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COMMUTING FLOW MODELS

Master thesis

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"I declare this thesis has been written by myself with

help of my supervisor and the referred literature."

"I would like to express special thanks

to my parents who have supported me in my study,

to my supervisor Doc. Boďa, to Prof. Möller, University of Regensburg

and Mr. Kušnier for their helpful advice."

Tatiana T.

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1 Introduction

Transportation modelling is now a widely spread discipline in urban planning. Its long history has its roots in social sciences and goes back into the second half of the nineteenth century. It was primary used to explain the phenomena of city growth, migration and other social problems concerning urban development.

The first models were based on gravitational interaction, derived from its similarity to the famous Newton's gravity law. A great "boom" of gravity models was in the sixties years of the twentieth century. Many models were created to explain traffic flows, urban systems, trade flows and other areas [18].

The development of urban models continued. Many spatial models have been developed. We now divide these models into many categories. Two main categories are spatial interaction models (represented by gravity model) and spatial choice model (represented by logit models).

As the popularity of spatial models has grown, various areas of use have been found for these models. The models are implemented not only in urban planning (road planning, migration, commuting flows, etc.), but also in international trade, marketing, production analysis and other socio-economic disciplines.

This thesis deals with different models that are used in transportation analysis. The aim of the thesis is to provide an overview over different types of commuting models and then to apply selected models on real commuting flow data.

Two types of models (gravity model and maximum entropy model) are used for estimation of commuting flows among selected regions in Slovakia. The models are estimated with different estimation techniques and then the results are compared.

The thesis is divided into 9 chapters. Introduction is followed by overview of models that have been invented for transportation analysis. Then the thesis proceeds with chapter about gravity model. The gravity model is described in detail and its variations and estimation techniques are introduced. Chapter 4 provides information on maximum entropy models and ways of their estimations. In the chapter 5, we discuss different issues concerning data: its source, problems in implementation and others. Chapter 6 is devoted to application of selected models and estimation results. These results are then compared in the following section. Bibliography overview is presented in the part 8 and symbols used in the thesis are explained in the next part.

Source data sets and estimated data is enclosed in Annex A and B. All data sets are enclosed in Excel data files and selected sets are printed at the end of thesis in Annex. In Annex C and D source codes of programs used in the thesis are enclosed.

2 Commuting Models - Overview

The aim of chapter two is to introduce basic concepts in commuting flow estimation theory.

Commuting flow models can be divided into two groups of models: spatial interaction models and spatial choice models.

First spatial interaction models were founded in the middle of 19th century and until now they remain widely used. Mostly used models are gravity model and maximum entropy model. These models need macro-data for estimation.

Spatial choice models have gained on popularity within last years. Models are based on the concept of utility maximisation in decision making process of each individual. The most important models are random utility models and logit models. Spatial choice models are formulated using microeconomic data (e.g. data on individual preferences).

2.1 Spatial Interaction Models

Spatial interaction models estimate flows according to spatial characteristics of commuting as for example distance between commuting cities and relative attraction of origin and destination city. Estimation of these models is made on the base of aggregate data (e.g. population of commuting regions[1]). We discuss in this chapter, two models: gravity model and maximum entropy model.

2.1.1 Gravity Model

The gravity model was introduced as the first model for commuting flows. It is a baseline for other commuting models. By now many other concepts have been developed, but gravity model remains very important for commuting flow theory and practice.

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[1] In the thesis, the notions region, city, county and area are used as synonyms for origin / destination commuting areas

Concept of gravity model is based on physical Newton's law. Newton's law computes the gravity attraction between two objects. The force of attraction is proportional to the product of the masses of the two objects and inversely proportional to the square of their distance

$$F = k \frac{m_1 m_2}{d_{12}^2} . \tag{2.1}$$

In the context of commuting flows we can interpret the force *F* as the number of commuters commuting between two cities, *m* represents the mass of attraction of the city (e.g. population), distance *d* between two objects as the distance between the cities and constant *k* is a scaling constant. The measure of exact square of distance is very limiting and therefore not appropriate for reality. We introduce constant γ as a frictional effect of distance. We get so-called basic gravity model [12]

$$T_{ij} = K \frac{W_i W_j}{d_{ij}^{\gamma}} , \tag{2.2}$$

where volume of flows between each pair of cities I and J (T_{ij}) is proportional to the product of population of the two cities $W_i.W_j$ and inversely proportional to distance between the two cities $d_{ij.}$

There are many variations of the basic gravity model. The variations are discussed in the chapter 3. These models take into consideration regional specifics, inflow and outflow constraints, different distance functions and other specific characteristics.

2.1.2 Maximum Entropy Model

Maximum entropy is used as a way of predicting traffic flows on the base of commuting flow matrix[2].

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[2] Also called O-D Matrix, Origin-destination matrix, or trip matrix

Because in practical estimation of commuting systems we often do not have enough information for description of the O-D matrix, we try to estimate it with help of probability theory. The number of trips from origin I to destination J is estimated as the most probable state of the commuting system given by observed total number of trips from origin J, on the total number of commuters commuting to the destination J and total amount of cost that commuters expense.

The probability of the system is represented by system's entropy (2.3)

$$W = H(N) = -\sum_{i=1}^{n}\sum_{j=1}^{n} T_{ij} . \ln T_{ij} \ . \qquad (2.3)$$

We maximise entropy function with unknown flows T_{ij} using additional constraints: marginal origin and destination flows and total commuting costs. And so we get estimated origin - destination commuting flow matrix.

The presented model represents classical form of maximum entropy concept. Other variations can be introduced. These variations are further explained in the chapter 4.

2.2 Spatial Choice Models

This section deals with concepts of spatial choice models. The spatial choice models try to explain commuting flows on individual decision level. They associate commuting with behavioural theory of decision making. Basic concepts in this theory are utility function and probability of choice of commuting to a given. Models are used for microeconomic data, which include factors influencing decision making process of each individual (e.g. income level, commuting distance, travel time, characteristics of individual (age, education, gender). There are two types of models non-hierarchical and hierarchical model. The latter describes decision-making process as staged process.

These models are difficult to implement because the utility function is not easy to calculate. Utility function can be computed using different variables as measures of utility: travel time, travel cost related to distance to work, wage and non-monetary factors [4]. With development of computation techniques and data surveys, these disadvantages can be overcome and modelling with spatial choice models is more effective.

2.2.1 Random Utility Model

Random utility model is a baseline for spatial choice models. It is built on the following assumption [12]. It tries to characterise decision choice of each commuter:

1. each commuter is faced with discrete set of choice alternatives - choice is made or not made (i.e. to travel to the city J or not to travel);
2. individual will decide for the option from all available options, which maximises his/her utility (e.g. to which city to commute);
3. choices are made in probabilistic fashion - each individual has a likelihood of making each decision (probability of each individual to choose a given city to commute);
4. utility of decision has 2 components - strict utility and stochastic utility.

Random utility model formula is based on utility maximisation[3] (U_{ij}), where utility function (2.4b) is composed of deterministic part V_{ij} and stochastic term E_{ij}

$$P_{ik} = \Pr[V_{ik} + E_{ik} > V_{ij} + E_{ij}], \forall k \neq j, j = 1,...,n, \qquad (2.4a)$$

$$U_{ij} = V_{ij} + E_{ij}. \qquad (2.4b)$$

V_{ij} represent strict utility components, which are represented by relevant observed variables and E_{ik} are stochastic components of the utility function.

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[3] Utility maximisation: an individual chooses that option which brings him/her maximum utility from all possibilities: $U_{ik} > U_{ij}$ for each j different from k, and for each i.

2.2.2 Non-hierarchical Logit Models (Multinomial Logit Model)

Logit models are derived from random utility model. Logit model uses decision choice as a function of utility of choosing one alternative over another. Logit model assumes that E_{ij} term in (2.4a) follows Weibull distribution.

Using this assumption we get McFadden's logit model [12]:

$$P_{ik} = \frac{\exp[V_{ik}(X_k, S_i)]}{\sum_{j=1}^{n} \exp[V_{ij}(X_j, S_i)]} \,, \qquad (2.5)$$

where X_i represents set of choice specific attributes (e.g. time, cost, distance) and S_i individual-specific attributes (income, car ownership).

McFadden's model computes probability of choosing an alternative k over other alternatives as a function of all alternatives and commuter's individual preferences.

Andersen [1] estimates function V_{ij} as linear function of selected characteristics of choice, e.g. travelling cost, wage difference, tax, etc.

$$V_{ij} = \sum_{k=1}^{I} \beta_k X_{kj/i} + \sum_{k=I}^{I1} \beta_k X_{kj} \,. \qquad (2.6)^4$$

2.2.3 Hierarchical Models - Nested Logit Models

Nested logit model is a hierarchical logit model, where each commuter chooses his/her travelling destination upon a hierarchy of choices. Probability of each decision is than a conditional probability.

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4 Constants can be added to extent the model, see Berglund, Lundqvist: Barrier in Travel models

Travel choice stages are [12]:

1. whether or not to make a trip (commuter commutes or not);

2. where to go (into which city to commute);

3. by what mode (which vehicle to use for transportation).

The estimation process begins with the last step in hierarchy follows to the start in order to ensure that the strict utilities are preserved throughout the process

$$P_{is} = \frac{\exp(V_{is})[\sum_{k\in s}^{n} \exp(V_{ik})]^{\sigma}}{\sum_{x}^{n} \exp(V_{is})[\sum_{k\in s}^{n} \exp(V_{ik})]^{\sigma}}, \tag{2.8}$$

where P_{is} is probability that decision maker will select a particular spatial cluster s to focus its decision in, σ represents the extent to which decision makers process their information hierarchically and ranges from 0 to 1, $\sum_{k\in s}^{n} \exp(V_{ik})$ represents attractiveness of a cluster as a function of individual alternatives available within a cluster [12].

Likelihood of selecting particular alternative k within selected cluster s is:

$$P_{ik\in s} = \frac{\exp(V_{ik})}{\sum_{k\in s}^{n} \exp(V_{ik})} \tag{2.9}$$

and probability of deciding for k from all alternatives:

$$P_{ik} = P_{is} P_{ik\in s}. \tag{2.10}$$

3 Gravity Models and Their Estimation

Basics of gravity models were introduced in the chapter 2. In this chapter we present modifications of original gravity model concept, its extensions and ways of estimation.

3.1 Classical Model - Constrained and Unconstrained

In general commuting flow between origin I and destination J can be represented as function of origin and destination characteristics and distance between them (3.1)

$$T_{ij} = F(f_i, f_j, c_{ij}) \text{ ,} \tag{3.1}$$

where T_{ij} represents commuting flow from origin I to destination, f_i, f_j represent attraction functions of origin / destination and c_{ij} is distance deterrence between them. It is expected that the number of commuters depends proportionally on f_i, f_j and adverse proportionally on c_{ij}.

The most widely spread approach to this basic concept is Gravity function (3.2) annalogical to (2.2). In this function, number of commuters depends on mass of origin and destination regions (W_i, W_j), function of region distance ($F(c_{ij})$) and a constant term (k)

$$T_{ij} = k \frac{W_i \, W_j}{F(c_{ij})} \text{ .} \tag{3.2}$$

The deterrence function $F(c_{ij})$ can have different forms. The form of deterrence function should be adjusted according to the data type, structure of commuting regions and commuting behaviour.

Mostly two types of deterrence function are used:

1. Power function: $F(c_{ij}) = c_{ij}^{\gamma}$, (3.3)

where γ represents parameter of distance deterrence;

2. Exponential function: $F(c_{ij}) = \exp(\gamma c_{ij})$, (3.4)

where γ represents parameter of distance deterrence.

Apart from the previous deterrence functions, Glen, Thorsen and Uboe [11] propose also logistic specification of deterrence function[5].

3.1.1 Gravity Model with Power Function

The classical unconstrained gravity equation with deterrence function in form of power function is given by the following formula (3.5)

$$T_{ij} = K \frac{W_i W_j}{c_{ij}^{\gamma}} \quad . \tag{3.5}$$

We often set additional constraints to the gravity model. These constraints force the model to leave the number of commuters commuting from a origin I and/or commuting to destination J pre-set. This number is computed from original data in the following way:

1. origin constraint $\sum_{j=1}^{n} T_{ij} = O_i$; (3.6)

2. destination constraint $\sum_{i=1}^{n} T_{ij} = D_j$. (3.7)

If only one of the constraints is met we call the model origin/destination constrained. If both constraints are set we call the model doubly constrained. According to type of constraint the model to meet the constraints [12].

1. origin constrained

$$T_{ij} = A_i O_i W_j c_{ij}^{-\gamma}, \qquad \text{where } A_i = [\sum_{j=1}^{n} W_j c_{ij}^{-\gamma}]^{-1}; \tag{3.8a, 3.8b}$$

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5 Thorsen J and col.: A network approach to commuting, Journal of regional science, Vol. 39,. No. 1, p. 73-101

2. destination constrained

$$T_{ij} = B_j D_j W_i c_{ij}^{-\gamma}, \qquad \text{where } B_j = [\sum_{i=1}^{n} W_i c_{ij}^{-\gamma}]^{-1}; \qquad (3.9a, 3.9b)$$

3. doubly constrained

$$T_{ij} = A_i O_i B_j D_j c_{ij}^{-\gamma}, \qquad (3.10)$$

where A_i and B_j are set in (3.8b), (3.9b).

3.1.2 Gravity Model with Exponential Function

Analogically to (3.5) we can formulate gravity model with exponential function in the following way:

$$T_{ij} = k \frac{W_i W_j}{e^{\gamma c_{ij}}}. \qquad (3.11)$$

The unconstrained model can be extended with origin and/or destination constraints (3.6), (3.7). Analogically to the constrained gravity models with power function we get following models [12]:

1. origin constrained

$$T_{ij} = A_i O_i W_j \exp(-\gamma c_{ij}), \text{ where } A_i = [\sum_{j=1}^{n} W_j \exp(-\gamma c_{ij})]^{-1}; \qquad (3.12a, 3.12b)$$

2. destination constrained

$$T_{ij} = B_j D_j W_i \exp(-\gamma c_{ij}), \text{ where } B_j = [\sum_{i=1}^{n} W_i \exp(-\gamma c_{ij})]^{-1}; \qquad (3.13a, 3.13b)$$

3. doubly constrained

$$T_{ij} = A_i O_i B_j D_j \exp(-\gamma c_{ij}), \qquad (3.14)$$

where A_i and B_j are set in (3.12b), (3.13b).

3.2 Extensions of Gravity Model

Classical model with power of W_i and W_j equal to 1 does not explain real commuting flows or other estimated flows satisfactorily. It is not realistic that attraction of origin and destination area is explained simply by their mass[6]. Therefore we estimate gravity model with the following formula:

$$T_{ij} = K \frac{W_i^{\alpha} W_j^{\beta}}{F(c_{ij})}, \quad (3.15)$$

where α and β represent parameters for attraction of origin/destination area and $F(c_{ij})$ is a chosen deterrence function.

Different effects influence commuting flows and if these effects are observed and data is available, it is useful to add more variables to the model in the same way as W_i and W_j are used. For example, commuting flow depending on number of employees in origin / destination and relative wage in origin/destination can form following gravity model formula [1]:

$$T_{ij} = K \frac{W_i^{\alpha} W_j^{\beta} RW_{ij}^{\delta}}{F(c_{ij})}. \quad (3.16)$$

3.2.1 The Model Constant

In special cases the constancy of constant K is not reasonable. This appears when there is an unbalance between flows between two areas. This phenomenon appears mostly in clusters or very attractive areas. For example Bratislava forms such cluster of working possibilities with other positive effects.

According to Howard [18], adding non-constant constant term k allows control of omitted variables, which cannot be observed or data is not available.

Another possible reason for giving up the constancy of constant k is border. Areas on both sides of borders often have different approach to commuting

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[6] Mass measure depends on context used, e.g. population of area, …

(because of language, culture and economic barriers or problems with working allowances). Commuting behaviour is different from the expected behaviour on base of distance and destination/origin masses. This can be seen for example in Bratislava region, where there is greater tendency to commute to Vienna then the other way.

3.2.2 Dummies

One of possible ways of model improvement, is addition of dummy variables. Dummies can be used to control heterogeneous or asymmetry effects. Depending on commuting flows data it is often useful to use dummies to control

- intercounty flows (dummies used for intercounty flows as addition to distance deterrence function);
- asymmetric commuting flows (this can be controlled also by non-constant constant term k);
- intervening opportunities (to control direct and indirect connections of origin/destination);
- border effects[7] ;
- clusters;
- other unexplained but existing effects, which cannot be explained by observed data variables.

3.3 Estimation of Gravity Models

The choice of model, which is to be estimated, and evaluated is very important as well. In previous section we have discussed many different types

[7] Further reading on different types of border dummies can be found in Wall [3]

of gravity models, which can be combined and therefore the choice of final best model is not easy.

Finding of proper model form and its estimation technique is the most difficult part of estimation process. Methods of estimation differ widely and no best technique can be found.

The chosen type of econometric estimator depends on data structure and data characteristics. In this section we mention several methods of estimation and their characteristics. We try to estimate gravity model (3.2) and its extensions by different estimation techniques. OLS, NLS (non-linear least squares) and Poisson estimation theory together with spatial econometrics approach are discussed. All these estimations can be performed by standard statistic software (e.g. Splus, Eviews, SPSS, SAS or other).

3.3.1 OLS Estimation

Mainly used estimation with OLS (ordinary least squares) draws back. Although OLS preserves its place as the most common estimator, other estimation techniques gain on popularity.

Because of its easy implementation and good results, OLS estimation is normally first choice from available estimation techniques.

Estimation by OLS supposes distribution of error term (μ_{ij}) in (3.2) in the following way:

$$T_{ij} = K \frac{W_i^{\alpha} . W_j^{\beta}}{F(c_{ij})} . \mu_{ij} \,. \tag{3.16}$$

This model supposes also, that stochastic term in (3.2) is not linear but multiple. By logarithmic transformation we get linear model, which is to be estimated:

$$\ln Tij = \ln(K) + \alpha \ln W_i + \beta \ln W_j - \gamma_1 \ln(F(c_{ij})) + \ln(\mu_{ij}) \,. \tag{3.17}$$

By linearization of model, we get additive stochastic term. This can be not realistic and can lead to biases in estimation. Another problem by OLS estimation is that it expects heterogeneity of stochastic term, this cannot be always the case and so GLS estimation is better, where wages are used to deal with heteroscedasticity. Other problems by OLS estimation (e.g. multicolinearity, autocorrelation) are to be handled, when they occur depending on chosen data set.

If the estimated data contains zeros (in our case it makes about 20% of observed flows), simple logarithmic transformation can not be used. Berkvist and Westin [17] propose use of Poisson model instead of OLS. Other possible solution to this problem is aggregation of data into larger areas, where zero flows do not occur, or omit these observations from estimation. Both approaches lead to lost of data and therefore are not very suitable. Wall [18] solves the zero flows by adding constant 1 to each flow, which leaves zero flows after logarithmisation unchanged to zero.

Two main types of gravity model have been presented: Model with power function (3.3) and model with exponential function (3.4). By logarithmising the classical model we get following terms in estimation function.

1. Power function: $\ln(F(c_{ij})) = \gamma.\ln(c_{ij})$; (3.18)

2. Exponential function: $\ln(F(c_{ij})) = \gamma.c_{ij}$. (3.19)

According to model specification additional variables can be added[8]. The choice of model variables depends on data availability and their estimation significance. Variables explaining commuting flows are mainly socio-economic variables e.g. county population, average wage rate, unemployment rate, the number of employees, and other.

As already mentioned in chapter 3.2.1 constant term K, ln (K) respectively can be extended to reflect omitted or unobserved variables. Then constant term

[8] See chapter 3.2

ln (K_{1i}) for each origin and ln (K_{2j}) for each destination is added [15, 16]. Multicolinearity then must be avoided by leaving term ln (K) omitted or one leaving ln (K) term in estimation formula but removing one term ln (K_{1i}) and ln(K_{2j}) from constant set. By omitting ln (K) and setting ln (K_{1i}) = k_{1i} and ln(K_{2j})=k_{2j} we get following estimation formula (3.20). Capital letters of variables are replaced by small letters, meaning logarithms of corresponding variables and stochastic term ln (μ_{ij}) = ν_{ij}. Estimation formula follows:

$$t_{ij} = \alpha.w_i + \beta.w_j - \gamma.f(c_{ij}) + k_{1i} + k_{2j} + \upsilon_{ij}\,, \tag{3.20}$$

where k_{1i} and k_{2j} are estimated as parameters, in form of dummy variables, where 1 is for each pair of corresponding constants to flows and zero otherwise.

Other types of dummies can be used. Then dummy terms are added to model (3.20) analogically to the mentioned constant terms.

3.3.2 Other Estimation Techniques

OLS estimation is easy to implement but has its disadvantages in form of pre-conditions set on data and on the estimated model form. Therefore we present other estimation methods.

3.3.2.1 NLS Estimation

In OLS estimation stochastic term was multiplicative (3.16), NLS estimation uses additive stochastic term (3.21) in (3.2)

$$T_{ij} = K\frac{W_i^{\alpha}.W_j^{\beta}}{F(c_{ij})} + \mu_{ij}\,. \tag{3.21}$$

The main advantage to the OLS model is, that this representation allows commuting flows T_{ij} to take on zero values. Analogically to OLS estimation, adding additional explanatory variables and/or dummy variables can extend simple model (3.21). Multiplicative dummy variables take on value 1 for no effect and non-zero non-unity value for effect explanation.

3.3.2.2 Count Models: Poisson Estimation

Count models are used when explained variable expresses non-negative whole number data. Both OLS and NLS estimation do not consider this main characteristic of commuting flows. They also suppose normal distribution of stochastic term. The mostly used count model is Poisson model, which uses Poisson distribution as basis for regression. Formula for count model regression follows:

$$E(T_{ij} \mid W_i, W_j, F(c_{ij})) = K \frac{W_i^{\alpha} . W_j^{\beta}}{F(c_{ij})} . \tag{3.22}$$

If Poisson regression is better than OLS or NLS estimation, it is to be tested on real data.

3.3.2.3 Spatial Econometrics

A relatively new discipline in econometrics is spatial econometrics. It uses standard econometric techniques for regression but this type of regression takes into account spatial distribution of evaluated data (spatial interaction - spatial autocorrelation and spatial structure - spatial heterogeneity of the surveyed data). Its implementations are in various spheres of economic research (e.g. demand analysis, international economics, transportation systems, regional economic…) and in other areas, which work with geographically dependent data sets.

This chapter does not explain the theory of spatial econometrics, because this area of research is very broad, It would like to explain its basic principles[9].

Spatial distribution effects mainly covariance matrix of stochastic terms. The matrix is no more diagonal (autocorrelation) and diagonal values are not constant (heteroskedasticity). Both effects appear often simultaneously. Spatial dependence of data can be implemented into linear models either by adding

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[9] Sources of the chapter 3.3.2.3 are in Anselin [6]

additional regressor - spatial lagged dependent variable Wy, where W is so called weight matrix and y is explained variable or can be represented in the error term structure ($E[\mu_i,\mu_j]\neq 0$).

Spatial lag model:

Autocorrelation of error terms due to spatial distribution of surveyed objects is solved by concept of weight matrix. This is seen in the formula (3.23):

$$y = \rho Wy + X\beta + \varepsilon \,, \tag{3.23}$$

where y is a explained variable vector, W is a weight matrix, ρ is a spatial autoregressive coefficient, β is a parameter vector, X is a matrix of explaining variables and ε is a error term vector.

Weight matrix is a very important term in this formula. It includes information on spatial distribution of explained variable and is exogenous. Weights w_{ij} are non-zero when two locations are "neighbouring". Concept of "neighbours" is general and its implementation states its values. For example two counties are "neighbouring", if e.g. two counties share a common border, two counties lie within a given distance, etc. Usually weights are normalised (i.e. $\sum_{j=1}^{n} w_{ij} = 1, \forall i = 1,...,n$).

Spatial error model:

This model deals with spatial autocorrelation, which means that error term covariance is not diagonal, i.e. $E[\varepsilon\varepsilon'] = \Omega$..

The estimated model then takes the following form:

$$y = X\beta + \varepsilon \,, \qquad \varepsilon = \lambda W\varepsilon + \mu \,, \tag{3.24}$$

where W is weight matrix, λ is autoregressive coefficient and μ is stochastic term.

Model (3.24) is equivalent to the model (3.25), which is spatial lag model with an additional term

$$y = \lambda Wy + X\beta - \lambda WX\beta + \varepsilon \, . \tag{3.25}$$

Spatial models are estimated with maximum likelihood estimation, or general method of moments or other estimation techniques [6]. We do not implement this method in the thesis.

4 Maximum Entropy Models and their Estimation

Maximum entropy model belongs to the same model family as the gravity model but has very different characteristics and estimation techniques.

The method sees the commuting system (Origin - Destination matrix) as a probabilistic system of flows, where each state occurs with a certain probability (entropy). Maximum entropy concept computes degree (entropy) of likelihood of a selected state of surveyed system. The estimated flows are flows in the system with maximum entropy.

There are two main approaches to the same problem. The first approach is non-stochastic, which tries to estimate number of commuters Tij on aggregate level with omitting stochastic terms. The second model sees flows as random numbers and estimates probability distribution of commuter flows and then can computes expected flows.

4.1 Classical Maximum Entropy Model

The first model - classical maximum entropy model tries to describe commuting system on aggregated deterministic level. We now present the model and it's possible way of computation.

4.1.1 Model

The commuting system can be described in two different states: micro and macrostate. Macrostate gives information on the number of commuters between cities. Each macrostate comprises many microstates - commuting trips of each individual.

The number of microstates (N) that forms a given macrostate is represented by the following equation with $T_{ij} \geq 0$ [5, 13, 12]:

$$N = \frac{T!}{\prod_{ij}^{n} T_{ij}!}, \tag{4.1}$$

where T is total number of individuals in system, T_{ij} represents number of individuals commuting from city I to J.

Entropy (W) of the system can be then computed by the formula (4.2)

$$W = H(N) = -\sum_{i=1}^{n}\sum_{j=1}^{n} T_{ij} . \ln T_{ij} . \tag{4.2}$$

In order to compute the model, we introduce cost constraint (4.3)

$$\sum_{i=1}^{n}\sum_{j=1}^{n} T_{ij} c_{ij} = C , \tag{4.3}$$

where C represents overall expenditure available for trips and c_{ij} is cost of commuting between two cities I and J. Further we can use origin and destination constraints (equation 4.4 represents origin and equation 4.5 destination constraint):

$$\sum_{j=1}^{n} T_{ij} = O_i , \tag{4.4}$$

$$\sum_{i=1}^{n} T_{ij} = D_j , \tag{4.5}$$

where O_i res. D_j is the total number of commuters leaving the city I or arriving to the city J respectively.

We estimate the system by maximizing entropy function under given constraints [13,5].

4.1.2 Model Estimation

Concept of maximising entropy uses Lagrange function for maximisation of function under constraints:

$$L = W + \sum_{i=1}^{n} \alpha_i (O_i - \sum_{j=1}^{n} T_{ij}) + \sum_{j=1}^{n} \beta_j (D_j - \sum_{i=1}^{n} T_{ij}) + \gamma (C - \sum_{i=1}^{n} \sum_{j=1}^{n} T_{ij} c_{ij}), \qquad (4.6)$$

where L is Lagrange function, W is entropy of T_{ij} distribution, α_i, β_j, and γ are Lagrange multipliers associated with the appropriate constraints.

By solving the maximisation of Lagrange function L (4.6) we get the following solution for the number of commuters T_{ij} between any given two cities:

$$T_{ij} = \exp(-\alpha_i - \beta_j - \gamma c_{ij}), \qquad (4.7)$$

where α_i and β_j can be computed from origin / destination constraints

$$\sum_{j=1}^{n} T_{ij} = O_i \Rightarrow \exp(-\alpha_i) = O_i [\sum_{j=1}^{n} \exp(-\beta_j - \gamma c_{ij})]^{-1}, \qquad (4.8)$$

$$\sum_{i=1}^{n} T_{ij} = D_j \Rightarrow \exp(-\beta_j) = D_j [\sum_{i=1}^{n} \exp(-\alpha_i - \gamma c_{ij})]^{-1}. \qquad (4.9)$$

By substituting to the equation (4.7) we get

$$T_{ij} = A_i O_i B_j D_j \exp(-\gamma c_{ij}), \qquad (4.10)$$

where A_i, B_j are so called scaling constants that need to be found by solving the system. γ represents parameter of distance deterrence, which is calibrated.

Formula (4.10) - solution of classical maximum entropy model resembles doubly constrained gravity model with exponential function. This means consistency of both approaches to the same problem from aggregate point of view.

Parameters of the model, which are needed for trip matrix computation can not be found by simple solving of the system. There are different approaches used to compute the maximisation problem.

One of them is iterative two step method [5]. Parameters α_I and β_j, are estimated with use of origin and destinations constraints and γ is calibrated according to cost constraint.

In the first step α_i and β_j are computed using so called row-column-balancing method usually "Bregman method".

From origin constraint (4.4) we get balancing factor for α_I (4.8) and from destination constraint (4.5) we get balancing factor for β_j (4.9). Computation consists of iterative balancing using (4.8) and (4.8) until origin (4.4) and destination (4.5) constraints are met.

In the second step factor β is computed. This is usually made with use of cost constraint (4.3) by Newton method.

Another method uses dual system of the model. It computes unconstrained minimum of the new model and then transforms results to get parameters of the primary model. This method detailly explained and practically used on real data in the section 6.2.

4.2 Disaggregated Maximum Entropy Model

Unlike the classical approach, where commuting flows are expected to be non-stochastic elements, the disaggregated model supposes that flows underlay stochastic processes. This is reasonable, as the number of commuters, mainly from large counties, is not constant over time.

4.2.1 Model

Adding entropy term to each Tij represents the randomness of commuting flows. $T_{ij} \approx p_{ijx_{ij}}$, where X_{ij} is stochastic variable, which takes non-negative whole number values and $x_{ij} \leq \min\{O_i, D_j\}$ and $p_{ijx} = P(X_{ij} = x_{ij})$ [5].

Analogically to classical maximum entropy problem for commuting flows we need to solve following maximisation problem:

$$\max H\{p_{ij}\} = -\sum_{ijx_{ij}} p_{ijx_{ij}} \ln p_{ijx_{ij}} \text{ ,} \tag{4.11}$$

under constraints:

$$\sum_{jx_{ij}}^{n} x_{ij} p_{ijx_{ij}} = O_i \text{ ,} \tag{4.12}$$

$$\sum_{ix_{ij}}^{n} x_{ij} p_{ijx_{ij}} = D_j \text{ ,} \tag{4.13}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \sum_{x_{ij}} x_{ij} p_{ijx_{ij}} = C \text{ ,} \tag{4.14}$$

$$p_{ijx_{ij}} \geq 0 \text{ , } \sum_{x_{ij}} p_{ijx_{ij}} = 1 \text{ .} \qquad \text{(4.15a, 4.15b)}$$

4.2.2 Model Parameter Estimation

Solution of the maximization gives following formula for probability [5]:

$$p_{ijk} = \frac{\exp\{(\tau_i + \delta_j - \lambda.c_{ij})x_{ik}\}}{\sum_{x_{ij}} \exp\{(\tau_i + \delta_j - \lambda.c_{ij})x_{ij}\}} \text{ .} \tag{4.16}$$

Formula (4.16) resembles formula of non-hierarchical logit model (2.5) in chapter 2. This means that disaggregated model is analogy to logit model. Both models are thereby consistent on disaggregated level.

The parameters τ_i, δ_j and λ resemble parameters α_i, β_j and γ from the classical concept.

Parameters from (4.16) need to be estimated according to model constraints. Computed probabilities (4.16) with estimated parameters have to meet constraints (4.12)-(4.15b). Calibration process is more complicated than in classical case, because there is no exact solution like "Bergman method" to iteration process. The system has to be solved numerically. There are different numerical methods developed for maximising function with constraints, which can be used for system solving [5].

5 Data and its Specifics for Slovakia

The first step in estimation of commuting flows is search for correct and complete source data. The data forms the background for estimation. The data in the thesis comprises commuting flows among selected Slovak counties.

Slovakia is divided into 8 Slovak regions and 81 counties. Region division of Slovakia is depicted at the Figure 5.1. We have selected 27 counties in the Western Slovakia for estimation[10]. These counties belong to four western regions - Bratislava, Nitra, Trnava and Trenčín regions (Table 5.1 and Figures 5.1 and 5.2). Each region comprises more counties (Table 5.1). Each county is a possible origin and destination of commuting flows.

Table 5.1 Selected counties for estimation

County	Slovak region	County	Slovak region
Bratislava (city)	Bratislava	Nové Mesto nad Váhom	Trenčín
Malacky	Bratislava	Partizánske	Trenčín
Pezinok	Bratislava	Považská Bystrica	Trenčín
Senec	Bratislava	Prievidza	Trenčín
Dunajská Streda	Trnava	Púchov	Trenčín
Galanta	Trnava	Trenčín	Trenčín
Hlohovec	Trnava	Komárno	Nitra
Piešťany	Trnava	Levice	Nitra
Senica	Trnava	Nitra	Nitra
Skalica	Trnava	Nové Zámky	Nitra
Trnava	Trnava	Šaľa	Nitra
Bánovce nad Bebravou	Trenčín	Topoľčany	Nitra
Ilava	Trenčín	Zlaté Moravce	Nitra
Myjava	Trenčín		

[10] In further text, selected region, or Western Slovakia region, or Western Slovakia are used as synonyms

Figure 5. 1 Slovakia and its regions

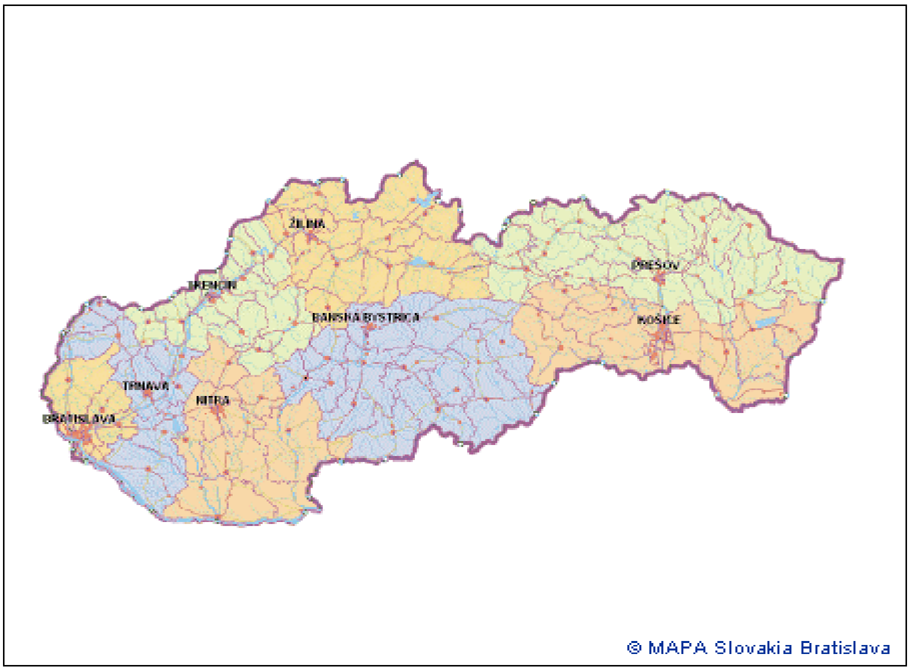

Figure 5. 2 Selected region of Western Slovakia

5.1 Commuting Flow and Socio-economic Data

Commuting flow data is provided by Štatistický úrad Slovenskej republiky every 5 or 10 years. The most recent data is from the year 1999 [7]. The data set represents the number of commuters between any two counties in Slovakia and between each county and foreign countries (aggregate). We have not taken into consideration flows between foreign countries and counties, because this would make the estimation very complicated. Although many counties border with Hungary, Austria or Czech Republic and these regions are exposed to a significantly high stream of commuters to foreign countries, we have omitted these observations.

Commuting flows for the region of Western Slovakia in the year 1999 can be found in the table 5.2 in the Annex A and is also printed at the end of the thesis in Annex. This table represents a 27x27 square matrix composed of number of commuters commuting from origin I to the destination J.

Economic and social information needed for estimation was provided by Štatistický úrad Slovenskej republiky [8]. The data is from the year 1999[11]. The data for each region comprise population, average monthly wage, unemployment rate, number of employees, number of unemployed. Overview on the socio-economic situation in Western Slovakia is in the Table 5.3.

Table 5.3 Basic socio-economic data on Western Slovakia regions (Year 1999)

Region	**Population**	**Economic active population**	**Number of employed persons**	**Unempoy-ment rate**	**Gross production**	**Average wage**
		in thousand		in %	in mil. SKK	in SKK
Slovakia	***5,393.40***	***2,573.00***	***2,132.10***	***16.20***	***1,839,200***	**10 961**
Bratislava	617.60	335.80	309.20	7.40	654 220	14 611
Trenčín	550.70	266.50	229.80	12.30	152 318	10 556
Trnava	609.70	288.40	252.40	11.40	154 987	10 134
Nitra	716.60	328.20	266.90	17.80	166 123	9 968

[11] When data from year 1999 was not available, we used data from 31.12.1999

Detailed information on selected Slovak counties can be found in the Table 5.4 in Annex A.

As it can be seen in the table 5.3, Bratislava region and its counties are the most prosperous parts in Western Slovakia. Average monthly wage reaches 1.3 times the Slovak average and is the highest among the surveyed regions. The unemployment rate is the lowest among the regions where the number of employees is the highest. On the other hand the region of Nitra has the highest population and is the least developed concerning employment situation. The average monthly wage in the region reaches less then 10 000 SKK whereby Slovak average is almost 11 000 SKK, here it can be seen a very high level of unemployment which is 2.4 times the level in Bratislava and higher then Slovak average.

From the given information, it can be assumed that the Bratislava region will attract a lot of commuters and Nitra region will distribute a lot of commuters into other regions and its attractiveness as a possible employment region is low.

5.2 Transportation Distance, Time and Cost Approximation

In the gravity model and maximum entropy model, information on distances, time and costs of transportation is needed. Distance detterence (which can be based on information on distances, time or cost of transportation) lessens the commuting flows between regions, because it builds a natural barrier between them.

5.2.1 Computation of Transportation Distance, Time and Cost

Distances in this thesis are calculated as distances between each pair of capital cities of counties on roads using the Autoroute software. Time and cost estimates are computed using this program as well. Time and costs are those of automobile transportation needed for travelling from the capital of one region to the capital of another region. Costs are computed in SKK using approximate fuel consumption and per litre fuel costs.

Distance is computed as the fastest route on road between cities with motorway usage. However, not all commuters use motorways as they are paid. Problematic areas are some border regions where the fastest and / or shortest distances are on roads which cross state borders (e.g. Komárno county, Skalica county). Where the fastest route could not be replaced by the shortest route not crossing state borders, the route was computed as the fastest route using only Slovak roads. Travelling costs and time were computed in the same way.

These calculations are not exact and not satisfactory as many commuters commute by public transport (rail or bus) and in Slovakia the rate of car transportation is relatively low comparing to Western Europe or USA. Car transportation and associated "road" costs, time and distances serve as an easy approximation because these are available between each pair of surveyed cities (counties) and rail /bus transportation is limited to railways or bus lines and do not connect cities equaly in the sence of equal possibitily to travel to each city.

Cross city distances, times and costs of transportation are shown in the tables 5.5, 5.6 and 5.7 in the Annex A. They are symmetric 27x27 matrices.

We have chosen not to consider intercounty commuting as "real" commuting. Intercounty distances, time and costs of transportation are set to zero[12]. The reason is, that commuters mostly do not consider time, costs and distances travelled to work within an area as "real" costs of commuting to work.

In this thesis used system of distance measuring between two cities as distance between two city centers is valid only when cities are seen as very small and distances traveled within cities are small as well. This is reasonable as we take city distances as measure for distances between counties and this measure is very approximate and we can abandon for intercity distances. But when talking about commuting distances between larger cities we should consider travelling in the cities as well. When intercity transportation is

[12] For the purpose of OLS estimation, zeros are set to 1, then ln(1)=0

calculated, the "Euclidean" distance can be used. This concept is introduced in the paper [11].

5.2.2 Intervening Opportunities

Not only distance is a measure for geographical deterrence of two cities. There is a difference of commuting behavior when two cities are situated next to each other or when there is another city situated between them or in a shorter distance from one city to this city then the distance between the cities themselves. This city is an intervening opportunity of commuting between cities.

This situation can be seen at Figure 5.4 a), b) and c). At 5.4a) two cities A and B lie next to each other and at b) and c) there is a city C which is the an intervening opportunity for commuting between cities A and B.

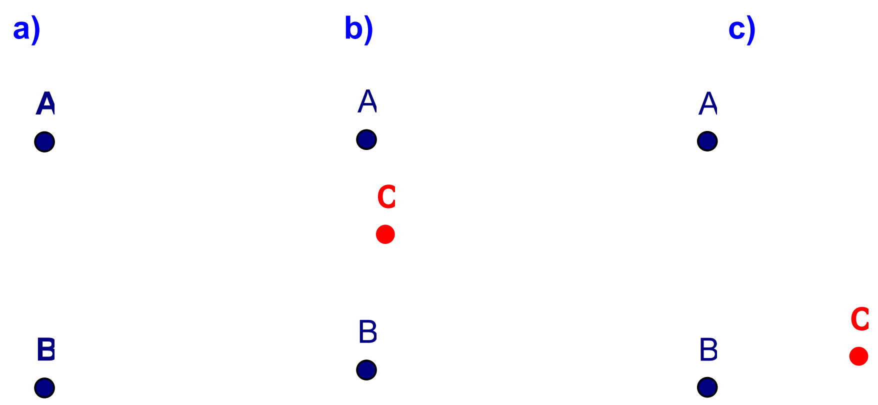

Figure 5.3: Intervening opportunities

Willingness to commute from city A to city B is greater when there is no city situated between them in the sense of previous paragraph. This third city C forms intervening opportunities. For a commuter it is more convenient and less costly to commute from city A to the city C then to the city B.

In our measure of distance we do not take into consideration the notion of intervening opportunities. This concept has to be implemented in the model itself.

5.3 Regional Specifics

Commuting behaviour can be influenced also by regional specifics. Different characteristics of regions cause different volume of commuting in similar regions. Regional characteristics need to be taken into consideration when trying to estimate trip matrix.

5.3.1 Bratislava Region

A specific region for commuting is Bratislava region. This region according to political division is divided into 8 counties. 5 counties (Bratislava I, II, … V) are situated within the city of Bratislava. These 5 counties are in this thesis aggregated to form a single unit - Bratislava. In this unit, commuting flows within the city (i.e. among counties Bratislava I to V) are considered as interregional flows and only flows from or to counties outside Bratislava city are considered for estimation.

Bratislava city has other specific characteristics. Bratislava is the capital of Slovakia with high level of industry and employment possibilities, the highest population in Slovakia. A very low unemployment rate and the highest wage rate in Slovakia are other positive factors for commuting. These factors make Bratislava a very attractive city for in-commuting. On the other side, Bratislava lies close to Austrian, Czech and Hungarian borders. Therefore there exist a relative high level of commuting outside Slovak borders.

5.3.2 Bordering Regions

Problematic areas are all counties situated next to either state or selected region borders. Here we can observe a high level of commuting to regions not included in survey. State borders form a different type of "commuting borders" then borders of our selected area with other Slovak counties.

State borders often lead to diminution of commuting flows, which would be higher when not existence of state borders. These borders are mainly a political obstacle for commuters as they need to cross state borders (time costs of commuting) and often need a special allowance to work in a foreign country.

Borders of the selected area (Western Slovakia) with the rest of Slovakia form only a formal border. They may cause greater errors in the results of our survey because commuters from counties bordering the selected area do not have a preference to work in Western Slovakia to other parts of Slovakia. This can be not included in our estimation and can lead to biased results for the bordering regions.

In the thesis we do not cope with this problem systematically and we simply omit these effects.

5.3.3 Zero and Low Level Flows

Slovak data is not very suitable for estimation, because there are many zero flows. Almost 20% of data are zeros, this means that commuting flow between two counties does not exist.

Slovak commuting behavior can be described as commuting averse. Only 14.2% workers commute to work to a county different from their home county. Number of destinations with commuting level less then 1% of working power in a county is 85% of all destinations and only 7% of destination has more then 5% commuters. Almost 84% of all data has commuting flow less then 100 persons. Compared to average intercounty commuting (average intercounty commuting is 20 549) this makes a great disproportion between commuting to different county and intercounty commuting.

Reasons for commuting aversion can be found in different socio-economic factors. Low wage rate compared to relatively high level of unemployment benefits and high travel costs are main reasons that lower commuting flows. A common statement explains willingness to commute: "Commuting will be the preferred choice of a worker whenever he can obtain an increase in wages greater than cost of commuting - workers apply for jobs according to a strategy that maximizes their expected payoffs (wages minus commuting costs) " [11].

5.3.4 Clustering and Attractiveness

Not equal distribution of labour force and labour opportunities in counties causes clustering effects. They have to be surveyed separately.

Cluster is characterised by commuting behaviour, where a region is seen either as a source of employment possibilities or as a source of working power. In the first case the region attracts commuters. The latter type of cluster provides labour force commuting to different locations. These clusters cannot be found only on geographic base. Commuting behaviour has to be surveyed to find such clusters. Clusters change commuting patterns and distances lose their deterrence function.

An example of cluster is Bratislava, where 98.3% of all commuters commuting from Bratislava commute to Bratislava but only 77.4% of all commuters commuting to Bratislava come from Bratislava. Clustering effect causes this difference. Bratislava is a large city with a great concentration of industry and other kinds of job opportunities. This makes Bratislava very attractive for commuting.

On the other hand Malacky, Pezinok and Senec, for example, attract only about 50% of their out commuters, whereby about 85% of their incommuters commute intercounty. These regions can be characterised as out-commuting. Interesting is that these regions are close to Bratislava, which is a very attractive commuting destination.

Attractiveness of a county for commuting can be characterized by ratio of outgoing and incoming commuters. If this ratio is significantly greater then 1 this county can be characterized as very attractive and on the other side if this ratio is very low (less than 0.7%) it means that the county has a great ratio of outgoing commuters to incoming commuters. For example, Púchov, Prievidza and Trenčín have a very balanced percentage of intercounty commuters coming to and departing from this region. In both directions ratio of intercounty out and in flows is close to 1. This makes these regions relatively neutral to commuting.

6 Commuting Models Estimation

We have described different models, which have been invented for commuting flow estimation. We have introduced their idea, fields of implementation, type of data needed for their estimation and their estimation techniques.

We now practically implement the estimation techniques of two model types, namely gravity model and maximum entropy. The models were explained in the chapters 3 and 4. We try to estimate commuting on base of data obtained from the Slovak Statistical Institute. The data covers commuting flows within the area of Western Slovakia in the year 1999.

Gravity model in the thesis is estimated with help of EVIEWS (EVIEWSv3.0) statistical software. For the purpose of maximum entropy model estimation, we have created a program in GAMS software. The program is enclosed in Annex C.

Firstly we introduce results from OLS, Poisson and NLS regressions of gravity model and then we compare these results with maximum entropy model estimation.

6.1 Gravity Model Estimation

Gravity model assumes that commuting flows between two areas depend on different variables that should be included in the model. The basic model is following:

$$T_{ij} = K \frac{W_i^{\alpha} . W_j^{\beta}}{F(c_{ij})} , \qquad (6.1)$$

where the terms in the equation 6.1 are the same as in the chapter 3 (equation 3.2). Additional variables can be added to this equation and distance deterrence function is to be set as well. The choice of relevant model variables is a question of estimation process. We use different estimation techniques to estimate the model on base of real observed commuting flows.

6.1.1 Model Variables

Into the gravity model, different variables come as explanatory variables. These variables may vary on the concept in which the model is used and on the data availability. The explained variable is in our case the number of commuters from region I to the region J (commuter) [13].

We have chosen different variables, which could be used as explanatory variables in the model. These variables are set into sets depending on their explanatory function.

Area's attraction can be measured by the following statistics:

- population of the area - population is the basic and mostly used characteristic, but does not take into consideration the age structure of the population and unemployment in the area (pop_from, pop_to);
- the number of employed persons in the area - this variable is an extension of population variable which includes age structure and employment in the area (emp_from, emp_to);
- the number of employed persons, which work in a given area and commute from and to an area within the selected Slovak region (Western Slovakia) - this statistic is the most suitable for explanation of mass of the area. It takes into consideration only the selected Slovak region (sum_from, sum_to).

Distance deterrence can be represented by

- travelling distance (dist): distance between capitals of counties;
- travelling time (time): variable based on distance, but takes into consideration different transportation speed on highways, local roads etc. ;

[13] In parentheses, the names of the variables are given, suffices _from defines that this variable is connected to the origin area, _to defines destination area variable and _rate defines ratio of _to and _from variables)

- travelling costs (cost): based on time and distance, takes into consideration economic costs of transportation.

Distance deterrence function is chosen from exponential and power function forms.

Potential additional variables:

- the number of companies in the area - it explains the working attraction of the area (nc_from, nc_to);
- the number of unemployed persons - this variable gives the perspective labour market in the area (un_from, un_to);
- the unemployment rate - this variable is very similar to the number of unemployed persons, but this measure stands for relative term which explains relative labour market situation (un_rate_from, un_rate_to, un_rate_rate);
- average wage - wage is very important for commuting habits, because wage is the "compensation" for commuting to work (av_wage_from, av_wage_to, av_wage_rate).

Dummy variables stand for unexplained characteristic, that cannot be explained by an observable variable. Potential dummy variables:

- border regions - in border regions, many people commute to regions that are not included in the survey (border, 1 when region lays on border of the selected region, 0 respectively);
- neighbouring regions: people tend to commute more to regions that are next to their origin region or to the same region (destination = origin) than to other regions, this phenomena can be explained by intervening opportunities approach (NBR, 1 when origin/destination regions are neighbouring or are the same, 0 respectively);
- inter-area commuting: commuters prefer commuting within their origin area mostly because of very good transportation conditions (IAC, 1 when origin region = destination region, 0 respectively);
- Bratislava - Bratislava is a special commuting region, which is very attractive for commuting as destination (BA, 1 when destination region is Bratislava, 0 respectively).

From all possible variables we have chosen set of these relevant variables, which are used in regression models (OLS, Poisson and NLS). The choice was made upon econometric research of the variables:

- sum_from - attraction variable of origin;
- sum_to - attraction variable of destination;
- time - distance deterrence function variable (deterrence function is used in power form);
- av_wage_rate - additional variable;
- BA - dummy for Bratislava region;
- NBR - Dummy for neighbouring regions.

The final choice of models is set upon the regression results, which are presented in tables in each section. In the tables we present estimated parameters for each variable with standard deviation in parentheses. * represents significance at 5% level and ** significance at 1% level in all tables of results. We expect in all final models (OLS, NLS and POISSON) coefficients sum_from, sum_to, av_wage_rate, NBR and BA to be positive and time negative. Further we expect lower sum_from coefficient than sum_to.

6.1.2 OLS Model Estimation

OLS estimation supposes linear form of regressors in regression equation and therefore by logarithmic transformation of (6.1) we get linear model (with power function deterrence), which is to be estimated:

$$\ln(T_{ij}) = \ln(K) + \alpha \ln W_i + \beta \ln W_j - \gamma \ln c_{ij} . \qquad (6.2)$$

The model is then fitted with use of OLS estimation, additional significant variables are added and the best model is chosen.

Because of the logarithmic form of OLS estimation equation and zero level of commuting flows (commuter) in many cases, we have used the transformation introduced in the chapter 3[14]

commuter1 = commuter +1. (6.3)

This transformation solves the problem of zero flows but on the other hand has many disadvantages (causes greater estimation errors by small flows) but on the other hand after logarithmic transformation, zero flows are set again to zero. Results of OLS estimation of different model variations are presented in the table 6.1. In all tables, shadowed models are the most suitable ones.

Table 6.1: OLS estimation results

Dependent Variable: LCOMMUTER1								
Method: Least Squares (OLS)								
Included observations: 729								
White Heteroskedasticity-Consistent Standard Errors & Covariance								
Variables	**Model 1**	**Model 2**	**Model 3**	**Model 4**	**Model 5**	**Model 6**	**Model 7**	**Model 8**
C	-9.629 (1.02)**	-10.147 (0.92)**	-10.91 (0.94)**	-5.995 (1.16)**	-11.367 (0.86)**	-7.064 (1.03)**	-7.454 (1.05)**	-8.344 (0.99)**
LSUM_from	0.693 (0.07)**	1.049 (0.08)**	0.633 (0.06)**	0.698 (0.07)**	0.971 (0.07)**	0.636 (0.07)**	0.985 (0.08)**	0.896 (0.07)**
LSUM_to	1.384 (0.08)**	1.076 (0.08)**	1.335 (0.07)**	0.995 (0.09)**	1.045 (0.07)**	0.917 (0.09)**	0.857 (0.09)**	0.795 (0.08)**
LTIME	-2.027 (0.05)**	-2.025 (0.05)**	-1.512 (0.06)**	-2.021 (0.05)**	-1.526 (0.05)**	-1.491 (0.06)**	-2.021 (0.05)**	-1.506 (0.05)**
LAV_WAGE _RATE		2.911 (0.34)**			2.753 (0.33)**		2.358 (0.36)**	2.120 (0.35)**
NBR			1.780 (0.15)**		1.727 (0.14)**	1.835 (0.14)**		1.781 (0.14)**
BA				2.241 (0.31)**		2.400 (0.30)**	1.600 (0.31)**	1.81 (0.30)**
R-squared	0.764	0.786	0.804	0.781	0.823	0.823	0.794	0.833
Adjusted R-squared	0.764	0.785	0.803	0.780	0.822	0.821	0.793	0.832
Akaike info criterion	3.186	3.091	3.007	3.116	2.905	2.909	3.058	2.851
Schwarz criterion	3.212	3.123	3.039	3.148	2.943	2.947	3.096	2.895
Log likelihood	-1157.41	-1121.84	-1091.11	-1130.96	-1052.87	-1054.30	-1108.61	-1032.08

[14] All variables have prefix "l" after logarithmic transformation e.g. ltime = ln(time)

We can see that the best fit is achieved in the Model 8, in which all relevant variables are included. This model has the best estimation statistics. The basic model (Model 1) is not satisfactory because of its relatively low fit and because of residuals, which are shifted from zero. This phenomenon is removed by adding dummy variable NBR. The dummy BA which is included in the Model 8 but excluded in model 5 contribute to the model not very much and therefore the Model 5 can be used as the result model. The dummy BA stands for commuting flows to Bratislava, which are underestimated in few cases. When we look at the resulting coefficients we can see that in the model 8 the coefficient for sum_to is less then the coefficient for sum_from, which does not resemble the theory, that home region is less attractive than the destination region. This is not the case in the model 5. Other coefficients in both models have similar form and correct signs.

The final Model 8, then estimates commuting flows in the following form:

$$T_{ij} = 0.000238 \frac{W_i^{0.89} W_j^{0.795} RW_{ij}^{2.12}}{c_{ij}^{1.51}} e^{1.78NBR+1.81BA} - 1 . \qquad (6.4)$$

The final Model 5, then estimates commuting flows in the following form:

$$T_{ij} = 0.000016 \frac{W_i^{0.97} W_j^{1.045} RW_{ij}^{2.75}}{c_{ij}^{1.52}} e^{1.73*NBR} - 1 . \qquad (6.5)$$

In equations (6.4) and (6.5): T_{ij} is the fitted commuting flow from region I to the region J, c_{ij} is the time of travelling between regions I and J, W_i, W_j are attractions of origin/destination regions, RW_{ij} is relative average wage of regions J and I and NBR and BA are dummies used to explain different commuting habits (with values 0 or 1 respectively).

Disadvantage in this approach is that fitted commuting flows may be negative (as result of estimated logarithmic form and transformation 6.3). The second disadvantage is that results are not whole numbers. These results need either to be rounded to get whole numbers or left in their not whole number form. But none of the estimation methods gives whole number results and therefore this disadvantage is not relevant in the comparison of methods. In our case, negative fitted values are very small and can be rounded up to zero. The estimation results can be found in Annex B.

6.1.3 Poisson Regression Estimation

We proceed gravitation model fitting by using Poisson regression. This estimation technique supposes regressed variable (commuter flows) in form of non-negative whole numbers and does not need logarithmic transformation for estimation. These are the main advantages in comparison to OLS estimation.

We have chosen the same relevant variables as in OLS and estimation results can be seen in the Table 6.2

Table 6.2: Poisson estimation results

Dependent Variable: COMMUTER					
Method: ML/QML - Poisson Count					
Included observations: 729					
QML (Huber/White) standard errors & covariance					
Variables	**Model 1**	**Model 2**	**Model 3**	**Model 4**	**Model 5**
C	0.991 (0.42)**	0.151 (0.21)	-0.969 (0.36)**	-0.513 (0.36)	-1.562 (0.48)**
LSUM_from	-0.398 (0.08)**	0.171 (0.06)*	0.204 (0.08)*	0.172 (0.07)*	0.206 (0.08)*
LSUM_to	1.289 (0.06)**	0.800 (0.06)**	0.777 (0.07)**	0.867 (0.06)**	0.836 (0.06)**
LTIME	-1.392 (0.03)**	-1.425 (0.02)**	-1.260 (0.04)**	-1.427 (0.02)**	-1.263 (0.04)**
LAV_WAGE _RATE		3.612 (0.36)**	3.465 (0.54)**	3.665 (0.36)**	3.520 (0.54)**
NBR			1.012 (0.20)**		1.007 (0.20)**
BA				-0.186 (0.08)*	-0.166 (0.10)
R-squared	0.988	0.993	0.993	0.996	0.996
Adjusted R-squared	0.988	0.993	0.993	0.996	0.996
Akaike info criterion	139.564	117.585	94.420	116.436	93.509
Schwarz criterion	139.589	117.616	94.458	116.474	93.553
Hannan-Quinn criter.	139.574	117.597	94.435	116.451	93.526
LR index (Pseudo-R2)	0.974	0.978	0.983	0.979	0.983

The basic model has the same problems with residuals as in OLS estimation and therefore needs to be extended. We have constructed similar models to OLS estimation and found out, that the best model is the Model 3, which has the same regressors as the final Model 5 in OLS regression. These models can be then compared. In Poisson regression higher fit has been achieved and the mentioned advantages speak in favour of Poisson model. The estimated parameters have correct signs and values. The difference between origin and destination areas is more obvious. The values of coefficients are more reasonable than those of OLS Models 8 and 5 (constant term and W_i and W_j coefficients).

The final fitted equation is in the following form:

$$T_{ij} = 0.37 \frac{W_i^{0.20} W_j^{0.78} RW_{ij}^{3.46}}{c_{ij}^{1.26}} e^{1.01*NBR} \quad . \tag{6.6}$$

6.1.4 NLS Estimation Results

NLS estimation was introduced in the chapter 3. It does not need logarithmic transformation of (6.1) for estimation (as in OLS estimation).

The estimation is made iteratively and convergence has to be achieved, because no exact computing method of coefficient estimation exists. Estimation of complicated models can therefore not be possible, when iteration process does not converge. Estimation results of different models are given in the Table 6.3.

Table 6.3 NLS estimation results

Dependent Variable: COMMUTER						
Method: Least Squares (NLS)						
Included observations: 729						
Variable	**Model 1**	**Model 2**	**Model 3**	**Model 4**	**Model 5**	**Model 6**
C	3.507 (0.208)**	1.693 (0.17)**	2.136 (0.13)**	1.165 (0.11)**	0.488 (0.05)**	0.880 (0.07)**
SUM_from	-0.196 (0.02)**	-0.144 (0.03)**	0.08 (0.03)	0.113 (0.03)**	0.057 (0.03)*	0.018 (0.03)
SUM_to	1.063 (0.02)**	1.020 (0.02)**	0.828 (0.03)**	0.805 (0.02)**	0.950 (0.02)**	0.984 (0.03)**
TIME	-1.179 (0.02)**	-1.091 (0.02)**	-1.329 (0.03)**	-1.224 (0.03)**	-1.243 (0.02)**	-1.347 (0.02)**
AV_WAGE _RATE			3.059 (0.25)**	2.726 (0.22)**	2.604 (0.19)**	2.883 (0.22)**
exp(NBR)		0.63 (0.08)**		0.55 (0.07)**	0.53 (0.06)**	
exp(BA)					-0.212 (0.01)**	-0.216 (0.01)**
R-squared	0.996	0.997	0.997	0.997	0.998	0.998
Adjusted R-squared	0.996	0.996	0.997	0.997	0.998	0.998
Akaike info criterion	14.729	14.628	14.507	14.401	14.051	14.189
Schwarz criterion	14.754	14.659	14.538	14.439	14.095	14.227
Log likelihood	-5364.75	-5326.82	-5282.75	-5243.09	-5114.64	-5165.89

In the basic model (Model 1) can be seen that estimated coefficient for origin attraction is negative which would indicate unattractiveness of home city. This not reasonable. Models 4 and 5 have the best results (all coefficients highly significant) and they are reasonable (according to the mentioned criteria). Discussion about better model from the Model 5 and 4 is similar to the discussion in chapter 6.1.2, though both models have correct signs of coefficients. In the Model 5 we can see, that despite our expectations the sign of the parameter BA is negative. It can be however probably reasonably explained. The attraction of Bratislava is included both in its size and relative wage. The wage difference and its great size and possibly neighbouring could overcome the effect of cluster. The fitted parameters are similar to Poisson regression and the fit is very good.

The resulting model forms according Model 4 (6.7) and Model 5 (6.7):

$$T_{ij} = 1.17 \frac{W_i^{0.11} W_j^{0.81} RW_{ij}^{2.73}}{c_{ij}^{1.23}} e^{0.80*NBR}, \qquad (6.7)$$

$$T_{ij} = 0.49 \frac{W_i^{0.06} W_j^{0.95} RW_{ij}^{2.60}}{c_{ij}^{1.24}} e^{0.77*NBR-0.31*BA}. \qquad (6.8)$$

6.2 Maximum Entropy Model

The principle of maximum entropy and its use in transportation analysis was introduced in the chapter 4. Maximum entropy uses known information on marginal commuting flows (row and column sums of commuters for each region, e.g. the out- and in-flows from / into each region) and information on cost of trips between regions.

We have used real data to compute marginal flows. These marginal flows are written in the Table 6.4. As cost of trip matrix, the transportation cost matrix (Table 5.7 in Annex A) has been used and total cost of trips C has been computed with use of real data.

Table 6.4 Marginal number of trips and a total cost of trips

	Region	Origin	Destination		Region	Origin	Destination
1	Bratislava	138 397	185 516	15	Nové Mesto / Váhom	15 332	14 884
2	Malacky	15 658	9 696	16	Partizánske	10 651	9 866
3	Pezinok	14 186	8 447	17	Považská Bystrica	13 795	12 593
4	Senec	12 343	7 040	18	Prievidza	38 443	38 013
5	Dunajská Streda	26 440	21 592	19	Púchov	13 111	13 284
6	Galanta	20 280	14 912	20	Trenčín	31 478	31 962
7	Hlohovec	12 407	11 115	21	Komárno	24 378	23 608
8	Piešťany	17 057	15 883	22	Levice	27 037	27 941
9	Senica	16 168	13 931	23	Nitra	43 226	44 671
10	Skalica	11 900	11 532	24	Nové Zámky	33 369	27 813
11	Trnava	36 787	35 654	25	Šaľa	12 355	10 652
12	B. nad Bebravou	9 841	8 758	26	Topoľčany	17 573	15 540
13	Ilava	16 928	16 530	27	Zlaté Moravce	9 473	8 128
14	Myjava	7 862	6 914		**Total costs**	**9 164 070**	

6.2.1 Computation of Maximum Entropy Model

Estimation of O-D matrix with help of maximum entropy concept is based on maximisation problem, which is introduced in the chapter 4. In this part of the thesis we described one possible way of solving this problem (by so-called Bregman's method). We have decided to use another estimation technique for solving of our problem. This computation technique is described in detail in this chapter and then applied on real data. Source codes of the developed program are in Annex C.

According to solution of classical maximum entropy problem [13], we have transformed the maximum entropy maximising system (4.2-4.5) by normalisation 6.9 into a normalised system (6.6.9a-6.13)

$$p_{ij} = \frac{T_{ij}}{\sum_{j=1}^{n}\sum_{i=1}^{n} T_{ij}} = \frac{T_{ij}}{T}, \tag{6.9}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij} = 1, \tag{6.9a}$$

$$H = -\sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}.\ln p_{ij}, \tag{6.10}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij} c_{ij} = C/T = c_0, \tag{6.11}$$

$$\sum_{j=1}^{n} p_{ij} = O_i/T = o_i, \tag{6.12}$$

$$\sum_{i=1}^{n} p_{ij} = D_j/T = d_j. \tag{6.13}$$

This maximisation problem with constraints can be solved by Lagrange function of the system and then by substituting Lagrange multipliers into the original system and constructing dual system to the original system. We then get unconstrained maximum entropy problem in the following minimalizing form (6.10)

$$H = \sum_{i=1}^{n} \alpha_i O_i + \sum_{j=1}^{n} \beta_i D_j + \gamma . C + \ln(\sum_{i=1}^{n} \sum_{j=1}^{n} \exp(-(\alpha_i + \beta_j + \gamma c_{ij})T)) . \quad (6.14)$$

By solving the (6.14) according to unknown parameters α_i, β_j, γ we get solution of our system in the following form (6.15) and after eliminating substituting T into the equation we get the solution (6.15a)

$$T_{ij} = T \frac{\exp(-(\alpha_i + \beta_j + \gamma c_{ij})T))}{\sum_{i=1}^{n} \sum_{j=1}^{n} \exp(-(\alpha_i + \beta_j + \gamma c_{ij})T))} , \quad (6.15)$$

$$T_{ij} = \exp(-\alpha'_i - \beta'_j - \gamma' c_{ij}) \frac{\sum_{j=1}^{n} D_j}{\sum_{i=1}^{n} \sum_{j=1}^{n} \exp(-\alpha'_i - \beta'_j - \gamma' c_{ij})} . \quad (6.15a)$$

The optimisation results (from 6.15a) can be found in the Table 6.5 and resulting commuting flow estimates are in the Table 6.11 in Annex B.

Table 6.5 Maximum entropy parameters

	Region	Alfa	Beta		Region	Alfa	Beta
1	Bratislava	3.60	-5.99	15	Nové m. / Váhom	-0.65	0.78
2	Malacky	2.51	-2.12	16	Partizánske	-0.93	1.53
3	Pezinok	3.56	-2.17	17	Považská Bystrica	-1.31	1.51
4	Senec	2.93	-1.32	18	Prievidza	-1.89	0.82
5	Dunajská Streda	1.12	-1.52	19	Púchov	-1.11	1.37
6	Galanta	0.71	-0.41	20	Trenčín	-1.32	0.59
7	Hlohovec	0.39	0.15	21	Komárno	-1.32	0.75
8	Piešťany	-0.46	0.46	22	Levice	-0.24	-0.50
9	Senica	-0.48	0.53	23	Nitra	-0.63	-0.45
10	Skalica	-0.21	0.38	24	Nové Zámky	-1.59	0.85
11	Trnava	0.61	-1.39	25	Šaľa	0.61	0.14
12	B. nad Bebravou	-1.02	1.64	26	Topoľčany	-1.02	1.00
13	Ilava	-1.14	1.22	27	Zlaté Moravce	-0.41	0.99
14	Myjava	-0.65	0.78		**Gamma**		**0.06**

A proper interpretation of these parameters is difficult because the equations (6.15 and 6.15a) include normalising factor (numerator in the equations 6.151 and 6.15a) which is not directly interpretable. Alfas and Betas characterise commuting habits between each pair of origin and destination regions. In gravity model, it can be achieved by additional constant terms, which would characterise each origin and destination area. This would also improve fit of the model. Gravity model on the other hand can include additional explanatory variables and maximum entropy uses only those variables that are included in the classic model.

We have achieved high correlation of real and estimated data 0.996. This correlation is comparable to that which was achieved by Poisson and NLS regression. But direct comparison of results is misleading. Estimated OD table from maximum entropy model is already normalised to meet origin and destination constraints and OD matrices from gravity model estimation by OLS, NLS and Poisson regression do not meet these constraints. We therefore used RAS (row-column approximate iterative process which results in doubly constrained OD matrix). The RAS process was programmed in Mathematica software and the program source code can be found in Annex D. After RAS transformation of result OD matrices of final OLS, NLS and Poisson models we have in all cases improved correlation of real and estimated data. These correlations before and after transformation are in the table 6.5. For better comparison we have chosen models with the same explanatory variables (basic model and extended model without BA dummy).

Table 6.5 Correlation of real and estimated OD matrices before and after RAS transformation

Model	**Estimation technique**	**Correlation**	
		Before RAS	**After RAS**
Classical	*OLS*	0.764	0.9986
	NLS	0.996	0.9982
	Poisson	0.988	0.9994
Extended	*OLS*	0.823	0.9994
	NLS	0.997	0.9993
	Poisson	0.993	0.9996

Here can be seen that correlation has been improved in all cases and in case of OLS estimation this improvement was very significant. In all cases correlation is comparable to the correlation of estimated and real OD matrix in maximum entropy model.

6.3 Final Comparison of Models

We have used all proposed models to estimate Origin-Destination matrix with use of original data. The results were described in the previous sections (6.1 and 6.2). In the following text, we would like to compare them. The selected estimation results can be found in Annex B in the Tables 6.6 - 6.11 and in Annex at the end of the thesis.

Maximum entropy estimation leads to a very good fit with original data, but this fit is comparable with other estimation methods. This method solves only the whole system and needs all column and row sums (origin and destination) and matrix of costs. Disadvantage of this model is that it gives no explicit solution for a given pair of regions., On the other hand its advantage is, that it is applicable on any commuting system without specifying additional regional characteristics. Maximum entropy also does not depend on coefficients that have been obtained in other regression.

On the contrary, gravity model needs for its best fit additional explanatory variables. And the fitted results depend on parameters that have been estimated in the regression. In gravity model, it is possible that the estimated parameters change, when we take into consideration a smaller part of the selected region or we change commuting conditions.

Maximum entropy model in this form does not answer to changeable environment as well, but can be used to any subregion without additional restrictions.

Another problematic task results from the choice of best estimation technique among gravity model estimation techniques. A very good correlation has been achieved in NLS and Poisson models. Their advantages and disadvantages were described in section 6.1 and do not need to be repeated.

Poisson regression leads to very good results. It does not give negative number results and is easy to estimate with possibility to use wide variety of regression variables. NLS estimation is rather complicated as convergence is not achieved in all cases and model variables need to be precociously selected.

Depending on purpose of commuting flow estimation, observable obtained data, we use either maximum entropy model or gravity model. From the gravity model estimation techniques, we recommend to use Poisson regression. It is easy to apply and its pre-conditions of non-negativity and whole number explained variable suit for commuting flows. Not forgetting its very good fit, we suppose that results gained in the thesis (final model 3 by Poisson regression) is the best for prediction of commuting flows in Western Slovakia region.

7 Conclusion

Since the beginning of urban analysis, many models have been developed to better understand processes in urban systems. The thesis has introduced various models used in regional analysis. Different model categories, source data needs, estimation techniques and functions have been described.

A further aim of the thesis was to estimate commuting flows among Slovak counties based on real commuting data from Slovak Statistical Institute [7]. In order to estimate the flows, two models were chosen: gravity model and maximum entropy model. As already discussed in this thesis, both models have many extensions and variations. Estimation techniques differ according to the model used. In the chapter 6, both models for estimation of commuting flows were applied. We have also tried to find their best possible model forms. The gravity model can be estimated using a variety of estimation techniques. Only three of them have been selected: OLS, Poisson and NLS estimation. We have implemented them and then compared the results. It has been found out, that the best results for gravity model are achieved by Poisson regression (model 3). Results of all used estimations are written in the chapter 6.

Further gravity model and maximum entropy model results were compared in the chapter 7. Gravity model and maximum entropy estimations show a very good fit with real data, which is similar in both cases. It can not be said, which method is better only comparing model results. Both models have their fields of use in different situations and therefore both models can find their application.

The gravity model results can be used for forecasting traffic flows between two areas, based on their socio-economic and travelling data. We do not need information on other areas in the system. We can predict traffic flows for each pair of origin and destination separately.

In maximum entropy model, we need aggregate data on traffic flows from each origin and to each destination in the system. Further information on overall travelling costs in the system and cost matrix of all origin-destination areas are needed. We can forecast the system as whole. This is useful when we can not obtain any additional information on determinants, which influence commuting behaviour (e.g. socio-economic data).

Forecasts made by both models can be used in urban planning, road construction, regional development projects and other projects.

Not all models introduced in this thesis were applied. We suppose that he created model forms, their estimation and so their results can be further improved.

In gravity model estimation, other estimation methods could be used (for example spatial econometrics) and then be compared to the used ones. More explanatory variables can be introduced into the model to better explain socio-economic situation influencing commuting patterns. A special focus can be set on border effects. Border effects play probably a very important role in commuting behaviour in Western Slovakia, as the most part if it lies next to state borders. Special discussion on the problem is to be a subject of another paper.

In entropy problem, the disaggregated model can be used for tracking traffic flow changes. This is important, as commuting flows vary in time and "dynamic" system could better describe the commuting system.

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Index of Used Signs

T_{ij} - number of commuters commuting from region I to region J

T - total number of commuters

d_{ij} - distance between origin I and destination J

t_{ij} - travelling time between origin I and destination J

c_{ij} - travelling cost between origin I and destination J

C - total travelling costs ($C = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}.T_{ij}$)

k - constant term

W_i - mass of region I (given by population, number of emplyees, …)

n - number of regions

W - entropy function (W = H(N))

N - number of microstates in entropy function

U_{ij} - utility function, gives utility of commuting from region I to region J

$F(c_{ij})$ - distance deterrence function, as function of commuting costs

O_i - origin constraint for region I ($O_i = \sum_{j=1}^{n} T_{ij}$)

D_j - destination constraint for region J ($D_j = \sum_{i=1}^{n} T_{ij}$)

μ_{ij} - stochastic term

ESTIMATION VARIABLES

commuter	-	number of commuters
commuter1	=	commuter -1
C	-	constant
BA	-	Bratislava
NBR	-	neighbouring region
sum_from	-	number of outgoing commuters from given region
sum_to	-	number of in-commuting commuters to given region
time	-	travelling time
av_wage_rate	-	rate of average wage between each pair of origin and destination

ANNEX