Fakulta matematiky, fyziky a informatiky Univerzity
Komenského v Bratislave

DIPLOMOVÁ PRÁCA

Bratislava 2004 Matej Maceáš

Fakulta matematiky, fyziky a informatiky Univerzity
Komenského v Bratislave

Ekonomická a finančná matematika

Dynamically optimal hedging strategy in the presence of transaction costs

Diplomová práca

Autor: Matej Maceáš
Vedúci diplomovej práce: Dr. Aleš Černý

Bratislava 2004

Čestne vyhlasujem, že som túto diplomovú prácu vypracoval samostatne, len s použitím uvedenej literatúry.

V Bratislave, 5. apríla 2004 ....

Matej Maceáš

Acknowledgements

The author would hereby like to thank Dr. Aleš Černý for his invaluable assistance and guidance throughout the writing of this thesis.

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1 Introduction

The decisions that investors and portfolio managers constantly have to make in the course of their work revolve around and are determined by two factors: return and risk. On the one hand, the investor will want to attain the highest possible return on their investment; on the other hand, they will want to minimise the risk of loss associated with the investment.

Unfortunately for the investor, risk and return are usually directly proportionate – higher return is accompanied by higher risk, while lower risk leads to lower returns. The investor’s ambitions to achieve high return at low risk thus contradict each other, and the investor has to settle on an acceptable compromise.

What is acceptable will, of course, depend on each particular investor’s preferences. For some investors, the prospect of a high return on their investment is worth taking a greater risk. For more conservative investors, e.g. pension fund managers, the primary concern is to eliminate or at least minimise as much as possible the risk of a significant decrease of the value of their portfolio, with the long-term character of their investments being relied upon to provide a reasonable level of return.

It is a variant of the latter approach that we will adopt in this thesis. First, we will obtain investment funds by selling stock options. Subsequently, we will create a portfolio consisting of cash (invested in a risk-free account) and stock, which will be periodically readjusted so as to minimise the risk of loss at the end of the investment horizon – the time of option expiry. Thus, the task at hand will be to maximise the value of the portfolio to the investor in the presence of a risk constraint.

2 Utility function

Notice that in the previous paragraph, portfolio value *to the investor* was mentioned, rather than portfolio value as such. The reason for this is that any given change in the value of the portfolio will be perceived differently by the investor depending on specific circumstances. Generally, the investor's utility can be described by the following premises. Firstly, greater wealth has more value to the investor than lesser wealth. In other words, having more is better than having less. Secondly, a negative change of wealth will be perceived as more dramatic than an equally large positive change. (In terms of utility function properties, the second premise is equivalent to saying that e.g. a \$500 change in the value of a portfolio worth \$1,000 is more significant than the same change in a portfolio worth \$100,000.)

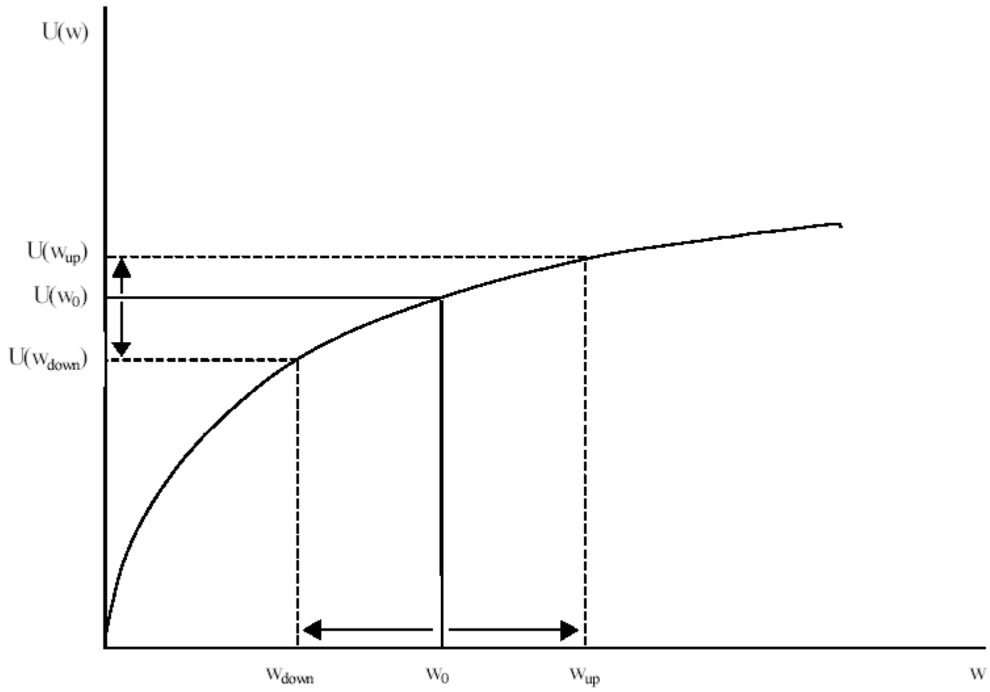

Figure 2.1 [1]: The change in utility caused by a decrease in wealth is greater than that caused by an equally large increase in wealth.

Mathematically, the first premise implies that the utility function must be strictly increasing, and the second premise implies that the function must be concave. There are several classes of functions that satisfy these two conditions, e.g. exponential functions, logarithmic functions, or n-th root functions.

However, since we will be interested in comparing the terminal level of wealth w_T relative to the initial level of wealth w_0, we will impose one additional requirement on the utility function, namely that the function looks the same regardless of the actual value of w_0. In mathematical notation, this may be written as

$$\frac{U(w)}{U(w_0)} = f(w - w_0) \tag{2.1}$$

where f is the utility function we seek. According to [1], only the exponential utility function satisfies this condition. The utility function that we will use will thus take the form

$$U(w) = -e^{-Aw},\ A > 0 \tag{2.2}$$

It can be easily shown that function (2.2) satisfies all three conditions:

- $U'(w) = Ae^{-Aw} > 0$
- $U''(w) = -A^2 e^{-Aw} < 0$
- $\frac{U(w)}{U(w_0)} = \frac{-e^{-Aw}}{-e^{-Aw_0}} = e^{-A(w-w_0)} = f(w - w_0)$

The utility function (2.2) is called the *constant absolute risk aversion (CARA)* utility ('constant' because it's invariant in w_0, and 'absolute' because it reacts to changes in absolute wealth), and coefficient A is called the *coefficient of absolute risk aversion*. The latter varies from one investor to another, and characterises the investor's attitude towards risk. Although investors are not consciously aware of their risk aversion coefficient, its value may be determined by examining their investment decisions, whether it be real (preferably) or hypothetical ones. The utility function can thus be calibrated to fit each investor's attitude towards risk.

Now that we have a suitable utility function, we may move on to precise mathematical formulation the problem that will be examined and solved in this thesis.

3 Formulation of the problem

The problem that we will attempt to solve stems from the following scenario. An investor with a negative exponential utility function and a given level of initial wealth sells a specified number of European call options. Subsequently, the investor creates a portfolio consisting of the options' underlying stock, and of a cash account that may be freely invested into or borrowed from at a single, fixed risk-free interest rate.

The task at hand is to determine the optimal quantity of stock that should be held in the portfolio at each time period before option expiry, so that the investor's utility of the net value of the portfolio (portfolio value less the debt represented by total option value at expiry) at the time of option expiry is maximised. The mathematical formulation of this problem is

$$\max_{\theta_0,\theta_1,\ldots,\theta_{T-1}} U\left(V_T - \alpha H_T\right) \tag{3.1}$$

$$U\left(x\right) = -\exp\left(-Ax\right) \tag{3.2}$$

where V_T is portfolio value at time T (the time of option expiry), H_T is the value of the option at time T, α is the number of options sold, θ_t is the quantity of stock held at time t, and $A > 0$ is the investor's coefficient of absolute risk aversion.

An important restriction placed upon the hedging process θ_t is that it must depend only on information that is already available at time t; in our case, that information is the evolution of the stock price, discussed below. (Obviously, if the optimal value of θ_t depended on future stock prices, which, due to their stochastic character, are not known, the entire model would be useless to the investor.) The stock price process generates a sequence of $\sigma-$algebras $\{F_t\}$. As the number of possible stock price paths increases in time, each $\sigma-$algebra in this sequence is "richer" than the previous one, i.e. $F_0 \subset F_1 \subset \ldots \subset F_T$. The latter property means that the sequence $\{F_t\}$ is a filtration, and the restriction mentioned at the start of this paragraph means that the process θ_t has to be adapted to this filtration, generated by the stock price process.

The next step is to specify a portfolio valuation formula. Because we will be working with discrete time, the value of the portfolio will be given by the first-order difference equation

$$V_{t+1} = R_f V_t + \theta_t S_t X_{t+1} - \kappa \left| \theta_{t+1} - \theta_t \right| S_{t+1} \quad (3.3)$$

where R_f is the risk-free appreciation factor of the cash account (i.e. the risk-free interest rate + 1), S is the price of the stock, X is the excess return of the stock (i.e. the difference between the actual return of the stock and the risk-free rate), and κ is the coefficient that determines what proportion of the transaction volume will be paid as transaction costs.

Since stock price plays a major role in the given problem, we must also create a realistic model of stock returns. In the classical model of Black and Scholes, a lognormal distribution of stock returns is assumed. However, this assumption does not hold true in real markets. In reality, stock return distributions have been found to exhibit fat tails, negative skewness, self-scaling, leverage and even some correlations in the increments of return [3]. Therefore, we will use the stock-return model described and used in [2], which is based on empirical observations, specifically on the weekly returns of the FTSE 100 index between years 1984 and 2001, and assume that these returns are independently distributed.

The construction of the log-return histogram is fairly simple. We take the historical data and divide the log-returns into a number of categories, depending on how branched-out we want the resulting tree to be. To get a trinomial tree, three categories would be needed. In our case, we shall use a slightly denser tree – one where each node at time t branches out to seven nodes at time $t+1$, hence we will use seven categories. One thing that is important to remember is that in order to get a recombinant tree, the log-returns must be spaced out regularly.

With that in mind, we'll divide the log-returns into those of –5% or less, those between –5% and –3%, and continuing in this manner until the last category of log-returns of more than 5%. This will give us the desired histogram, from which we can calculate the objective probabilities of the underlying asset returns.

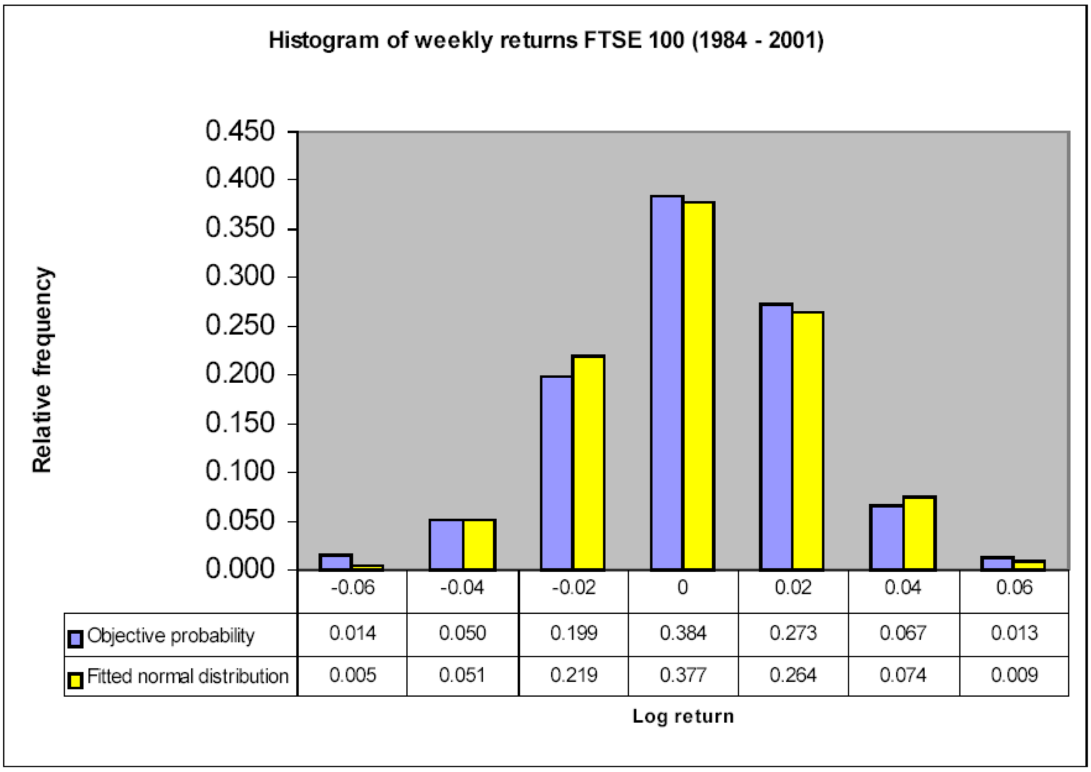

Figure 3.1 [2]: Histogram of weekly log returns on the FTSE 100 index

The advantage of this method of stock-price model construction is threefold. Firstly, as has been mentioned already, the method is simple in that it comprises some basic processing of historical data. Secondly, since we are dealing with real data, we can assume that the resulting model will be quite realistic in that it will describe reasonably well the probability distributions exhibited by real-world financial markets. No extra calibration is needed to achieve this. Finally, this method is very general. Provided that we're dealing with developed markets (i.e. sufficiently long time series are available), the investor can use the historical data of that index, which they think will best approximate the behaviour of the underlying asset's price. This method also makes it possible (once again, given a sufficient quantity of data) to create a model for daily or monthly returns (depending on the investor's preferred hedging frequency), and to use any number of log-return categories (for example, we could use a tree with 11 branches from each node, and this would not qualitatively change the character of the model).

4 Dynamic optimisation

Looking at the maximisation problem (3.1)-(3.3), we can see that the optimal value of θ_t will be influenced by several factors. The most obvious one is the stock price S – the value of the sold options, i.e. the debt, directly depends on this variable. Then there's the value of the portfolio, V_t. Finally, the transaction costs come into play. Since the overall cost of the transaction will be directly proportional to the change in stock holdings between times $t-1$ and t, the optimal value of θ_t will also depend on θ_{t-1}. The magnitude of the latter effect will be determined by the value of κ. The higher this constant, the greater the role of θ_{t-1} in determining the optimal θ_t.

To solve our optimisation problem, we will apply the principles of dynamic programming. Instead of trying to compute the optimal values of θ_t for all times at once, we will reduce the problem to a set of one-period optimisation tasks, starting at the terminal time T and working backwards to $t=0$. Not only is this approach effective and relatively easy to implement, but it also provides the added benefit of being able to react to deviations from the optimal sequence of control variables. If, for example, the investor were to stray from the optimal path of θ_t values, the algorithm given by dynamic programming will adapt to this fact and will provide results that are optimal with regard to the new situation.

For clarity's sake, we shall denote the two sets of terms in equation (3.3) as follows:

$$\tilde{V}_{t+1} \triangleq R_f V_t + \theta_t S_t X_{t+1} \tag{4.1}$$

$$C_{t+1} \triangleq \kappa \left|\theta_{t+1} - \theta_t\right| S_{t+1} \tag{4.2}$$

to get

$$V_{t+1} = \tilde{V}_{t+1} - C_{t+1} \tag{4.3}$$

While this shorter notation doesn't bring anything new in saying that the value of the portfolio is given by the difference of the value of the assets held and the transaction cost associated with rehedging, it does motivate us to examine the special case of C_T, i.e. the transaction costs at time T.

The formula for C_T essentially describes how the debt, in the form of European call options in short position, is repaid. There are several possibilities:

- Repayment in kind, i.e. if the options end up in-the-money, the investor must hand over an appropriate number of stocks. In this case,

$$\theta_T = \alpha \tag{4.4}$$

(the number of options sold). If the options are not exercised, the investor converts the stock to cash, i.e.

$$\theta_T = 0 \tag{4.5}$$

The transaction costs formula is then

$$C_T = \kappa \left| \alpha - \theta_{T-1} \right| S_T , \qquad S_T > K \tag{4.6}$$

$$C_T = \kappa \left| \theta_{T-1} \right| S_T , \qquad S_T < K \tag{4.7}$$

- Repayment in cash, where all stock is converted to cash regardless of whether the option ends up in-the-money or out-of-money. Thus the formula for C_T is

$$C_T = \kappa \left| \theta_{T-1} \right| S_T , \qquad \forall S_T \tag{4.8}$$

- Other, more theoretical cases, for example

$$\theta_T = \theta_{T-1} \tag{4.9}$$

$$C_T \equiv 0 \tag{4.10}$$

In all of these variants, the terminal transaction costs are a function of only θ_{T-1} and S_T. Generally, we can write

$$C_T = C_T \left(\theta_{T-1}, S_T \right) \tag{4.11}$$

We know that $S_T = S_{T-1}\rho$, where $\rho = X + R_f$ is a random variable with a known probability distribution given by our model of stock price returns. We can rewrite

$$C_T = C_T \left(\theta_{T-1}, S_{T-1}\rho \right) \tag{4.12}$$

Hence at time $T-1$, we have all necessary knowledge to compute the terminal transaction costs. This will prove to be important in that it will help provide an 'anchor' for the recursive set of problems described below.

Now we can continue by rewriting the original optimisation problem (3.1) as a recursive set of more simple, one-period optimisation problems. We will start with the last period $t = T$, find the one-period solution, and use it to calculate the solution for period $t = T - 1$. Then we'll use the solution from period $t = T - 1$ to obtain the solution for $t = T - 2$. Following this pattern, we'll find the solutions for all periods down to $t = 0$. According to Bellman's principle of optimality, when we take together all the solutions of the partial problems, we get the optimal solution of the original multi-period problem (see [4]).

The first step is to write down the value function J_T. This function will be identical to the function we are trying to maximise:

$$J_T = U\left(V_T - \alpha H_T\right) \tag{4.13}$$

By substituting the utility function (3.2) into the value function (4.13) and normalising the coefficient of absolute risk aversion $A = 1$, we obtain

$$J_T = -\exp\left(-\left(V_T - \alpha H_T\right)\right) \tag{4.14}$$

Moving on to period $t = T - 1$, we'll get the first of recursive one-period optimisation problems

$$J_{T-1} = \max_{\theta_{T-1}} E_{T-1}\left[J_T\right] \tag{4.15}$$

Substituting (4.14) into (4.15), simplifying the inner term of the exponential function and applying the equivalence of optimisation problems $\max\left(-f\right) \Leftrightarrow \min f$, we can write

$$J_{T-1} = \min_{\theta_{T-1}} E_{T-1}\left[\exp\left(\alpha H_T - V_T\right)\right] \tag{4.16}$$

Using the recursive portfolio value formulae (3.3) and (4.3), we can rearrange

$$\alpha H_T - V_T = \alpha H_T - \tilde{V}_T + C_T = \alpha H_T - \left(R_f V_{T-1} + \theta_{T-1} S_{T-1} X_T\right) + C_T \tag{4.17}$$

and expressing this in terms of $\tilde{V}_{T-1}$ we get

$$\alpha H_T - V_T = \alpha H_T - R_f\left(\tilde{V}_{T-1} - C_{T-1}\right) - \theta_{T-1} S_{T-1} X_T + C_T \tag{4.18}$$

Just like the terminal transaction cost, the unit debt H_T is also a function only of quantities that are known at time $t = T - 1$. Indeed, the value H_T of a call option with strike price K is given by the formula

$$H_T = \max\left\{S_T - K, 0\right\} \tag{4.19}$$

where once again $S_T = S_{T-1}\rho$, so we can write

$$H_T = H_T\left(S_{T-1}\rho, K\right) \tag{4.20}$$

Substituting (4.18) into (4.16), we get

$$J_{T-1} = \min_{\theta_{T-1}} E_{T-1}\left[\exp\left(\alpha H_T - R_f \tilde{V}_{T-1} + R_f C_{T-1} - \theta_{T-1} S_{T-1} X_T + C_T\right)\right] \tag{4.21}$$

Because the term $-R_f \tilde{V}_{T-1}$ is both deterministic and at the same time is not a function of θ_{T-1}, we can take it out of both the expected value operator $E_{T-1}[\bullet]$ and the minimisation operator $\min_{\theta_{T-1}}(\bullet)$ and write

$$J_{T-1} = \exp\left(-R_f \tilde{V}_{T-1}\right) \min_{\theta_{T-1}} E_{T-1}\left[\exp\left(\alpha H_T + R_f C_{T-1} - \theta_{T-1} S_{T-1} X_T + C_T\right)\right] \tag{4.22}$$

Finally, we will take into account the function arguments given by (4.2), (4.12) and (4.20) to obtain a simplified notation of the expected value term in (4.22)

$$J_{T-1} = \exp\left(-R_f \tilde{V}_{T-1}\right) \min_{\theta_{T-1}} g_{T-1}\left(\theta_{T-1}, \theta_{T-2}, S_{T-1}\right) \tag{4.23}$$

From this result it is apparent that the optimal value of θ_{T-1} does not depend on the value of $\tilde{V}_{T-1}$, which will make the calculation less complicated.

Now let's move to the next period, $t = T-2$. Analogically as in (4.15), the value function J_{T-2} will be calculated as

$$J_{T-2} = \min_{\theta_{T-2}} E_{T-2}\left[J_{T-1}\right] \tag{4.24}$$

In compliance with what has been mentioned earlier in this chapter, the solution of problem (4.15) is needed to find the solution of problem (4.24). We will therefore assume that the former is known at this point, and denote θ^*_{T-1} the value of θ_{T-1} at which the minimal value $g^*_{T-1}\left(\theta^*_{T-1}, \theta_{T-2}, S_{T-1}\right)$ of function $g_{T-1}\left(\theta_{T-1}, \theta_{T-2}, S_{T-1}\right)$ is obtained. Hence,

$$J_{T-2} = \min_{\theta_{T-2}} E_{T-2}\left[\exp\left(-R_f \tilde{V}_{T-1}\right) g^*_{T-1}\left(\theta^*_{T-1}, \theta_{T-2}, S_{T-1}\right)\right] \tag{4.25}$$

By applying (4.1), then (4.3) to (4.25), we get an expression of J_{T-2} first as a function of V_{T-2} and subsequently of $\tilde{V}_{T-2}$

$$\begin{aligned} -R_f \tilde{V}_{T-1} &= -R_f\left(R_f V_{T-2} + \theta_{T-2} S_{T-2} X_{T-1}\right) = \\ &= -R_f^2\left(\tilde{V}_{T-2} - C_{T-2}\right) - R_f \theta_{T-2} S_{T-2} X_{T-1} \end{aligned} \tag{4.26}$$

Once again, the term $-R_f^2\tilde{V}_{T-2}$ is neither stochastic nor a function of θ_{T-2}, and can be taken out of the expected value and minimisation operators to obtain

$$J_{T-2} = \exp\left(-R_f^2\tilde{V}_{T-2}\right)\min_{\theta_{T-2}} E_{T-2}\left[\exp\left(R_f^2 C_{T-2} - R_f\theta_{T-2}S_{T-2}X_{T-1}\right)g_{T-1}^*\right] \quad (4.27)$$

where $g_{T-1}^* = g_{T-1}^*\left(\theta_{T-1}^*, \theta_{T-2}, S_{T-1}\right)$. We can see that the $E[\bullet]$ term in (4.27) is a function of θ_{T-2}, θ_{T-3} (because of the C_{T-2} term) and S_{T-2}. Therefore we can write

$$J_{T-2} = \exp\left(-R_f^2\tilde{V}_{T-2}\right)\min_{\theta_{T-2}} g_{T-2}\left(\theta_{T-2}, \theta_{T-3}, S_{T-2}\right) \quad (4.28)$$

Similarly, we can derive

$$J_{T-3} = \exp\left(-R_f^3\tilde{V}_{T-3}\right)\min_{\theta_{T-3}} E_{T-3}\left[\exp\left(R_f^3 C_{T-3} - R_f^2\theta_{T-3}S_{T-3}X_{T-2}\right)g_{T-2}^*\right] \quad (4.29)$$

where $g_{T-2}^* = g_{T-2}^*\left(\theta_{T-2}^*, \theta_{T-3}, S_{T-2}\right)$.

Comparing formulae (4.27) and (4.29), it is obvious that the value functions J_t follow a pattern. This is good news, because we don't have to manually derive the value function individually for each period. Instead, we can use the general formula

$$J_{T-t} = \exp\left(-R_f^t\tilde{V}_{T-t}\right)\min_{\theta_{T-t}} g_{T-t}\left(\theta_{T-t}, \theta_{T-t-1}, S_{T-t}\right) \quad (4.30)$$

$$g_{T-t} = E_{T-t}\left[\exp\left(R_f^t C_{T-t} - R_f^{t-1}\theta_{T-t}S_{T-t}X_{T-t+1}\right)g_{T-t+1}^*\left(\theta_{T-t+1}^*, \theta_{T-t}, S_{T-t}\rho\right)\right] \quad (4.31)$$

right down to time $t = 0$, where the iterations will stop with the final value function

$$J_0 = \exp\left(-R_f^T\tilde{V}_0\right)\min_{\theta_0} E_0\left[\exp\left(R_f^T C_0 - R_f^{T-1}\theta_0 S_0 X_1\right)g_1^*\left(\theta_1^*, \theta_0, S_0\rho\right)\right] \quad (4.32)$$

where $C_0 = C_0\left(\theta_0, \theta_{init}, S_0\right)$, θ_{init} being a known initial quantity of stocks held in the portfolio.

The values of g_t will have to be computed numerically at each node of the three-dimensional state-time grid (θ_{t-1} and S_t are the state variables).

5 Numerical implementation

Now that we have the necessary formulae, we can proceed with the numerical implementation of the dynamic optimisation algorithm specified in the previous chapter. We do not yet know whether the functions $g_t(\theta_t, \theta_{t-1}, S_t)$ are convex, and therefore if the algorithms of convex programming can be applied to correctly solve the problem at hand.

The first step in the numerical implementation will be to assign a specific value to all constants: the transaction cost coefficient κ, the risk-free interest rate and hence the appreciation factor R_f, the investment horizon T, the number of options sold α, as well as the arbitrary values of the initial stock price S_0 and the option strike price K. Once the initial stock price is set, we can construct the recombinant stock-price tree that will form two of the three dimensions of the state-time grid mentioned at the end of the previous chapter. Finally, the function determining the terminal transaction costs has to be selected.

Staying with the setup used in [2], let us assume that the risk-free interest rate is 4% per annum. Because we will be working with a unit time period of one week, we need to transform this annual interest rate to a weekly interest rate. The latter can be calculated as $1.04^{1/52} - 1 = 0.075\%$ per week. Hence $R_f = 1.00075$.

For simplicity's sake, let us assume that we initially sell one option: $\alpha = 1$. To examine how the results change for different values of α, and whether there happens to be a simple relationship between the latter and the results for $\alpha = 1$, is a separate problem in itself and will be addressed at a later stage.

For the other constants in our model, we will use the following values: $T = 5$, i.e. the option will expire in 5 weeks' time. This number is not only fairly realistic, but also provides for a sufficient number of portfolio readjustments without making the whole computation unnecessarily long; $\kappa = 0.01$, i.e. the cost of rehedging the portfolio will be 1% of the overall transaction volume; $S_0 = 100$; and $K = S_0$, i.e. the option will begin at-the-money.

As per what has been written in chapter 3, we will construct a stock price tree using the equidistant log-returns -0.06, -0.04, …, 0.06. Hence we can write

$$S_{t+1} = \left\{ e^{-0.06}, e^{-0.04}, e^{-0.02}, e^{0.00}, e^{0.02}, e^{0.04}, e^{0.06} \right\} S_t \tag{5.1}$$

and the resulting stock price tree is depicted in Figure 5.1 below.

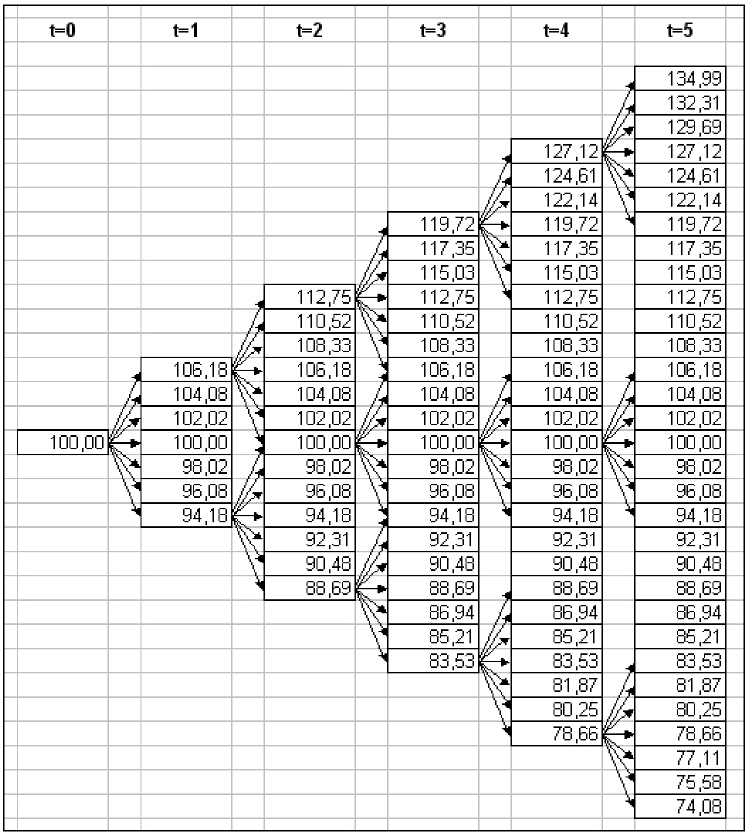

Figure 5.1: The stock price tree.

As for the terminal transaction cost function, we shall use repayment in kind. The reason behind this choice is that it will be quite easy and intuitive to judge whether the obtained results make sense. Obviously, if the option ends up in-the-money, we will need the portfolio to contain one unit of stock at time T. If the option ends up out-of-money, we will theoretically need zero units of stock ('theoretically' because due to the transaction costs, it may well turn out to be disadvantageous to reduce the quantity of stock held to zero).

We now have all the information necessary to start examining the convexity of $g_4\left(\theta_4, \overline{\theta}_3, \overline{S}_4\right)$ as a function of θ_4. While it is not feasible to examine the shape of the function for all possible combinations of (θ_3, S_4), we can plot the graphs for such combinations of values of (θ_3, S_4) that will cover several out-of-money, at-the-money and in-the-money stock price eventualities for several different values of the state variable θ_3. Specifically, these combinations will be $\theta_3 = \{0, 0.25, 0.5, 0.75, 1\}$ and $S = \{127.12, 112.75, 100.00, 88.69, 78.66\}$. Several 'random' sets of (θ_3, S_4) values will also be examined.

The resulting graphs (Figure 5.2) show that the function is convex, and attains a minimum within the interval $[0,1]$ (this is the interval of possible quantities of stock needed to hedge a single stock option). This makes it possible to use Mathematica's built-in function *FindMinimum*, which is relatively fast and accurate. When *FindMinimum* returns a definite value, this value is guaranteed to correspond to at least a local minimum of the function being examined[1]. Because we are dealing with a convex function, the optimal values θ_4^* yielded by *FindMinimum* that minimise the value of g_4 correspond to global minima.

Below are some examples of what the function g_4 looks like. The two numbers above each graph correspond to the values of (θ_3, S_4) for which the graph had been plotted.

[1] See [6]. It should also be noted that the *FindMinimum* function can use both gradient and non-gradient methods, so smoothness of the optimised function is not a requirement.

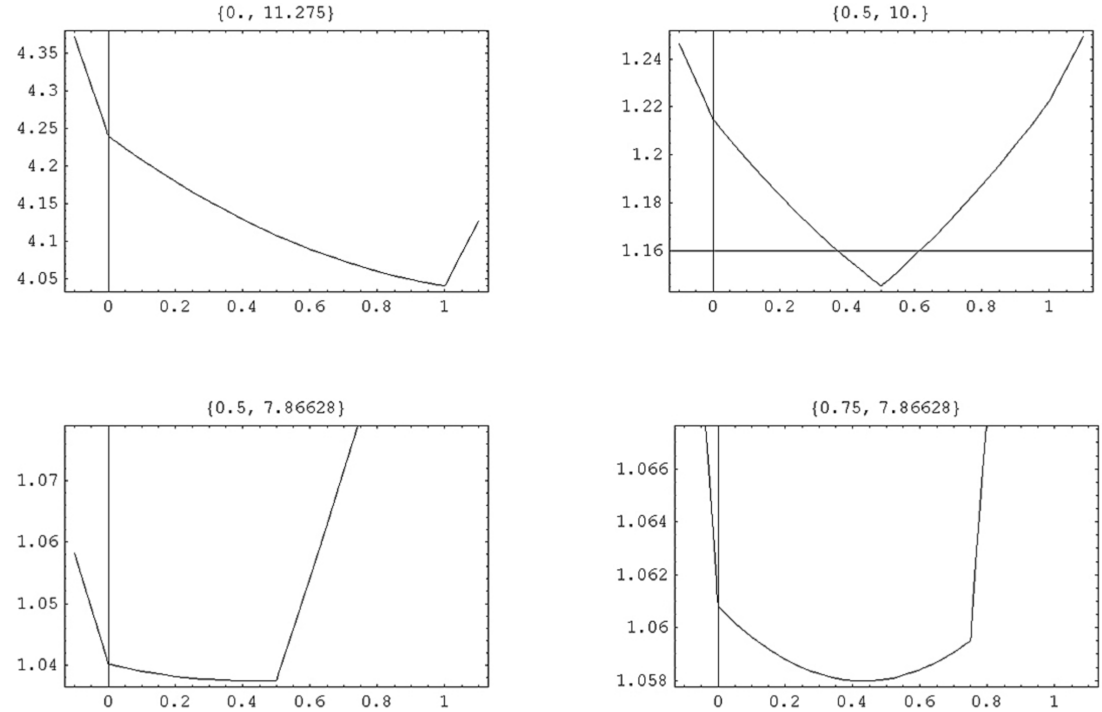

Figure 5.2: Some examples of function $g_4\left(\theta_4,\overline{\theta}_3,\overline{S}_4\right)$.

Unfortunately, it turns out that functions g_3, …, g_0 do not behave as 'nicely' as g_4 (see Figures 5.3, 5.4), and therefore the optimisation function *FindMinimum* cannot be generally relied upon to find the correct minimum in the appropriate interval. We will therefore at this point implement a brute force algorithm that will search for the global minimum within the interval $[0,1]$. The algorithm will evaluate the functional values at a given set of equidistant points, and pick the point in which the functional value is the lowest. This very simple algorithm is guaranteed to find the minimum with arbitrary precision; the downside of the algorithm is its slow speed of convergence to the optimal solution, and the fact that the optimum is identified only after the algorithm has run through the entire interval $[0,1]$, i.e. many calculations are conducted even though theoretically they are no longer necessary.

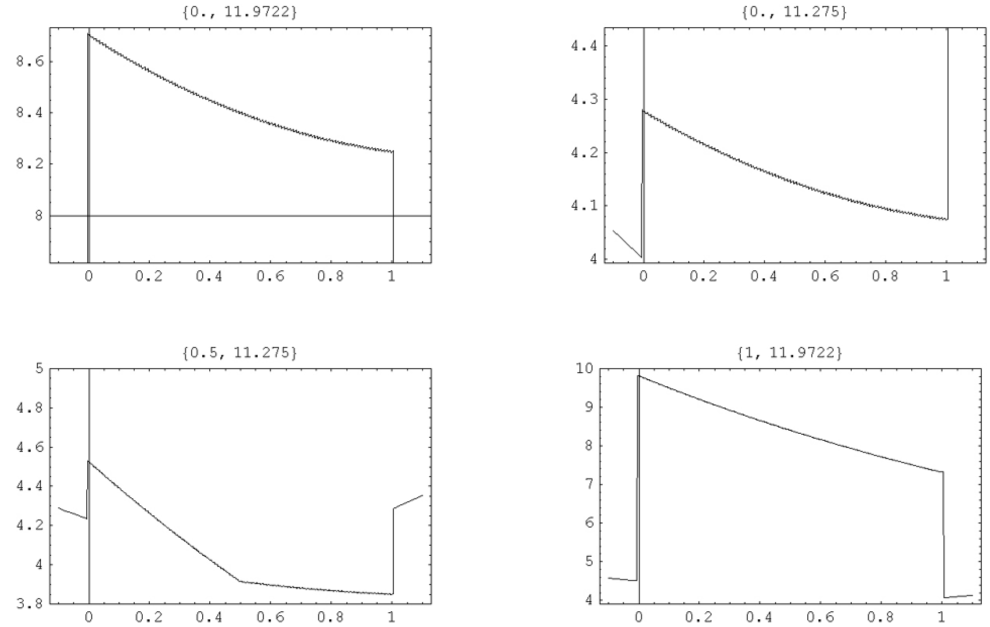

Figure 5.3: Examples of function $g_3\left(\theta_3, \overline{\theta}_2, \overline{S}_3\right)$

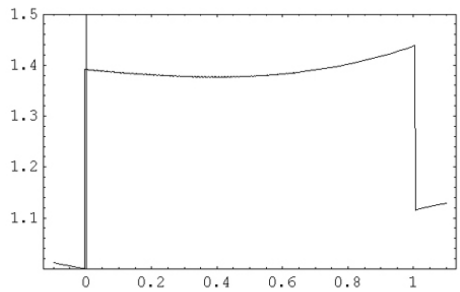

Figure 5.4: Function $g_0\left(\theta_0, \theta_{init} = 0\right)$

6 Results for α = 1

Using the algorithm and constant values described in the previous chapter, we have acquired a set of results from which an optimal hedging strategy can be determined. At this point, writing down specific strategies would not be very meaningful, because the strategy will depend on the specific path followed by the stock price. Instead, in the tables below we will illustrate two important attributes of the calculated strategies.

The first one is that the results yielded by the algorithm satisfy the obvious fact that as the stock price increases into-the-money, the quantity of stocks held increases, up to the value of $\theta = 1$ in nodes where the option is bound to end up in-the-money. Analogically, for out-of-money stock prices, the strategy yields smaller values of θ. The reason why this result is important is that it suggests that the calculations have been conducted correctly and that the results are reasonable. Obviously, if the strategies didn't behave this way, it would be an indication that an error had occurred somewhere in the process (whether in the mathematical model or the numerical implementation).

The second important result shown in the tables is that the optimal quantity of stocks held in the portfolio changes for different values of the previous quantity of the same. In other words, transaction costs do indeed come into play in determining the optimal hedging strategy. This once again indicates that the results are qualitatively correct and that attempting to calculate an optimal strategy in the presence of transaction costs is a meaningful task.

In the tables below, the optimal values of θ_t are given for specific stock prices (in rows; "*n* up" and "*n* down" means a node *n* levels above and below the at-the-money level, respectively) and for selected values of θ_{t-1} (in columns) at each time t. The values are given for $S_0 = 100$, $K = 100$, $\alpha = 1$, $\kappa = 0.01$, $R_f = 1.00075$. Because we have assumed the initial value of held stocks to be zero ($\theta_{init} = 0$), there is no comparison of θ_0 for different values of θ_{init}, and thus the tables start at time $t = 1$. The optimal value of θ_0 is $\theta_0^* = 0.51$.

t=1	0	0,25	0,51	0,75	1
3 down	0,14	0,25	0,28	0,28	0,28
	0,26	0,26	0,39	0,39	0,39
	0,38	0,38	0,5	0,5	0,5
even	0,5	0,5	0,51	0,62	0,62
	0,62	0,62	0,62	0,73	0,73
	0,72	0,72	0,72	0,75	0,84
3 up	0,82	0,82	0,82	0,82	0,94

Table 6.1: Selected optimal values of θ_1

t=2	0	0,28	0,4	0,82	1
6 down	0	0,05	0,05	0,05	0,05
	0	0,08	0,08	0,08	0,08
	0	0,14	0,14	0,14	0,14
3 down	0,09	0,24	0,24	0,24	0,24
	0,22	0,28	0,35	0,35	0,35
	0,35	0,35	0,48	0,48	0,48
even	0,5	0,5	0,51	0,61	0,61
	0,63	0,63	0,63	0,74	0,74
	0,74	0,74	0,74	0,82	0,87
3 up	0,85	0,85	0,85	0,85	0,97
	0,93	0,93	0,93	0,93	0,99
	0,98	0,98	0,98	0,98	0,99
6 up	0,99	0,99	0,99	0,99	0,99

Table 6.2: Selected optimal values of θ_2

t=3	0	0,24	0,4	0,74	1
9 down	0	0,04	0,04	0,04	0,04
	0	0,04	0,04	0,04	0,04
	0	0,04	0,04	0,04	0,04
6 down	0	0,04	0,04	0,04	0,04
	0	0,04	0,04	0,04	0,04
	0	0,08	0,08	0,08	0,08
3 down	0,01	0,16	0,16	0,16	0,16
	0,15	0,24	0,29	0,29	0,29
	0,32	0,32	0,45	0,45	0,45
even	0,49	0,49	0,51	0,6	0,6
	0,64	0,64	0,64	0,74	0,76
	0,78	0,78	0,78	0,78	0,91
3 up	0,89	0,89	0,89	0,89	0,99
	0,97	0,97	0,97	0,97	0,99
	0,99	0,99	0,99	0,99	0,99
6 up	0,99	0,99	0,99	0,99	0,99
	0,99	0,99	0,99	0,99	0,99
	0,99	0,99	0,99	0,99	0,99
9 up	0,99	0,99	0,99	0,99	0,99

Table 6.3: Selected optimal values of θ_3

t=4	0	0,29	0,4	0,78	1
12 down	0	0,04	0,04	0,04	0,04
	0	0,04	0,04	0,04	0,04
	0	0,04	0,04	0,04	0,04
9 down	0	0,04	0,04	0,04	0,04
	0	0,04	0,04	0,04	0,04
	0	0,04	0,04	0,04	0,04
6 down	0	0,04	0,04	0,04	0,04
	0	0,04	0,04	0,04	0,04
	0	0,04	0,04	0,04	0,04
3 down	0	0,04	0,04	0,04	0,04
	0	0,16	0,16	0,16	0,16
	0,24	0,29	0,36	0,36	0,36
even	0,47	0,47	0,51	0,58	0,58
	0,68	0,68	0,68	0,78	0,8
	0,86	0,86	0,86	0,86	1
3 up	0,99	0,99	0,99	0,99	1
	1	1	1	1	1
	1	1	1	1	1
6 up	1	1	1	1	1
	1	1	1	1	1
	1	1	1	1	1
9 up	1	1	1	1	1
	1	1	1	1	1
	1	1	1	1	1
12 up	1	1	1	1	1

Table 6.4: Selected optimal values of θ_4

Apart from $S_0 = 100$, we have also run the algorithm for other starting values of the initial stock price, with interesting results. For $S_0 = 10$, the differences between θ_t^* at various values of θ_{t-1} are noticeably larger than in the case of $S_0 = 100$. For $S_0 = 1000$, such differences appear to be virtually non-existent.

There may be several related reasons as to why this is happening. One is a numerical reason and it is that the differences are there, but do not manifest themselves within the first two decimal places. Another reason, an economical one, may be that the investor is much more conservative when dealing with greater wealth, and is willing to bear the price of the transaction costs in return for the certainty of having a fully hedged portfolio. This latter hypothesis is also supported by the fact that we are using a utility function with *absolute* risk aversion, which means that the strategy is bound to differ when the stock price (and subsequently the portfolio price and the overall wealth) is in the tens and when it is in the thousands.

This also suggests that the method will yield different results when dealing, for example, in British pounds and Japanese yen. While some may view such a lack of unit invariance as a significant shortcoming of the method, it should be stressed that both the root and the solution of this apparent problem lie with the utility function. Remember that we are using a very specific utility function, $U(x) = -\exp(-x)$. In general, the constant absolute risk aversion utility function may be parameterised, $U(x) = -b\exp(-ax)$, and the parameters $a, b > 0$ can be tailored to reflect the particular scenario that needs to be addressed. In other words, by setting these parameters according to a specific investor's situation (which needs to be done because of the very individual factor of attitude to risk), both the issue of currency and of personal risk aversion are dealt with.

7 General values of α

Now that we have the results for the simplified case of $\alpha = 1$, it is time to examine how the strategy changes for other values of α. Since real-life investors deal in fairly large quantities of stocks and options at a time, it is quite important to find a way to calculate the optimal hedging strategy for any given value of α.

In complete markets (i.e. markets where it is possible to use available assets to perfectly replicate future states), the calculation of the optimal hedging strategy in the case of multiple options, given the optimal hedging strategy for a single-option case, is extremely simple and straightforward. All that is needed is to multiply the values of the single-option hedge by the appropriate number of options.

The scenario examined in this thesis does not feature a complete market. We therefore cannot take for granted that the elegant solution described above will work in our case; yet if the appropriate multiples of the single-option optimal hedging strategy turned out to be reasonably close to the true multiple-option optimal hedging strategy, we could consider the multiplication method as a viable way of generalizing the available results.

The reason why we would want to do this as opposed to doing a whole set of calculations with the given higher value of α is because of the increased extent of the calculations. When calculating to two decimal places with an increment of 0.01, the number of calculations at each node of the state-space grid is 10.000 (function value at θ_t = 0, 0.01, …, 1 for each θ_{t-1} = 0, 0.01, …, 1) for $\alpha = 1$, but increases to 1.000.000 (θ_t = 0, 0.01, …, 10 for each θ_{t-1} = 0, 0.01, …, 10) for $\alpha = 10$. For $\alpha = 100$ the number of cycles per node would reach 100.000.000, etc. Such a rapid increase in the number of calculation cycles per node must necessarily lead to extremely long processing times which would render the entire method inefficient for practical application.

However, in order to compare the results yielded by the multiplication method to those of the proper calculation, at least several test runs had to be made with increased values of α. Specifically, the optimal hedging strategy for $\alpha = 2$, $\alpha = 5$, and $\alpha = 10$ have been calculated. Even though 10 is still nowhere near the realistic number of stocks and options that are traded at a time, we had to stop at this value because the presumption of rapidly increasing calculation times had, unsurprisingly, turned out to be very true. On a computer equipped with a 1GHz Duron processor, with 100% of the processing power dedicated to the task at hand, the time needed to complete the calculation increased from a matter of minutes for $\alpha = 1$, through over 2 hours for $\alpha = 5$, up to 8 hours for $\alpha = 10$. Even if we allowed for greater processing power, computations for higher values of α would simply not be feasible, even more so if these higher values were coupled with even more realistic (i.e. more branched-out) stock-price trees and more frequent hedging times / longer investment horizons (all leading to a higher number of nodes).

To gauge the difference between the true optimal multiple-option hedge and the appropriate multiple of the optimal single-option hedge, we calculated the relative errors of the latter, as

$$err = \frac{\left|\theta^*_\alpha - \alpha\theta^*_{\alpha=1}\right|}{\theta^*_\alpha} \tag{7.1}$$

where θ^*_α is the optimal value of θ for the given value of α, and $\theta^*_{\alpha=1}$ is the optimal value of θ for $\alpha = 1$. These err values are shown in Table 7.1 below. As before, the rows indicate which node of the state-space grid we are in, and the columns give the value of θ_{t-1}. Values of 'n/a' indicate division by zero. $S = 100$, $\alpha = 2$.

It should be noted that the error values can be viewed in more than one way. If we assume that the investor always stays with the optimal hedge, then obviously some groups of the error values can be dismissed as having resulted from unattainable situations. For example, given that the value of $\theta^*_0 = 1.04$, the entire 0, 0.5, 1.5, and 2 columns in Table 7.1 ($t = 1$), are irrelevant to our purpose (notice that this effectively removes the by far largest of the errors, 0.364).

However, an important attribute of dynamic programming is that even if we stray off the optimal track, the algorithm will adapt to the new situation and yield results that are optimal with regard to the new conditions. From this point of view, it makes sense to consider all error values, not only the ones attainable by always remaining on the optimal path.

t=1	0	0,5	1,02	1,5	2
3 down	0,364	0,000	0,000	0,000	0,000
	0,175	0,175	0,054	0,054	0,054
	0,084	0,084	0,075	0,075	0,075
even	0,038	0,038	0,019	0,097	0,097
	0,008	0,008	0,008	0,098	0,098
	0,014	0,014	0,014	0,000	0,105
3 up	0,038	0,038	0,038	0,038	0,112

t=2	0	0,56	1,02	1,64	2
6 down	n/a	0,111	0,111	0,111	0,111
	1,000	0,111	0,111	0,111	0,111
	1,000	0,125	0,125	0,125	0,125
3 down	0,514	0,040	0,040	0,040	0,040
	0,241	0,034	0,014	0,014	0,014
	0,125	0,125	0,067	0,067	0,067
even	0,029	0,029	0,010	0,089	0,089
	0,008	0,008	0,008	0,104	0,104
	0,021	0,021	0,021	0,065	0,130
3 up	0,049	0,049	0,049	0,037	0,121
	0,051	0,051	0,051	0,051	0,042
	0,037	0,037	0,037	0,037	0,005
6 up	0,005	0,005	0,005	0,005	0,005

t=3	0	0,48	1,02	1,48	2
9 down	n/a	1,000	1,000	1,000	1,000
	n/a	1,000	1,000	1,000	1,000
	n/a	1,000	1,000	1,000	1,000
6 down	n/a	1,000	1,000	1,000	1,000
	n/a	0,143	0,143	0,143	0,143
	1,000	0,158	0,158	0,158	0,158
3 down	0,926	0,158	0,158	0,158	0,158
	0,412	0,059	0,049	0,049	0,049
	0,158	0,158	0,059	0,059	0,059
even	0,039	0,039	0,000	0,091	0,091
	0,008	0,008	0,008	0,088	0,118
	0,047	0,047	0,047	0,047	0,145
3 up	0,047	0,047	0,047	0,047	0,100
	0,037	0,037	0,037	0,037	0,005
	0,000	0,000	0,000	0,000	0,005
6 up	0,005	0,005	0,005	0,005	0,005
	0,005	0,005	0,005	0,005	0,005
	0,005	0,005	0,005	0,005	0,005
9 up	0,005	0,005	0,005	0,005	0,005

t=4	0	0,58	1,02	1,56	2
12 down	n/a	1,000	1,000	1,000	1,000
	n/a	1,000	1,000	1,000	1,000
	n/a	1,000	1,000	1,000	1,000
9 down	n/a	1,000	1,000	1,000	1,000
	n/a	1,000	1,000	1,000	1,000
	n/a	1,000	1,000	1,000	1,000
6 down	n/a	1,000	1,000	1,000	1,000
	n/a	1,000	1,000	1,000	1,000
	n/a	1,000	1,000	1,000	1,000
3 down	n/a	1,000	1,000	1,000	1,000
	1,000	0,179	0,179	0,179	0,179
	0,284	0,134	0,014	0,014	0,014
even	0,069	0,069	0,000	0,084	0,084
	0,023	0,023	0,023	0,114	0,143
	0,036	0,036	0,036	0,036	0,143
3 up	0,021	0,021	0,021	0,021	0,000
	0,000	0,000	0,000	0,000	0,000
	0,000	0,000	0,000	0,000	0,000
6 up	0,000	0,000	0,000	0,000	0,000
	0,000	0,000	0,000	0,000	0,000
	0,000	0,000	0,000	0,000	0,000
9 up	0,000	0,000	0,000	0,000	0,000
	0,000	0,000	0,000	0,000	0,000
	0,000	0,000	0,000	0,000	0,000
12 up	0,000	0,000	0,000	0,000	0,000

Table 7.1: Errors of the hedge yielded by multiplying the single-option optimal hedge by α relative to the true multiple-option optimal hedge.

We will combine these two approaches; we will consider all possible states, keeping in mind that some are less probable than others, or to put it differently, that some result only from extreme deviations from the continuously optimal hedging strategy.

In Table 7.1, especially at times $t=3$ and $t=4$, we can see that there are three distinct groups of errors. These groups clearly correspond to three stock price ranges. In the nodes where it is already clear that the options will end up in-the-money, the relative error is close to zero ($t=3$) or zero ($t=4$). Then there are such nodes that if they have been reached, the option is bound to finish out-of-money. In these nodes, the error seems to be very large. However, a closer inspection will reveal that this is simply because at these nodes, θ_t^* has the same value regardless of the value of α. Finally, there is the middle range of nodes in which the options can end up in-the-money, at-the-money, or out-of-money.

The zero-error nodes do not pose any problem. In these nodes, $\theta_t^* = \alpha$, which is obviously the correct result. If the options are bound to finish in-the-money, the investor will need the same quantity of stocks as the number of outstanding options.

The high-error nodes also don't pose a problem, because an easy work-around solution exists. In these nodes, we can simply define the multiple-option optimal value as equal to the single-option optimal value. If the investor has the certainty that the options will not be realised, then very little or no stock needs to be held in the portfolio, regardless of the number of options sold at the beginning of the investment.

The nodes around the at-the-money level are thus the only ones where the errors can be considered problematic, as they often reach percentage values in the tens (or even higher, in the less-probable circumstances mentioned earlier in this chapter). This motivates us to try to find a different algorithm – one that would yield results with a lower relative error, while not leading to a significant increase in computing time.

In our existing algorithm, there is a very simple relationship between the precision at which the optimal hedge values are calculated and the total number of calculation cycles. Because there is a trade-off between the number of cycles and the total time required to calculate the results, we can decrease the calculation time by decreasing the absolute precision. This translates into increasing the size of the increment used by the brute force algorithm.

The new algorithm will thus be obtained by adapting the size of the increment to the actual value of α. To keep the number of cycles (and hence the calculation time) constant, the size of the increment should be directly tied to α. Specifically, when the increment for $\alpha = 1$ is 0.01, the increment for $\alpha = 2$ will be 0.02, etc. In general, we shall use an increment of 0.01α. While this causes a decrease of the absolute precision of the computations as α increases, the precision of the calculated optimal value relative to the number of units the investor is dealing with remains constant. (Adapting the precision to a particular investor's desires is quite straightforward: the size of the increment can easily be changed for instance to 0.001α or 0.05α, depending on whether precision or speed is of greater essence.)

Table 7.2 below gives the errors of the proportional-increment algorithm relative to the original fixed-increment algorithm. In order to allow a direct comparison between the values herein and those in Table 7.1 (multiple-option hedge as a multiple of the single-option optimal hedge), the input values have been kept the same.

Notice that in Table 7.2, no values are given for time $t = 4$. The reason for this is that just like the original algorithm, by virtue of target function convexity at time $t = 4$ the new adjusted algorithm can use Mathematica's built-in optimum-seeking function *FindMinimum*, which yields results that can be considered exact for our purposes. The optimal hedge values at $t = 4$ are thus the same in the original algorithm and the new algorithm (which means that the relative errors at all nodes are zero).

This latter characteristic of the new algorithm is very convenient. Remember that $t = 4$ in our case corresponds to $t = T - 1$. In a certain sense, portfolio readjustments at this time play a crucial role, for this is the last chance to rehedge the portfolio. The portfolio created at this point in time will be the portfolio the investor will have at the time of option expiry. Having precise results at $t = 4$ is therefore an advantage.

t=1	**0**	**0,5**	**1,02**	**1,5**	**2**
3 down	0,000	0,000	0,000	0,000	0,000
	0,016	0,016	0,000	0,000	0,000
	0,012	0,012	0,011	0,011	0,011
even	0,000	0,000	0,000	0,009	0,009
	0,008	0,008	0,008	0,008	0,008
	0,000	0,000	0,000	0,000	0,000
3 up	0,000	0,000	0,000	0,000	0,006

t=2	**0**	**0,56**	**1,02**	**1,64**	**2**
6 down	n/a	0,111	0,111	0,111	0,111
	0,333	0,000	0,000	0,000	0,000
	0,000	0,000	0,000	0,000	0,000
3 down	0,027	0,000	0,000	0,000	0,000
	0,000	0,000	0,014	0,014	0,014
	0,000	0,000	0,000	0,000	0,000
even	0,010	0,010	0,010	0,000	0,000
	0,008	0,008	0,008	0,000	0,000
	0,007	0,007	0,007	0,000	0,000
3 up	0,000	0,000	0,000	0,000	0,006
	0,006	0,006	0,006	0,006	0,000
	0,005	0,005	0,005	0,005	0,005
6 up	0,005	0,005	0,005	0,005	0,005

t=3	0	0,48	1,02	1,48	2
9 down	n/a	0,000	0,000	0,000	0,000
	n/a	0,000	0,000	0,000	0,000
	n/a	0,000	0,000	0,000	0,000
6 down	n/a	0,000	0,000	0,000	0,000
	n/a	0,143	0,143	0,143	0,143
	0,000	0,053	0,053	0,053	0,053
3 down	0,037	0,000	0,000	0,000	0,000
	0,020	0,020	0,016	0,016	0,016
	0,000	0,000	0,012	0,012	0,012
even	0,000	0,000	0,000	0,000	0,000
	0,008	0,008	0,008	0,000	0,000
	0,007	0,007	0,007	0,007	0,006
3 up	0,000	0,000	0,000	0,000	0,000
	0,005	0,005	0,005	0,005	0,005
	0,000	0,000	0,000	0,000	0,005
6 up	0,005	0,005	0,005	0,005	0,005
	0,005	0,005	0,005	0,005	0,005
	0,005	0,005	0,005	0,005	0,005
9 up	0,005	0,005	0,005	0,005	0,005

Table 7.2: Errors of the proportional-increment algorithm relative to the true multiple-option optimal hedge.

A direct comparison of the values in Tables 7.1 and 7.2 will quickly show that the algorithm using increments proportional to α yields noticeably better results than the method of multiplying the single-option optimal hedge by α, while not requiring extra computation time.

8 Comparison with the Black-Scholes delta hedge

In order to judge the benefits of our hedging strategy and the taking into account of the transactions cost, we will compare it to the continuous Black-Scholes delta hedge. The latter can be calculated very easily as

$$\Delta = \frac{\partial V}{\partial S} \tag{8.1}$$

from the well-known option valuation formula

$$V = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \tag{8.2}$$

where

$$d_1 = \frac{\log\frac{S}{K} + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \tag{8.3}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \tag{8.4}$$

(Note: the V in (8.1) and (8.2) is the value of the option, a quantity different from the portfolio value V_t.) The value of the volatility σ can be calculated from the stock price return probability distribution as

$$\sigma = \sqrt{E\left[\left(\frac{S_{i+1}}{S_i} - \mu\right)^2\right]} \tag{8.5}$$

where μ is the expected stock price return. In our scenario, the historical volatility is $\sigma = 0.0216709$. The risk-free interest rate is $r = 0.00075$, and the strike price is $K = S_0$. The resulting values of the Black-Scholes delta hedge are shown below in Table 8.2.

The idea now is to take specific stock price walks, substitute the appropriate values of Δ_t into formula (3.3), calculate the portfolio values for the optimal transaction costs hedge and those for the Black-Scholes delta hedge, and compare the results. The Black-Scholes delta will be calculated without incorporating transaction costs, but the transaction costs *will* be included in the portfolio value calculations. In other words, the delta values which do not assume the existence of transaction costs will be substituted into a formula which does include transaction costs. The reason behind this choice is to be able to evaluate the benefit of including the transaction costs in the calculations as opposed to ignoring them.

At first, we will take a look at several specific cases where the price of the underlying stock evolves in a straight-forward manner (e.g. the price remains constant, or increases or decreases monotonously). Then, we will start examining the more realistic cases where the stock price fluctuates up and down. This can be done either by inspection of several model cases, or by performing a large number of Monte Carlo experiments and summing up the differences between the portfolio value for the transaction costs optimal hedge and the Black-Scholes delta hedge.

t=0	t=1	t=2	t=3	t=4
				1,0000
				1,0000
				1,0000
			1,0000	1,0000
			1,0000	1,0000
			1,0000	1,0000
		0,9995	1,0000	1,0000
		0,9970	0,9996	1,0000
		0,9864	0,9963	0,9999
	0,9299	0,9533	0,9784	0,9976
	0,8447	0,8738	0,9146	0,9707
	0,7096	0,7296	0,7633	0,8336
0,5405	0,5362	0,5314	0,5256	0,5181
	0,6555	0,3249	0,2782	0,1901
	0,2027	0,1618	0,1073	0,0359
	0,0979	0,0643	0,0291	0,0032
		0,0201	0,0054	0,0001
		0,0049	0,0007	0,0000
		0,0009	0,0001	0,0000
			0,0000	0,0000
			0,0000	0,0000
			0,0000	0,0000
				0,0000
				0,0000
				0,0000

Table 8.2: The continuous Black-Scholes delta hedge

The Monte Carlo approach has better overall informational value, because for a sufficiently large number of runs it is likely to cover all possible paths, while incorporating their respective probabilities. The reason why we choose to do the special monotonous stock-price evolution experiments separately is to discover any exceptional behaviours that could be taking place.

At this point, the selling price of the option is assumed to be the Black-Scholes price according to formula (8.2), without any premium.

The results for monotonous stock price evolution for $S_0 = 100$, $\alpha = 1$ are given in Table 8.3. We can see that in the cases where the option ended up in-the-money, the Black-Scholes delta hedge yields slightly better results than the transaction costs hedge. This is not surprising: the Black-Scholes delta hedge implies larger transactions earlier on while the stock is cheaper, whereas the transaction costs hedge leads to stock purchase at a later time. Similarly, if the stock price monotonously decreases to out-of-money values, the Black-Scholes delta hedge leads to more favourable results, selling the stock earlier on, before its price falls. Here the relative difference between the two hedges is greater, but the absolute values are much smaller than when the stock price increases. In other words, neither hedge allows the portfolio value to go too deeply into negative figures.

Interestingly, if the stock price remains at the same level throughout the duration of the investment period, the transaction costs hedge gives a result that is about 10% better than the Black-Scholes hedge. Another interesting attribute of these results is that the relative advantage of the Black-Scholes hedge over the transaction costs hedge decreases as the rate of stock price change increases. When the stock price increases by 1 node at a time, the Black-Scholes hedge is the better by 8%; however, if the increase is 3 nodes at a time, the difference is only about 3%. A similar effect (though with much higher percentages) can be observed when the stock price decreases. The loss with the Black-Scholes hedge is only 57% of that of the transaction costs hedge when the stock price decreases by 1 node at a time, but this value rises to 66% when the stock price decreases by 3 nodes at a time. In other words, the transaction costs hedge is more sensitive to greater changes in the stock price.

These results might look pessimistic in terms of the worth of the transaction costs hedging strategy we calculated, but let it be repeated that these are only seven stock price paths out of a possible 16807 ($= 7^5$) and the probability of any of these special cases taking place is very low.

S_i / S_{i-1}	V_5 with transaction costs hedge	V_5 with Black-Scholes hedge	$\frac{V_5^{BS}}{V_5^{TC}}$
exp(0.06)	31.3791	32.4293	1.0335
exp(0.04)	19.3679	20.311	1.0487
exp(0.02)	8.7771	9.4755	1.0796
1	0.9368	0.8481	0.9053
exp(-0.02)	-1.9950	-1.1412	0.5720
exp(-0.04)	-3.1901	-1.9451	0.6097
exp(-0.06)	-4.0623	-2.6648	0.6560

Table 8.3: A comparison of portfolio values given constant changes in the stock price

A more informative view is offered by the results of Monte Carlo experiments, summarised in Table 8.4. For each given combination of S_0 and α, 100.000 random stock price paths were simulated and the corresponding hedging strategies and portfolio values calculated. Portfolio values at time T were added up separately for the transaction costs hedge and the continuous Black-Scholes delta hedge, and at the end of the experiment, the ratio of these two sums was computed.

As the table shows, the hedging strategy that incorporates transaction costs is clearly more favourable for the investor.

$S_0 \setminus \alpha$	1	10	100	1000
1	1.3105	1.2520	1.1605	1.1215
10	1.2279	1.1437	1.1147	1.1143
100	1.1574	1.1292	1.1257	1.1189
1000	1.1286	1.1314	1.1259	1.1281

Table 8.4: The ratio of $\sum V_5^{TC}$ to $\sum V_5^{BS}$

We can see that the ratios range from 1.1143 to 1.3105, in other words that the ending value of the portfolio hedged using the transaction costs hedge was 11% to 31% higher than the value of the benchmark portfolio. It can also be seen from the table that these values are not random, but seem to depend on the overall volume of the transaction. Notice that the numbers in the diagonals running from bottom-left to top-right tend to be very similar. This is very convenient, because if we wanted to calculate similar values for higher transaction volumes, a single combination of S_0 and α should give us sufficient information about all other combinations of these two values that together give the appropriate volume.

Another important finding is that even though initially the relative advantage of the transaction costs hedge decreases from 31% for volumes of 1 to 23%-25% for volumes of 10 and 14%-16% for volumes of 100, at higher volumes it seems to settle around values of 11%-13%.

Because we are using the stock to hedge the option rather than as an investment, and because our scenario is that of an incomplete market which does not allow perfect replication of the mean value process, it is equally (if not more) important to determine and compare the hedging error of both the transaction costs hedge and the Black-Scholes delta hedge. This can be done using the same Monte-Carlo procedure that was used to calculate the portfolio values at $t = T$.

The average square hedging error will give us information about the overall magnitude of the difference between the portfolio values and the value of the debt H_5 (the value of options that will be exercised) at $t = T$

$$\overline{\varepsilon}^2 = \frac{1}{n}\sum\left(V_5 - H_5\right)^2 \tag{8.9}$$

where n is the number of runs of the Monte Carlo experiment.

However, the average square error does not discern between positive and negative errors, which are qualitatively different. Obviously, $V_5 - H_5 > 0$ is much better news for the investor than $V_5 - H_5 < 0$, even though both mean that the hedging wasn't perfect. We will therefore also evaluate the simple average error

$$\overline{\varepsilon} = \frac{1}{n}\sum\left(V_5 - H_5\right) \tag{8.10}$$

which should give us an idea about how positive or negative an error can be expected. The results of the experiment for several combinations of S_0 and α (this time concentrating on volumes of 10.000 up to 1.000.000) are given in Table 8.5.

The values in the table reveal several things. Firstly, both errors depend on volume and not on the specific combination of S_0 and α. Secondly, the average error and the average square error increase linearly and quadratically, respectively, with the transaction volume. Thirdly, the error ratios for the two hedging strategies remains constant (what small differences there are can be attributed to the random nature of Monte Carlo experiments), with the transaction costs hedge coming out as the slightly better hedge both in terms of the expected profit $V_5 - H_5$ and in how well it copies the ideal self-financing portfolio.

Volume	Error	Transaction costs hedge	Black-Scholes hedge	Ratio
$S = 1000$	$\overline{\varepsilon}$	-112.897	-129.466	0.8720
$\alpha = 10$	$\overline{\varepsilon^2}$	22925.2	24604.3	0.9318
$S = 100$	$\overline{\varepsilon}$	-113.714	-129.446	0.8785
$\alpha = 100$	$\overline{\varepsilon^2}$	23042.0	24596.5	0.9368
$S = 10$	$\overline{\varepsilon}$	-114.520	-129.098	0.8871
$\alpha = 1000$	$\overline{\varepsilon^2}$	23007.5	24432.1	0.9417
$S = 1000$	$\overline{\varepsilon}$	-1125.27	-1285.88	0.8751
$\alpha = 100$	$\overline{\varepsilon^2}$	$2.2968 \text{x} 10^6$	$2.4275 \text{x} 10^6$	0.9462
$S = 100$	$\overline{\varepsilon}$	-1138.95	-1290.59	0.8825
$\alpha = 1000$	$\overline{\varepsilon^2}$	$2.3051 \text{x} 10^6$	$2.4410 \text{x} 10^6$	0.9443
$S = 1000$	$\overline{\varepsilon}$	-11301.9	-12918.8	0.8748
$\alpha = 1000$	$\overline{\varepsilon^2}$	$2.3062 \text{x} 10^8$	$2.4567 \text{x} 10^8$	0.9387

Table 8.5: Hedging errors of the transaction costs hedge and the continuous Black-Scholes delta hedge

While the percentage that describes the relative advantage of the transaction costs hedge over the Black-Scholes hedge may seem optimistic (the expected loss of the former is only about 87% that of the latter), the fact of the matter is that in the current scenario, both hedging strategies generate loss. This can be put down to the fact that the Black-Scholes price at which we sell the options does not assume the existence of transaction costs, and the options are thus undervalued – the investor sells them at a price that is less than fair.

To counterbalance the negative impact of transaction costs, a premium over the Black-Scholes price has to be charged when selling the options. The relationship of this premium and the optimal quantity of options sold, and its subsequent effect on the investor's utility, will be examined in the next chapter.

9 Optimisation of investor's utility

As has been mentioned at the end of the preceding chapter, the Black-Scholes option valuation formula has yielded prices at which the option was undervalued and the entire deal generated loss. In this chapter, we will examine what happens when we increase the selling price of the option (charge a premium over the Black-Scholes price), and the effect of the selling price on the number of options the investor should sell to maximise their utility.

The first step is to find a benchmark level of utility that will allow us to judge whether a specific selling strategy is advantageous for the investor or not. In our scenario, we have one risk-free asset (the bank account with a weekly interest of 0.00075%) and one risky asset (the stock). The obvious alternative to investing in the risky asset (selling an option and hedging it with stock) is to invest all funds in the risk-free asset, i.e. not to sell any options at all. The benchmark utility will then be

$$U\left(w_0\right) = -\exp\left(-w_0 R_f^5\right) \tag{9.1}$$

where w_0 is the initial wealth of the investor. Because the value of w_0 has no effect on the dynamically optimal hedging strategy, we have in all our calculations let initial wealth be zero, i.e. the only funds available to the investor are those gained from selling options. We will adhere to this assumption. With no initial wealth and no option sales, the investor's wealth remains zero throughout the investment period, and the resulting utility is $U\left(0\right) = -\exp\left(0\right) = -1$. Thus, if the expected utility of a particular selling strategy turned out to be lower than –1, we know that that strategy is to be avoided – the investor would be better off 'not doing anything'. If the expected utility of a selling strategy is greater than –1, we will consider that strategy an acceptable (but not yet optimal) one.

Our approach to finding optimal combinations of α (the number of options sold) and the premium charged per option will be to construct a matrix of expected utilities with increasing volume αS_0 (with the initial stock price fixed at $S_0 = 100$ [1]) in one

[1] This is just an arbitrary value; relevant information for *any* stock price can be acquired from the matrix. As the results in chapter 8 have shown, it is the volume αS_0 that is important, not the specific

dimension, and increasing premium in the other dimension. This will allow us to determine the minimum premium needed for a given α to lead to a utility greater than -1, as well as the optimal α for a given premium.

The expected utilities will be calculated from Monte Carlo experiments, with the number of runs set to 10.000 to facilitate the speed of the computations.

$\alpha S_0 \setminus \sigma_h +$	0.000	0.005	0.010	0.015	0.020	0.025	0.030
0.39	-1,0034	-1,0017	-1,0000	-0,9983	-0,9965		
0.78	-1,0072	-1,0037	-1,0005	-0,9970	-0,9931		
1.56	-1,0156	-1,0085	-1,0016	-0,9940	-0,9875		
3.13	-1,0316	-1,0168	-1,0026	-0,9892	-0,9751		
6.25	-1,0665	-1,0388	-1,0086	-0,9803	-0,9548		
12.50	-1,1454	-1,0820	-1,0224	-0,9696	-0,9148		
25.00	-1,3388	-1,1956	-1,0667	-0,9556	-0,8543	-0,7675	-0,6865
50.00	-1,8985	-1,5249	-1,2228	-0,9701	-0,7879	-0,6229	-0,4957
100.00	-4,6235	-2,9064	-1,8706	-1,2068	-0,7574	-0,4893	-0,3175
200.00	-45,0633	-19,5364	-8,6793	-3,2548	-1,3368	-0,5298	-0,2053
400.00			-7,59E+02	-80,9752	-15,7818	-3,2575	-0,4706
800.00					-4,94E+04	-8,74E+03	-81,6281

Table 9.1: Expected utility from the transaction costs hedge[2] at various volumes and premiums

To increase the selling price of the option, we will gradually increase the implied volatility σ in formulae 8.3 and 8.4. The historical volatility, calculated from the distribution of stock returns, is $\sigma_h = 0.0216709$. For the first set of calculations, we will increase this value in steps of 0.005. This leads to relatively large increases in option price: the base Black-Scholes price in our scenario is 2.12, while the price with the historical volatility increased by 0.005 is 2.57, which corresponds to a 21% increase. These large increases are needed to quickly determine the price range in which the selling strategy turns profitable (in terms of utility); once that information is available, a finer examination will follow.

combination of these two values. Hence, if the stock price is 100 and the optimal volume turns out to be 100, the optimal number of options to be sold is 1; if the stock price is 50 and the optimal volume is 100, two options should be sold, etc.

[2] In order to compare even more thoroughly the advantage of the transaction costs hedge over the continuous Black-Scholes delta hedge, we have also calculated the expected utility of the latter. In all cases, the transaction costs hedge yielded a better utility than the Black-Scholes hedge.

The data in Table 9.1 reveal several facts. Firstly, if we sell the option at the Black-Scholes price, the expected utility increases as α decreases, but it does not exceed –1 (see Figure 9.1). This indicates that the optimal selling strategy is not to sell any options at all – the investor can expect to be better off doing nothing. The minimal premium required to make the investment worth it starts between $\sigma_h + 0.010$ (the corresponding selling price is $P = 3.01$) and $\sigma_h + 0.015$ ($P = 3.45$), and increases with the volume αS_0 (see also Table 9.2).

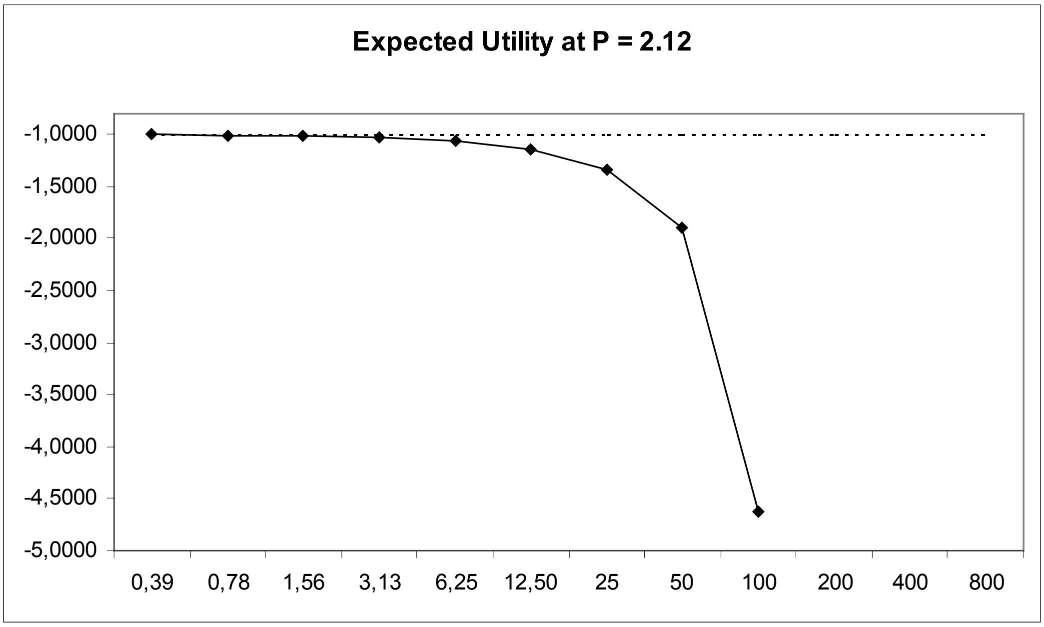

Figure 9.1: Expected utility for different volumes at option selling price P = 2.12 (corresponding to the Black-Scholes price)

Secondly, when we move to prices at which the risky investment becomes worthwhile, many different volumes lead to profit, but some more so than others. The volume that maximises the utility increases with the premium (see Figures 9.2 and 9.3). The optimal volumes for each premium are summarised in Table 9.3.

Volume	0.39	0.78	1.56	3.13	6.25	12.50	25.00	50.00	100.00
Minimal premium σ_h+	0.011	0.011	0.011	0.012	0.012	0.012	0.013	0.015	0.017
Corresponding price	3.10	3.10	3.10	3.19	3.19	3.19	3.28	3.45	3.63
Increase over B-S price	46%	46%	46%	50%	50%	50%	55%	63%	71%

Table 9.2: Minimal premium needed to make given investment worthwhile

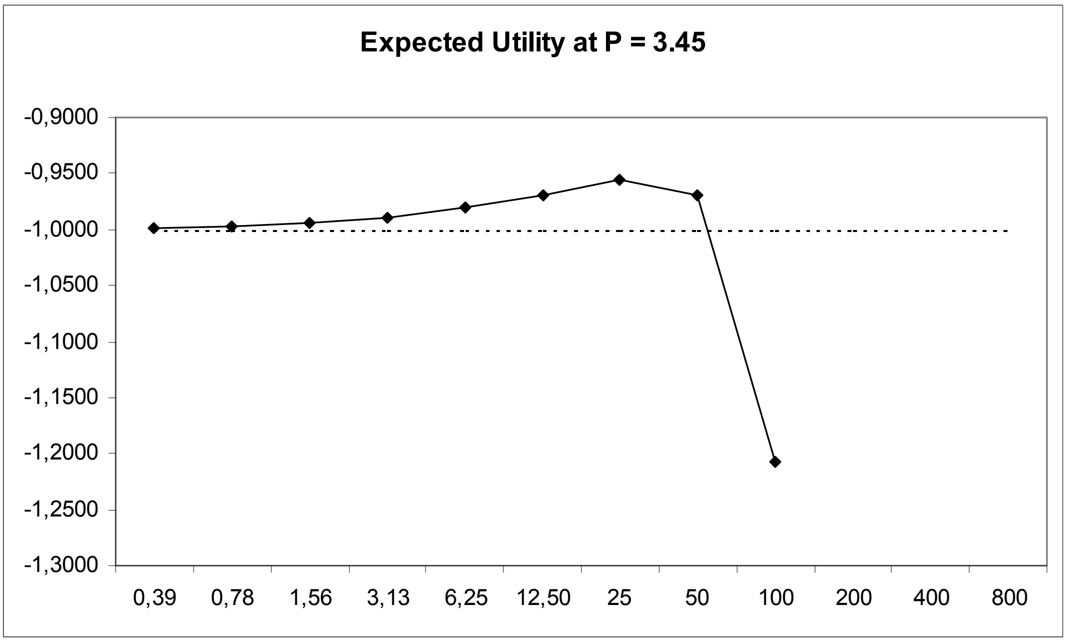

Figure 9.2: Expected utility for different volumes at option selling price P = 3.45 (Black-Scholes price with historical volatility increased by 0.015). The optimal volume is 25.

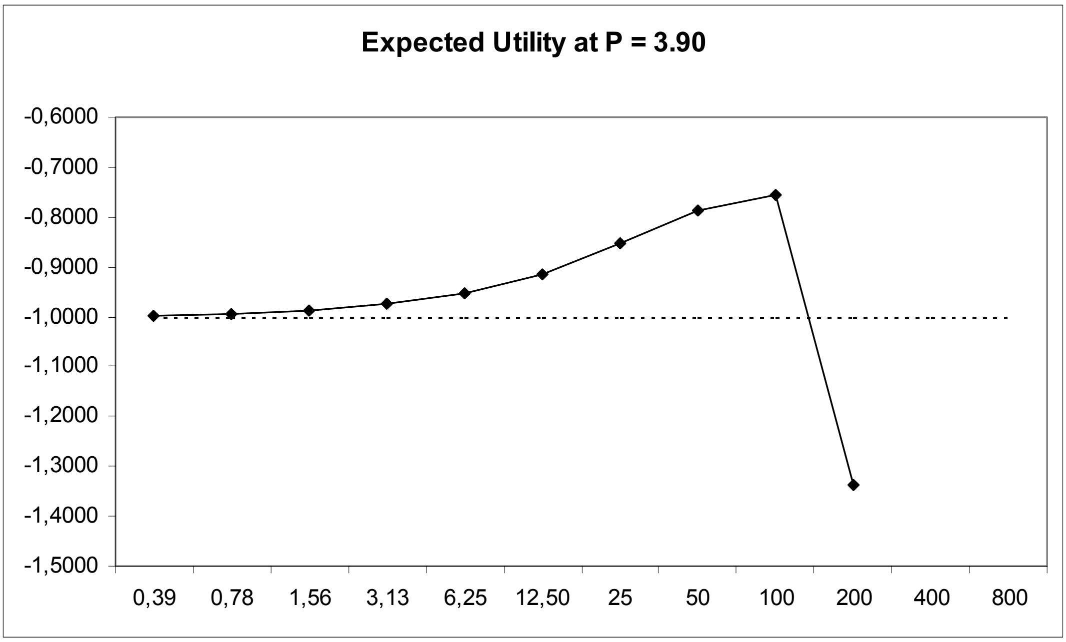

Figure 9.3: Expected utility for different volumes at option selling price P = 3.90 (Black-Scholes price with historical volatility increased by 0.020). The optimal volume is 100.

Premium	0	0.005	0.010	0.015	0.020	0.025	0.030
Optimal volume	0	0	0	25	100	100	200

Table 9.3: Optimal volumes for given premiums

The values in Table 9.3 are obviously only approximate in terms of optimality. For example, if the implied volatility were $\sigma_h + 0.015$, the volume of 25 gives better results than any of the other volumes we examined, but the *real* optimal volume at that premium can be practically anywhere between 12.5 and 50. This, however, is not a problem: once we know the price (and hence the premium) that we can sell the option for on the market, we can run a number of calculations that would allow us to determine with much greater accuracy[3] the interval wherein the optimal volume lies.

In addition to evaluating the value of the investment in terms of utility, we will also look at another tool, the Sharpe ratio, which investors use to decide whether or not an investment is favourable. The two quantities used in the Sharpe ratio are return and risk of the investment, represented by expected return and its standard deviation, respectively. The formula is

$$SR = \frac{E[X]}{\sqrt{Var[X]}} \tag{9.2}$$

In chapter 8, we calculated the average error (equation 8.10) and the average square error (equation (8.9)) of the two hedging strategies being compared. In the Monte Carlo experiments that gave us the data shown in Table 9.1, we also calculated these two errors, and this information can be directly used to calculate the Sharpe ratios:

$$SR = \frac{\overline{\varepsilon}}{\sqrt{\overline{\varepsilon^2} - \left(\overline{\varepsilon}\right)^2}} \tag{9.3}$$

The results, presented in the same volume-premium matrix as the expected utility, are shown in Table 9.4.

Unlike the expected utilities, the Sharpe ratio does not seem to give an immediate insight into which investments are good and which should be avoided. For example, those investments that have a negative Sharpe ratio correspond to investments with a utility of less than –1; but there are also some investments that fulfil the latter, yet their Sharpe ratio is positive.

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[3] The increased accuracy can come from a higher number of examined volumes, as well as from a higher number of runs in the Monte Carlo simulations used to calculate the expected utility.

However, when we fix the premium and take a look at the Sharpe ratios at different volumes, the values exhibit an interesting behaviour – the volumes at which the Sharpe ratio is the highest correspond to the volumes at which expected utility is the highest. An example of this is shown below in Figure 9.4.

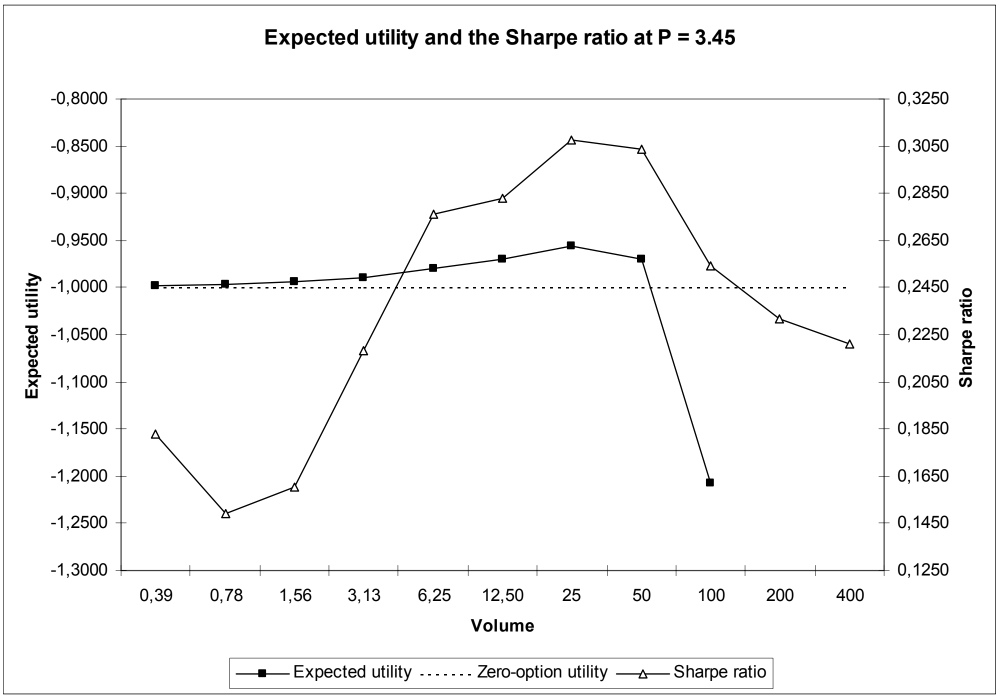

Figure 9.4: Expected utility and the Sharpe ratio

$\alpha S_0 \setminus \sigma_h +$	0.000	0.005	0.010	0.015	0.020	0.025	0.030
0.39	-0,3496	-0,1621	0,0100	0,1830	0,3859		
0.78	-0,2936	-0,1538	-0,0134	0,1492	0,3075		
1.56	-0,3427	-0,1821	-0,0165	0,1603	0,3166		
3.13	-0,4995	-0,2621	-0,0157	0,2180	0,4629		
6.25	-0,7097	-0,4049	-0,0555	0,2760	0,5852		
12.50	-0,8506	-0,4621	-0,0752	0,2830	0,6842		
25.00	-1,0041	-0,5633	-0,1178	0,3075	0,7374	1,1583	1,5885
50.00	-1,1093	-0,6419	-0,1810	0,3039	0,7407	1,2393	1,7204
100.00	-1,1818	-0,6971	-0,2246	0,2543	0,7584	1,2404	1,7158
200.00	-1,2390	-0,7506	-0,2620	0,2316	0,7088	1,2040	1,6995
400.00			-0,2704	0,2209	0,6842	1,1522	1,6315
800.00					0,6681	1,1186	1,5745

Table 9.4: Sharpe ratios

Now that we have found that relatively large option price premiums are required to make the investment pay off, it would be interesting to gauge what part in those premiums is played by the transaction costs. In all the computations presented in this chapter, the transaction costs coefficient was set to $\kappa = 0.01$. To measure the effect of the transaction costs, we will carry out the same set of computations with $\kappa = 0.00$, and compare the results.

In Table 9.5 a matrix similar to Table 9.1 is shown[4]. In anticipation of much faster shifts from non-favourable utilities (less than –1) to favourable ones, we decided to test premiums corresponding to implied volatility increments of 0.001. The number of runs in the Monte Carlo simulations was once again 10.000. In line with expectations, the results show that much lower premiums are needed to make the investments worthwhile when $\kappa = 0.00$ as compared to the case of $\kappa = 0.01$.

The smaller and medium volumes (0.39 up to 12.50) are good even if no premium is charged – the Black-Scholes price in this case can be considered fair enough (see Figure 9.5). For further volumes up to 100, the required volatility premium stays below 0.004 (compared to 0.017 when $\kappa = 0.01$). Volumes higher than 100 can be considered very high in terms of the premium required to make the investment favourable, and we have not pursued the calculations into higher implied volatility values. Nevertheless, even if we do not know the exact point at which the expected utility from these investments exceeds –1, we could compare the premiums needed to achieve certain sub-optimal utility values[5].

The results are summarised in Table 9.6, which also offers a direct comparison with the values given in Table 9.2.

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[4] As before, the calculations were made for both the transaction costs hedge and the Black-Scholes delta hedge. While the transaction costs hedge once again always led to better utility than the Black-Scholes hedge, it was interesting to see that for volumes of 400 and 800, the expected profit of the latter was higher than that of the transaction costs hedge.

[5] For example, with no transaction costs, a volume of 200 will lead to a utility of –3.66 at zero premium. With transaction costs, similar utility can be observed slightly below a premium of 0.015. The extra volatility premium required by the transaction costs for this investment volume would thus be approximately 0.015.

$\alpha S_0 \setminus \sigma_h +$	0.000	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008
0.39	-0,9958	-0,9960	-0,9942	-0,9955	-0,9940	-0,9949	-0,9940	-0,9950	-0,9938
0.78	-0,9954	-0,9936	-0,9939	-0,9931	-0,9927	-0,9912	-0,9915	-0,9904	-0,9896
1.56	-0,9929	-0,9907	-0,9920	-0,9895	-0,9892	-0,9865	-0,9838	-0,9830	-0,9836
3.13	-0,9892	-0,9862	-0,9854	-0,9810	-0,9790	-0,9763	-0,9740	-0,9721	-0,9700
6.25	-0,9885	-0,9788	-0,9771	-0,9724	-0,9670	-0,9595	-0,9549	-0,9498	-0,9443
12.50	-0,9866	-0,9747	-0,9687	-0,9552	-0,9457	-0,9374	-0,9236	-0,9167	-0,9049
25.00	-1,0007	-0,9781	-0,9616	-0,9383	-0,9174	-0,8959	-0,8735	-0,8595	-0,8404
50.00	-1,0508	-1,0045	-0,9700	-0,9238	-0,8907	-0,8509	-0,8126	-0,7653	-0,7397
100.00	-1,3502	-1,2347	-1,1164	-1,0207	-0,9328	-0,8577	-0,7845	-0,7216	-0,6540
200.00	-3,6594	-2,7379	-2,5020	-2,0577	-1,7410	-1,4779	-1,2238	-0,9632	-0,8052
400.00	-1,39E+02	-72,6974	-49,8850	-39,2396	-27,7159	-19,6819	-12,2437	-8,4522	-5,5132
800.00	-1,22E+06	-1,66E+06	-9,17E+05	-2,69E+05	-1,44E+05	-7,62E+04	-8,94E+04	-1,79E+04	-1,96E+04

Table 9.5: Expected utility with zero transaction costs

Volume	0.39	0.78	1.56	3.13	6.25	12.50	25.00	50.00	100.00
Min. premium ($\kappa = 0.01$)	0.011	0.011	0.011	0.012	0.012	0.012	0.013	0.015	0.017
Min. premium ($\kappa = 0.00$)	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.004
Difference	0.011	0.011	0.011	0.012	0.012	0.012	0.012	0.013	0.013
Price ($\kappa = 0.01$)	3.10	3.10	3.10	3.19	3.19	3.19	3.28	3.45	3.63
Price ($\kappa = 0.00$)	2.12	2.12	2.12	2.12	2.12	2.12	2.21	2.30	2.48
Difference	0.98	0.98	0.98	1.07	1.07	1.07	1.07	1.15	1.15
Increase over B-S price ($\kappa = 0.01$)	46%	46%	46%	50%	50%	50%	55%	63%	71%
Increase over B-S price ($\kappa = 0.00$)	0%	0%	0%	0%	0%	0%	4%	8%	17%
Difference	46%	46%	46%	50%	50%	50%	51%	55%	54%

Table 9.6: The effect of transaction costs on the required option price premium

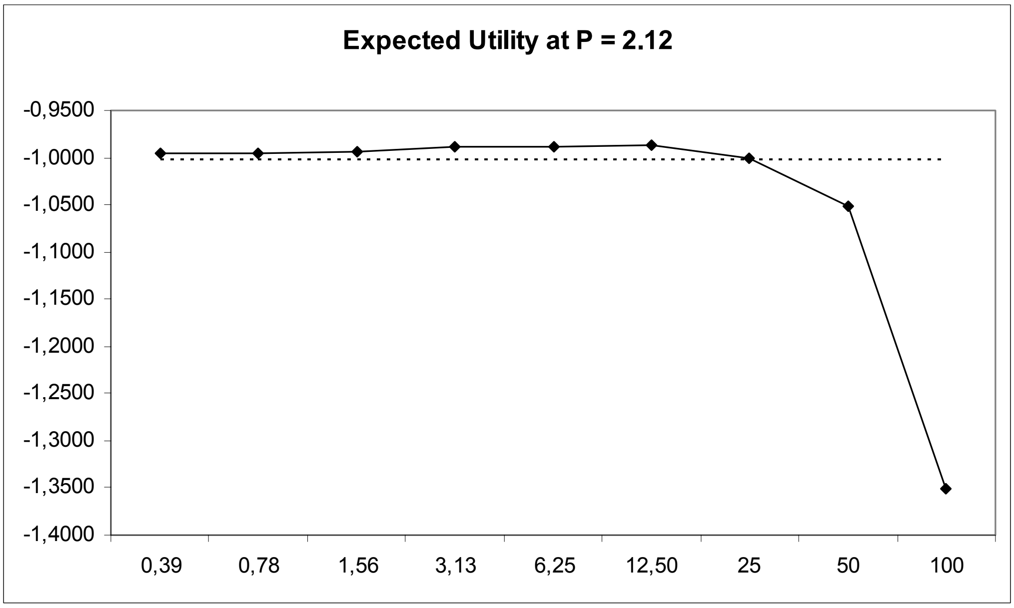

Figure 9.5: Expected utility for different volumes at option selling price P = 2.12 (corresponding to the Black-Scholes price) with no transaction costs

Another method which we can utilise to measure the effect of transaction costs is to compare the premiums at which a given volume becomes optimal. For instance, in the case without transaction costs, a volume of 25 is optimal when the implied volatility is $\sigma_h + 0.002$; with transaction costs, the corresponding implied volatility is $\sigma_h + 0.015$ (possibly slightly less). The difference would thus be approximately

0.013. Similarly, a volume of 100 is optimal at premiums of 0.006 and 0.020 with and without transaction costs, respectively, suggesting a difference of about 0.014.

Finally, we can try to estimate the extra premium required by the transaction costs via the Sharpe ratio. By fixing the volume and plotting the Sharpe ratios obtained at different levels of premium for both the transaction-costs case and the no-transaction-costs case, as shown in Figures 9.6 and 9.7[6], the magnitude of the horizontal shift of the two lines can give us some idea of the extra premium required for the transaction-costs Sharpe ratios to reach the same level as the no-transaction-costs Sharpe ratios.

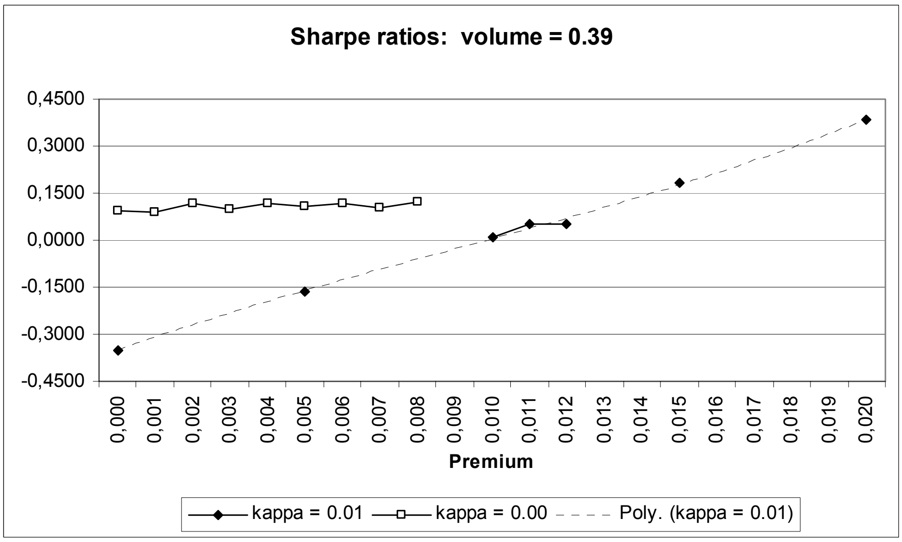

Figure 9.6: A comparison of Sharpe ratios of investments with and without transaction costs

It is interesting to see that for the lowest volume, the no-transaction-costs Sharpe ratio as a function of premium is close to (but not quite) constant. As the volume increases, the slope of this function increases as well, until it seems to settle at parallel to the transaction-costs Sharpe ratio line at volumes of 25 and higher. This change of slope directly proportional to volume implies a phenomenon in line with the observations summarised in Table 9.6, namely that the premium required for smaller investment volumes is smaller than that required in the case of larger volumes. The plots also show that the horizontal shift generally falls within the 0.010 – 0.015 interval, which is consistent with both the minimal premium shifts and the optimal volume shifts.

[6] The trendline give for the transaction-costs case is a polynomial function of degree 3, calculated by MS Excel. While there may exist other regression types that are more suitable, the polynomial trendline used here appears quite reasonable for our purposes.

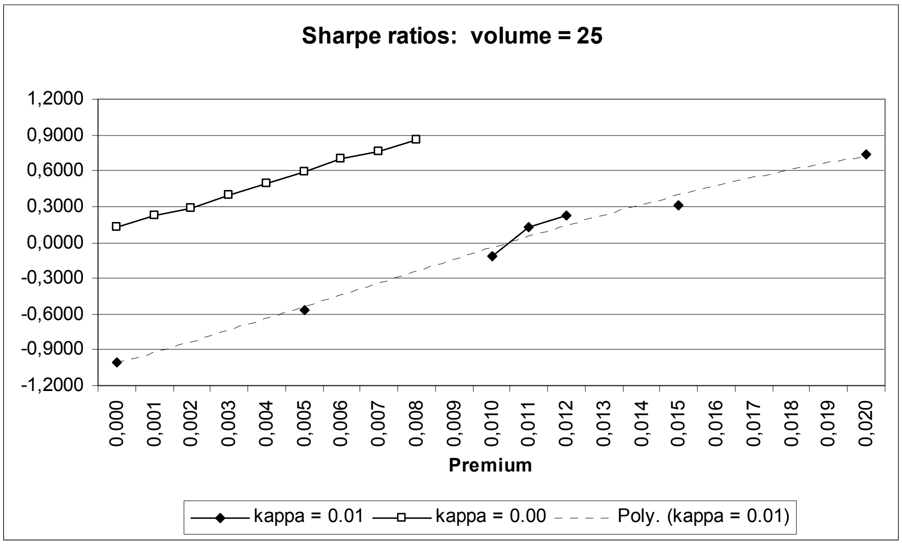

Figure 9.7: A comparison of Sharpe ratios of investments with and without transaction costs

10 Relationship between the generalised Sharpe ratio, transaction costs and the option price premium

In this chapter, we will try to extend and refine the results presented in chapter 9. The objective is to find the relationship between the transaction costs (represented by the coefficient κ), the price (represented by the premium expressed as the difference between implied volatility and historical volatility) at which the investor is able to sell the option on the market, and the benefit of the investment to the investor (represented by the generalised Sharpe ratio).

The model we use and the approach we have adopted to solve the dynamic optimisation problem carries with it three separate sources of risk:

- The discrete stock price changes of the multinomial tree make it impossible to hedge the option perfectly, even with no transaction costs.
- The transaction costs increase the price of the hedging portfolio, as well as the hedging error[1].
- The numerical implementation of the optimisation algorithm required that the continuous interval $[0,\alpha]$ of possible values of θ_t be reduced to a finite set of values at which the target function was evaluated. This means that the hedging strategy used may in fact be slightly sub-optimal.

The first of the risks is very much model-dependent. To mitigate this risk, we could, for example, change the heptanomial tree to one with a greater number of branches, resulting in lower jumps in stock price. The risk would naturally still be there, but it would be lower. It is of interest that while decreasing the hedging interval should intuitively lead to smaller stock price jumps (because the price is given less time to change, as it were), available data suggest that as the hedging interval approaches zero, these jumps cannot be avoided if a viable stock return distribution is to be obtained.

[1] The negative effect of transaction costs could be directly seen in Table 8.3 in that when the stock price happens to increase or decrease monotonously, the presence of the transaction costs leads to buying or selling stock later than would be optimal, implying lower utility at the end of the investment.

Given enough processing power and/or time, the numerical risk could also be lowered, by expanding the set of θ_t values being examined.

This leaves the risk posed by transaction costs. The transaction costs are determined exogenously, and have to be accepted into the model as they are. By examining the overall risk of the model as well as isolating the risk of the transaction costs, we will hopefully gain information that will enable the investor to act in a variety of market conditions.

In chapter 9, we compared the situation where there were no transaction costs with the case where $\kappa = 0.01$. To find a reasonably general relationship between transaction costs, option price premium, and investor's utility, we will have to carry out calculations for several other values of κ. The cases we will examine will be $\kappa = 0$, $\kappa = 0.001$, $\kappa = 0.005$, $\kappa = 0.01$, $\kappa = 0.02$ and $\kappa = 0.05$. These values can be expected to sufficiently cover the realistic range of transaction costs.

In addition to the broader range of transaction cost coefficients, there will be two more differences as compared to the earlier calculations. As the title of this chapter suggests, we will use a generalised form of the Sharpe ratio as opposed to the classic Sharpe ratio (9.2). The generalised Sharpe ratio, discussed in greater detail in [1], in our case takes the form

$$SR_G = \sqrt{\left(E\left[U\left(\alpha\right)\right]\right)^{-2} - 1} \tag{10.1}$$

One of the advantages of this indicator is that its calculation only requires the expected utility of the investment to be known.

To explain the relationship between the classic Sharpe ratio and the generalised Sharpe ratio, we shall list several main properties of the latter, as given in [1]. In general, the generalised Sharpe ratio is defined as

$$1 + SR_\gamma^2 \triangleq \left(1 + \frac{IP_\gamma}{\gamma}\right)^{2\gamma} \tag{10.2}$$

where γ is the risk aversion coefficient in hyperbolic absolute risk aversion (*HARA*) utility (the *CARA* utility is obtained as a limit of the *HARA* utility with $\gamma \to \infty$) and IP is the investment potential, a quantity that measures the percentual increase of the investor's wealth from the risky asset as compared to investing in a riskless asset. Four properties of the generalised Sharpe ratio follow:

1. SR_{-1} is the classic Sharpe ratio
2. $SR_{\gamma} \approx SR_{-1}$ for small risks
3. in the continuous-time Black-Scholes model, the SR_{γ} of the optimal trading strategy is the same for all γ
4. for normally distributed returns, SR_{γ} with exponential utility is identical to SR_{-1}.

The second feature in which the upcoming calculations will differ from those in chapter 9 is that instead of Monte Carlo experiments, we will determine the expected utility using the recursive set of value functions J_t, derived in chapter 4. Looking back to equations (4.13), (4.15) and (4.24), it becomes obvious that the expected utility of the optimal investment is equal to the final value function (4.32) multiplied by –1, so that

$$E\left[U\left(\tilde{V}_0\right)\right] = -J_0 = -\exp\left(R_f^T \tilde{V}_0\right) g_0^*\left(\theta_0^*, \theta_{init}, S_0\right) \tag{10.3}$$

where $T = 5$ and the initial portfolio value prior to subtracting transaction costs is

$$\tilde{V}_0 = \alpha P \tag{10.4}$$

P being the price at which the investor sells the option. Because the optimal values g_t^* are, by necessity, calculated along with the optimal values θ_t^*, the value of g_0^* is known, allowing the expected utility to be calculated by simple multiplication as a function of the number of options sold and the premium at which they are sold. This method of calculating expected utility is therefore faster and more reliable than Monte Carlo experiments.

The expected utilities and corresponding Sharpe ratios (generalised) calculated at different levels of volume and premium, and for the transaction cost coefficient values listed earlier, are given in Tables 10.1 – 10.6. Sharpe ratios of "–0" indicate that the expected utility does not exceed –1 (this is the utility when the investor does not sell any options). In Tables 10.5 and 10.6, the utilities for the highest volumes are not given, as these volumes required very large premiums to pay off. So far, the data in these tables are only an extension of those presented in chapter 9, allowing us to find the optimal α for a variety of premiums.

The data that will give us new insight into the effect of transaction costs on the optimal investment comes with the ability to compute with arbitrary precision the premium required to reach a given Sharpe ratio value. We have chosen to look for these premiums in steps of 0.0001, i.e. the precision is one order of magnitude higher than in the calculations of chapter 9. Additionally, if we fix the initial stock price at $S_0 = 100$, a premium increase of 0.0001 roughly corresponds to a 0.01 increase in the Black-Scholes price of the option; this value is suitable in that it is the smallest monetary fraction in most of the world's leading currencies.

We have chosen to examine the minimal premiums required to achieve a Sharpe ratio of 0 and of 0.5. The meaning of the former is obvious: when the investment's Sharpe ratio reaches 0, the investment becomes as good as 'not doing anything'; it is the breaking point beyond which the investment becomes a worthwhile venture. The value of 0.5 is an arbitrary one, reflecting some fictional requirement of the investor[2].

Table 10.7 gives the premium values required for the described Sharpe ratios to be achieved. We can see that even when $\kappa = 0$, the higher volumes require a certain premium for the investment to pay off. Because the transaction costs in this case are absent and thus cannot pose any risk, the premiums in question are apparently caused by the other two risks discussed earlier in this chapter.

[2] In the real world, the investor could have several investment possibilities to choose from, and the choice would be made according to the Sharpe ratios of the investments; in order to exceed the Sharpe ratios of the other investments, the option sale would have to have a Sharpe ratio of 0.5.

$\alpha S_0 \setminus \sigma +$	0,000	0,002	0,004	0,006	0,008	0,010	0,012	0,014	0,016	0,018	0,020
0,39	-0,9962	-0,9955	-0,9948	-0,9941	-0,9934	-0,9927	-0,9920	-0,9914	-0,9907	-0,9900	-0,9893
0,78	-0,9951	-0,9938	-0,9924	-0,9910	-0,9896	-0,9882	-0,9869	-0,9855	-0,9841	-0,9827	-0,9814
1,56	-0,9932	-0,9905	-0,9877	-0,9850	-0,9822	-0,9795	-0,9768	-0,9741	-0,9713	-0,9686	-0,9659
3,13	-0,9903	-0,9848	-0,9794	-0,9739	-0,9685	-0,9631	-0,9578	-0,9524	-0,9471	-0,9419	-0,9366
6,25	-0,9879	-0,9769	-0,9661	-0,9554	-0,9448	-0,9343	-0,9240	-0,9137	-0,9036	-0,8936	-0,8837
12,5	-0,9891	-0,9673	-0,9460	-0,9251	-0,9047	-0,8848	-0,8653	-0,8462	-0,8275	-0,8093	-0,7914
25	-1,0005	-0,9569	-0,9152	-0,8753	-0,8371	-0,8006	-0,7657	-0,7323	-0,7003	-0,6698	-0,6406
50	-1,0565	-0,9664	-0,8840	-0,8086	-0,7396	-0,6765	-0,6188	-0,5660	-0,5177	-0,4735	-0,4331
100	-1,3400	-1,1213	-0,9382	-0,7850	-0,6567	-0,5494	-0,4597	-0,3846	-0,3217	-0,2692	-0,2252
200	-3,5431	-2,4809	-1,7368	-1,2159	-0,8511	-0,5957	-0,4169	-0,2918	-0,2042	-0,1430	-0,1001
400	-120,4030	-59,0297	-28,9328	-14,1784	-6,4971	-3,4036	-1,6674	-0,8168	-0,4001	-0,1960	-0,0960
800	-8,06E+06	-1,94E+06	-4,66E+05	-1,12E+05	-2,69E+04	-6,45E+03	-1,55E+03	-3,71E+02	-89,0917	-21,3782	-5,1299
1600	-1,89E+19	-1,09E+18	-6,30E+16	-3,63E+15	-2,09E+14	-1,21E+13	-6,94E+11	-4,00E+10	-2,30E+09	-1,33E+08	-7,63E+06
0,39	0,0874	0,0952	0,1024	0,1091	0,1155	0,1215	0,1273	0,1320	0,1373	0,1425	0,1475
0,78	0,0994	0,1119	0,1240	0,1351	0,1454	0,1550	0,1635	0,1722	0,1805	0,1885	0,1956
1,56	0,1172	0,1388	0,1583	0,1752	0,1912	0,2057	0,2192	0,2321	0,2449	0,2567	0,2681
3,13	0,1403	0,1764	0,2062	0,2331	0,2571	0,2795	0,3001	0,3201	0,3389	0,3566	0,3741
6,25	0,1570	0,2188	0,2672	0,3091	0,3468	0,3816	0,4138	0,4448	0,4741	0,5023	0,5297
12,5	0,1489	0,2622	0,3427	0,4105	0,4709	0,5266	0,5793	0,6297	0,6785	0,7258	0,7724
25	-0	0,3035	0,4403	0,5525	0,6535	0,7484	0,8400	0,9299	1,0193	1,1086	1,1987
50	-0	0,2660	0,5288	0,7276	0,9100	1,0886	1,2695	1,4565	1,6526	1,8602	2,0811
100	-0	-0	0,3689	0,7892	1,1484	1,5209	1,9319	2,4001	2,9432	3,5776	4,3264
200	-0	-0	-0	-0	0,6169	1,3483	2,1803	3,2779	4,7940	6,9211	9,9398
400	-0	-0	-0	-0	-0	-0	-0	0,7063	2,2906	5,0031	10,3686
800	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0
1600	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0

Table 10.1: Expected utility (top half of the table) and generalised Sharpe ratio (bottom half of the table) as a function of investment volume (rows) and option price premium (columns) at $\kappa = 0$

$\alpha S_0 \setminus \sigma +$	0,000	0,002	0,004	0,006	0,008	0,010	0,012	0,014	0,016	0,018	0,020
0,39	-0,9992	-0,9985	-0,9978	-0,9971	-0,9964	-0,9957	-0,9950	-0,9943	-0,9936	-0,9930	-0,9923
0,78	-0,9985	-0,9971	-0,9957	-0,9943	-0,9929	-0,9916	-0,9902	-0,9888	-0,9874	-0,9860	-0,9847
1,56	-0,9973	-0,9946	-0,9918	-0,9890	-0,9863	-0,9836	-0,9808	-0,9781	-0,9754	-0,9726	-0,9699
3,13	-0,9962	-0,9907	-0,9852	-0,9797	-0,9743	-0,9689	-0,9635	-0,9581	-0,9528	-0,9475	-0,9422
6,25	-0,9982	-0,9872	-0,9763	-0,9654	-0,9547	-0,9441	-0,9337	-0,9233	-0,9131	-0,9030	-0,8929
12,5	-1,0076	-0,9854	-0,9637	-0,9425	-0,9217	-0,9014	-0,8815	-0,8621	-0,8431	-0,8245	-0,8063
25	-1,0358	-0,9907	-0,9475	-0,9062	-0,8667	-0,8289	-0,7927	-0,7581	-0,7251	-0,6934	-0,6632
50	-1,1300	-1,0337	-0,9455	-0,8649	-0,7911	-0,7236	-0,6618	-0,6053	-0,5537	-0,5064	-0,4632
100	-1,5307	-1,2808	-1,0717	-0,8967	-0,7502	-0,6276	-0,5251	-0,4393	-0,3675	-0,3075	-0,2572
200	-4,6321	-3,2434	-2,2707	-1,5895	-1,1127	-0,7788	-0,5451	-0,3815	-0,2670	-0,1869	-0,1308
400	-210,2660	-103,0870	-50,5270	-24,7605	-12,1321	-5,9438	-2,9118	-1,4264	-0,6987	-0,3423	-0,1677
800	-2,62E+07	-6,29E+06	-1,51E+06	-3,63E+05	-8,72E+04	-2,09E+04	-5,02E+03	-1,20E+03	-2,89E+02	-69,3812	-16,6487
1600	-1,98E+20	-1,14E+19	-6,61E+17	-3,81E+16	-2,20E+15	-1,27E+14	-7,29E+12	-4,20E+11	-2,42E+10	-1,39E+09	-8,01E+07
0,39	0,0400	0,0548	0,0664	0,0763	0,0851	0,0930	0,1004	0,1072	0,1137	0,1189	0,1248
0,78	0,0548	0,0763	0,0930	0,1072	0,1198	0,1304	0,1410	0,1509	0,1603	0,1691	0,1770
1,56	0,0736	0,1043	0,1289	0,1496	0,1673	0,1834	0,1988	0,2128	0,2260	0,2390	0,2511
3,13	0,0874	0,1373	0,1740	0,2046	0,2312	0,2554	0,2778	0,2990	0,3186	0,3375	0,3556
6,25	0,0601	0,1616	0,2217	0,2701	0,3117	0,3492	0,3835	0,4160	0,4465	0,4758	0,5043
12,5	-0	0,1728	0,2770	0,3546	0,4209	0,4803	0,5357	0,5878	0,6378	0,6863	0,7336
25	-0	0,1373	0,3375	0,4666	0,5756	0,6749	0,7690	0,8602	0,9497	1,0392	1,1285
50	-0	-0	0,3444	0,5803	0,7732	0,9539	1,1328	1,3150	1,5039	1,7028	1,9133
100	-0	-0	-0	0,4936	0,8814	1,2405	1,6207	2,0449	2,5307	3,0945	3,7572
200	-0	-0	-0	-0	-0	0,8054	1,5380	2,4230	3,6094	5,2562	7,5796
400	-0	-0	-0	-0	-0	-0	-0	-0	1,0239	2,7449	5,8786
800	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0
1600	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0

Table 10.2: Expected utility (top half of the table) and generalised Sharpe ratio (bottom half of the table) as a function of investment volume (rows) and option price premium (columns) at $\kappa = 0.001$

$\alpha S_0 \backslash \sigma +$	0,000	0,002	0,004	0,006	0,008	0,010	0,012	0,014	0,016	0,018	0,020
0,39	-1,0015	-1,0008	-1,0001	-0,9994	-0,9987	-0,9980	-0,9973	-0,9966	-0,9959	-0,9952	-0,9946
0,78	-1,0031	-1,0017	-1,0003	-0,9989	-0,9975	-0,9961	-0,9948	-0,9934	-0,9920	-0,9906	-0,9892
1,56	-1,0066	-1,0038	-1,0010	-0,9983	-0,9955	-0,9927	-0,9899	-0,9872	-0,9844	-0,9817	-0,9790
3,13	-1,0146	-1,0089	-1,0033	-0,9977	-0,9922	-0,9867	-0,9812	-0,9757	-0,9703	-0,9649	-0,9596
6,25	-1,0318	-1,0204	-1,0091	-0,9979	-0,9869	-0,9759	-0,9651	-0,9544	-0,9438	-0,9334	-0,9230
12,5	-1,0712	-1,0476	-1,0245	-1,0019	-0,9798	-0,9582	-0,9371	-0,9164	-0,8962	-0,8764	-0,8571
25	-1,1670	-1,1162	-1,0675	-1,0210	-0,9764	-0,9339	-0,8931	-0,8542	-0,8169	-0,7813	-0,7472
50	-1,4398	-1,3171	-1,2048	-1,1020	-1,0080	-0,9220	-0,8433	-0,7713	-0,7055	-0,6453	-0,5902
100	-2,5264	-2,1141	-1,7689	-1,4800	-1,2382	-1,0359	-0,8667	-0,7251	-0,6066	-0,5075	-0,4245
200	-13,0765	-9,1560	-6,4101	-4,4873	-3,1410	-2,1986	-1,5388	-1,0770	-0,7538	-0,5276	-0,3693
400	-1854,1000	-909,0070	-445,5410	-218,3350	-106,9790	-52,4120	-25,6763	-12,5780	-6,1615	-3,0182	-1,4785
800	-2,69E+09	-6,47E+08	-1,55E+08	-3,73E+07	-8,96E+06	-2,15E+06	-5,16E+05	-1,24E+05	-2,97E+04	-7,13E+03	-1,71E+03
1600	-2,63E+24	-1,52E+23	-8,76E+21	-5,05E+20	-2,91E+19	-1,68E+18	-9,66E+16	-5,56E+15	-3,20E+14	-1,84E+13	-1,06E+12
0,39	-0	-0	-0	0,0347	0,0510	0,0633	0,0736	0,0827	0,0908	0,0983	0,1043
0,78	-0	-0	-0	0,0469	0,0708	0,0886	0,1024	0,1155	0,1273	0,1381	0,1482
1,56	-0	-0	-0	0,0584	0,0952	0,1215	0,1432	0,1616	0,1787	0,1940	0,2082
3,13	-0	-0	-0	0,0679	0,1256	0,1647	0,1967	0,2246	0,2493	0,2722	0,2932
6,25	-0	-0	-0	0,0649	0,1635	0,2236	0,2714	0,3128	0,3502	0,3844	0,4169
12,5	-0	-0	-0	-0	0,2041	0,2986	0,3725	0,4368	0,4950	0,5495	0,6010
25	-0	-0	-0	-0	0,2212	0,3828	0,5037	0,6087	0,7061	0,7989	0,8895
50	-0	-0	-0	-0	-0	0,4199	0,6373	0,8252	1,0045	1,1838	1,3678
100	-0	-0	-0	-0	-0	-0	0,5756	0,9497	1,3106	1,6978	2,1329
200	-0	-0	-0	-0	-0	-0	-0	-0	0,8717	1,6101	2,5164
400	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0
800	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0
1600	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0

Table 10.3: Expected utility (top half of the table) and generalised Sharpe ratio (bottom half of the table) as a function of investment volume (rows) and option price premium (columns) at $\kappa = 0.005$

$\alpha S_0 \setminus \sigma+$	0,000	0,002	0,004	0,006	0,008	0,010	0,012	0,014	0,016	0,018	0,020
0,39	-1,0035	-1,0028	-1,0022	-1,0015	-1,0008	-1,0001	-0,9994	-0,9987	-0,9980	-0,9973	-0,9966
0,78	-1,0072	-1,0058	-1,0044	-1,0030	-1,0016	-1,0002	-0,9988	-0,9975	-0,9961	-0,9947	-0,9933
1,56	-1,0150	-1,0122	-1,0094	-1,0066	-1,0038	-1,0010	-0,9982	-0,9954	-0,9926	-0,9899	-0,9871
3,13	-1,0316	-1,0259	-1,0202	-1,0145	-1,0089	-1,0033	-0,9977	-0,9921	-0,9866	-0,9811	-0,9757
6,25	-1,0668	-1,0550	-1,0433	-1,0317	-1,0203	-1,0090	-0,9978	-0,9867	-0,9758	-0,9650	-0,9543
12,5	-1,1453	-1,1201	-1,0954	-1,0713	-1,0477	-1,0246	-1,0020	-0,9799	-0,9583	-0,9371	-0,9165
25	-1,3374	-1,2791	-1,2233	-1,1700	-1,1190	-1,0701	-1,0235	-0,9788	-0,9361	-0,8953	-0,8562
50	-1,9055	-1,7431	-1,5945	-1,4585	-1,3340	-1,2202	-1,1161	-1,0208	-0,9337	-0,8540	-0,7811
100	-4,5491	-3,8066	-3,1850	-2,6648	-2,2295	-1,8653	-1,5605	-1,3056	-1,0922	-0,9137	-0,7644
200	-45,4807	-31,8452	-22,2948	-15,6071	-10,9247	-7,6467	-5,3521	-3,7460	-2,6218	-1,8350	-1,2843
400	-2,62E+04	-1,28E+04	-6,29E+03	-3,08E+03	-1,51E+03	-7,39E+02	-3,62E+02	-1,77E+02	-86,9209	-45,5786	-20,8574
800	-7,52E+11	-1,81E+11	-4,34E+10	-1,04E+10	-2,50E+09	-6,01E+08	-1,44E+08	-3,46E+07	-8,31E+06	-1,99E+06	-4,78E+05
1600	-2,19E+29	-1,26E+28	-7,30E+26	-4,20E+25	-2,42E+24	-1,40E+23	-8,04E+21	-4,63E+20	-2,67E+19	-1,54E+18	-8,84E+16
0,39	-0	-0	-0	-0	-0	-0	0,0347	0,0510	0,0633	0,0736	0,0827
0,78	-0	-0	-0	-0	-0	-0	0,0490	0,0708	0,0886	0,1034	0,1163
1,56	-0	-0	-0	-0	-0	-0	0,0601	0,0962	0,1223	0,1432	0,1622
3,13	-0	-0	-0	-0	-0	-0	0,0679	0,1264	0,1654	0,1972	0,2246
6,25	-0	-0	-0	-0	-0	-0	0,0664	0,1647	0,2241	0,2718	0,3132
12,5	-0	-0	-0	-0	-0	-0	-0	0,2036	0,2982	0,3725	0,4365
25	-0	-0	-0	-0	-0	-0	-0	0,2093	0,3757	0,4976	0,6034
50	-0	-0	-0	-0	-0	-0	-0	-0	0,3835	0,6092	0,7994
100	-0	-0	-0	-0	-0	-0	-0	-0	-0	0,4448	0,8435
200	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0
400	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0
800	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0
1600	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0

Table 10.4: Expected utility (top half of the table) and generalised Sharpe ratio (bottom half of the table) as a function of investment volume (rows) and option price premium (columns) at $\kappa = 0.01$

$\alpha S_0 \setminus \sigma +$	0,000	0,002	0,004	0,006	0,008	0,010	0,012	0,014	0,016	0,018	0,020
0,39	-1,0058	-1,0051	-1,0044	-1,0037	-1,0030	-1,0023	-1,0016	-1,0009	-1,0002	-0,9995	-0,9988
0,78	-1,0119	-1,0105	-1,0091	-1,0077	-1,0063	-1,0049	-1,0035	-1,0021	-1,0007	-0,9993	-0,9979
1,56	-1,0248	-1,0220	-1,0192	-1,0163	-1,0135	-1,0107	-1,0079	-1,0050	-1,0023	-0,9995	-0,9967
3,13	-1,0537	-1,0479	-1,0421	-1,0363	-1,0305	-1,0248	-1,0191	-1,0134	-1,0078	-1,0022	-0,9966
6,25	-1,1230	-1,1106	-1,0983	-1,0861	-1,0741	-1,0622	-1,0504	-1,0387	-1,0272	-1,0158	-1,0046
12,5	-1,2899	-1,2615	-1,2337	-1,2065	-1,1799	-1,1539	-1,1284	-1,1035	-1,0792	-1,0554	-1,0321
25	-1,7268	-1,6516	-1,5796	-1,5107	-1,4448	-1,3818	-1,3215	-1,2639	-1,2088	-1,1560	-1,1056
50	-3,2589	-2,9811	-2,7269	-2,4943	-2,2815	-2,0868	-1,9087	-1,7459	-1,5969	-1,4606	-1,3359
100	-14,0117	-11,7246	-9,8102	-8,2080	-6,8672	-5,7453	-4,8066	-4,0212	-3,3642	-2,8145	-2,3546
200	-499,5890	-349,8080	-244,9010	-171,4390	-120,0040	-83,9967	-58,7912	-41,1484	-28,7997	-20,1568	-14,1077
400											
800											
1600											
0,39	-0	-0	-0	-0	-0	-0	-0	-0	-0	0,0316	0,0490
0,78	-0	-0	-0	-0	-0	-0	-0	-0	-0	0,0374	0,0649
1,56	-0	-0	-0	-0	-0	-0	-0	-0	-0	0,0316	0,0814
3,13	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0	0,0827
6,25	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0
12,5	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0
25	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0
50	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0
100	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0
200	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0
400											
800											
1600											

Table 10.5: Expected utility (top half of the table) and generalised Sharpe ratio (bottom half of the table) as a function of investment volume (rows) and option price premium (columns) at $\kappa = 0.02$

$\alpha S_0 \setminus \sigma +$	0,000	0,002	0,004	0,006	0,008	0,010	0,012	0,014	0,016	0,018	0,020
0,39	-1,0123	-1,0116	-1,0109	-1,0102	-1,0095	-1,0088	-1,0081	-1,0074	-1,0067	-1,0060	-1,0053
0,78	-1,0253	-1,0238	-1,0224	-1,0210	-1,0196	-1,0181	-1,0167	-1,0153	-1,0139	-1,0125	-1,0111
1,56	-1,0531	-1,0502	-1,0472	-1,0443	-1,0414	-1,0385	-1,0356	-1,0327	-1,0299	-1,0270	-1,0241
3,13	-1,1169	-1,1107	-1,1046	-1,0984	-1,0923	-1,0863	-1,0802	-1,0742	-1,0682	-1,0623	-1,0564
6,25	-1,2822	-1,2680	-1,2539	-1,2400	-1,2263	-1,2127	-1,1993	-1,1860	-1,1728	-1,1598	-1,1469
12,5	-1,7851	-1,7457	-1,7073	-1,6696	-1,6328	-1,5968	-1,5616	-1,5272	-1,4935	-1,4605	-1,4283
25	-3,5880	-3,4317	-3,2821	-3,1390	-3,0021	-2,8712	-2,7460	-2,6262	-2,5116	-2,4021	-2,2973
50	-15,4303	-14,1149	-12,9112	-11,8099	-10,8024	-9,8807	-9,0375	-8,2662	-7,5608	-6,9155	-6,3253
100	-3,65E+02	-3,06E+02	-2,56E+02	-2,14E+02	-1,79E+02	-1,50E+02	-1,25E+02	-1,05E+02	-87,6735	-73,3475	-61,3624
200											
400											
800											
1600											
0,39	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0
0,78	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0
1,56	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0
3,13	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0
6,25	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0
12,5	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0
25	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0
50	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0
100	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0	-0
200											
400											
800											
1600											

Table 10.6: Expected utility (top half of the table) and generalised Sharpe ratio (bottom half of the table) as a function of investment volume (rows) and option price premium (columns) at $\kappa = 0.05$

	$\kappa = 0$		$\kappa = 0.001$		$\kappa = 0.005$		$\kappa = 0.01$		$\kappa = 0.02$		$\kappa = 0.05$	
αS_0 \ required SR_G	0	0.5	0	0.5	0	0.5	0	0.5	0	0.5	0	0.5
0.39	0.0000	0,3167	0.0000	0,3259	0,0044	0,3331	0,0102	0,3394	0,0167	0,3464	0,0352	0,3666
0.78	0.0000	0,1540	0.0000	0,1589	0,0045	0,1657	0,0104	0,1717	0,0170	0,1785	0,0359	0,1978
1.56	0.0000	0,0753	0.0000	0,0783	0,0048	0,0850	0,0107	0,0910	0,0177	0,0979	0,0372	0,1176
3.13	0.0000	0,0366	0.0000	0,0387	0,0052	0,0453	0,0112	0,0513	0,0188	0,0589	0,0397	0,0799
6.25	0.0000	0,0179	0.0000	0,0198	0,0057	0,0257	0,0117	0,0317	0,0209	0,0409	0,0447	0,0647
12.5	0.0000	0,0091	0,0007	0,0107	0,0062	0,0162	0,0122	0,0222	0,0229	0,0329	0,0520	0,0621
25	0,0001	0,0051	0,0016	0,0066	0,0070	0,0120	0,0131	0,0181	0,0246	0,0296	0,0574	0,0624
50	0,0013	0,0038	0,0028	0,0053	0,0082	0,0107	0,0145	0,0170	0,0265	0,0290	0,0614	0,0640
100	0,0033	0,0046	0,0048	0,0061	0,0104	0,0117	0,0170	0,0183	0,0297	0,0309	0,0663	0,0675
200	0,0071	0,0078	0,0086	0,0093	0,0145	0,0151	0,0215	0,0221	0,0349	0,0355	0,0735	0,0741
400	0,0135	0,0138	0,0150	0,0154	0,0211	0,0215	0,0286	0,0289	0,0429	0,0432	0,0840	0,0844
800	0,0223	0,0225	0,0240	0,0241	0,0305	0,0306	0,0384	0,0385	0,0536	0,0537	0,0964	0,0965
1600	0,0312	0,0312	0,0328	0,0329	0,0395	0,0395	0,0474	0,0475	0,0628	0,0629	0,1058	0,1059

Table 10.7: Minimum premium required to reach the required Sharpe ratio

By calculating the ratios of the $\kappa = 0$ premiums and their appropriate counterparts with transaction costs, we can obtain information about what part of the overall risk is posed by the transaction costs and what part by is posed by the multinomial tree model and its numerical implementation. The ratios are shown in Table 10.8 for volumes of 25 and higher (for lower volumes, the ratio is 0).

	$\kappa = 0.001$	$\kappa = 0.005$	$\kappa = 0.01$	$\kappa = 0.02$	$\kappa = 0.05$
25	0.06	0.01	0.00	0.00	0.00
50	0,46	0,16	0,09	0,05	0,02
100	0,69	0,32	0,19	0,11	0,05
200	0,83	0,49	0,33	0,20	0,10
400	0,90	0,64	0,47	0,31	0,16
800	0,93	0,73	0,58	0,42	0,23
1600	0,95	0,79	0,66	0,50	0,29

Table 10.8: Ratio of premiums for $\kappa = 0$ to those for $\kappa \neq 0$

We can see that the contribution of the model risk to the total risk increases with investment volume and decreases as transaction costs become higher. Both of these effects are logical and unsurprising.

To calculate the transaction costs risk, we will assume that the model risk remains constant at various levels of κ (there does not appear to be any reason to assume otherwise), and subtract the $\kappa = 0$ premiums from the $\kappa \neq 0$ ones. The resulting transaction costs premiums are given in Table 10.9. It is immediately apparent that the order of magnitude of the premiums approximately corresponds to that of the transaction cost coefficient. While hoping to find a simple relationship along the lines of '$\kappa = 0.012$ implies a transaction cost premium of 0.012' would be overly optimistic, the data do provide motivation to explore the dynamics of the relationship. In other words, if we know the premium for a given level of transaction costs, and then double that level of transaction costs, how close to twice the original premium will the new premium be?

	$\kappa = 0.001$	$\kappa = 0.005$	$\kappa = 0.01$	$\kappa = 0.02$	$\kappa = 0.05$
0.39	0,0000	0,0044	0,0102	0,0167	0,0352
0.78	0,0000	0,0045	0,0104	0,0170	0,0359
1.56	0,0000	0,0048	0,0107	0,0177	0,0372
3.13	0,0000	0,0052	0,0112	0,0188	0,0397
6.25	0,0000	0,0057	0,0117	0,0209	0,0447
12.5	0,0007	0,0062	0,0122	0,0229	0,0520
20	0,0015	0,0069	0,0130	0,0245	0,0573
50	0,0015	0,0069	0,0132	0,0252	0,0601
100	0,0015	0,0071	0,0137	0,0264	0,0630
200	0,0015	0,0074	0,0144	0,0278	0,0664
400	0,0015	0,0076	0,0151	0,0294	0,0705
800	0,0017	0,0082	0,0161	0,0313	0,0741
1600	0,0016	0,0083	0,0162	0,0316	0,0746

Table 10.9: Net transaction costs premiums

The above question and a partial answer are illustrated in Figure 10.1, showing premium versus volume plots at several transaction costs levels, along with theoretical premium levels calculated as multiples of real premium levels. We can see that the premium at $\kappa = 0.01$ is close to, but certainly not equal to, twice the premium at $\kappa = 0.005$. Similarly, the premium at $\kappa = 0.02$ is relatively close to twice the premium at $\kappa = 0.01$. While this matter would have to be investigated in greater detail before any final conclusions could be drawn, the data hint at the possible existence of the following rule of thumb: in the absence of a better tool, if the transaction costs change, the investor can get the approximate values of the new minimal required premium by multiplying the old premium using the same factor by which the transaction costs changed.

At this point, we are still looking at the dependence of premiums on both transaction costs and volume. It is clear from the presented data as well as from Figure 10.1 that if we fix the transaction costs, higher volumes will require higher premiums. We could search for a more precise description of this relationship by plotting the premium against the logarithm of the volume and doing regression. However, we shall not follow this problem further, because the investment volume is a reaction of the investor to the situation on the market rather than a determining factor (the investor will see what the transaction costs are and what price the options can be sold at, and choose the optimal investment volume based on that). Instead, we shall concentrate on finding the premium as a function of transaction costs, given a required Sharpe ratio value.

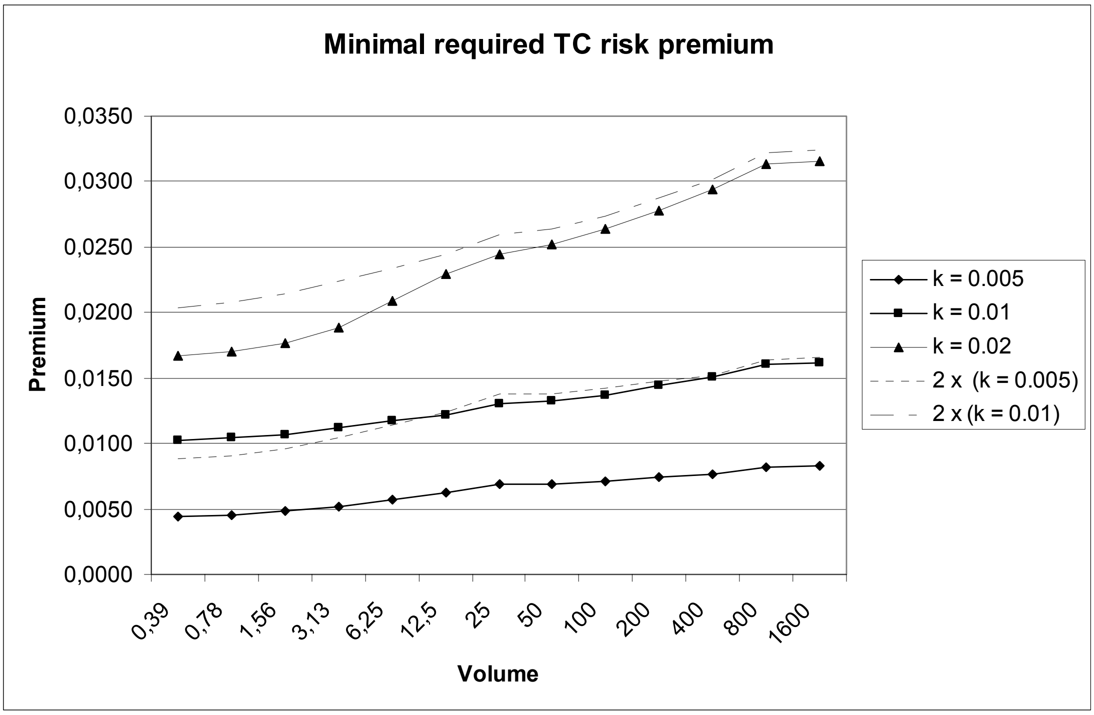

Figure 10.1: Minimal required transaction costs premium as a function of volume, for different values of κ (denoted in the graph legend as 'k').

To find the premium required for $SR_G = 0$ as a function of transaction costs, we will take the premiums at $\alpha S_0 = 0.39$ (smallest required premium at each value of κ) and $\alpha S_0 = 1600$ (largest required premium), and plot them against κ. Then, using regression, we can determine the 'cone' wherein the minimal required premium will fall (for volumes between 0.39 and 1600). See Figure 10.2. The range is relatively wide, but it does offer some idea as to what the working interval is. The trendlines were obtained by 3$^{\text{rd}}$-order polynomial regression, and the equations are $y = 40.299x^3 - 5.7634x^2 + 1.6799x + 0.0312$ with $R^2 = 1$ for the top curve and $y = 416.95x^3 - 35.279x^2 + 1.4576x - 0.0016$ with $R^2 = 0.9995$ for the bottom curve. The top curve is practically linear, and we could simplify the equation to $y = 1.4893x + 0.0319$ with $R^2 = 0.9993$ (a very slight decrease). In the case of the bottom curve, linear regression leads to $y = 0.6945x + 0.0014$ with $R^2 = 0.9856$.

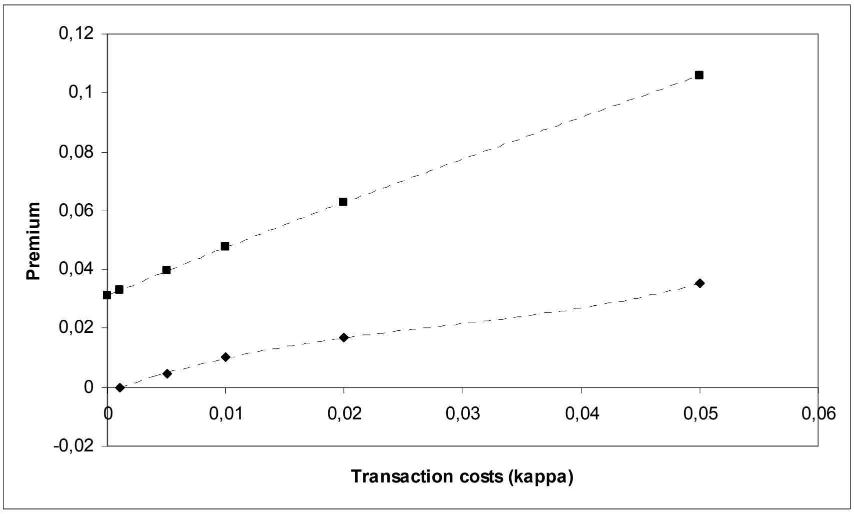

Figure 10.2: Total minimal required premium as a function of transaction costs

To obtain the relationship between transaction costs and the transaction costs premium (as opposed to the total premium), we will do a similar procedure, but this time subtracting the premiums at $\kappa = 0$ from those at $\kappa \neq 0$. The result is shown in Figure 10.3. The cone wherein the required premium lies is noticeably narrower than in the previous case.

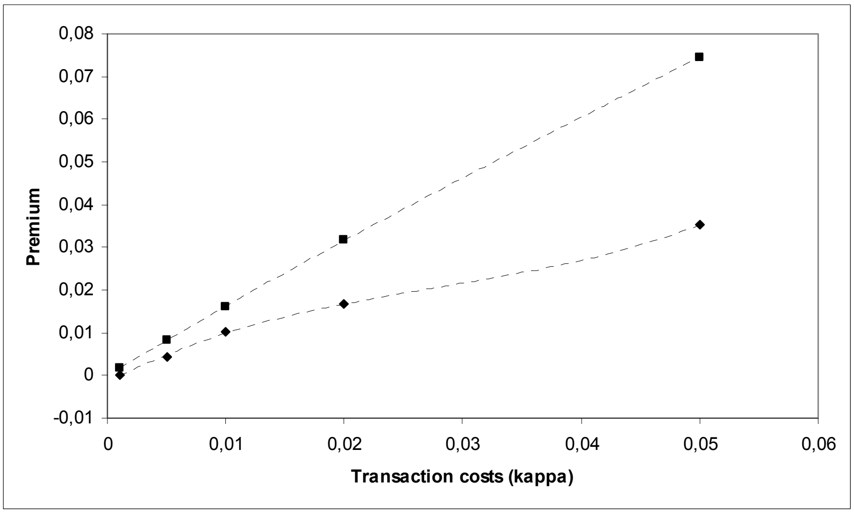

Figure 10.3: Minimal required transaction costs premium as a function of transaction costs

The equations are $y = 46.189x^3 - 6.2045x^2 + 1.6879x - 6 \times 10^{-5}$ with $R^2 = 1$ (top curve) and the same equation as before for the bottom curve (the minimal required premium $\kappa = 0$ at the smallest volume is 0, hence the bottom curve remains unchanged). We could once again linearise the top curve, to get $y = 1.4815x + 0.001$ with $R^2 = 0.9994$. All these regression results were obtained using MS Excel.

Let us now examine what happens when we fix the required Sharpe ratio at $SR_G = 0.5$. In Table 10.7, we can see that for each value of κ the premium is a convex function of volume. The optimal investment will correspond to the volume where the premium is the lowest[3]. We will plot the premiums corresponding to the optimal investment against the transaction costs to find the relationship between the two. Both total required premium and premium related to transaction costs will be examined. The latter will once again be determined by subtracting the premium at $\kappa = 0$ from the premiums at $\kappa \neq 0$.

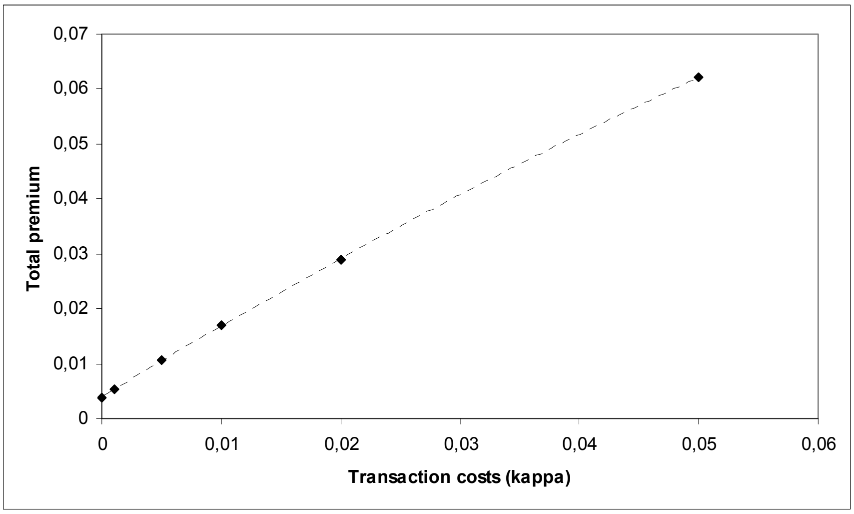

Figure 10.4: Total premium required to reach $SR_G = 0.5$ as a function of transaction costs

Figure 10.4 shows the total premium. The 2nd-order polynomial trendline equation and its R-squared value are $y = -3.2151x^2 + 1.3229x + 0.004$ and 1, respectively. If we used linear regression instead, the equation would be $y = 1.1602x + 0.0047$ with $R^2 = 0.9987$.

[3] All volume-premium combinations lead to the same Sharpe ratio and utility. To be able to sell the option on a competitive market, the investor will have to keep the price as low as possible – this can be done by choosing the volume where the required premium is the lowest. Alternatively, the investor can be looking for an investment with a Sharpe ratio of 0.5 *or higher*; if the market price of the option is fixed at a certain level, choosing the investment with the lowest required premium to achieve a Sharpe ratio of 0.5 will give the investor the most *extra* premium, ultimately leading to a higher Sharpe ratio; or, if the market price is lower than the premium required to achieve a Sharpe ratio of 0.5, choosing the investment with the lowest required premium will lead as close to the originally desired Sharpe ratio as possible.

Figure 10.5 shows the premium required by the presence of transaction costs. The trendline equation is $y = -2.9823x^2 + 1.3086x + 0.0003$ with $R^2 = 1$. Linear regression would lead to $y = 1.1503x + 0.0012$ with $R^2 = 0.999$.

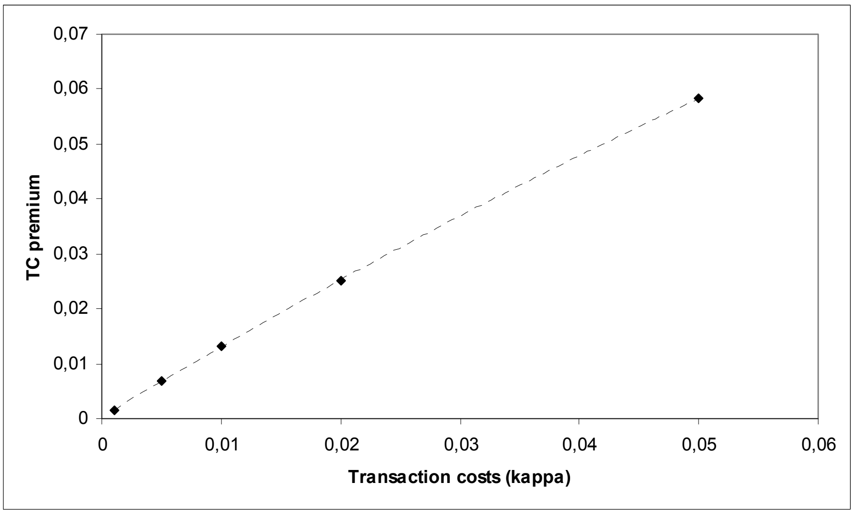

Figure 10.5: Transaction costs related premium required to reach $SR_G = 0.5$ as a function of transaction costs

Up to this point, the basic time unit in our calculation was one week. This is quite valid, and a natural consequence of using a weekly distribution of stock returns and rehedging the portfolio once a week. In the real world, however, the convention is to quote returns, volatilities, etc. on a per annum basis. This common basis allows the markets to quickly compare very different investments. For this reason, we shall convert our results to a yearly base.

In the case of implied volatility, the time unit conversion is very straightforward – the weekly volatility simply needs to be multiplied by a factor of $\sqrt{52}$ (if monthly data had been used, the factor would be $\sqrt{12}$, etc.)[4].

The per annum convention also applies to the Sharpe ratio. Here the conversion is only slightly more complex than in the case of volatility. Let $SR_G(\tau)$ be the generalised Sharpe ratio of an investment with time to expiry equal to τ, and let $\tau = 1$ correspond to one year. Then, according to [1],

[4] For example, if the weekly historical volatility of the data in our model is 0.02167, the yearly historical volatility will be 0.15627.

$$SR_G(\tau) = \sqrt{\exp\left(SR_G^2(1)\tau\right) - 1} \tag{10.5}$$

What we will do now is to decide what yearly Sharpe ratio that we want to achieve, and convert this value to the Sharpe ratio corresponding to our time to expiry by substituting $\tau = 5/52$ (i.e. five weeks) into (10.5). This value will then be used in calculating the required premium.

Table 10.10 shows the premium (in per annum units of volatility) required to achieve the given Sharpe ratio (per annum equivalent) at the given level of transaction costs, assuming optimal investment volumes. Table 10.11 offers equivalent information, except that the implied volatility rather than the premium is listed.

$SR_G \setminus \kappa$	0	0.001	0.005	0.01	0.02	0.05
0.25	0,0000	0,0000	0,0447	0,0880	0,1428	0,2834
0.50	0,0000	0,0130	0,0526	0,0959	0,1659	0,3180
1.00	0,0159	0,0274	0,0656	0,1096	0,1925	0,3844

Table 10.10: Option price premium as a function of transaction costs and the generalised Sharpe ratio

$SR_G \setminus \kappa$	0	0.001	0.005	0.01	0.02	0.05
0.25	0,1564	0,1564	0,2011	0,2444	0,2992	0,4398
0.50	0,1564	0,1694	0,2091	0,2523	0,3223	0,4744
1.00	0,1723	0,1838	0,2220	0,2660	0,3490	0,5408

Table 10.11: Implied volatility as a function of transaction costs and the generalised Sharpe ratio

11 Conclusion

The goal of this thesis was to derive and calculate a dynamically optimal portfolio hedging strategy leading to maximisation of a fictional investor's utility in an incomplete market with three assets: stock, stock options, and cash. We have modelled the market using an empirically observed distribution of stock returns and volume-based transaction costs associated with buying and selling of stock.

In chapters 2 and 3, we selected a realistic utility function describing the investor's benefit from a given level of wealth, and formulated the optimisation problem as maximisation of the investor's utility from the net value of their assets at the end of the investment.

Chapter 4 is dedicated to finding a mathematical algorithm that would allow the optimal hedging strategy to be calculated. The algorithm was derived using Bellman's principle of optimality as a set of recursive one-period optimisation problems the solution of which yields an optimal multi-period hedging strategy.

The numerical implementation of the obtained optimisation algorithm is described in chapter 5. Due to the fact that with the exception of a single period the target function was not convex, a simple brute-force algorithm was used to calculate the optimal hedge. The results yielded by this algorithm for the simplified case when the investor sells a single option ($\alpha = 1$) are given in chapter 6. Subsequently in chapter 7 the algorithm is adjusted to effectively deal with larger quantities of options.

In chapter 8, the transaction costs hedge is compared to the classic continuous-time Black-Scholes hedge using Monte Carlo experiments. The transaction costs hedge has been found to be the better of the two in terms of final portfolio value, expected profit and hedging error. The expected profit and the hedging error are shown to depend on the volume αS_0 rather than on the specific combination of α and S_0, which has an important positive consequence on the usability of the optimal hedging algorithm. It also becomes apparent that in the presence of transaction costs the Black-Scholes price of the option is lower than fair and that a premium has to be charged to make the investment pay off.

In chapter 9, option price premiums are introduced and the optimal investment α is found for various values of the premium. The optimal investment is determined by comparing utility values as well as values of the Sharpe ratio, both of which are calculated with the help of Monte Carlo experiments. An initial investigation of the effect of transaction costs on the required premium is carried out by comparing minimal premiums needed for the investment to pay off, premiums needed for a given investment volume to become optimal, and Sharpe ratio values.

The search for the relationship between transaction costs, option price premium and Sharpe ratio values is extended and finalised in chapter 10. A new criterion of investment valuation, the generalised Sharpe ratio, is introduced and calculated at different levels of transaction costs and premiums. The contribution of several sources of risk is identified, and several dependencies between transaction costs and premiums are inferred. Finally, a table is given with information (in units compliant with real-world conventions) about the premium that is required to obtain a given Sharpe ratio level at the given level of transaction costs.

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