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A DELAY DIFFERENTIAL EQUATION MODEL OF OSCILLATIONS OF EXCHANGE RATES

DIPLOMOVÁ PRÁCA

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Týmto prehlasujem, že predkladanú diplomovú prácu som vypracoval samostatne, len s pomocou konzultácií a literatúry uvedenej v zozname.

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1. INTRODUCTION

Over the last thirty years, a number of economic literature has been devoted to the topic of modeling foreign exchange rates. Several important events have influenced this effort: the demise of the Breton Woods system of 'fixed but adjustable' exchange rates in the early 1970s, the following period of free floating, the establishment of the European Monetary System in 1979 with resulting target zones regimes, the foundation of the European Monetary Union and finally the adoption of common currency Euro in 2002.

A number of models have been introduced to explain the behavior of the exchange rate either within a band or under the free floating regime. Perhaps the most cited models have been the *flexible price monetary model* (see for example Mussa [13]) and the *sticky-price monetary model* (due to Dornbusch [5], stochastic modification by Miller and Weller [12]). Most of other models somehow utilize these concepts.

Instead of discussing specific models in detail, we point out one common feature – under all models, the exchange rate turns out to be the function of economic fundamentals, expectations and random disturbances. In symbols, we have

$$S = S(\text{economic fundamentals, expectations, random influences}).$$

where S denotes the foreign exchange rate (that is, the amount of domestic currency needed to purchase one unit of foreign currency). Let $\theta(t)$ denote the state of economic fundamentals characterizing the economy at time t. Then the general model of exchange rate is given by

$$S(t) = f(\theta(t)) + x(t) \tag{1.1}$$

where the term $f(\theta(t))$ holds for the *'natural' exchange rate*, the rate that would prevail if the other factors were not present. Howerer, these factors, including expectations, speculation, or random disturbances are always present. They are reflected by the term $x(t)$. As already mentioned, various ways of modeling $x(t)$ have been adopted.

The aim of this paper is to propose an interesting way of modeling component $x(t)$ under the assumption of free floating exchange rate regime. The model will take into account the expectations of economic agents in such a way that we will be able to demonstrate that even when economic factors causing the change of the exchange rate are absent, psychology itself may lead to oscillations.

The rest of the paper is organized as follows. Section 1.1 introduces the idea of the model – a simple retarded (delay) functional differential equation (RFDE) model. Chapter 2 introduces the basic theory of RFDEs. Chapter 3 analyzes qualitative properties of the model. In Chapter 4 the results of numerical simulations of the model are given. In Chapter 5 we introduce several real data observations. Chapter 6 concludes.

1.1 The proposed model

We will work within the framework given by (1.1). Our aim is to model the variable $x(t)$, the deviation from the 'natural' exchange rate. In this paper we consider the the delay differential equation model[1] given by

$$\dot{x}(t) = a(x(t) - x(t-1)) - b|x(t)|x(t) \tag{1.2}$$

where a, $b > 0$ are parameters. We briefly explain the logic behind the two right-hand side terms in (1.2).

• The first term, $a(x(t) - x(t-1)) \approx a(S(t) - S(t-1))$:

As long as the exchange rate increases (decreases), it is worthwhile to purchase (sell) foreign currency. Hence, the demand for foreign currency goes up (down) and the exchange rate continues to increase (decrease).

• The second term, $-b|x(t)|x(t)$:

As long as the absolute deviation $|x(t)|$ from the 'natural' exchange rate increases, a growing number of agents realize that the tendency will turn back. Moreover, with increasing absolute deviation, each agent trades growing quantities. Then for increasing positive (decreasing negative) deviation the demand for foreign currency goes down (up) propertionately to the square of the deviation. Hence, the absolute deviation $|x(t)|$ is expected to stop increasing.

The two effects are put together. We have: As long as the exchange rate increases (decreases), the first term in (1.2) pushes the domestic currency to further depreciation (appreciation) until outweighed by the effect of the second term. Then the exchange rate stops increasing (decreasing) and the currency starts to appreciate (depreciate). Intuitively, this process should lead to oscillations.

In what follows we will analyze model (1.2). We will be, of course, primarily interested in the qualitative behavior of the model near the only equilibrium $\overline{x} = 0$. At the end of this paper we will test whether the observed data are consistent with the implications of the model.

[1] The idea of modeling economic variable by a RFDE model is not entirely new. For example, MACKAY [10] has proposed a RFDE model of commodity price fluctuations.

2. RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS – SELECTED TOPICS

Recall that the model under consideration is of the form

$$\dot{x}(t) = f(x(t), x(t-1)) \tag{2.1}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. Equation (2.1) is an example of retarded functional differential equation (RFDE). To be able to analyze qualitative properties of such a model, at least a brief survey into the theory of RFDE is necessary.

This chapter summarizes several basic notions relating delay differential equations. The emphasis is put on the topics that are pertinent to the analysis given in the Chapter 3. As most results are given without proofs, we recommend HALE [7], [8] or DIEKMANN, VAN GILS, VERDUYN LUNEL AND WALTHER [4] for further reference.

2.1 Initial value problem, general autonomous equation

Equation (2.1) is not sufficient to determine $x(t)$. To fix x, we need an initial condition. We adopt the most convenient way of prescribing x on the interval $[-1, 0]$. Then we have the *initial value problem*

$$\begin{cases} \dot{x}(t) = f(x(t), x(t-1)), & \text{for} \quad t \geq 0, \\ x(t) = \varphi(t), & \text{for} \quad -1 \leq t \leq 0, \end{cases} \tag{2.2}$$

where

$$\varphi \in C([-1, 0], \mathbb{R})$$

is a given *initial function*.

It is well-known that given a continuous function f satisfying Lipschitz condition and a continuous initial function φ, there exists an open interval $I \supset [-1, 0]$ and a continuous function $x : I \to \mathbb{R}$, differentiable for $t > 0$, $t \in I$ and such that (2.2) is satisfied. The solution x is uniquely determined by φ and is either unbounded for $t > 0$ or can be extended to an interval containing $[-1, \infty)$. (The method of steps and the elementary theory of ordinary differential equations provide us with a proof.)

We will not confine ourselves to a specific system (2.2) and in what follows we will work within a more general framework. For a fixed $h > 0$ and for a given function $x(t)$, we denote x_t the function defined on the interval $[-h, 0]$ by

$$x_t(\theta) = x(t + \theta), \qquad \text{for} \quad -h \leq \theta \leq 0. \tag{2.3}$$

Let us assume that x_t is, for all t for which it is defined, a continuous mapping from $[-h, 0]$ into $\mathbb{R}$. In symbols, we have $x_t \in \mathcal{C}$, where $\mathcal{C} = C([-h, 0], \mathbb{R})$ equipped with the supremum norm.

Let $F : \mathcal{C} \to \mathbb{R}$ be a continuous (nonlinear) mapping. The class of autonomous RFDEs we want to consider is

$$\dot{x}(t) = F(x_t) \tag{2.4}$$

with an initial condition $x_0 = \varphi \in \mathcal{C}$.

Remark. Equation (2.4) is clearly a generalization of equation (2.1). By putting $h = 1$ and $F(x_t) = f(x_t(0), x_t(-1))$ we obtain the same equation.

In the rest of this chapter we introduce the theory needed to analyze the qualitative behavior of solutions of delay differential equations near an equilibrium. As for ordinary differential equations (ODE), such an analysis starts with an analysis of the linearized system. Therefore, we first concentrate on linear autonomous delay equations.

2.2 Linear autonomous RFDE

A linear autonomous RFDE has the form

$$\dot{x}(t) = Lx_t \tag{2.5}$$

where L is a continuous linear function mapping $\mathcal{C}$ into $\mathbb{R}$. Then, according to the *Riesz representation theorem*, there exists a bounded variation function η, defined on $[-h, 0]$ with real values such that for all $\phi \in \mathcal{C}$

$$L\phi = \int_{-h}^{0} \phi(\theta)\, d\eta(\theta). \tag{2.6}$$

Let $x(\varphi)$ be the unique solution to (2.5) with the initial function $\varphi \in \mathcal{C}$ at zero. Then the *solution operator* $T(t) : \mathcal{C} \to \mathcal{C}$ is defined by the relation

$$x_t(\varphi) = T(t)\varphi.$$

It can be shown that $T(t)$, $t \geq 0$ is a strongly continuous semigroup of linear operators on $\mathcal{C}$ on $[0, \infty)$. Its *infinitesimal generator* A is defined as

$$A\varphi = \lim_{t \to 0^+} \frac{1}{t} \left(T(t)\varphi - \varphi\right) \tag{2.7}$$

whenever this limit exits. The operator A is given by

$$A\varphi(\theta) = \begin{cases} d\varphi(\theta)/d\theta, & \text{for} \quad -h \leq \theta < 0, \\ L\varphi = \int_{-h}^{0} \varphi(\theta)\, d\eta(\theta), & \text{for} \quad \theta = 0. \end{cases} \tag{2.8}$$

The point spectrum of the operator A, denoted $\sigma_p(A)$, plays an important role in determining the behavior of solutions to (2.5). Let us denote $\mathcal{M}_\lambda(A)$ the *generalized eigenspace* corresponding to a given *eigenvalue* $\lambda \in \sigma_p(A)$. We have the following.

Lemma 2.1. (cf. [7, p. 168]) *If A is defined by equation (2.8), then the spectrum of A consists of the point spectrum only and λ is in $\sigma_p(A)$ if and only if λ satisfies the characteristic equation*

$$\Delta(\lambda) = 0 \tag{2.9}$$

where

$$\Delta(\lambda) = \lambda - \int_{-h}^{0} e^{\lambda\theta}\, d\eta(\theta).$$

The roots of equation (2.9) have real parts bounded above and for any $\lambda \in \sigma_p(A)$ the generalized eigenspace $\mathcal{M}_\lambda(A)$ is finite dimensional. Finally, $A\mathcal{M}_\lambda(A) \subseteq \mathcal{M}_\lambda(A)$, or in other words, $\mathcal{M}_\lambda(A)$ is invariant under A.

Remark. As for ODE, the characteristic equation (2.9) can be also obtained by seeking nontrivial solutions of (2.1) of the form $x(t) = x(0)e^{\lambda t}$.

Our next task is to analyze the action of $T(t)$ on $\mathcal{M}_\lambda(A)$ for a given $\lambda \in \sigma_p(A)$. Let $\{\phi_1, \dots, \phi_{m_\lambda}\}$ be a basis of $\mathcal{M}_\lambda(A)$. Since $\mathcal{M}_\lambda(A)$ is invariant under A, there exists a $m_\lambda \times m_\lambda$ matrix B_λ such that

$$A\Phi_\lambda = \Phi_\lambda B_\lambda \tag{2.10}$$

where $\Phi_\lambda = (\phi_1, \dots, \phi_{m_\lambda})$. The only eigenvalue of the matrix B_λ is λ. We also have

$$\frac{d}{dt}T(t)\Phi_\lambda = T(t)A\Phi_\lambda = T(t)\Phi_\lambda B_\lambda.$$

The action of $T(t)$ on Φ_λ is given by $T(t)\Phi_\lambda = \Phi_\lambda e^{tB_\lambda}$.

We can reinterpret this in the following way. Let $\phi \in \mathcal{M}_\lambda(A)$ be arbitrary. Define $z(t)$ to be the m_λ vector of coordinates of $T(t)\phi$ with respect to the basis Φ_λ. Then

$$\dot{z}(t) = B_\lambda z(t). \tag{2.11}$$

Hence, the restriction of $T(t)$ to the finite dimensional subspace $\mathcal{M}_\lambda(A)$, invariant under A, is generated by the ordinary differential equation (2.11).

Repeating the above process for all eigenvalues we obtain the following result.

Theorem 2.2. (cf. [7, p. 170]) *Let $\Lambda = \{\lambda_1, \dots, \lambda_p\}$ be a finite set of eigenvalues of A and let*

$$\Phi_\Lambda = (\Phi_{\lambda_1}, \dots, \Phi_{\lambda_p}), \qquad B_\Lambda = diag(B_{\lambda_1}, \dots, B_{\lambda_p}),$$

where Φ_{λ_j} is a basis for the generalized eigenspace $\mathcal{M}_{\lambda_j}(A)$ and B_{λ_j} is the matrix defined by (2.10), $j = 1, 2, \dots, p$. Finally, let $\mathcal{M}_\Lambda(A)$ be the $m_{\lambda_1} + \dots + m_{\lambda_p}$ dimensional subspace spanned by Φ_Λ. Then the action of the semigroup $T(t)$ on $\mathcal{M}_\Lambda(A)$ for $t \in \mathbb{R}$ is given by

$$T(t)\Phi_\Lambda = \Phi_\Lambda e^{tB_\Lambda}.$$

Furthermore, there exists a subspace Q_Λ of $\mathcal{C}$ such that $T(t)Q_\Lambda \subseteq Q_\Lambda$ for all $t \geq 0$ and

$$\mathcal{C} = P_\Lambda \oplus Q_\Lambda$$

where $P_\Lambda = \{\phi \in \mathcal{C} : \phi = \Phi_\Lambda a,\ \text{for some vector } a\}$.

As a corollary we obtain that the stationary point of the system (2.5) is asymptotically stable if $\operatorname{Re}\lambda < 0$ for all $\lambda \in \Lambda$ and unstable if there exists $\lambda \in \Lambda$ such that $\operatorname{Re}\lambda > 0$.

2.3 Nonlinear autonomous RFDE

In this section we consider the nonlinear autonomous delay equation

$$\dot{x}(t) = F(x_t) \tag{2.12}$$

where F is a continuous and differentiable nonlinear mapping from $\mathcal{C}$ into $\mathbb{R}$. Let us assume that $\overline{x}$ is the stationary point of (2.12) and denote

$$L = DF(\overline{x}). \tag{2.13}$$

Then the stability properties of the stationary point $\overline{x}$ can be derived from the properties of the associated linearized problem

$$\dot{y}(t) = Ly_t. \tag{2.14}$$

More precisely, we have

Theorem 2.3. (cf. [4, p. 240]) *Let Λ be the set of roots of the characteristic equation corresponding to the linearized problem (2.14). Then the stationary point $\overline{x}$ of (2.12) is*

(i) unstable if Re $\lambda > 0$ for some root $\lambda \in \Lambda$,
(ii) (locally) exponentially stable if Re $\lambda < 0$ for all roots $\lambda \in \Lambda$.

Although this result is valuable, it does not explain the situations when there are eigevalues with both zero and negative real parts. In such cases we have to employ the centre manifold analysis.

2.4 Centre manifold analysis

Let $\overline{x} = 0$ be a stationary point of (2.12) and let L be the linear continuous operator defined by (2.13). Then (2.12) can be rewritten as

$$\dot{x} = Lx_t + g(x_t) \tag{2.15}$$

where $g \in C(\mathcal{C}, \mathbb{R})$ and

$$g(x_t) = F(x_t) - Lx_t, \qquad g(\overline{x}) = 0, \qquad Dg(\overline{x}) = 0.$$

In addition, let us assume that g is C^1 and that the linear part of (2.15),

$$\dot{x} = Lx_t \tag{2.16}$$

has m eigenvalues (counting multiplicity) with zero real parts, all other eigenvalues having negative real parts. In such a situation, Hale [8] has shown that there exists an m-dimensional invariant submanifold of the state space $\mathcal{C}$, the *centre manifold*, and that long term behavior of the solutions to the nonlinear equation is well approximated by the flow on this manifold. We briefly summarize the main result.

Let $P \subset \mathcal{C}$ be the m-dimensional subspace spanned by the solutions to (2.16) corresponding to the m zero real part eigenvalues and let Φ be its basis. Then there exists a splitting of the space $\mathcal{C} = P \oplus Q$ such that both P and Q are invariant under the flow $T(t)$ associated with the linear system (2.16).

On P, the flow $T(t)$ is equivalent to an ordinary differential equation

$$\dot{z}(t) = Bz(t)$$

with $z \in \mathbb{R}^m$ and B the $m \times m$ matrix satisfying

$$A\Phi = \Phi B. \tag{2.17}$$

(See Section 2.2 for a detailed discussion of this topic.) The eigenvalues of B correspond to the m eigenvalues of (2.16) with zero real part. The centre manifold introduced above is given by

$$M_g = \{\phi \in \mathcal{C} : \phi = \Phi z + h_g(z),\ z \text{ in neighborhood of zero in } \mathbb{R}^m\}$$

where $h_g : \mathbb{R}^m \to Q$ is such that $h_g(z) = o(z)$ for $z \to 0$. In addition, z satisfies the ordinary differential equation

$$\dot{z} = Bz + cg(\Phi z + h_g(z)). \tag{2.18}$$

In equation (2.18), B is given by (2.17) and c is determined from the solution to the equation adjoint to (2.15). Specifically,

$$c = \Psi(0)$$

where Ψ is the basis for the invariant subspace P^* of the adjoint problem corresponding to P, such that

$$\langle \Psi, \Phi \rangle = I_m. \tag{2.19}$$

In equation (2.19) symbol I_m stands for the $m \times m$ identity matrix and the bilinear form $\langle \cdot, \cdot \rangle$ is defined by

$$\langle \psi, \phi \rangle = \psi(0)\phi(0) - \int_{-h}^{0} \int_{0}^{\theta} \psi(\xi - \theta)[d\eta(\theta)]\phi(\xi)\, d\xi \tag{2.20}$$

for $\phi \in \mathcal{C}$, $\psi \in \mathcal{C}^* = C\left([0, h], \mathbb{R}\right)$.

Remark. The normalized Ψ (in the sense of (2.19)) can be obtained as follows:

$$\Psi = \langle \widetilde{\Psi}, \Phi \rangle^{-1} \widetilde{\Psi} \tag{2.21}$$

where $\widetilde{\Psi}$ is any basis for the invariant subspace of the adjoint problem.

3. ANALYSIS OF THE MODEL

This chapter analyzes some qualitative properties of the proposed model. We make use of the theory introduced in the previous chapter. We consider the delay differential equation

$$\dot{x}(t) = a(x(t) - x(t-1)) - b\,|x(t)|\,x(t), \qquad \text{for} \quad t \geq 0 \tag{3.1}$$

with the initial condition

$$x(t) = \varphi(t), \qquad \text{for} \quad -1 \leq t \leq 0 \tag{3.2}$$

where $a, b > 0$ and $\varphi \in \mathcal{C} = C\left([-1, 0], \mathbb{R}\right)$ provided by the supremum norm.

Let $\overline{x} = 0$ denote the only equilibrium of (3.1). We are interested whether this steady state is stable or unstable. The most of this chapter is devoted to this issue.

3.1 Existence, uniqueness and boundedness of the solution

The general result on existence and uniqueness of the solution x was given in the previous chapter. We proceed with the property of its boundedness.

Lemma 3.1. *Let $\|\varphi\| < 2a/b$. Then the solution $x(t)$ of the initial value problem (3.1) – (3.2) is bounded.*

Proof. Let $M = 2a/b > 0$ and let $\mathcal{M}$ be the set of all t satisfying $|x(t)| = M$. For the sake of contradiction, suppose that $x(t)$ is not bounded. Then there exists $t > 0$ such that $|x(t)| > M$. As $x(t)$ is continuous, the set $\mathcal{M}$ is not empty and it is closed. Then there exists the smallest element of $\mathcal{M}$, denoted t_0, i.e.

$$t_0 = \min_{t>0} \mathcal{M} = \min_{t>0} \left\{t : \; |x(t)| = M\right\}.$$

We have either $x(t_0) = M$ or $x(t_0) = -M$. Let us analyze the first case. The choice of t_0 implies

$$\dot{x}(t_0) \geq 0. \tag{3.3}$$

Substituting $x(t_0) = M$ into (3.1), we obtain

$$\dot{x}(t_0) = a(M - x(t_0 - 1)) - bM^2 < 2Ma - bM^2 = M(2a - bM) = 0,$$

what contradicts (3.3). Similarly, $x(t_0) = -M$ leads both to $\dot{x}(t_0) \leq 0$ and

$$\dot{x}(t_0) = a(-M - x(t_0 - 1)) + bM^2 > -2aM + bM^2 = M(-2a + bM) = 0.$$

The contradiction is apparent. □

Remark. The boundedness of the solution $x(t)$ implies its extendability to an interval containing $[-1, \infty]$.

3.2 Characteristic equation – location of the roots

The analysis of the qualitative behavior of solutions near an equilibrium starts with an analysis of the linearized equation. This reduces to the analysis of the characteristic equation. Recall that the model (3.1) is of the form

$$\dot{x}(t) = f(x(t), x(t-1)).$$

The linearization at the equilibrium $\overline{x}$ is given by

$$\dot{x}(t) = D_1 f(\overline{x}, \overline{x})x(t) + D_2 f(\overline{x}, \overline{x})x(t-1).$$

Hence, the linearization of (3.1) at the only steady state $\overline{x} = 0$ yields

$$\dot{x}(t) = ax(t) - ax(t-1) \tag{3.4}$$

with the corresponding characteristic equation

$$F(a, \lambda) = \lambda - a + ae^{-\lambda} = 0. \tag{3.5}$$

A particularly relevant question is how the location of the roots in the complex plane depends on the parameter a. We are principally interested in the number of the right half-plane roots of (3.5). Writing $\lambda = \mu + i\nu$, we obtain two real equations

$$\begin{aligned} \mu - a + ae^{-\mu}\cos\nu &= 0, \\ \nu - ae^{-\mu}\sin\nu &= 0. \end{aligned} \tag{3.6}$$

Lemma 3.2. *Let $\nu_0 \neq k\pi$, $k \in \mathbb{Z}$. Then $a > 0$, $\mu \geq 0$ implies $F(a, \mu + i\nu_0) \neq 0$.*

PROOF. For the sake of contradiction, suppose that there are real numbers $a > 0$, $\mu \geq 0$ such that $F(a, \mu + i\nu_0) = 0$. As $\sin\nu_0 \neq 0$, we can substitute $p = \frac{\nu_0}{\sin\nu_0}$. From (3.6) we obtain

$$a = pe^{\mu}. \tag{3.7}$$

In view of (3.7) we have $p > 0$. Moreover, $p > 1$ since $\sin\nu_0 < |\nu_0|$. Substituting (3.7) into the first equation of (3.6) yields

$$\mu - pe^{\mu} + p\cos\nu_0 = 0. \tag{3.8}$$

Let the real-valued function g be defined by

$$g(x) = x - pe^{x} + p\cos\nu_0.$$

We can easily verify that $g(x)$ is decreasing on the interval $[0, \infty)$ as

$$g'(x) = 1 - pe^{x} \leq 1 - p < 0$$

holds for all $x \geq 0$. Since

$$g(0) = -p + p\cos\nu_0 = p(\cos\nu_0 - 1) < 0,$$

there is no nonnegative root x of the equation $g(x) = 0$. This contradicts (3.8). □

Lemma 3.3. *Let $\nu_0 = k\pi$, $k \in \mathbb{Z} \setminus \{0\}$. Then there are no roots of $F(a, \mu + i\nu_0) = 0$.*

PROOF. Substituting $\sin \nu_0 = 0$ into the second equation of (3.6) we obtain $\nu_0 = 0$. Hence the statement of the lemma is evident. □

The previous two lemmas clearly demonstrate the non-existence of right half-plane roots of equation (3.5) of the form $\lambda = \mu + i\nu$, $\nu \neq 0$. However, we still have to discuss the case $\nu = 0$.

To find possible purely real right half-plane roots, we have to examine the real-valued function

$$h(\mu) = \mu - a + ae^{-\mu}$$

on the interval $[0, \infty)$. First observe that $h(0) = 0$ for all $a > 0$. We have

$$h'(\mu) = 1 - ae^{-\mu} = \begin{cases} > 0 & \text{for} \quad \mu > \ln a, \\ = 0 & \text{for} \quad \mu = \ln a, \\ < 0 & \text{for} \quad \mu < \ln a, \end{cases}$$

i.e., h is decreasing on $(-\infty, \ln a)$ and increasing on $(\ln a, \infty)$. To obtain another (except $\mu = 0$) positive root of $h(\mu) = 0$, it is sufficient that

$$\ln a > 0.$$

This is satisfied for all $a > 1$. The following theorem summarizes the conclusions of this section.

Theorem 3.4. *The number of zero and right half-plane roots of (3.5) depends on the parameter a in the following way:*

- $a < 1$*: simple zero root, all other roots have negative real parts;*
- $a = 1$*: double zero root, all other roots have negative real parts;*
- $a > 1$*: simple zero root and simple positive real root, all other roots have negative real parts.*

Combining Theorem 3.4 and Theorem 2.3 we obtain the following result.

Corollary 3.5. *The stationary point $\overline{x} = 0$ of (3.1) is unstable for $a > 1$.*

On the other hand, Theorem 2.3 is not able to decide on stability of $\overline{x}$ for $a \leq 1$. In this case, the characteristic equation (3.5) has either single or double zero root while all other roots have negative real parts. This is precisely the situation in which the centre manifold analysis (see Section 2.4) has to be used.

3.3 Centre manifold analysis

We first show that the presumptions of the technique are satisfied. Equation (3.1) can be rewritten into

$$\dot{x} = Lx_t + g(x_t)$$

where $L : \mathcal{C} \to \mathbb{R}$ and $g : \mathcal{C} \to \mathbb{R}$ are given by

$$L\phi = a\phi(0) - a\phi(-1), \qquad \text{and} \qquad g(\phi) = -b\,|\phi(0)|\,\phi(0). \tag{3.9}$$

The operator L can be expressed in the integral form (2.6) by putting

$$\eta(\theta) = \begin{cases} 0, & \text{for} \quad \theta = -1, \\ -a, & \text{for} \quad -1 < \theta < 0, \\ 0, & \text{for} \quad \theta = 0. \end{cases} \tag{3.10}$$

Lemma 3.6. *The function g given by (3.9) is C^1.*

PROOF. Let $L_\phi : \mathcal{C} \to \mathbb{R}$ be a bounded linear operator given by

$$L_\phi h = -2b|\phi(0)|h(0). \tag{3.11}$$

We show that $L_\phi h$ is the (Fréchet) differential of the mapping g at the point $\phi \in \mathcal{C}$. We are supposed to prove that

$$V = \lim_{\|h\|\to 0} \frac{\|g(\phi+h) - g(\phi) - L_\phi h\|}{\|h\|} = 0.$$

We have

$$U = g(\phi+h) - g(\phi) = -b|\phi(0)+h(0)|(\phi(0)+h(0)) + b|\phi(0)|\phi(0). \tag{3.12}$$

We are interested in the value of expression (3.12) for $\|h\| \to 0$. Hence, without loss of generality, we can assume that $|\phi(0)| > |h(0)|$ for $\phi(0) \neq 0$. Then we have

$$U = \begin{cases} -b\,(\phi(0)+h(0))^2 + b\phi^2(0) \;=\; -2b\phi(0)h(0) - bh^2(0), & \text{for} \quad \phi(0) > 0, \\ -b|h(0)|h(0), & \text{for} \quad \phi(0) = 0, \\ \;\; b\,(\phi(0)+h(0))^2 - b\phi^2(0) \;=\; 2b\phi(0)h(0) + bh^2(0), & \text{for} \quad \phi(0) < 0. \end{cases} \tag{3.13}$$

Substituting (3.11) and (3.13) we obtain

$$V = \lim_{\|h\|\to 0} \frac{\|-2b|\phi(0)|h(0) - b|h(0)|h(0) - L_\phi h\|}{\|h\|} = \lim_{\|h\|\to 0} \frac{bh^2(0)}{\|h\|} = 0.$$

Therefore

$$Dg(\phi)h = L_\phi h = -2b|\phi(0)|h(0)$$

is the Fréchet differential of g at the point ϕ. As Dg is continuous, the function g is C^1. □

We also have $g(\overline{x}) = 0$ and $Dg(\overline{x}) = 0$. This, together with Theorem 3.4 and Lemma 3.6, allows us to employ the centre manifold analysis for $a < 1$ and $a = 1$. We treat both cases separately.

3.3.1 Single zero eigenvalue, $a < 1$.

In this case, the dimension of the centre manifold is 1. The basis $\Phi = (\phi_1)$ of the eigenspace P is given by

$$\phi_1(\theta) = 1, \qquad \text{for} \quad -1 \le \theta \le 0.$$

As $A\phi_1 = \phi_1 B$ and

$$A\phi_1(\theta) = \begin{cases} d\phi_1(\theta)/d\theta = 0, & \text{for} \quad -1 \le \theta < 0, \\ L\phi_1 = a - a = 0, & \text{for} \quad \theta = 0, \end{cases}$$

we have $B = 0$. Let $\widetilde{\Psi} = (\widetilde{\psi}_1)$ be the basis of the invariant subspace P^* of the adjoint problem given by

$$\widetilde{\psi}_1(\theta) = 1, \qquad \text{for} \quad 0 \leq \theta \leq 1.$$

The normalized basis $\Psi = (\psi_1)$ can be derived from

$$\psi_1 = \left[\langle \widetilde{\psi}_1, \phi_1 \rangle\right]^{-1} \widetilde{\psi}_1 = \left[\langle \widetilde{\psi}_1, \phi_1 \rangle\right]^{-1}. \tag{3.14}$$

We simplify the bilinear form (2.20) for η given by (3.10). We have

$$\langle \psi, \phi \rangle = \psi(0)\phi(0) - a \int_{-1}^{0} \psi(\xi + 1)\phi(\xi)\, d\xi \tag{3.15}$$

for given $\phi \in \mathcal{C}$ and $\psi \in \mathcal{C}^* = C([0,1], \mathbb{R})$. We employ (3.14) and (3.15) to obtain

$$\psi_1 = \left[\widetilde{\psi}_1(0)\phi_1(0) - a \int_{-1}^{0} \widetilde{\psi}_1(\xi + 1)\phi_1(\xi)\, d\xi\right]^{-1} = \left[1 - a \int_{-1}^{0} d\xi\right]^{-1} = \frac{1}{1-a}.$$

Then $c = \Psi(0) = \dfrac{1}{1-a}$. Substituting all necessary elements into (2.18) we obtain the ordinary differential equation

$$\dot{z} = Bz + cg(\phi_1 z + h_g(z)) = \frac{-b}{1-a}\, |z|\, z + o(z^2), \qquad \text{for} \quad z \to 0. \tag{3.16}$$

Recall from Section 2.4 that the stability properties of this ODE are the same as of the original RFDE. We postpone the analysis of this system to Section 3.4.

3.3.2 Double zero eigenvalue, $a = 1$

In this case, the dimension of the centre manifold is 2. The basis $\Phi = (\phi_1, \phi_2)$ of the eigenspace P is given by

$$\phi_1(\theta) = 1, \qquad \phi_2(\theta) = \theta, \qquad \text{for} \quad -1 \leq \theta \leq 0. \tag{3.17}$$

As $A\phi_1 = 0$ and

$$A\phi_2(\theta) = \begin{cases} d\phi_2(\theta)/d\theta = 1, & \text{for} \quad -1 \leq \theta < 0, \\ L\phi_2 = a = 1, & \text{for} \quad \theta = 0, \end{cases}$$

we have $A\Phi = (0, \mathbf{1})$. Recall that $A\Phi = \Phi B$. Hence

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{3.18}$$

Let $\widetilde{\Psi} = \mathrm{col}(\widetilde{\psi}_1, \widetilde{\psi}_2)$ be the basis of the eigenspace P^* given by

$$\widetilde{\psi}_1(\theta) = 1, \qquad \widetilde{\psi}_2(\theta) = \theta, \qquad \text{for} \quad 0 \leq \theta \leq 1.$$

We make use of formula (3.15) to obtain

$$\begin{aligned}
\langle\widetilde{\psi}_1, \phi_1\rangle &= 1 - \int_{-1}^{0} d\xi &&= 0,\\
\langle\widetilde{\psi}_2, \phi_1\rangle &= 0 - \int_{-1}^{0} (\xi+1)\, d\xi &&= -\frac{1}{2},\\
\langle\widetilde{\psi}_1, \phi_2\rangle &= 0 - \int_{-1}^{0} \xi\, d\xi &&= \frac{1}{2},\\
\langle\widetilde{\psi}_2, \phi_2\rangle &= 0 - \int_{-1}^{0} (\xi+1)\xi\, d\xi &&= -\frac{1}{6}.
\end{aligned} \tag{3.19}$$

Substituting (3.19) into (2.21) we have

$$\Psi = \left[\langle\widetilde{\Psi}, \Phi\rangle\right]^{-1} \widetilde{\Psi} = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 1/6 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \widetilde{\psi}_1 \\ \widetilde{\psi}_2 \end{pmatrix}.$$

The normalized basis Ψ is therefore given by

$$\Psi(\theta) = \begin{pmatrix} 2/3 & -2 \\ 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \theta \end{pmatrix} = \begin{pmatrix} 2/3 - 2\theta \\ 2 \end{pmatrix}. \tag{3.20}$$

As a conclusion we obtain

$$c = \Psi(0) = \begin{pmatrix} 2/3 \\ 2 \end{pmatrix}. \tag{3.21}$$

Let $z = (x, y)^T$. Then

$$(\Phi z)(\theta) = x\phi_1(\theta) + y\phi_2(\theta) = x + \theta y, \qquad \text{for} \quad -1 \leq \theta \leq 0. \tag{3.22}$$

Substituting (3.9), (3.18), (3.21) and (3.22) into (2.18) gives, for the dynamical system on the centre manifold, the 'explicit' expression

$$\begin{aligned}
\dot{x} &= y - \frac{2b}{3}\,|x|\,x + o\left((|x| + |y|)^2\right),\\
\dot{y} &= -2b|x|x + o\left((|x| + |y|)^2\right),
\end{aligned} \tag{3.23}$$

for $(x, y) \to (0, 0)$. The analysis of the stability properties of this system is given in the following section.

3.4 Stability analysis – Lyapunov's second method

Our next task is to analyze the local stability of the systems (3.16) and (3.23). We employ Lyapunov's second method as outlined in Appendix. We first analyze equation (3.16),

$$\dot{z} = \frac{-b}{1-a}\,|z|\,z + o(z^2) \qquad \text{for} \quad z \to 0. \tag{3.24}$$

Let us consider the (positive definite) function $V(z) = z^2$. We have

$$\dot{V}(z) = \frac{d}{dt} z^2(t) = 2z\dot{z} = -\frac{2b}{1-a}\,|z|\,z^2 + w(z),$$

where $w(z) = o(|z|^3)$. In other words, for any $\varepsilon > 0$, there exists a $\delta_\varepsilon > 0$ such that $0 < |z| < \delta_\varepsilon$ implies $|w(z)| < \varepsilon|z|^3$. Let us take $\varepsilon = 2b/(1-a)$. Then there exists δ_ε such that

$$\dot{V}(z) < -\frac{2b}{1-a}|z|z^2 + \varepsilon|z|^3 = 0 \qquad \text{for} \quad 0 < |z| < \delta_\varepsilon$$

We see that $V(z)$ is a positive definite Lyapunov function for the system (3.16) such that $\dot{V}(z) < 0$ on some neighborhood of zero. Hence, according to Theorem 7.5, the zero steady state of (3.16) is asymptotically stable.

We now turn our attention to the system (3.23). Recall that it is given by

$$\begin{aligned} \dot{x} &= y - \frac{2b}{3}\,|x|\,x + \text{hot}, \\ \dot{y} &= -2b|x|x + \text{hot}, \end{aligned} \tag{3.25}$$

where hot $= o\left((|x| + |y|)^2\right)$ for $(x, y) \to (0, 0)$.

Right at the beginning of the analysis we have to admit that we have not completely resolved the question of stability of the zero equilibrium of (3.25). Nevertheless, the results below may be of some interest.

Lemma 3.7. *The zero equilibrium of the truncated system*

$$\begin{aligned} \dot{x} &= y - \frac{2b}{3}\,|x|\,x, \\ \dot{y} &= -2b|x|x, \end{aligned} \tag{3.26}$$

is asymptotically stable.

Proof. We consider the (positive definite) scalar function V,

$$V(x, y) = |x|x^2 + \frac{3}{4b}y^2. \tag{3.27}$$

By differentiating with respect to (3.26) we obtain

$$\dot{V}(x, y) = 3|x|x\dot{x} + \frac{3}{2b}y\dot{y} = 3|x|xy - 2b|x|^2x^2 - 3y|x|x = -2bx^4 \leq 0.$$

Then V is a positive definite Lyapunov function for the system (3.26) on the region $\Omega = \mathbb{R}^2$. We also have

$$E = \{(x, y) \in \Omega \,|\, \dot{V}(x, y) = 0\} \;=\; \{(x, y) \in \mathbb{R}^2 \,|\, x = 0\}.$$

It is obvious that the origin is the only invariant subset of E with respect to (3.26). This allows us to apply Theorem 7.6. Hence, the zero solution of (3.26) is asymptotically stable. □

Remark. Applying the same procedure to the system (3.25) does not resolve the stability question. We obtain

$$\dot{V}(x, y) = -2bx^4 + o\left((|x| + |y|)^2(|x|^2 + |y|)\right).$$

There is no region $\Omega \subseteq \mathbb{R}^2$ containing the origin such that $\dot{V}(x, y) \leq 0$ for all $(x, y) \in \Omega$.

Lemma 3.8. *There exists some region $\Omega \subseteq \mathbb{R}^2$ containing the origin such that within $\Omega \setminus \{0\}$ the solution to (3.26) rotates clock-wisely around the zero steady state.*

Proof. We use the transformation

$$x = r\sin\omega, \qquad y = r\cos\omega, \qquad r \geq 0, \qquad \omega \in [0, 2\pi)$$

to obtain

$$\dot{r} = r\cos\omega\sin\omega - \frac{2}{3}br^2|\sin\omega|\sin\omega\,(\sin\omega + 3\cos\omega) + o(r^2), \tag{3.28}$$

and

$$\dot{\omega} = \cos^2\omega + \frac{2}{3}br|\sin\omega|\sin\omega\,(3\sin\omega - \cos\omega) + o(r) \tag{3.29}$$

for $r > 0$. Let us concentrate our attention on equation (3.29). For $w \in \{\pi/2,\, 3\pi/2\}$ it reduces to

$$\dot{\omega} = 2br > 0.$$

For $w \notin \{\pi/2,\, 3\pi/2\}$ and for r small enough, the term $\cos^2\omega > 0$ dominates the right-hand side of (3.29). Therefore, there exists $r_0 > 0$ such that

$$\dot{\omega} > 0 \qquad \text{on} \qquad \{(r, \omega) \in (0, r_0) \times [0, 2\pi)\}.$$

This proves the statement of the lemma. □

Unfortunately, neither Lemma 3.7 nor Lemma 3.8 decides on the local stability behavior of the zero equilibrium of (3.25).

3.5 Conclusions

The following theorem summarizes the conclusions of this chapter.

Theorem 3.9. *Let $a, b > 0$ be given parameters and let $\varphi \in \mathcal{C}$ be a given initial function such that $\|\varphi\| < 2a/b$. Then there exists unique solution $x(t)$ to the initial value problem (3.1) – (3.2). This solution exists for all $t \geq 0$ and is continuous and bounded. The only stationary point $\overline{x} = 0$ of the system (3.1) is*

(i) unstable for $a > 1$,

(ii) asymptotically stable for $a < 1$.

The behavior of the system near the zero equilibrium for $a = 1$ remains an open question. We hope to tackle this problem in the near future.

4. NUMERICAL SIMULATIONS

The previous chapter has been almost entirely devoted to the (local) stability analysis of the equilibrium of the equation

$$\dot{x}(t) = a(x(t) - x(t-1)) - b|x(t)|x(t), \tag{4.1}$$

where $a,\ b > 0$. We have concluded that the stationary point $\overline{x} = 0$ is locally asymptotically stable for $a < 1$ and unstable for $a > 1$. We now continue our analysis of the model with several numerical simulations.

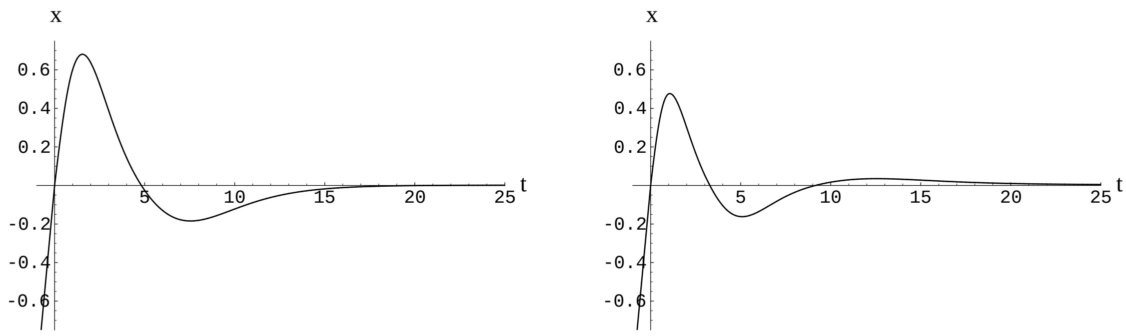
We use simple Euler method[1] with step size $h = 0.005$ to obtain numerical solutions to equation (4.1). Throughout this chapter, we work with the initial condition[2]

$$\varphi(\theta) = \theta \qquad \text{for} \qquad -1 \leq \theta \leq 0.$$

We treat qualitatively different cases $a < 1$, $a = 1$ and $a > 1$ separately.

4.1 Asymptotically stable trajectories, $a < 1$

Asymptotically stable trajectories are expected for $a < 1$. The following results are obtained for $a = 0.8$.



(i) for $b = 0.5$, $M = 3.2$ (ii) for $b = 1.5$, $M = 1.07$

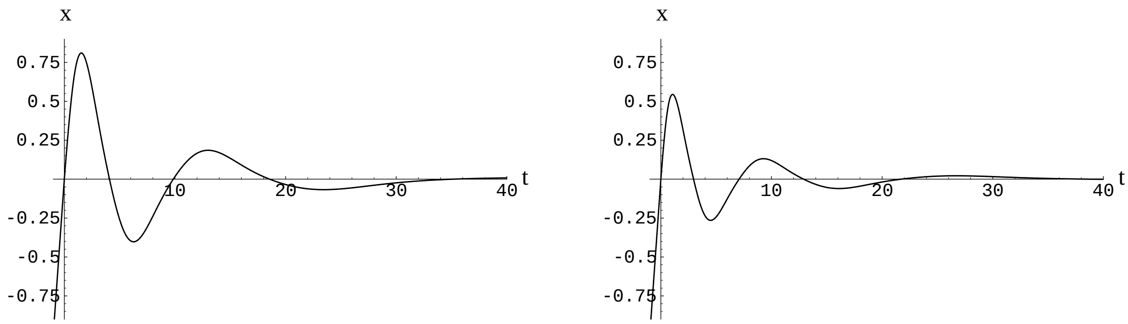
Figure 4.1: Numerical solutions for $a = 0.8$

The above figure is consistent with the conclusions of Chapter 3. We see that the simulations indicate asymptotically stable trajectories.

[1] We have also used more demanding predictor-corrector (PECECE) scheme with 4 step Adams-Bashford predictor and 4 step Adams-Moulton corrector. We have obtained the same results.

[2] We have also worked with other initial conditions. Althought the choice of this condition has affected the solution trajectories a lot, it has not changed the asymptotic behavior of the solutions.

Parameter b seems to affect the amplitude of the oscillations in the following way: the larger b, the smaller the amplitude. This is consistent with Lemma 3.1. (Recall that we have proved that the solution is bounded above by $M = 2a/b$.) The same conclusions can be drawn from the simulations for $a = 0.9$.



(i) for $b = 0.5$, $M = 3.6$ (ii) for $b = 1.5$, $M = 1.2$

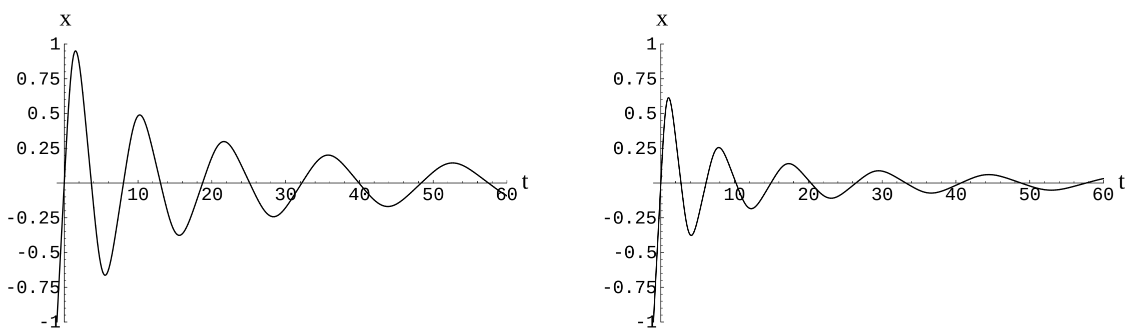
Figure 4.2: Numerical solutions for $a = 0.9$

The convergence to the zero steady state now appears to be slower.

A number of other simulations have been made for various values of parameters $0 < a < 1$ and $b > 0$ and various initial functions φ. Although the results have been different in many aspects, all simulations have indicated globally asymptotically stable trajectories. The speed of the convergence has been affected by the parameter a – the larger a, the slower the convergence (holding b and φ constant).

4.2 The boundary point $a = 1$

Numerical solutions to the initial value problem (3.1) – (3.2) for $a = 1$ are of particular interest to us – keep in mind that in this case, we have not resolved the question of the zero steady state stability yet (see Chapter 3). Two simulations are depicted on the following graphs.



(i) for $b = 0.5$, $M = 4$ (ii) for $b = 1.5$, $M = 1.33$

Figure 4.3: Numerical solutions for $a = 1$

The solutions are very similar to those obtained for values $a < 1$ close to 1. However, the convergence to the zero steady state, if any, is slower. A larger interval is necessary to determine the behavior of the solutions near zero for $t \to \infty$.

The following figure depicts the solution on the interval $[0, 300]$.

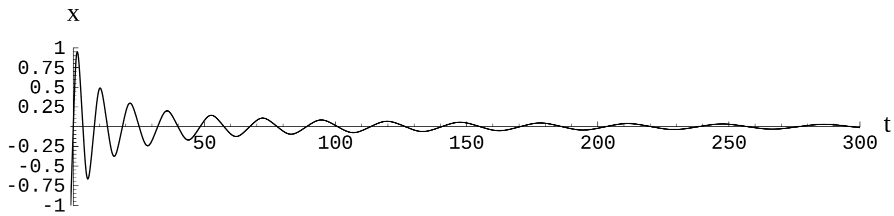
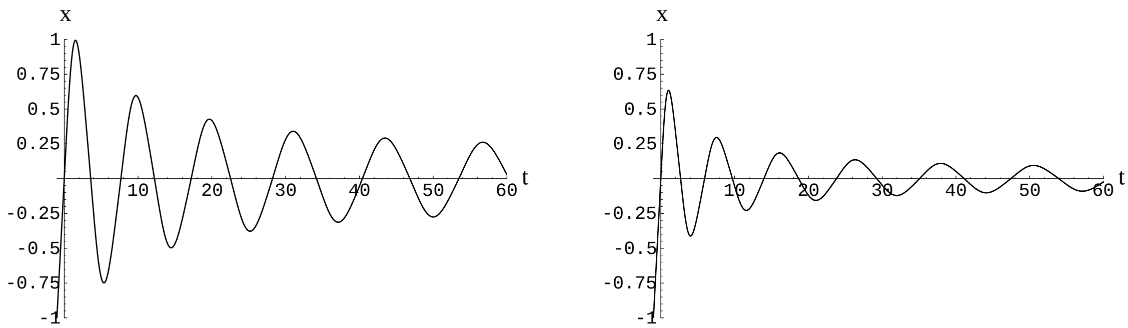


Figure 4.4: Numerical solution for $a = 1$ and $b = 0.5$

The above simulation indicates globally asymptotically stable trajectory. The solution appears to be oscillatory. Of course, this observation is nothing more than the hypothesis about the asymptotic behavior.

4.3 Locally unstable trajectories, $a > 1$

Recall that for the case $a > 1$, the solutions to (4.1) are expected to exhibit locally unstable behavior. In this section we verify this conjecture. The following results are obtained for the case $a = 1.03$.



(i) for $b = 0.5$, $M = 4.12$

(ii) for $b = 1.5$, $M = 1.37$

Figure 4.5: Numerical solutions for $a = 1.03$

Again, we can observe that the larger the parameter b, the smaller are the amplitudes of the oscillations. On the other hand, the interval $[0, 60]$ is not sufficient to determine the asymptotic behavior of the solutions. We use larger interval.

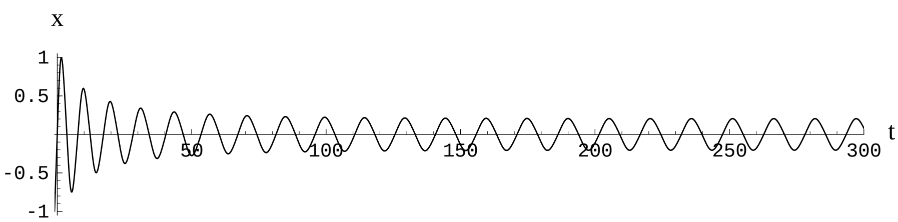
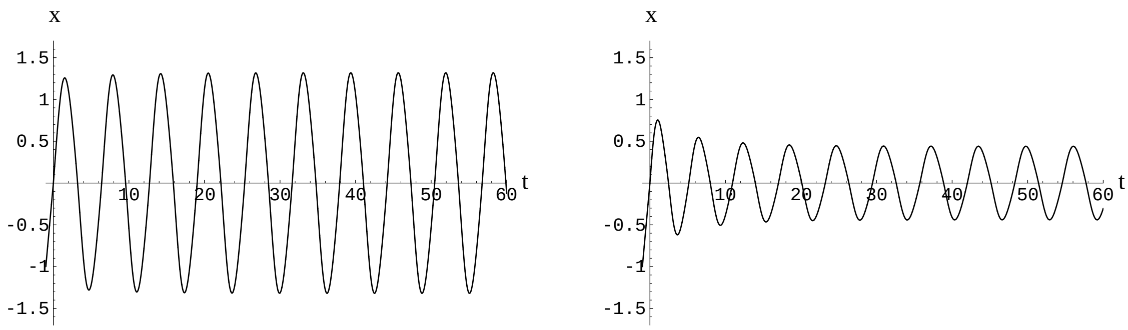


Figure 4.6: Numerical solution for $a = 1.03$ and $b = 0.5$

We are now able to contrast figures 4.4 and 4.6. In the latter case, the trajectory seems to be (locally) unstable with periodic oscillations.

The local instability (and periodicity) result is more apparent for larger values of the parameter a, such as $a = 1.2$.



(i) for $b = 0.5$, $M = 4.8$

(ii) for $b = 1.5$, $M = 1.6$

Figure 4.7: Numerical solutions for $a = 1.2$

As in the previous section, a number of simulations in addition to those depicted here have been made. All indicate locally unstable and (after some time) globally stable periodic trajectories. The periods and the amplitudes appear to depend on a and b in the following way: The larger a, the larger the amplitude and the smaller the period length of the oscillation. The larger b, the smaller the amplitude. The effect of b on the period length is not very significant.

Remark. The outlined dependence allows us to construct oscillations with different periods and amplitudes. For example, relatively large period length of approximately 25 can be obtained for $a = 1.01$.

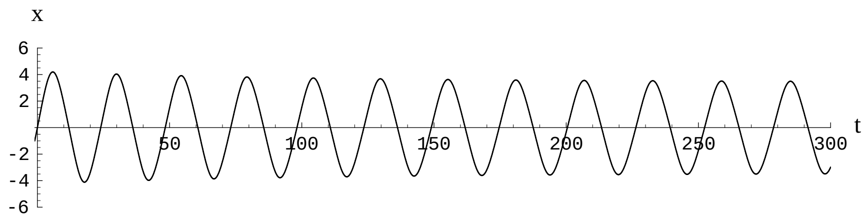


Figure 4.8: Numerical solution for $a = 1.01$ and $b = 0.01$

The amplitude of the oscillations can be easily controlled by b – for $b = 0.1$ the amplitude would be approximately 0.35, for $b = 1$ the amplitude would be approximately 0.035.

4.4 Conclusions

The results obtained by numerical simulations are consistent with the theoretical results given by Theorem 3.9. For $a < 1$ (but also for $a = 1$), numerical simulations indicate a globally asymptotically stable trajectory. For $a > 1$, numerical simulations indicate a globally stable periodic trajectory. Both the amplitude and the period of the oscillations are determined by the parameters a and b.

The case $a > 1$ is the most interesting one from the viewpoint of a possible economic application of the model – the zero steady state is unstable, indicating periodic oscillations. We will be in search of such a behavior while analyzing real data in the following chapter.

5. REAL DATA OBSERVATIONS

This chapter introduces several real data observations. It summarizes the effort to find any evidence that would support the idea of our model. On the other hand, we have to stress that it is beyond the scope of this paper to provide rigorous econometric analysis of the issue.

Recall that we wish to model market exchange rate $S(t)$ by

$$S(t) = f(\theta(t)) + x(t), \tag{5.1}$$

where $f(\theta(t))$ holds for 'natural' exchange rate given the state of fundamentals $\theta(t)$ and where $x(t)$ is driven by

$$\dot{x}(t) = a(x(t) - x(t-1)) - b|x(t)|x(t). \tag{5.2}$$

Numerical simulations (see Chapter 4) indicate that such a model implies periodic oscillations for the variable $x(t)$ (in the case $a > 1$). Our task is to explore whether observed data exhibit behavior consistent with this conclusion.

We will work with the daily exchange rates S_k on the eurodollar market in the year 2002. The discrete form of (5.1) can be rewritten into

$$x_k = S_k - f(\theta_k). \tag{5.3}$$

We are interested whether the time series x_k contains oscillations consistent with the model. A common way of identifying oscillations of various frequencies in the list of data is to find its discrete Fourier transformation (DFT), or spectrum. The following paragraph briefly introduces the idea of this approach.

Let f_k, $k = 0, 1, \ldots, N-1$ be a sample of N values such that the series extended outside the range $\{0, \ldots, N-1\}$ is N-periodic. Then the DFT is defined as follows

$$F(w) = \sum_{k=0}^{N-1} f_k e^{\frac{2\pi i k w}{N}}, \qquad \text{for} \qquad w = 0, 1, \ldots, N-1. \tag{5.4}$$

The amplitude of the oscillation with a period length N/w (i.e., one that occurs w times in the range N) is given by

$$b_w = \left| \frac{2}{N} F(w) \right|, \qquad \text{for} \qquad w = 1, 2, \ldots, N/2.$$

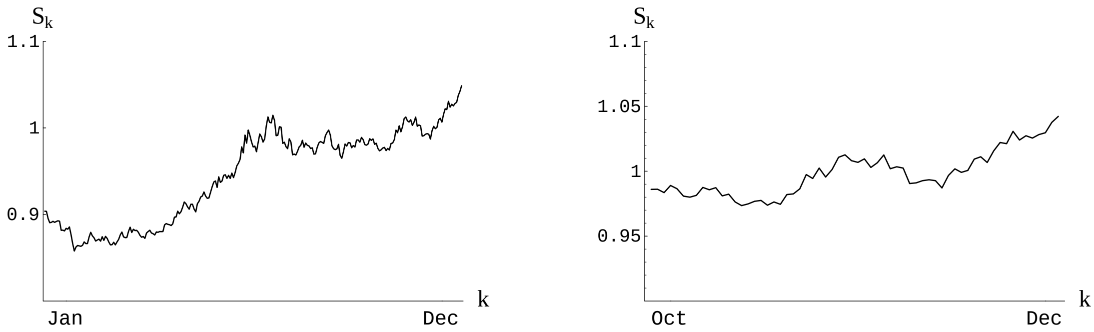
This allows us to identify any relevant oscillations in the time series f_k simply by calculating coefficients b_w for various frequencies w.

We apply this procedure on the data set x_k. Since it is given by (5.3), we have to estimate $f(\theta_k)$ first. Unfortunately, this is not simple as the concept of the 'natural' exchange rate is only theoretical. Several possible approaches can be adopted to tackle the problem of estimating $f(\theta_k)$. We will be trying the following three

(i) We assume that $f(\theta(t))$ is constant within a certain time interval (such as one quarter);
(ii) We assume that $f(\theta(t))$ has a linear trend and apply the linear regression to estimate it;
(iii) We employ the moving average technique to obtain $f(\theta(t))$.

5.1 Case One – Assumption of constant 'natural' exchange rate

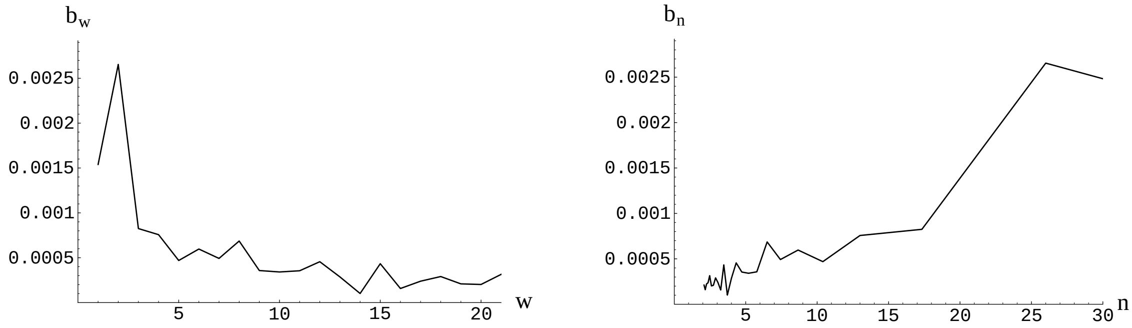
In this section we assume that the 'natural' exchange rate does not change within a certain, relatively short interval. There is some logic behind this assumption: The period of two or three months is too short to cause any significant change of economic fundamentals. The following graphs depict the data we will be working with.



(i) Year 2002, 255 working days (ii) Winter 2002, 64 working days

Figure 5.1: USD/EUR daily exchange rates

Let us concentrate our attention on the last quarter of the year 2002. The assumption of a fixed 'natural' exchange rate within this period allows us to calculate the Fourier transformation directly for the time series S_k (the subtraction of the constant term $f(\theta_k)$ would result only in the change of the zero frequency term $F(0)$ in (5.4)). The result of the application the DFT method on a suitable subinterval[1] is shown on the following two graphs.



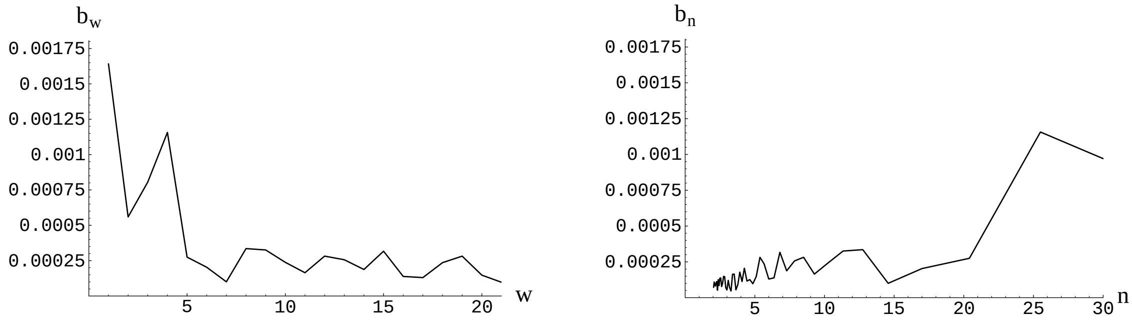
(i) Amplitude against frequency (ii) Amplitude against period

Figure 5.2: DFT analysis – power spectrum for the last 52 working days of 2002

The spectrum diagrams indicate that there is a relatively significant periodic oscillation with frequency 2 (in an interval 52 working days long) corresponding to a 26 working days period.

[1] Keep in mind that the method makes sense only if applied on a periodic time series. Therefore, we have employed the DFT procedure not only on the whole series of 64 winter data, but also on a number of relatively large subintervals of this period. Figure 5.2 depicts interesting result obtained for a period of 52 working days.

We might ask whether such a period is observed also in the case of a larger interval. The answer is, surprisingly, yes. While examining last 102 working days[2] of the year 2002, we have encountered the following behavior.



(i) Amplitude against frequency (ii) Amplitude against period

Figure 5.3: DFT analysis – power spectrum for the last 102 working days of 2002

Again, the diagrams indicate possible periodic oscillation with 25-26 working days long period. Without exaggerating the significance of this result we can conclude this section with an observation that there might be some support for our model.

5.2 Case Two – 'Natural' exchange rate estimated by linear regression

The assumption of a linear trend of observed economic variable belongs to the one most commonly used. We use this approach in this section.

We apply the linear regression on the data S_k to obtain

$$f(\theta_k) \approx 0.86055 + 0.000664244k \tag{5.5}$$

with relatively high coefficient of determination $R^2 = 0.85$. Substituting (5.5) into (5.3) for $k = 0, 1, \ldots, 254$ we obtain a time series of 255 deviations of exchange rates x_k from its 'natural' level. The data are shown on the following figure.

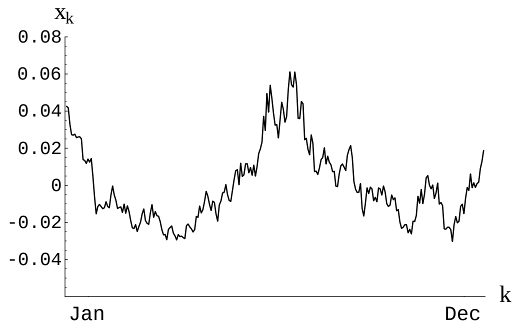
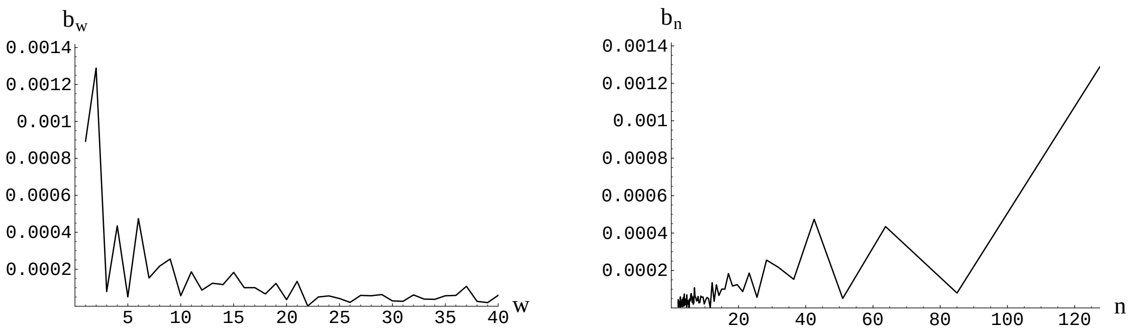


Figure 5.4: Deviations of exchange rates from estimated 'natural' level, 2002

The application of the DFT method on these data yields the following results.

[2] Of course, to do this, the winter data were not enough.



(i) Amplitude against frequency (ii) Amplitude against period

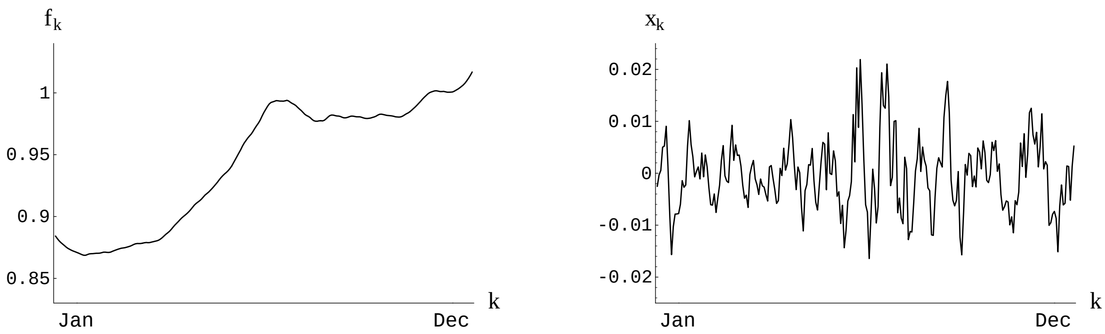
Figure 5.5: DFT analysis – power spectrum for deviations, 255 values

The above spectrum diagram indicates several relatively significant periodic oscillations (with periods 128, 43 and 64 working days, respectively). Unfortunately, we have to admit that further examination has not verified this result. More precisely, different subintervals have indicated different oscillations.

5.3 Case Three – 'Natural' exchange rate estimated by MA technique

An alternative way of smoothing out local fluctuations in a given time series of economic data is to use an (m-term) moving average technique. The approach allows us, if properly chosen parameter m, to eliminate any cyclical or random influences. The smoothed data should therefore well approximate the 'natural' exchange rate.

We will work with a simple 20-term moving average[3]. We subtract the estimated 'natural' exchange rates, $f_k \approx f(\theta_k)$, from the observed market rates S_k to obtain fluctuations x_k. We wish to analyze whether these deviations exhibit any periodic behavior. Both the natural exchange rates and the fluctuations are depicted on the following figure.



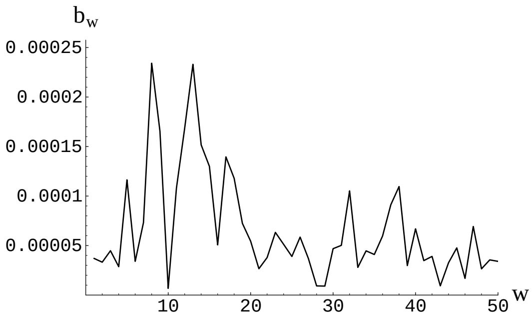
(i) 'Natural' exchange rate (ii) Fluctuations

Figure 5.6: Estimated 'natural' exchange rates and resulting fluctuations

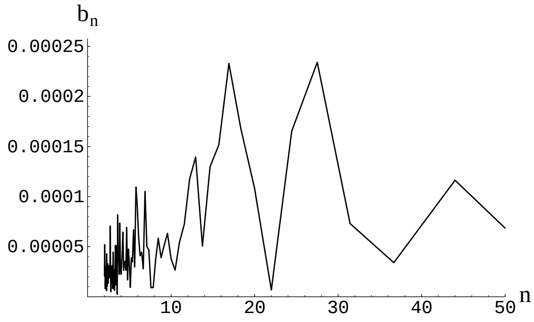
It is beyond the scope of this paper to present all calculations that have been made. We therefore only summarize our effort. We have employed the DFT method on a large number of different time series – subintervals of the series depicted on Figure 5.7 (ii). Although we have obtained

[3] We have worked with a number of different m-term moving averages. The results obtained for m near 20 has appeared to be the most revealing.

many different results, several ones were quite similar. This group has indicated two relatively significant periodic oscillations with period lengths 16-18 and 26-28 working days. One of these results is depicted on the following figure.



(i) Amplitude against frequency



(ii) Amplitude against period

Figure 5.7: DFT analysis – power spectrum for fluctuations, 220 values

Similar power spectra have been observed for a number of other subintervals[4]. On the other hand, we see that the identified oscillations are not very dominant.

5.4 Conclusions

We are now able to conclude this chapter. We have made a large number of experiments (far much more than those mentioned in this paper) employing different methods on different time intervals. We were interested whether the estimated deviations x_k exhibit behavior consistent with the model, i.e. whether the variable x_k periodically oscillates around zero.

We have to admit that none of the observations has fully supported this hypothesis. No truly dominant periodic oscillation has been identified. On the other hand, two or three relatively significant oscillations have been observed – perhaps the most significant one being that with a period approximately 25 working days long. The problem of possible more oscillations of different period lengths could be resolved by adding more processes of the form (5.2) with suitably chosen parameters[5] a and b.

[4] As well as for other values of m.

[5] Keep in mind that a and b determine the period and the amplitude of the oscillations

6. CONCLUSIONS

In this paper we have introduced a delay differential equation model of oscillations of exchange rate. The model is given by

$$\dot{x}(t) = a(x(t) - x(t-1)) - b|x(t)|x(t), \tag{6.1}$$

where $x(t)$ denotes the deviation of the market exchange rate from the 'natural' exchange rate. We have examined how the stability of the only steady state $\overline{x} = 0$ depends on the parameters a and b. For $a > 1$, the stationary point $\overline{x}$ is locally unstable and numerical simulations indicate a globally stable periodic trajectory. For $a \leq 1$, we have employed the centre manifold analysis. We have proved that the equilibrium $\overline{x}$ is locally asymptotically stable for $a < 1$. Numerical simulations indicate that this stability is global.

From the mathematical viewpoint, two interesting questions remain open: (i) the proof of the stability or instability of the zero steady state for $a = 1$ and (ii) the proof of the periodicity of the solutions for $a > 1$. We hope to address these questions in the near future.

From the viewpoint of a possible economic application of the model, only a little evidence has been found in favor of it. Nevertheless, this is not surprising – the model is too simple. To tackle the problem of possible more oscillations of different frequencies, more processes of the form (6.1) can be added. Another improvement of the model could be adding random term either in equation (6.1) or directly in equation (1.1). On the other hand, such an improvement would make the analysis of the model much more difficult.

APPENDIX

Lyapunov's second method

In the analysis of the stability of the systems (3.16) and (3.23) (see Section 3.4), we have made use of Lyapunov's second method (also called *direct method*). This appendix contains a brief summary of the theory employed. For a more detailed account, including the proofs of the following results, we refer for example to Brauer, Nohel [2].

Let $U \subset \mathbb{R}^n$ be open, $f \in C^1(U, \mathbb{R}^n)$. We will consider autonomous differential system

$$\dot{x} = f(x). \tag{7.1}$$

Throughout this appendix we assume that U contains the origin and that $\overline{x} = 0$ is an isolated fixed point of (7.1).

Definition 7.1. *The scalar function $V(x)$ is said to be positive definite on the set $\Omega \in \mathbb{R}^n$ if and only if $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$ and $x \in \Omega$.*

Definition 7.2. *The scalar function $V(x)$ is said to be negative definite on the set $\Omega \in \mathbb{R}^n$ if and only if $-V(x)$ is positive definite on Ω.*

Definition 7.3. *The scalar function $V \in C^1(\mathbb{R}^n, \mathbb{R})$ is said to be Lyapunov function at $\overline{x}$ for the system (7.1) if there exists some region $\Omega \subseteq R^n$ containing the origin such that $x \in \Omega$ implies $\dot{V}(x) \leq 0$, where*

$$\dot{V}(x) = \frac{d}{dt}V(x(t)) = \frac{\partial V}{\partial x_1}(x) f_1(x) + \cdots + \frac{\partial V}{\partial x_n}(x) f_n(x). \tag{7.2}$$

We are now ready to introduce the main results.

Theorem 7.4. *Let there exist a positive definite Lyapunov function $V(x)$ for the system (7.1) on some region Ω containing the origin. Then the zero solution of (7.1) is stable.*

Theorem 7.5. *Let there exist a positive definite Lyapunov function $V(x)$ for the system (7.1) such that $\dot{V}(x)$ is negative definite on some region Ω containing the origin. Then the zero solution of (7.1) is asymptotically stable.*

Theorem 7.6. *Let there exist a positive definite Lyapunov function $V(x)$ for the system (7.1) on some region Ω containing the origin. Let the origin be the only invariant subset (with respect to (7.1)) of the set $E = \{x \in \Omega \mid \dot{V}(x) = 0\}$. Then the zero solution of (7.1) is asymptotically stable.*

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