vartheta(t)} \neq 0$, we can divide both sides by $e^{i\vartheta(t)}$ to obtain

$$\dot{r}(t) + ir(t)\dot{\vartheta}(t) = (\alpha + i\beta)r(t).$$

Equating the real and the imaginary parts we obtain (7.14).

Example. Consider the system

$$\begin{aligned} \dot{x} &= x_1 - 2x_2 \\ \dot{x}_2 &= 2x_1 + x_2. \end{aligned}$$

Here

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix},$$

its eigenvalues being $1 \pm 2i$, hence the origin is an unstable focus. The trajectories are the origin and ellipses encircling it. Since $\dot{x}_1 < 0$ for $x_1 = 0$, $x_2 > 0$, the solution point moves counterclockwise along the nonzero trajectories.

7.6. Affine equations

If A is regular, then the equation

$$\dot{x} = Ax + b$$

has a unique stationary solution $\hat{x} = -A^{-1}b$. The transformation $z = x - \hat{x}$ moves $\hat{x}$ to the origin, z satisfying the linear differential equation (7.1). Therefore, we can classify the stationary solutions according to **7.2**.

Note that by a sufficiently small change of the coefficients of A in all cases except of the center case the type of the equation (or, its stationary point) is not changed. A center can be turned into both a stable and an unstable focus by an arbitrary small change of the equation. We say that node, saddle and focus are robust in the class of affine equations and that center is not.

7.7. Nonlinear equations

Consider the equation (6.3) with $x \in U \subset \mathbb{R}^2$, U open, $f \in C^1(U, \mathbb{R}^2)$. Assume that $\hat{x}$ is a stationary point of (6.3) and that $A = Df(\hat{x})$ is regular. If $x(t)$ is a solution of (6.3), then $z(t) = x - \hat{x}$ satisfies the equation

$$\dot{z} = Az + o(z)$$

(cf. 6.2). Since $o(z)$ is small compared to Az for z small, it can be expected that, locally near $\hat{x}$, the phase portrait of (6.3) will be "like" that of (7.1), provided the latter is robust. This turns out to be true in the following sense: there exists a nonlinear local C^1 change of coordinates which carries the trajectories of (6.3) near $\hat{x}$ into the trajectories of (7.1) at 0. In particular, the separatrices of a nonlinear saddle are curves emanating from the stationary point tangent to the separatrices of the linearization.

7.8 The method of isoclines

The results of **7.3 - 7.6** allow us to obtain patches of the phase portrait of a differential equation

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned} \tag{7.15}$$

in the neighborhoods of its stationary solutions $(\hat{x}, \hat{y})$ which are solutions of the system of equations

$$\begin{aligned} f(\hat{x}, \hat{y}) &= 0 \\ g(\hat{x}, \hat{y}) &= 0. \end{aligned}$$

By an *isocline* we understand the set of those points (x, y) at which the tangents of the trajectories of (7.15) are the same, i.e. $g(x, y) \neq 0$ and $f(x, y)/g(x, y)$ is the same or $f(x, y) \neq 0$ and $g(x, y)/f(x, y)$ is the same.

Isoclines help to understand the global phase portrait of (7.15). Of particular importance are the y - *isocline* $((g(x, y) = 0)$ and the x - *isocline* $(f(x, y) = 0)$.

Those isoclines separate regions in which f and g have different signs which determine whether x or y decreases or increases along trajectories. We demonstrate the use of the method of isoclines on several examples.

Example. Consider the system

$$\begin{aligned}\dot{x} &= (2 - x - y)x \\ \dot{y} &= (3 - 3x - y)y. \end{aligned} \tag{7.16}$$

The x- isoclines are the lines $x = 0$ and $x+y = 2$, the y- isoclines are the lines $y = 0$ and $3x + y = 3$. The two x- isoclines have four points of intersection with the y-isoclines, namely $A : (0, 0)$, $B : (0, 3)$, $C : (2, 0)$ and $D : (1/2, 3/2)$. In the quadrant $x > 0,\ y > 0$, the isoclines separate four open regions in which the components $(\dot{x}, \dot{y})$ of the direction vector have the same sign:

- above both the isoclines $x + y = 2$, $3x + y = 3$ one has $\dot{x} < 0, \dot{y} < 0$, i. e., the direction vector points SW,
- above $x + y = 2$ and below $3x + y = 3$ one has $\dot{x} < 0$, $\dot{y} > 0$, the direction vector points NW,
- below $x + y = 0$ and above $3x + y = 3$ one has $\dot{x} > 0$, $\dot{y} < 0$, the direction vector points SE,
- below both isoclines $x + y = 2$, $3x + y = 3$ one has $\dot{x} > 0$, $\dot{y} > 0$, the direction vector points NE.

7.9. Example. The Volterra-Lotka equations

The Volterra-Lotka equations represent the simplest model of the dynamics of two or more interacting populations.

The table below represents three possible types of mutual relationship of two species X, Y: the plus (minus) sign means that an increase of the quantity of one of the species is beneficial (harmful, respectively) for the other one:

	$X \to Y$	$Y \to X$
competition	$-$	$-$
parasitism (predator Y - prey X)	$+$	$-$
symbiosis	$+$	$+$.

Since there is no good way to measure the mutual influence of species quantitatively, *Volterra-Lotka* equations postulate its most simple, i.e. linear form. That is, the general V-L system of equations for 2 species has the form

$$\begin{aligned}\dot{x} &= (a + bx + cy)x \\ \dot{y} &= (d + ex + fy)y \end{aligned} \tag{7.17}$$

In the predator - prey case we have $a > 0,\ c < 0,\ d < 0$ (the predator is dying out without its prey), $e > 0$; for competing species we have $a, d > 0,\ c, e < 0$. In both cases we have $b, f = 0$ or < 0 depending on whether we assume intraspecies competition or not.

The isoclines of the V-L equations are straight lines:

$$a + bx + cy = 0, \quad x = 0 \quad \text{(the } x\text{-isoclines)} \tag{7.18}$$

$$d + ex + fy = 0, \quad y = 0 \quad \text{(the } y\text{-isoclines)} \tag{7.19}$$

Because of the linearity of (7.18), (7.19) stationary points are easily determined.

Let us note that once for a solution $(x(t), y(t))$ we have $x(t_0) = 0$ or $y(t_0) = 0$ for some t_0, then $x(t) \equiv 0$ resp. $y(t) \equiv 0$. This follows immediately from the fact that $x(t)$ solves the linear equation

$$\dot{x} = q(t)x$$

with $q(t) = a + bx(t) + cy(t)$ the only solution of which satisfying $x(t_0) = 0$ is $x(t) \equiv 0$; a similar conclusion we can draw for the coordinate axis $y = 0$ We say that the coordinate axes are *invariant* sets of the equation (7.17). Because of uniqueness of trajectories (Theorem 7.1), no solution can enter the quadrant $x \geq 0$, $y \geq 0$ from outside, neither can a solution leave it. This is important for the biological interpretation of the equations because negative x or y make no sense in the biological context.

The invariance of the coordinate axes has another important consequence: one of the eigenvectors of the linearization of the equation at a stationary point on an axis is the axis itself. For the origin, which is always a stationary point, both the eigenvectors lie in the axes.

Example. Consider the system (7.16). The linearization matrix of the right-hand sides is

$$\begin{pmatrix} 2 - 2x - y & -x \\ -3y & 3 - 3x - 2y \end{pmatrix}.$$

Substituting for (x, y) the coordinates of the four stationary points A, B, C, D we obtain the linearization matrices at them :

$$A: \qquad \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix},$$

eigenvalues $(2, 3)$, corresponding eigenvectors $(1, 0)$, $(0, 1)$.

$$B: \qquad \begin{pmatrix} -1 & 0 \\ -9 & -3 \end{pmatrix},$$

eigenvalues $-1, -3$, corresponding eigenvectors $(3, -8)$, $(0, 1)$.

$$D: \qquad \begin{pmatrix} -1/2 & -1/2 \\ -9/2 & -3/2 \end{pmatrix},$$

eigenvalues $-1 \pm \sqrt{17/8}$, corresponding eigenvectors $(-1 + \sqrt{17/4}, 1)$, $(-1 - \sqrt{17/4}, 1)$. The case of C, which is analogous to B, is left to the reader.

We see that:

- A is an unstable node with all non-point trajectories except of the halfaxes y tangent to the x-axes at A
- B is a node with two trajectories lying in the y-axes, all the remaining non-point trajectories to the other eigenvector at B
- D is a saddle with the stable separatrix pointing SW/NE, the unstable one pointing NW/SE.

Again, the case of the point C is left to the reader. In Figs. 5 and 6 the phase portrait of the equation is diplayed, arrowas representing the tangent vectors of the trajectories. Whereas in Fig. 5 the arrows have the same length, in Fig. 6 their length is prorpotional to the magnitude of the tangent vectors.

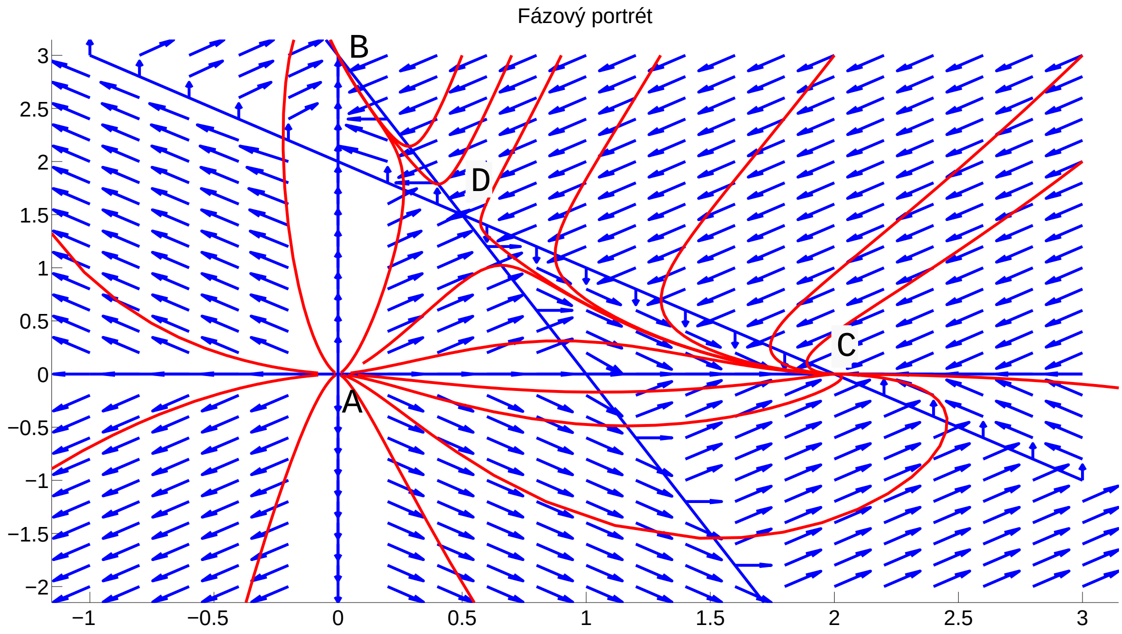

Fig. 5. Competition 1

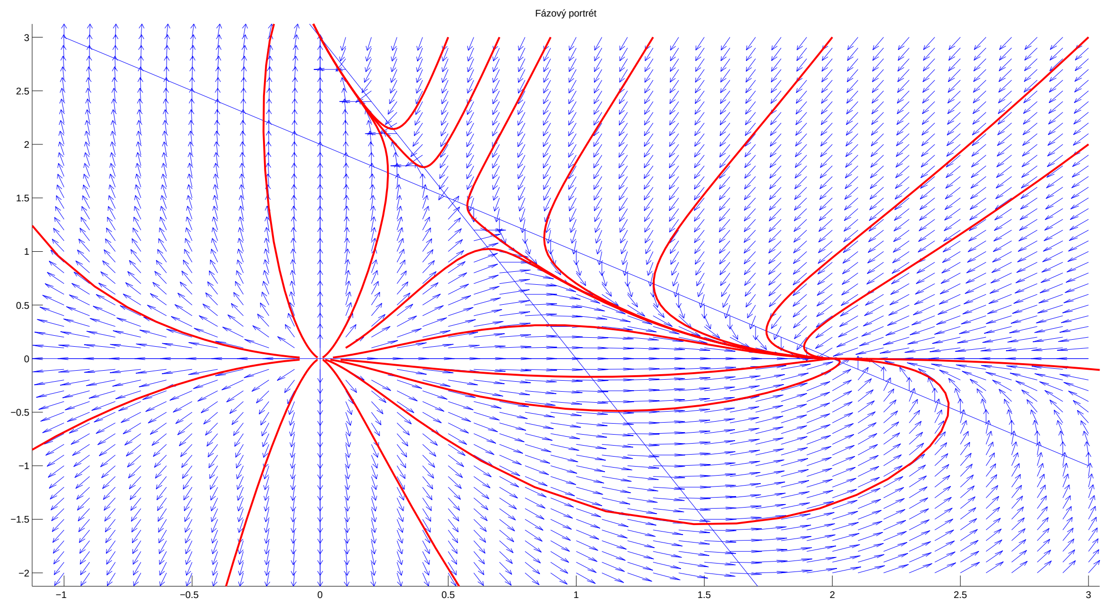

Fig. 6. Comeptition 2

7.10. Example. The predator-prey equation

This model is the most famous one among the V-L models. Why, we explain later in 8.2. If we neglect intraspecies competition, the model reads:

$$\begin{aligned} \dot{x} &= (a - by)x \\ \dot{y} &= (-c + dx)y \end{aligned} \tag{7.20}$$

(a, b, c, d having different meanings than in (7.17).)

The isoclines are the coordinate axes $x = 0$, $y = 0$ and their parallels $y = a/b$, $x = c/d$. An inspection of the signs of $\dot{x}$ and $\dot{y}$ shows that the trajectories wind counterclockwise around the stationary point $(\hat{x}, \hat{y}) = (\frac{c}{d}, \frac{a}{b})$, which is the intersection of the two non-zero $\dot{x}$- and $\dot{y}$-isoclines.

The linearization matrix of (7.17) at $(\hat{x}, \hat{y})$ is:

$$\begin{pmatrix} a - b\hat{y} & -b\hat{x} \\ d\hat{y} & -c + d\hat{x} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{bc}{d} \\ \frac{ad}{b} & 0 \end{pmatrix}$$

the eigenvalues of which are $\pm i\sqrt{ac}$. Hence, by 7.6, we are not in a position to decide the character and the stability of $(\hat{x}, \hat{y})$ from the linearized equation. In 8.2 we show that a special property of the predator-prey equations allows us to resolve the problem.

7.11. Example. Goodwin's growth cycle

There is an economic model leading to the predator - prey equations of 7.10. We consider an economy with an aggregated product the amount of which we denote by Y. Further, we denote by N the labor supply, L its employed part and K the amount of invested capital. We assume

(i) Malthus growth of labor supply,

$$\dot{N} = \beta N \tag{7.21}$$

(ii) steady productivity progress,

$$(Y/L)^{\dot{}} = \alpha(Y/L) \tag{7.22}$$

(iii) production is a linear function of the invested capital,

$$Y = kK, \tag{7.23}$$

(iv) the real wage increase rate grows linearly with $v = L/N$, the employment ratio (due to the increase of negotiating power of labor),

$$\dot{w} = (-\gamma + \rho v)w \tag{7.24}$$

(v) all wages are consumed while all the rest of the product is invested,

$$\dot{K} = Y - Lw. \tag{7.25}$$

For $u = wL/Y$, the workers share of the product, we obtain from (7.24), (7.22)

$$\begin{aligned} \dot{u} &= \dot{w}\frac{L}{Y} + w(\frac{L}{Y})^{\cdot} = \\ &= (-\gamma + \rho v)w\frac{L}{Y} + w\alpha\frac{L}{Y} = \\ &= [-(\alpha+\gamma) + \rho v]\, u. \end{aligned} \tag{7.26}$$

For v, we have

$$\dot{v} = (L/N)^{\cdot} = \frac{\dot{L}}{N} - \frac{L}{N}\frac{\dot{N}}{N} = (\frac{\dot{L}}{L} - \beta)v \tag{7.27}$$

by (7.21). Further, we have

$$\frac{\dot{Y}}{L} - \frac{\dot{L}}{L}\cdot\frac{Y}{L} = \alpha\frac{Y}{L},$$

hence

$$\frac{\dot{L}}{L} = \frac{\dot{Y}}{Y} - \alpha. \tag{7.28}$$

From (7.21) and (7.25) we obtain

$$\begin{aligned} \frac{\dot{Y}}{Y} &= \frac{\dot{K}}{kY} = \frac{1}{kY}\,[Y - Lw] = \\ &= \frac{1}{k}\,[1-u]\,. \end{aligned} \tag{7.29}$$

From (7.27), (7.28), (7.29) we obtain

$$\dot{v} = \left[\frac{1}{k} - \alpha - \beta - \frac{1}{k}u\right] v. \tag{7.30}$$

If $1/k > \alpha + \beta$, (7.27) and (7.30) are equations of predator - prey type. Therefore, this model is used to explain fluctuations in the unemployment rate and the welfare of the labor force.

8. Integrals of differential equations

8.1. The concept of integral

Consider a differential equation

$$\dot{x} = f(x) \tag{8.1}$$

with $x \in \mathbb{R}^n$, $f \in C^1, (U, \mathbb{R}^n)$, $U \subset \mathbb{R}^n$ open.

By an *integral* of (8.1) we understand a function $I \in C^1(U, \mathbb{R})$ which is constant on the trajectories of (8.1). Note that if $x(t)$ is a solution of (8.1) we have

$$\frac{d}{dt}I(x(t)) = DI(x(t))\dot{x}(t) = DI(x(t))f(x(t)),$$

hence I is an integral of (8.1) if and only if

$$DI(x)f(x) = 0 \qquad \text{for } x \in U.$$

Remarks. 1. Every equation has a continuum of integrals $I(x) \equiv c$, $c \in \mathbb{R}$. An integral is called *nontrivial* if it is not constant.
2. In case a nontrivial integral satisfies certain nondegeneracy conditions, its level sets $I(x) = c$ for $c \in \mathbb{R}$ are hypersurfaces of dimension $n-1$ and every trajectory (8.1) is contained in one hypersurface. In this sense the existence of an integral "reduces" the dimension of the differential equation. If $n = 2$, the equation $I(x) = c$ can already be considered as an implicit equation for trajectories (cf. 8.2, 8.3).
3. In the case $n = 2$ an integral can sometimes be found by solving (7.2) and (7.3) by the method of separated variables 2.6 (cf. Example 8.2).
4. In case (8.1) admits a collection of $n-1$ "independent" integrals (a concept not be specified here) $I_1, \ldots, I_{n-1}$, the equations $I_1(x) = c_1, \ldots, I_{n-1} = c_{n-1}$ define the trajectories as in Remark 2. In such a case the equation (8.1) is called integrable. Historically, it was a great achievement of the young differential and integral calculus when Newton was able to integrate the 12-dimensional two body problem and thus justify Kepler's empirical planet movement laws.

8.2. Example. The predator - prey equation

We consider the system of equations (7.16)

$$\begin{aligned} \dot{x} &= (a - by)x \\ \dot{y} &= (-c + dx)y \end{aligned} \tag{8.2}$$

$a, b, c, d > 0$.

Away from the lines $y = b/a$ and $x = 0$, the trajectories of (8.2) are graphs of solutions of the equation

$$\frac{dy}{dx} = \frac{(-c + dx)y}{(a - by)x}.$$

This is an equation with separated variables and, as such, can be solved to obtain the implicit expression of its solutions:

$$G(x) + H(y) = \gamma \tag{8.3}$$

with $G(x) = dx - c\ln x$, $H(y) = by - a\ln y$ in the quadrant $x > 0$, $y > 0$. By solving for dx/dy away from the lines $y = 0$ and $x = d/c$ we can check that (8.3) holds also along the lines $y = b/a$ and $x = 0$.

Hence, $G(x) + H(y)$ is an integral of (8.2).

We now show what conclusions can be drawn from the presence of this integral for the phase portrait of (8.2).

As in 7.10 denote $\hat{x} = c/d$, $\hat{y} = a/b$ the components of the nonzero stationary point of (8.2). We have

$$G'(x) = d - c/x < 0 \text{ for } 0 < x < \hat{x}, \ > 0 \text{ for } x > \hat{x} \tag{8.4}$$

$$G(0) = G(\infty) = \infty \tag{8.5}$$

as well as

$$H'(y) = b - a/y < 0 \text{ for } y < \hat{y}, > 0 \text{ for } y > \hat{y} \tag{8.6}$$

$$H(0) = H(\infty) = \infty. \tag{8.7}$$

From (8.4) - (8.7) it follows that G and H attain their minima over $(0, \infty)$ in $\hat{x}, \hat{y}$, respectively and for each $\alpha > G(\hat{x})$ and $\beta > H(\hat{y})$ each of the equations $G(x) = \alpha$, $H(y)$ has precisely two solutions, one $< \hat{x}$, the other $> \hat{x}$ for the former and one $< \hat{y}$, the other $> \hat{y}$ for the latter. Consequently, the trajectories of (8.3) which wind around $(\hat{x}, \hat{y})$ by 7.10 have to be closed, and, thus, $(\hat{x}, \hat{y})$ is a center.

The important message is that periodic fluctuations in nature (and by 7.11, economy as well) should not be considered as something pathological.

8.3. Conservative equations with one degree of freedom

A physical system is called *conservative* if it conserves some kind of energy. Of course, if this system is modelled by a differential equation, this energy is its integral.

The mechanical energy of a mechanical system is the sum of its kinetic energy and its potential energy. We restrict ourselves to *mechanical systems of one degree of freedom*, i.e. systems moving on a line, or, more generally, on a curve (where they have to be kept by external force if the curve is not a line). Consider a point of unit mass moving along a line and denote $x = x(t)$ the longitudinal coordinate of its position at time t. The second Newton's law asserts that acceleration is equal to force to be denoted by f, i.e.,

$$\ddot{x} = f. \tag{8.7}$$

We assume that $f = f(x)$, i.e. the force acting on the point depends only on its position.

Denote

$$U(x) = -\int_a^x f(\xi)d\xi;$$

for a fixed but arbitrary $a \in \mathbb{R}$. We call U the *potential energy.* The mechanical energy H of the point is the sum of its potential energy and its *kinetic* energy $T(\dot{x}) = (\dot{x})^2/2$. We have

$$\frac{d}{dt}H(x(t)) = \frac{d}{dt}\left[\frac{1}{2}\dot{x}^2(t) + U(x(t))\right] = \\ = \left[\ddot{x}(t) - f(x(t))\right]\dot{x}(t) = 0,$$

hence H is an integral of (8.7) and (8.7) is conservative.

The special form of the energy integral H of (8.1) enables us to draw the phase portraits of the two dimensional system

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= f(u) = -U'(u) \end{aligned} \tag{8.8}$$

corresponding to (8.7) comfortably. To this end we observe that the stationary points of (8.8) lie on the line $v = 0$ and recall that trajectories of (8.8) are parts of the level sets of the integral

$$H(u, v) = \frac{1}{2}v^2 + U(u) \tag{8.9}$$

where U is (any) primitive function of f.

By E_γ we denote the level set $H = \gamma$ of (8.9). Note that E_γ consists of the graphs of the functions $v = \pm\sqrt{2(\gamma - U(u))}$. The following obvious observation we formulate as

Lemma.

(i) The level sets of H are symmetric with respect to the line $v = 0$, i.e.

$$(u, v) \in E_\gamma \qquad \textit{implies} \qquad (u, -v) \in E_\gamma.$$

(ii) There is a v such that $(u, \pm v) \in E_\gamma$ if and only if $U(u) \leq \gamma$; one has $v = 0$ if $U(u) = \gamma$ and $v \neq 0$ if $U(u) < \gamma$.

(iii) $(\hat{u}, \hat{v})$ is a stationary point of (8.8) if and only if $\hat{v} = 0$ and $\hat{u}$ is a stationary point of U (i.e. $U'(\hat{u}) = 0$); $(\hat{u}, \hat{v})$ is a saddle if $U''(\hat{u}) < 0$ and a center if $U''(\hat{u}) > 0$.

(iii) Along the trajectories one has $dv/du > (<) 0$ if $v > 0$, $f(u) >$ (respectively, $<$)0

Proof. Conclusion (i) follows immediately from the fact that F is a function of v^2.

For (ii) we note that if $U(u) \leq \gamma$ we have $\gamma - U(u) \geq 0$, hence $\frac{v^2}{2} = \gamma - U(u)$ has a nonnegative solution which vanishes precisely if $U(u) = \gamma$.

For (iii) observe that $\dot{u} = 0$ precisely if $v = 0$ while $\dot{v} = 0$ if and only if $f(u) = -U'(u) = 0$. The linearization of (8.8) at a stationary point is

$$\begin{pmatrix} 0 & 1 \\ f'(\hat{u}) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -U''(\hat{u}) & 0 \end{pmatrix},$$

its characteristic equation being

$$\lambda^2 + U''(\hat{u}) = 0.$$

This equation has roots $\lambda_{1,2} = \pm\left[-U''(\hat{u})\right]^{1/2}$ of different sign if $U''(\hat{u}) < 0$ and two conjugate pure imaginary parts $\lambda_{1,2} = \pm i\left[U''(\hat{u})\right]^{1/2}$ if $U''(\hat{u}) > 0$. To see that in the latter case $(\hat{u}, 0)$ is indeed a center, observe that the trajectories in the neighborhood of $(\hat{u}, 0)$ have to be closed by (i).

8.4. Example. The harmonic oscillator

In 6.2. we have solved the equation of the harmonic oscillator

$$\ddot{y} + \omega^2 y = 0.$$

This is an conservative equation of type (8.7) and, therefore, by (8.3), admits the integral

$$H(y, \dot{y}) = \frac{1}{2}\dot{y}^2 + \frac{1}{2}\omega^2 y^2.$$

That is, the energy level sets E_γ of the corresponding planar system

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= -\omega^2 u, \end{aligned}$$

given by

$$E_\gamma = \{(u,v) : \omega^2 u^2 + v^2 = \gamma\}$$

are ellipses encircling the origin for $\gamma > 0$, whereas $E_\gamma = (0,0)$ for $\gamma = 0$.

Since the level curves do not contain a stationary points for $\gamma > 0$, each E_γ for $\gamma > 0$ consists of a single trajectory which is periodic by 6.1, Theorem.

This conclusion can of course been drawn also from the explicit formula of the solution from 4.5.

8.5. The pendulum equation

The equation for the ideal pendulum is

$$\ddot{\varphi} + a^2 \sin\varphi = 0, \tag{8.10}$$

φ being the angle it creates to its lower vertical position and $a^2 = g/l$, l is the length of the pendulum and g is the gravitational constant.

Consider the planar system

$$\begin{aligned} \dot{\varphi} &= \psi \\ \dot{\psi} &= -a \sin\varphi. \end{aligned} \tag{8.11}$$

corresponding to (8.10). Note that φ and $\varphi + 2\pi$ represent the same points in space. Hence, we can restrict our attention to the interval $[-\pi, \pi]$. The potential energy $U(\varphi) = -a\cos\varphi$ has $U'(\varphi) = 0$ for $\varphi = 0, \pm\pi$ and $U''(0) > 0$ while $U''(\pm\pi) < 0$. Therefore, the stationary points of (8.11) are the center $(0,0)$ (representing the downward equilibrium position) and the saddle $(\pm\pi, 0)$ (the upward equilibrium position). Consequently, the downward equilibrium $(0,0)$ is stable (but not asymptotically), whereas the upper one $(\pm\pi, 0)$ is unstable.

The energy level sets E_γ are defined by the equation

$$H(\varphi, \psi) = \frac{1}{2}\psi^2 - a\cos\varphi = \gamma.$$

Note that

$$H(\varphi, \psi) \geq -a\cos\varphi \geq -a, \tag{8.12}$$

$$H(\varphi, 0) \leq a. \tag{8.13}$$

The level sets E_γ are of several types (Fig. 7):

(i) For $\gamma < -a$ we have $E_\gamma = \emptyset$ by (8.12).

(ii) For $\gamma = -a$, E_γ is the point $(0,0)$.

(iii) For $-a < \gamma < a$, E_γ has two points of intersection φ_1, φ_2 with the axis $\psi = 0$, which are defined by $\cos\varphi_{1,2} = \frac{a}{\gamma}$, $-\pi < \varphi_1 < 0 < \varphi_2 < \pi$. There is no stationary point on E_γ, hence E_γ is a single closed trajectory. The corresponding movements oscillate around the downward equilibrium without reaching the upward unstable one.

(iv) For $\gamma = a$, E_γ is the union of the two stationary points $(\pm\pi, 0)$ (geometrically representing the same point in space) and the trajectories connecting them. That is, E_a consists of the upward equilibrium and the movements which make one turn (in either direction) in infinite time and tend to this equilibrium for $t = \pm\infty$.

(v) For $\gamma > a$, by (8.13) and the Lemma,(iii), E_γ does not intersect the axis $\psi = 0$. Hence, E_γ has two components, one with $\psi > 0$, the other with $\psi < 0$, defined by

$$\psi = \psi(\varphi) = \pm\sqrt{\gamma - 2a\cos\varphi} \qquad \text{for} \qquad \varphi \in [-\pi, \pi]\,.$$

They satisfy $\psi(-\pi) = \psi(\pi)$ and represent movements completing full turns around the point at which the pendulum is fixed.

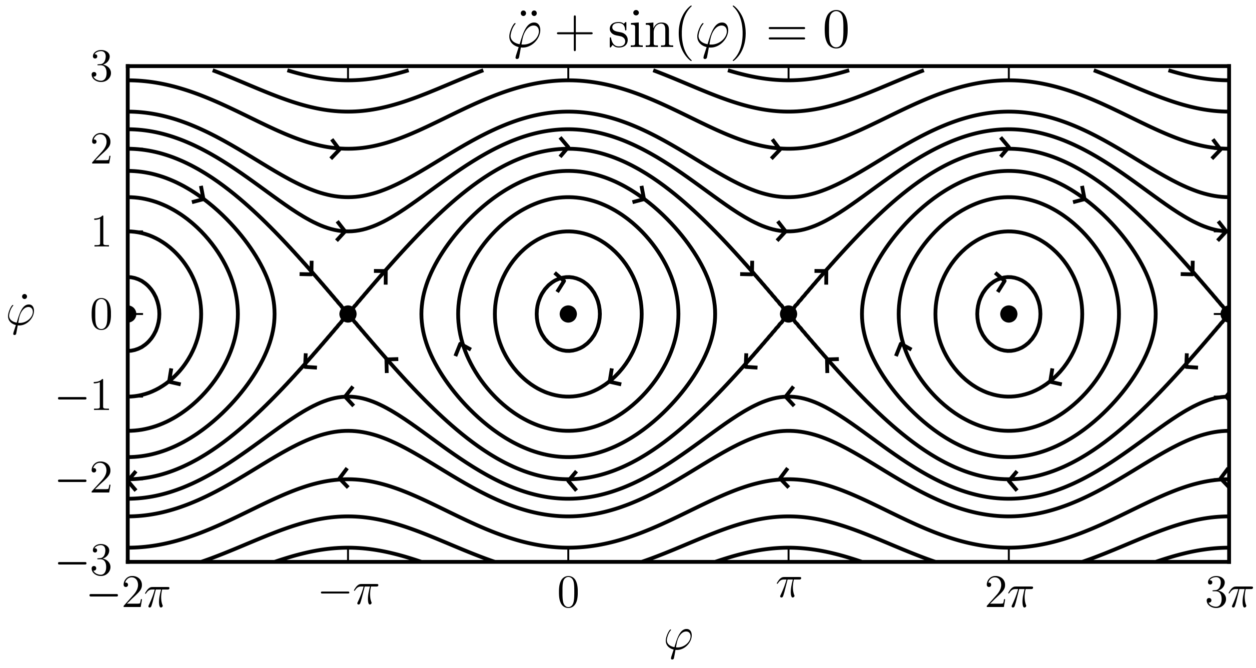

Fig. 7. Pendulum