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NONLINEAR THEORY OF RATIONAL EXPECTATIONS

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Prehlasujem, že som diplomovú prácu
vypracoval samostatne a použil som
iba literatúru uvedenú v zozname.

V Bratislave 17.apríla 2005 Miroslav Zámečník

Ďakujem vedúcemu diplomovej práce,
prof. RNDr. Pavlovi Brunovskému, DrSc.
za všestrannú pomoc a cenné rady pri tvorbe
tejto práce. Ďakujem svojim rodičom,
ktorí mi boli veľkou oporou počas celého
vysokoškolského štúdia.

Contents

1 Introduction

To model the behavior of agents whose decisions are based upon their expectations, Blanchard and Kahn [1] introduce a linear difference equation model. The state of the economy is modelled by a vector with components partitioned into predetermined and non-predetermined ones, agents having the possibility to choose the latter. It is argued that rationality of expectations drives them to choose those components in such a way that the economy moves along a trajectory tending to steady state.

In [1] a dimension condition is derived to allow agents to choose the non-predetermined variables uniquely. Somewhat hidden in the text one finds a regularity condition under which this dimension conclusion is valid. The principal goal of this thesis was to relate the theory of rational expectations to dichotomy theory well-known in theory of dynamical systems. We show that the dimension condition together with the regularity one can be naturally formulated in the language of dichotomies and, further, can be extended to nonlinear systems. The importance of this result lies in the fact that linearity is as a rule an idealisation.

In literature, models with one predetermined and one non-predetermined variable are usually considered and there is virtually no multi-dimensional model available. The second goal of the thesis was to establish a higher dimensional model which would allow us to apply the theory.

At the beginning of Chapter 2 we introduce the basis of the theory of exponential dichotomies and then we adjust this theory for linear and nonlinear difference model with predetermined and non-predetermined variables. At the end of Section 2.1 we present a theorem giving conditions of existence of a unique bounded solution when a vector of predetermined variables is given in terms of exponential dichotomies. In Section 2.2 the theory is extended to nonlinear systems. Following the theory of exponential dichotomies we present the sufficient conditions for an existence of unique stable solution of the nonlinear system when a vector of predetermined variables near the steady state of the system is given. The main aim of this section is to argue a linearization, the technique common used by solving the difference models at all.

In order to apply the existing theory to a model with more variables, we consider a very simple two-sectoral model of economy in Chapter 3. In this model we do not admit any uncertainty and hence the rational expectations reduce to the special case of perfect foresight.

2 Exponential dichotomies

In connection with the theory of rational expectations this chapter deals with the properties of difference system consisting of the predetermined and non-predetermined variables. From point of view of this chapter, the main difference between these two types of variables is that the vector of predetermined variables at time 0 is given, whereas the vector of the non-predetermined variables is not. This feature allows the representative agents to choose the optimal vector of non-predetermined variables at time 0 in the rational expectations models. The optimal vector could be defined as the one which leads to the bounded solution of the whole system. In this chapter we deal with the uniqueness of this vector as well.

2.1 Linear systems

Let us consider a homogeneous system of linear difference equations

$$\mathbf{u_{j+1}} = \mathbf{A_j u_j} \tag{2.1}$$

where $\mathbf{u_j} \in R^p$, $j \in Z_0^+$ and the matrix $\mathbf{A_j}$ (n×n) is for each $j \in Z_0^+$ invertible.

Definition 1 *For each i, $j \in Z_0^+$ we define the transition matrix by the formula*

$$\mathbf{\Phi(j,\, i)} := \begin{cases} \mathbf{A_{j-1} \dots A_i} & \text{if } j > i \\ \mathbf{I} & \text{if } j = i \\ \mathbf{A_j^{-1} \dots A_{i-1}^{-1}} & \text{if } j < i. \end{cases}$$

Definition 2 *We say that system* (2.1) *has an exponential dichotomy on Z_0^+, if there exist constants $K > 0$, $\alpha > 0$ and a family of projections $\mathbf{Q_j}$, $j \in Z_0^+$, which satisfy the following three conditions:*

- $\mathbf{Q_{j+1} A_j} = \mathbf{A_j Q_j}$ for all $j \in Z_0^+$.
- $|\mathbf{\Phi(j,\, i) Q_i}| \le K e^{-\alpha(j-i)}$ for $i \le j$.
- $|\mathbf{\Phi(j,\, i)(I - Q_i)}| \le K e^{-\alpha(i-j)}$ for $j \le i$.

Remark 2.1 *One can see that if system* (2.1) *has exponential dichotomy on the interval* (q, ∞)*, where* $q \in Z_0^+$*, then system* (2.1) *has also exponential dichotomy on* Z_0^+*.*

Remark 2.2 *If system* (2.1) *has exponential dichotomy on* Z_0^+*, system* (2.1) *has no non-trivial solution which is simultaneously satisfying:*

- *There exists a constant* $L > 0$ *such that the inequality* $|\mathbf{u_j}| < L$ *holds true for each* $j \in Z_0^+$*.*
- $\mathbf{Q_0 u_0} = \mathbf{0}$.

Theorem 2.1 *Let* $\mathbf{A_j}$ *be a constant matrix* $\mathbf{A}$ *(for each* $j \in Z_0^+ : \mathbf{A_j} = \mathbf{A}$*). Then the following statement holds:*
The system (2.1) *has exponential dichotomy on* Z_0^+ *if and only if all eigenvalues of the matrix* $\mathbf{A}$ *lie off the unit circle.*

Remark 2.3 *If* $\mathbf{A_j}$ *are equal to a constant matrix* $\mathbf{A}$ *for all* j*, then the family of projections* $\mathbf{Q_j}$ *is independent of* j *(for all* $j \in Z_0^+ : \mathbf{Q_j} = \mathbf{Q}$*).*

Remark 2.4 *The range of matrix* $\mathbf{Q}$ *is the sum of the generalized eigenspaces of* $\mathbf{A}$ *corresponding to the eigenvalues inside the unit circle and the nullspace of this matrix* $\mathbf{Q}$ *is the sum of the generalized eigenspaces of* $\mathbf{A}$ *corresponding to the eigenvalues outside the unit circle.*

Theorem 2.2 *Let system* (2.1) *admit exponential dichotomy on* Z_0^+ *with constants* K, α *and the family of projections* $\mathbf{Q_j}$ *and let* $\{\mathbf{h_j}\}_{j=0}^{\infty}$ *be a bounded sequence of vectors from* R^p*. Also let* $\boldsymbol{\xi}$ *be arbitrary,* $\boldsymbol{\xi} \in R^p$*. Then the nonhomogeneous system of linear difference equations*

$$\mathbf{u_{j+1}} = \mathbf{A_j u_j} + \mathbf{h_j} \tag{2.2}$$

has a unique solution $\mathbf{u_j}$ *which is bounded on* Z_0^+ *and satisfies* $\mathbf{Q_0 u_0} = \mathbf{Q_0}\boldsymbol{\xi}$*.*

Moreover,

$$|\mathbf{u}_j| \le K|\boldsymbol{\xi}| + K(1 + e^{-\alpha})(1 - e^{-\alpha})^{-1} \sup_{i \in Z_0^+} |\mathbf{h_i}|$$

and

$$\mathbf{u_0} = \mathbf{Q_0}\boldsymbol{\xi} - \sum_{i=0}^{\infty} (\mathbf{I} - \mathbf{Q_0})\boldsymbol{\Phi}(\mathbf{0},\, \mathbf{i+1})\mathbf{h_i}. \tag{2.3}$$

Proof: Define $\mathbf{u_j} := \mathbf{\Phi(j,\, 0)Q_0}\boldsymbol{\xi} + \sum_{i \in Z_0^+} \mathbf{G(j,\, i+1)h_i}$, where $\mathbf{G(j,\, i)}$ is the so-called Green's function defined by

$$\mathbf{G(j,\, i)} := \begin{cases} \mathbf{\Phi(j,\, i)Q_i} & \text{if } j \geq i \\ -\mathbf{\Phi(j,\, i)(I - Q_i)} & \text{if } j < i. \end{cases}$$

This vector $\mathbf{u_j}$ is well-defined because, thanks to the inequalities in the definition of exponential dichotomy, the following inequality holds for all $i,\, j \in Z_0^+$:

$$|\mathbf{G(j,\, i)}| \leq K e^{-\alpha|j-i|}. \tag{2.4}$$

By using this property it is easy to see that:

$$\sum_{i \in Z_0^+} |\mathbf{G(j,\, i+1)h_i}| \leq K \sum_{i \in Z_0^+} e^{-\alpha|j-i-1|} \sup_{j \in Z_0^+} |\mathbf{h_j}| \leq K(1+e^{-\alpha})(1-e^{-\alpha})^{-1} \sup_{j \in Z_0^+} |\mathbf{h_j}|. \tag{2.5}$$

Now, by the definition of $\mathbf{u_0}$ we can write

$$\mathbf{u_0} = \mathbf{\Phi(0,\, 0)Q_0}\boldsymbol{\xi} + \sum_{i=0}^{\infty} \mathbf{G(0,\, i+1)h_i} =$$

$$= \mathbf{Q_0}\boldsymbol{\xi} - \sum_{i=0}^{\infty} \mathbf{\Phi(0,\, i+1)(I - Q_{i+1})h_i} = \mathbf{Q_0}\boldsymbol{\xi} - \sum_{i=0}^{\infty} \big(\mathbf{\Phi(0,\, i+1)} - \mathbf{\Phi(0,\, i+1)Q_{i+1}}\big)\mathbf{h_i} =$$

$$= \mathbf{Q_0}\boldsymbol{\xi} - \sum_{i=0}^{\infty} \mathbf{(I - Q_0)\Phi(0,\, i+1)h_i}.$$

We showed that vector $\mathbf{u_0}$ as defined above satisfies the equality (2.3).

For each $i \in Z_0^+$ the sequence of matrices $\{\mathbf{X_j}\}$ defined by $\mathbf{X_j} = \mathbf{G(j,\, i)}$ solves the system

$$\mathbf{X_{j+1}} = \mathbf{A_j X_j} + \mathbf{H_j^{(i)}}, \tag{2.6}$$

where

$$\mathbf{H_j^{(i)}} := \begin{cases} \mathbf{I} & \text{if } j+1 = i \\ \mathbf{0} & \text{otherwise.} \end{cases} \tag{2.7}$$

Now, for the sequence $\mathbf{u_j}$ defined above we can write:

$$\mathbf{u_{j+1}} = \mathbf{\Phi(j+1,\, 0)Q_0}\boldsymbol{\xi} + \sum_{i \in Z_0^+} \mathbf{G(j+1,\, i+1)h_i} = \mathbf{A_j\Phi(j,\, 0)Q_0}\boldsymbol{\xi} +$$

$$+\sum_{i\in Z_0^+,\, i\le j-1}\Big(\mathbf{A_jG(j,\, i+1)h_i}+\big(\mathbf{A_jG(j,\, j+1)+I}\big)\mathbf{h_j}\Big)+\sum_{i\in Z_0^+,\, i\ge j+1}\mathbf{A_jG(j,\, i+1)h_i}=$$

$$=\mathbf{A_j\Phi(j,\, 0)Q_0}\boldsymbol{\xi}+\mathbf{A_j}\Big(\sum_{i\in Z_0^+}\mathbf{G(j,\, i+1)h_i}\Big)+\mathbf{h_j}=\mathbf{A_ju_j}+\mathbf{h_j}.$$

We have shown that $\mathbf{u_j}$ is a solution of (2.2). This solution has the property $\mathbf{Q_0u_0}=\mathbf{Q_0}\boldsymbol{\xi}$:

$$\mathbf{Q_0u_0}=\mathbf{Q_0}\Big(\mathbf{Q_0}\boldsymbol{\xi}+\sum_{i\in Z_0^+}\mathbf{G(0,\, i+1)h_i}\Big)=\mathbf{Q_0}\boldsymbol{\xi}+\mathbf{Q_0}\Big(\sum_{i\in Z_0^+}-\mathbf{\Phi(0,\, i+1)(I-Q_{i+1})h_{i+1}}\Big)=$$

$$=\mathbf{Q_0}\boldsymbol{\xi}.$$

The uniqueness of this solution follows from remark 2.2 and the estimate

$$|\mathbf{u_0}|\le K|\boldsymbol{\xi}|+K(1+e^{-\alpha})(1-e^{-\alpha})\sup_{j\in Z_0^+}|\mathbf{h_j}|$$

follows from (2.5) and the equality (2.3). ■

The next theorem shows that very small perturbations of the original system keep the property of exponential dichotomy with the same rank, only the constants of exponential dichotomy changes. This theorem can be considered as a motivation for investigation of nonlinear difference systems and does not have any special importance for next conclusions.

Theorem 2.3 *Let $\mathbf{A_i}$, $i\in Z_0^+$ be invertible square matrices of dimension p such that $|\mathbf{A_i}^{-1}|\le M$ and system (2.1) has exponential dichotomy on Z_0^+ with constants K, α and family of projections $\mathbf{Q_i}$.*
Also let $0<\delta<\alpha$ and $\mathbf{D_i}$, $i\in Z_0^+$ be invertible square matrices of dimension p simultaneously satisfying
$|\mathbf{D_i}|<M^{-1}, \quad 2K(1-e^{-\alpha})(1-e^{\alpha})|\mathbf{D_i}|\le 1, \quad 2Ke^{\alpha}(e^{-\delta})(e^{\delta}-1)^{-1}|\mathbf{D_i}|\le 1.$
Then the square matrix $\mathbf{A_i}+\mathbf{D_i}$ is invertible for each $i\in Z_0^+$ and perturbed system

$$\mathbf{u_{i+1}}=(\mathbf{A_i}+\mathbf{D_i})\mathbf{u_i} \tag{2.8}$$

has exponential dichotomy on Z_0^+ with constants $2K(1+e^{\delta})(1-e^{-\delta})^{-1}$, $\alpha-\delta$ and a family of projections with the same rank as for original system (2.1).

Now we present one theorem from linear algebra which will be helpful in our effort to treat properly with difference systems.

Theorem 2.4 *Let V be an l-dimensional vector subspace in R^p. Let X, S be a k-dimensional resp. (p-k)-dimensional subspace of R^p. Then the following statement holds true:*

For each vector $\mathbf{x} \in X$ there exists a unique vector $\mathbf{s} \in S$ such that $\mathbf{x} + \mathbf{s} \in V$ if and only if $V \cap S = \{\mathbf{0}\}$ and $k = l$ simultaneously.

Proof: Without loss of generality we can assume that the space X is generated by the vectors $\mathbf{e_1} = (1, 0, \dots, 0)^T, \mathbf{e_2} = (0, 1, 0, \dots, 0)^T, \dots \mathbf{e_k} = (\underbrace{0, \dots, 0, 1}_{k}, 0, \dots, 0)^T$ and the space S is generated by the vectors $\mathbf{e_{k+1}} = (\underbrace{0, \dots, 0}_{k}, 1, 0, \dots, 0)^T$, $\mathbf{e_{k+2}} = (\underbrace{0, \dots, 0}_{k}, 0, 1, 0, , \dots, 0)^T$, $\dots \mathbf{e_p} = (0, \dots, 0, 1)^T$. We know that $\mathbf{x} \in X$ can be expressed: $\mathbf{x} = (x_1, x_2, \dots, x_k, \underbrace{0, \dots, 0}_{p-k})^T$. In the same way we can write
$\mathbf{s} = (0, \dots, 0, s_{k+1}, s_{k+2}, \dots, s_p)^T$.

Let the vector space V be generated by the set of vectors:

$\mathbf{v^{(1)}} = (v_1^{(1)}, v_2^{(1)}, \dots, v_l^{(1)})^T$, $\mathbf{v^{(2)}} = (v_1^{(2)}, v_2^{(2)}, \dots, v_l^{(2)})^T$, $\dots$ $\mathbf{v^{(l)}} = (v_1^{(l)}, v_2^{(l)}, \dots, v_l^{(l)})^T$

Now it is easy to see that the property $\mathbf{x} + \mathbf{s} \in V$ is equivalent to this statement: There exists a set of constants $c_1, \dots, c_l$ which solves the system of linear equations:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ s_{k+1} \\ \vdots \\ s_p \end{pmatrix} = c_1 \begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \\ \vdots \\ v_p^{(1)} \end{pmatrix} + c_2 \begin{pmatrix} v_1^{(2)} \\ v_2^{(2)} \\ \vdots \\ v_p^{(2)} \end{pmatrix} + \dots + c_l \begin{pmatrix} v_1^{(l)} \\ v_2^{(l)} \\ \vdots \\ v_p^{(l)} \end{pmatrix}. \tag{2.9}$$

Now we denote $\mathbf{c} = (c_1, c_2, \dots, c_l)$, $\tilde{\mathbf{x}} = (x_1, x_2, \dots, x_k)$ and

$$\tilde{\mathbf{V}} = \begin{pmatrix} v_1^{(1)} & \dots & v_1^{(l)} \\ \vdots & & \vdots \\ v_l^{(1)} & \dots & v_l^{(l)} \end{pmatrix}.$$

Then the first k equations of the system (2.9) related to the vector space X can be written in the form

$$\tilde{\mathbf{x}} = \tilde{\mathbf{V}}.\mathbf{c}\,. \tag{2.10}$$

" $\Rightarrow$ " :

The statement "for each vector $\mathbf{x} \in X$ there exists a unique vector $\mathbf{s} \in S$ such that the vector $\mathbf{x}+\mathbf{s}$ is from the space V" means that the system (2.10) has a unique solution $\mathbf{c} \in R^l$ for each vector $\mathbf{x} \in X$. We know that the columns of the matrix $\tilde{\mathbf{V}}$ are linear independent and they have to generate the vector space R^k. In such case $k = l$ and no vector from vector space V lies in the vector space S.

" $\Leftarrow$ ":

We suppose $k = l$ and $S \cap V = \{\mathbf{0}\}$, so the columns of the matrix $\tilde{\mathbf{V}}$ are linear independent. The matrix $\tilde{\mathbf{V}}$ is then invertible. This means that for each $\mathbf{x} \in X$ there exists a unique solution $\hat{\mathbf{c}} = (\hat{c}_1, \hat{c}_2, \ \ldots\ , \hat{c}_k)$ of the system (2.10). Thus the vector $\mathbf{s} \in S$ which satisfies the condition $(\mathbf{x}+\mathbf{s}) \in V$ is unique by (2.9). ■

For further purposes we will consider the system of linear difference equations

$$\begin{pmatrix} \mathbf{x_{j+1}} \\ \mathbf{s_{j+1}} \end{pmatrix} = \mathbf{B_j} \begin{pmatrix} \mathbf{x_j} \\ \mathbf{s_j} \end{pmatrix} \tag{2.11}$$

where $j \in Z_0^+, \mathbf{x_j} \in R^k, \mathbf{s_j} \in R^{p-k}$ and the square matrix $\mathbf{B_j}$ of dimension p is invertible for each $j \in Z_0^+$. Denote by X the k-dimensional subspace in R^p generated by the vectors $\mathbf{e_1} = (1,0,\ldots,0)^T, \mathbf{e_2} = (0,1,0,\ldots,0)^T, \ \ldots,\ \mathbf{e_k} = (\underbrace{0,\ldots,0,1}_{k},0,\ldots,0)^T$ and by S the $(p-k)$ - dimensional subspace in R^p generated by the vectors $\mathbf{e_{k+1}} = (\underbrace{0,\ldots,0}_{k},1,0,\ldots,0)^T$, $\mathbf{e_{k+2}} = (\underbrace{0,\ldots,0}_{k},0,1,0,\ldots,0)^T, \ \ldots,\ \mathbf{e_p} = (0,\ldots,0,1)^T$.

Definition 3 *Let $\mathbf{C}$ be a square matrix of dimension p and $\Re(\mathbf{C})$ the range of the matrix $\mathbf{C}$. Let $\mathbf{y}$ be a vector from R^p. We define set $\Omega(\mathbf{C},\mathbf{y})$ as the set of all vectors $\mathbf{z}$ from S satisfying $\mathbf{y}+\mathbf{z} \in \Re(\mathbf{C})$.*

Remark 2.5 *Notice that in general case the set $\Omega(\mathbf{C},\mathbf{y})$ can be empty.*

Theorem 2.5 *Consider the nonhomogeneous system of linear difference equations*

$$\begin{pmatrix} \mathbf{x_{j+1}} \\ \mathbf{s_{j+1}} \end{pmatrix} = \mathbf{B_j} \begin{pmatrix} \mathbf{x_j} \\ \mathbf{s_j} \end{pmatrix} + \mathbf{h_j} \tag{2.12}$$

derived from (2.11). Assume that (2.11) admits exponential dichotomy on Z_0^+ *with projections* $\mathbf{Q_i}$ $(i \in Z_0^+)$. *For* $\mathbf{\hat{x}_0} \in X$ *denote* $\beta(\mathbf{\hat{x}_0})$ *the set of all* $\mathbf{\hat{x}_0} + \mathbf{\hat{s}_0}$, $\mathbf{\hat{s}_0} \in S$ *such that the solution of (2.12) with* $\mathbf{x_0} = \mathbf{\hat{x}_0}$, $\mathbf{s_0} = \mathbf{\hat{s}_0}$ *is bounded.*

Then

$$\beta(\mathbf{\hat{x}_0}) = \{\mathbf{\hat{x}_0} + \mathbf{z} \mid \mathbf{z} \in \Omega(\mathbf{Q_0}, \mathbf{\hat{x}_0} + \mathbf{q})\}$$

where $\mathbf{q} = \sum_{i=0}^{\infty} (\mathbf{I} - \mathbf{Q_0})\mathbf{\Phi}(\mathbf{0}, \mathbf{i+1})\mathbf{h_i}$.

Proof: We prove $\beta(\hat{x}_0) \subset \{\mathbf{\hat{x}_0} + \mathbf{z} \mid \mathbf{z} \in \Omega(\mathbf{Q_0}, \hat{x}_0 + \mathbf{q})\}$ first.
Let $\mathbf{x_0^\star} \in X, \mathbf{s_0^\star} \in S$ be such that $\mathbf{x_0^\star} + \mathbf{s_0^\star} \in \beta(\mathbf{\hat{x}_0})$. Thus $\mathbf{x_0^\star} + \mathbf{s_0^\star} = \mathbf{\hat{x}_0} + \mathbf{w}$ for some $\mathbf{w} \in S$. Denote by $\mathbf{x_j^\star} + \mathbf{s_j^\star}$ the solution of (2.12) with starting point $\mathbf{x_0^\star} + \mathbf{s_0^\star}$.

Put $\boldsymbol{\xi} = \mathbf{\hat{x}_0} + \mathbf{w} + \mathbf{q}$. Now we use theorem 2.2. It says that there exists a unique bounded solution $\mathbf{x_j^{\star\star}} + \mathbf{s_j^{\star\star}}$ ($\mathbf{x_j^{\star\star}} \in X, \mathbf{s_j^{\star\star}} \in S$) of system (2.12) which satisfies:

$$\mathbf{Q_0}(\mathbf{x_0^{\star\star}} + \mathbf{s_0^{\star\star}}) = \mathbf{Q_0}\boldsymbol{\xi}. \tag{2.13}$$

It is easy to see that

$$\mathbf{Q_0 q} = \mathbf{Q_0}\Big(\sum_{i=0}^{\infty} (\mathbf{I} - \mathbf{Q_0})\mathbf{\Phi}(\mathbf{0}, \mathbf{i+1})\mathbf{h_i}\Big) = \sum_{i=0}^{\infty} \mathbf{Q_0}(\mathbf{I} - \mathbf{Q_0})\mathbf{\Phi}(\mathbf{0}, \mathbf{i+1})\mathbf{h_i} = \mathbf{0}.$$

Now, we can write

$$\mathbf{Q_0}(\mathbf{x_0^{\star\star}} + \mathbf{s_0^{\star\star}}) = \mathbf{Q_0}\boldsymbol{\xi} = \mathbf{Q_0}(\mathbf{\hat{x}_0} + \mathbf{w} + \mathbf{q}) = \mathbf{Q_0}(\mathbf{\hat{x}_0} + \mathbf{w}) = \mathbf{Q_0}(\mathbf{x_0^\star} + \mathbf{s_0^\star}).$$

Since by theorem 2.2 the solution $(\mathbf{x_j^{\star\star}} + \mathbf{s_j^{\star\star}})$ is unique in the sense of (2.13), we obtain the identity $(\mathbf{x_j^{\star\star}} + \mathbf{s_j^{\star\star}}) = (\mathbf{x_j^\star} + \mathbf{s_j^\star})$ for all $j \in Z_0^+$. The last equality is equivalent to the equalities $\mathbf{x_j^{\star\star}} = \mathbf{x_j^\star}$, $\mathbf{s_j^{\star\star}} = \mathbf{s_j^\star}$ for all $j \in Z_0^+$.

The equality $\mathbf{x_0^\star} + \mathbf{s_0^\star} = \mathbf{Q_0}\boldsymbol{\xi} - \mathbf{q}$ follows from identity (2.3). Then

$$\mathbf{x_0^\star} + \mathbf{s_0^\star} = \mathbf{Q_0}(\mathbf{\hat{x}_0} + \mathbf{q} + \mathbf{w}) - \mathbf{q} = \mathbf{\hat{x}_0} + \mathbf{w} \quad \Rightarrow \quad \mathbf{Q_0}(\mathbf{\hat{x}_0} + \mathbf{q} + \mathbf{w}) = \mathbf{\hat{x}_0} + \mathbf{q} + \mathbf{w}$$

The latter equality says that $\mathbf{\hat{x}_0} + \mathbf{q} + \mathbf{w} \in \Re(\mathbf{Q_0})$, so in the sense of definition 3, we have $\mathbf{w} \in \Omega(\mathbf{Q_0},\, \mathbf{\hat{x}_0} + \mathbf{q})$. Thus

$$(\mathbf{x_0^\star} + \mathbf{s_0^\star}) = (\mathbf{\hat{x}_0} + \mathbf{w}) \in \{\mathbf{\hat{x}}_0 + \mathbf{z} \mid \mathbf{z} \in \Omega(\mathbf{Q_0}, \mathbf{\hat{x}}_0 + \mathbf{q})\}$$

and therefore $\beta(\mathbf{\hat{x}_0}) \subset \{\mathbf{\hat{x}}_0 + \mathbf{z} \mid \mathbf{z} \in \Omega(\mathbf{Q_0},\, \mathbf{\hat{x}}_0 + \mathbf{q})\}$.

The first part of proof is now complete.

Now we will show that $\{\mathbf{\hat{x}}_0 + \mathbf{z} \mid \mathbf{z} \in \Omega(\mathbf{Q_0},\, \mathbf{\hat{x}}_0 + \mathbf{q})\} \subset \beta(\mathbf{\hat{x}_0})$. Let $\boldsymbol{\alpha} \in \{\mathbf{\hat{x}}_0 + \mathbf{z} \mid \mathbf{z} \in \Omega(\mathbf{Q_0},\, \mathbf{\hat{x}}_0 + \mathbf{q})\}$ be chosen arbitrarily. It means that

$$\boldsymbol{\alpha} = \mathbf{\hat{x}_0} + \mathbf{r} \tag{2.14}$$

for some $\mathbf{r} \in S$. Put

$$\boldsymbol{\rho} = \boldsymbol{\alpha} + \mathbf{q}. \tag{2.15}$$

Then we can see that $\boldsymbol{\rho} \in \Re(\mathbf{Q_0})$ following from the definition of $\Omega(\mathbf{Q_0},\, \mathbf{\hat{x}_0} + \mathbf{q})$. In such case we can write

$$\mathbf{Q_0}\boldsymbol{\rho} = \boldsymbol{\rho}. \tag{2.16}$$

Now we use the theorem 2.2 which says that there exists a unique solution $(\mathbf{\breve{x}_j} + \mathbf{\breve{s}_j})$ of the system (2.12) such that $\mathbf{\breve{x}_j} \in X$, $\mathbf{\breve{s}_j} \in S$ and

$$\mathbf{Q_0}(\mathbf{\breve{x}_0} + \mathbf{\breve{s}_0}) = \mathbf{Q_0}\boldsymbol{\rho}.$$

The equality

$$\mathbf{\breve{x}_0} + \mathbf{\breve{s}_0} = \mathbf{Q_0}\boldsymbol{\rho} - \mathbf{q}$$

follows from identity (2.3). By (2.15) and (2.16) we can see

$$\mathbf{\breve{x}_0} + \mathbf{\breve{s}_0} = \mathbf{Q_0}\boldsymbol{\rho} - \mathbf{q} = \boldsymbol{\rho} - \mathbf{q} = \boldsymbol{\alpha}$$

and with respect to (2.14) we obtain $\mathbf{\breve{x}_0} = \mathbf{\hat{x}_0}$. Thus $\boldsymbol{\alpha} \in \beta(\mathbf{\hat{x}_0})$. ■

Theorem 2.6 *Let system (2.11) admit exponential dichotomy on Z_0^+ with projections $\mathbf{Q_i}$ $(i \in Z_0^+)$ and $\{\mathbf{h_i}\}_{i=0}^\infty$ be a bounded sequence of vectors from R^p. Assume the range* *of the projection (matrix)* $\mathbf{Q_0}$ *has dimension l. Let* $\mathbf{\hat{x}_0} \in X$ *be chosen arbitrarily. Then there exists a unique solution* $\mathbf{\tilde{x}_j} + \mathbf{\tilde{s}_j}$, $(\mathbf{\tilde{x}_j} \in X,\ \mathbf{\tilde{s}_j} \in S)$ *of the nonhomogeneous system*

$$\begin{pmatrix} \mathbf{x_{j+1}} \\ \mathbf{s_{j+1}} \end{pmatrix} = \mathbf{B_j} \begin{pmatrix} \mathbf{x}_j \\ \mathbf{s_j} \end{pmatrix} + \mathbf{h_j} \tag{2.17}$$

which is bounded on Z_0^+ *and satisfies* $\mathbf{\tilde{x}_0} = \mathbf{\hat{x}_0}$ *if and only if k=l and there exists no nontrivial vector* $\mathbf{s}^\star \in S$ *which for some bounded solution* $\mathbf{y_j}$ *of the system (2.11) satisfies* $\mathbf{s}^\star = \mathbf{y_0}$.

Proof: We prove "⇒" part first.

We suppose that for all $\mathbf{\hat{x}_0} \in X$ there exists a unique bounded solution $(\mathbf{\tilde{x}_j} + \mathbf{\tilde{s}_j})$, $\mathbf{\tilde{s}_j} \in S$, of system (2.17) which satisfies $\mathbf{\tilde{x}_0} = \mathbf{\hat{x}_0}$. Under this assumption by theorem 2.5 the set $\Omega(\mathbf{Q_0},\ \mathbf{\hat{x}_0} + \mathbf{q})$ contains exactly one element. Denote it by $\mathbf{\hat{s}_0}$, $\mathbf{\hat{s}_0} \in S$. In such case there cannot exist bounded solution $\mathbf{y_j}$ of (2.11) satisfying $\mathbf{y_0} = \mathbf{s}^\star$ for some $\mathbf{s}^\star \neq \mathbf{0}$, because the set $\Omega(\mathbf{Q_0},\ \mathbf{\hat{x}_0} + \mathbf{q})$ would then also contain another element $(\mathbf{\hat{s}_0} + \mathbf{y_0}) \in S$.

The set $\Omega(\mathbf{Q_0},\ \mathbf{\hat{x}_0} + \mathbf{q})$ has exactly one element for each $\mathbf{\hat{x}_0} \in X$. We know that for each $\mathbf{w} \in R^p$ there exists a unique vector $\mathbf{v} \in S$ such that $\mathbf{w} + \mathbf{v} \in \Re(\mathbf{Q_0})$. Now, the equality $k = l$ follows from theorem 2.4 which we can use in terms of $V = \Re(\mathbf{Q_0})$.

The part "⇐" of the proof:

Now we suppose that there exists no nontrivial vector $\mathbf{s}^\star$ from S such that for some bounded solution $\mathbf{y_j}$ of (2.11) the equality $\mathbf{y_0} = \mathbf{s}^\star$ holds true. Let $\mathbf{r_j}$ and $\mathbf{t_j}$ be two bounded solutions of (2.17) satisfying $\mathbf{r_0} = \mathbf{\hat{x}_0} + \mathbf{s_0^r}$ and $\mathbf{t_0} = \mathbf{\hat{x}_0} + \mathbf{s_0^t}$ for some $\mathbf{s_0^r},\ \mathbf{s_0^t} \in S$. Then $\mathbf{r_j} - \mathbf{t_j}$ is a solution of the related homogeneous system (2.11). This solution has to be trivial according the assumption and hence $\mathbf{s_0^r} = \mathbf{s_0^t}$. Then $\mathbf{r_j} = \mathbf{t_j}$ for all $j \in Z_0^+$. We showed that the set $\beta(\mathbf{\hat{x}_0})$ as defined in theorem 2.5 cannot have more than one element. The latter fact can be by theorem 2.5 written in the form: $\Re(\mathbf{Q_0}) \cap S = \{\mathbf{0}\}$.

Because we assume that $k = l$ and know that $\Re(\mathbf{Q_0}) \cap S = \{\mathbf{0}\}$, theorem 2.5 says that the set $\Omega(\mathbf{Q_0},\ \mathbf{\hat{x}_0} + \mathbf{q})$ has exactly one element for each $\mathbf{\hat{x}_0} \in X$. It means (in the sense of theorem 2.5) that the system (2.17) has a unique bounded solution $(\mathbf{\tilde{x}_j} + \mathbf{\tilde{s}_j})$, $\mathbf{\tilde{x}_j} \in X$, $\mathbf{\tilde{s}_j} \in S$, which satisfies $\mathbf{\tilde{x}_0} = \mathbf{\hat{x}_0}$ for any $\mathbf{\hat{x}_0} \in X$.

2.2 Nonlinear systems

Lemma 2.1 *Let S, V be subspaces of* R^p *such that for each* $\mathbf{x} \in R^p$ *there exists a unique vector* $\mathbf{s}(\mathbf{x}) \in S$ *which satisfy* $\mathbf{x} + \mathbf{s}(\mathbf{x}) \in V$*. Then there exists a positive constant* $L_V > 0$ *such that the estimate*

$$| \mathbf{x_0} + \mathbf{s}(\mathbf{x_0}) | \leq L_V . | \mathbf{x_0} |$$

holds true for each $\mathbf{x_0} \in X$*.*

Proof: In the case described above we can define a linear mapping $\gamma : \mathbf{x_0} \longrightarrow \mathbf{x_0} + \mathbf{s}(\mathbf{x_0})$ which assigns the unique vector $\mathbf{x_0} + \mathbf{s}(\mathbf{x_0}) \in V$ to each $\mathbf{x_0} \in R^p$. This operator can be represented as the square matrix $\mathbf{C} = \{\mathbf{c_{ij}}\}$ of dimension p. This matrix $\mathbf{C}$ maps the space R^p into the space V. We can write

$$\gamma(\mathbf{x_0}) = \mathbf{C}\mathbf{x_0} = \mathbf{x_0} + \mathbf{s}(\mathbf{x_0}).$$

Now put $L_V = | \mathbf{C}|^* + 1$, where

$$| \mathbf{C} |^* := \sup_{1 \leq i \leq p,\, 1 \leq j \leq p} \{|c_{ij}|\}.$$

Then

$$| \mathbf{x_0} + \mathbf{s}(\mathbf{x_0}) | = | \mathbf{C}\mathbf{x_0} | \leq | \mathbf{C} |^* . | \mathbf{x_0} | \leq L_V . | \mathbf{x_0} |$$

and the proof is complete. ■

Now we investigate the system of nonlinear difference equations

$$\mathbf{y_{j+1}} = G(\mathbf{y_j}) \tag{2.18}$$

where $\mathbf{y_j} = (\mathbf{x_j} + \mathbf{s_j})$[1] $\in R^p$, $j \in Z_0^+$ and the vector function $G : R^p \longrightarrow R^p$ is differentiable at a point $\theta \in R^p$. Let $\boldsymbol{\theta}$ be a fixed point of G, i.e. $G(\boldsymbol{\theta}) = \boldsymbol{\theta}$. Without loss of generality we can assume that $\boldsymbol{\theta} = \mathbf{0}$. Thus we can write

$$G(\boldsymbol{\eta}) = \mathbf{B}.\boldsymbol{\eta} + g(\boldsymbol{\eta}) = \mathbf{B}.\boldsymbol{\eta} + o(|\boldsymbol{\eta}|) \tag{2.19}$$

[1] $\mathbf{x_j}$ is a vector from X, $\mathbf{s_j}$ is a vector from S, where X and S are as defined in the previous section.

where $\mathbf{B}$ is a square matrix of dimension p and the mapping $g : R^p \longrightarrow R^p$ satisfies

$$\lim_{\boldsymbol{\eta_1},\,\boldsymbol{\eta_2}\to\mathbf{0}} \frac{|g(\boldsymbol{\eta_1}) - g(\boldsymbol{\eta_2})|}{|\boldsymbol{\eta_1} - \boldsymbol{\eta_2}|} = \mathbf{0}. \tag{2.20}$$

Under these assumptions let the system

$$\mathbf{u_{j+1}} = \mathbf{B}\mathbf{u_j} \tag{2.21}$$

have exponential dichotomy on Z_0^+ with constants K, α and a family of projections $\mathbf{Q_j}$. In such case $\mathbf{Q_j} = \mathbf{Q}$ for each $j \in Z_0^+$.

Denote by W the sum of the generalised eigenspaces of $\mathbf{B}$ corresponding to the eigenvalues of $\mathbf{B}$ inside the unit circle. Then the vector space W is the range of the matrix $\mathbf{Q}$ ($W = \Re(\mathbf{Q})$). Denote by L_W a constant for this space W in the language of lemma 2.1.

Theorem 2.7 *Let the space W be of dimension*[2] *k and $W \cap S = \{\mathbf{0}\}$. Then there exists a positive constant σ such that for all $\hat{\mathbf{x}}_\mathbf{0} \in X$, $|\hat{\mathbf{x}}_\mathbf{0}| \le \sigma$ the system (2.18) has a unique bounded solution $\tilde{\mathbf{y}}_\mathbf{j}$ satisfying $\tilde{\mathbf{y}}_\mathbf{0} = \hat{\mathbf{x}}_\mathbf{0} + \mathbf{s}$ for some $\mathbf{s} \in S$.*

Proof: Put

$$\gamma := \min\left\{ \frac{1}{6.L_W.K.|\sum_{i=0}^{\infty}(\mathbf{I}-\mathbf{Q})\boldsymbol{\Phi}(\mathbf{0},\,\mathbf{i}+\mathbf{1})|^*},\ \frac{1}{10.K.(1+e^{-\alpha}).(1-e^{-\alpha})^{-1}} \right\}. \tag{2.22}$$

Following from (2.19) there exists a positive constant δ such that for each vector $\boldsymbol{\eta} \in R^p$, $|\boldsymbol{\eta}| \le \delta$ the following inequality holds:

$$|g(\boldsymbol{\eta})| \le \gamma.|\boldsymbol{\eta}|\,. \tag{2.23}$$

[2]The space X generated by the vectors $\mathbf{e_1} = (1, 0, \ldots, 0)^T, \mathbf{e_2} = (0, 1, 0, \ldots, 0)^T, \ldots \mathbf{e_k} = (\underbrace{0, \ldots, 0, 1}_{k}, \underbrace{0, \ldots, 0}_{p-k})^T$ is also of dimension k.

Put

$$\sigma := \frac{\delta}{6.L_W.K}.$$

Let $\hat{\mathbf{x}}_\mathbf{0}$ be any vector from the space X such that

$$|\hat{\mathbf{x}}_\mathbf{0}| \le \frac{\delta}{6.L_W.K} = \sigma. \tag{2.24}$$

We will show that there exists a unique bounded solution $\tilde{\mathbf{x}}_\mathbf{j} + \tilde{\mathbf{s}}_\mathbf{j}$ ($\tilde{\mathbf{x}}_\mathbf{j} \in X$, $\tilde{\mathbf{s}}_\mathbf{j} \in S$) of (2.18) which satisfies $\tilde{\mathbf{x}}_\mathbf{0} = \hat{\mathbf{x}}_\mathbf{0}$.

Denote by $\ell^\infty(Z_0^+)$ the Banach space of R^p-valued bounded sequences $\mathbf{u} = \{\mathbf{u}_\mathbf{j}\}_{j\in Z_0^+}$ with the norm

$$\|\mathbf{u}\| := \sup_{j\in Z_0^+} |\mathbf{u}_\mathbf{j}|.$$

Then $\theta = \{\mathbf{u} \in \ell^\infty(Z_0^+) : \|\mathbf{u}\| \le \delta\}$ is a closed subset of $\ell^\infty(Z_0^+)$ and hence a complete metric space.

We define a mapping of θ into itself. Let $\mathbf{y} \in \theta$. Then the sequence of vectors $\{g(\mathbf{y}_\mathbf{i})\}_{i=0}^\infty$ is bounded and by theorem 2.6 the nonhomogeneous system

$$\mathbf{z}_{\mathbf{j+1}} = \mathbf{B}\mathbf{z}_\mathbf{j} + g(\mathbf{y}_\mathbf{j}) \tag{2.25}$$

has a unique bounded solution $\mathbf{z}_\mathbf{j} = \mathbf{x}_\mathbf{j} + \mathbf{s}_\mathbf{j}$ ($\mathbf{x}_\mathbf{j} \in X$, $\mathbf{s}_\mathbf{j} \in S$), such that $\hat{\mathbf{x}}_\mathbf{o} = \mathbf{x}_\mathbf{0}$. Now consider

$$\boldsymbol{\xi} = \hat{\mathbf{x}}_\mathbf{0} + \sum_{\mathbf{i=0}}^{\infty} (\mathbf{I} - \mathbf{Q})\boldsymbol{\Phi}(\mathbf{0},\, \mathbf{i+1})\mathbf{g}(\mathbf{y}_\mathbf{i}) + \mathbf{t},$$

where

$$\mathbf{t} \in \Omega(\mathbf{Q},\, \hat{\mathbf{x}}_\mathbf{0} + \sum_{\mathbf{i=0}}^{\infty} (\mathbf{I} - \mathbf{Q})\boldsymbol{\Phi}(\mathbf{0},\, \mathbf{i+1})\mathbf{g}(\mathbf{y}_\mathbf{i}))^3.$$

We can see that

$$\mathbf{Q}\boldsymbol{\xi} = \mathbf{Q}(\hat{\mathbf{x}}_\mathbf{0} + \sum_{\mathbf{i=0}}^{\infty} (\mathbf{I} - \mathbf{Q})\boldsymbol{\Phi}(\mathbf{0},\, \mathbf{i+1})\mathbf{g}(\mathbf{y}_\mathbf{i}) + \mathbf{t}) = \mathbf{Q}(\hat{\mathbf{x}}_0 + \mathbf{t}).$$

[3] The set $\Omega(\mathbf{Q},\, \hat{\mathbf{x}}_\mathbf{0} + \sum_{\mathbf{i=0}}^{\infty} (\mathbf{I} - \mathbf{Q})\boldsymbol{\Phi}(\mathbf{0},\, \mathbf{i+1})\mathbf{g}(\mathbf{y}_\mathbf{i}))$ contains exactly one element by theorem 2.5 because $dim\{X\} = dim\{W\}$ and $W \cap S = \{\mathbf{0}\}$.

The bounded solution $\mathbf{z_j}$ of (2.25) is the one as by theorem 2.2 satisfies $\mathbf{Q\xi} = \mathbf{Qz_0}$ because $\mathbf{z_0}$ is unique in the sense of the form $\mathbf{z_0} = \mathbf{\hat{x}_0} + \mathbf{\hat{s}_0},\ \mathbf{\hat{s}_0} \in S$. With respect to the estimate by theorem 2.2 we can write the estimate

$$|\mathbf{z_j}| \leq K|\mathbf{\hat{x}_0} + \sum_{\mathbf{i=0}}^{\infty}(\mathbf{I}-\mathbf{Q})\mathbf{\Phi}(\mathbf{0},\ \mathbf{i}+\mathbf{1})\mathbf{g}(\mathbf{y_i}) + \mathbf{t}| + K(1+e^{-\alpha})(1-e^{-\alpha})^{-1}\sup_{i\in Z_0^+}|g(\mathbf{y_i})|. \tag{2.26}$$

Then

$$|\mathbf{z_j}| \leq K|\mathbf{\hat{x}_0} + \sum_{\mathbf{i=0}}^{\infty}(\mathbf{I}-\mathbf{Q})\mathbf{\Phi}(\mathbf{0},\ \mathbf{i}+\mathbf{1})\mathbf{g}(\mathbf{y_i}) + \mathbf{t}| +$$
$$+ K(1+e^{-\alpha})(1-e^{-\alpha})^{-1}\frac{\delta}{10.K.(1+e^{-\alpha})(1-e^{-\alpha})^{-1}}$$

and finally we obtain

$$|\mathbf{z_j}| \leq \frac{\delta}{10} + K.|\mathbf{\hat{x}_0} + \sum_{\mathbf{i=0}}^{\infty}(\mathbf{I}-\mathbf{Q})\mathbf{\Phi}(\mathbf{0},\ \mathbf{i}+\mathbf{1})\mathbf{g}(\mathbf{y_i}) + \mathbf{t}|. \tag{2.27}$$

By (2.22) we know

$$|\mathbf{\hat{x}_0}| \leq \frac{\delta}{6.L_W.K}. \tag{2.28}$$

And hence with respect to (2.22) we have

$$\sum_{\mathbf{i=0}}^{\infty}(\mathbf{I}-\mathbf{Q})\mathbf{\Phi}(\mathbf{0},\ \mathbf{i}+\mathbf{1})\mathbf{g}(\mathbf{y_i}) \leq \gamma.\delta.|\sum_{\mathbf{i=0}}^{\infty}(\mathbf{I}-\mathbf{Q})\mathbf{\Phi}(\mathbf{0},\ \mathbf{i}+\mathbf{1})\mathbf{g}(\mathbf{y_i})| \leq$$

$$\leq \frac{\delta}{6.L_W.K.|\sum_{\mathbf{i=0}}^{\infty}(\mathbf{I}-\mathbf{Q})\mathbf{\Phi}(\mathbf{0},\ \mathbf{i}+\mathbf{1})\mathbf{g}(\mathbf{y_i})|}.|\sum_{\mathbf{i=0}}^{\infty}(\mathbf{I}-\mathbf{Q})\mathbf{\Phi}(\mathbf{0},\ \mathbf{i}+\mathbf{1})\mathbf{g}(\mathbf{y_i})| \leq$$

$$\leq \frac{\delta}{6.L_W.K}. \tag{2.29}$$

From (2.28) and (2.29) we obtain the inequality

$$|\mathbf{\hat{x}_0} + \sum_{\mathbf{i=0}}^{\infty}(\mathbf{I}-\mathbf{Q})\mathbf{\Phi}(\mathbf{0},\ \mathbf{i}+\mathbf{1})\mathbf{g}(\mathbf{y_i})| \leq \frac{\delta}{3.L_W.K}.$$

Thus

$$|\mathbf{\hat{x}_0} + \sum_{\mathbf{i=0}}^{\infty}(\mathbf{I}-\mathbf{Q})\mathbf{\Phi}(\mathbf{0},\ \mathbf{i}+\mathbf{1})\mathbf{g}(\mathbf{y_i}) + \mathbf{t}| \leq \frac{\delta}{3.K}. \tag{2.30}$$

From (2.27) and (2.30) we have

$$|\mathbf{z_j}| \leq \frac{\delta}{10} + \frac{\delta}{3} \leq \delta \tag{2.31}$$

for each j from Z_0^+. Then $\mathbf{z} = \{\mathbf{z_j}\}_{j \in Z_0^+} \in \theta$ and therefore the mapping $\pi : \mathbf{y} \longrightarrow \mathbf{z}$ maps θ into itself.

Moreover the mapping π is a contraction. For if $\mathbf{y_1}$, $\mathbf{y_2} \in \theta$ and $\mathbf{z_1}$, $\mathbf{z_2}$ are their images, then $\mathbf{w_j} = \mathbf{z_{1j}} - \mathbf{z_{2j}}$ is a bounded solution of the system

$$\mathbf{w_{j+1}} = \mathbf{B}\mathbf{w_j} + \mathbf{g}(\mathbf{y_{1j}}) - \mathbf{g}(\mathbf{y_{2j}}). \tag{2.32}$$

Put

$$\boldsymbol{\xi}_\mathbf{k} = \hat{\mathbf{x}}_\mathbf{0} + \sum_{\mathbf{i=0}}^{\infty} (\mathbf{I} - \mathbf{Q})\boldsymbol{\Phi}(\mathbf{0},\, \mathbf{i+1})\mathbf{g}(\mathbf{y_{ki}}) + \mathbf{t_k}$$

where $\mathbf{t_k} \in \Omega(\mathbf{Q},\, \hat{\mathbf{x}}_\mathbf{0} + \sum_{\mathbf{i=0}}^{\infty} (\mathbf{I} - \mathbf{Q})\boldsymbol{\Phi}(\mathbf{0},\, \mathbf{i+1})\mathbf{g}(\mathbf{y_i}))^4$ and $k \in \{1,\, 2\}$. Thus $\boldsymbol{\xi}_\mathbf{1}$, $\boldsymbol{\xi}_2 \in \Re(\mathbf{Q}) = W$ and by lemma 2.1

$$|\boldsymbol{\xi}_k| \leq L_W.|\boldsymbol{\xi}_k - \mathbf{t}_k| \tag{2.33}$$

for $k \in \{1,\, 2\}$. The vector $\mathbf{w_0}$ satisfies

$$\mathbf{Q}\mathbf{w_0} = \mathbf{Q}(\mathbf{z_{10}} - \mathbf{z_{20}}) = \mathbf{Q}\mathbf{z_{10}} - \mathbf{Q}\mathbf{z_{20}} = \mathbf{Q}\boldsymbol{\xi}_\mathbf{1} - \mathbf{Q}\boldsymbol{\xi}_\mathbf{2} = \mathbf{Q}(\boldsymbol{\xi}_\mathbf{1} - \boldsymbol{\xi}_\mathbf{2})$$

and hence the solution $\mathbf{w}_j$ of (2.25) is the one which is by theorem 2.2 unique in the sense of the equality $\mathbf{Q}\mathbf{w_0} = \mathbf{Q}(\boldsymbol{\xi}_\mathbf{1} - \boldsymbol{\xi}_\mathbf{2})$. Then by the inequality from theorem 2.2 we have for each j in Z_0^+:

$$\begin{aligned}
|\mathbf{w_j}| \leq & K|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| + K(1+e^{-1})(1-e^{-1})^{-1} \sup_{i \in Z_+^0} |g(\mathbf{y_{1i}}) - g(\mathbf{y_{2i}})| \\
|\mathbf{w_j}| \leq & K|\sum_{i=0}^{\infty} (\mathbf{I} - \mathbf{Q})\boldsymbol{\Phi}(\mathbf{0},\, \mathbf{i+1})\big(g(\mathbf{y_{1i}}) - g(\mathbf{y_{2i}})\big) + \mathbf{t_1} - \mathbf{t_2}| + \\
& + 2.K(1+e^{-1})(1-e^{-1})^{-1}.\gamma.\|\mathbf{y_1} - \mathbf{y_2}\|.
\end{aligned}$$

Indeed, we can write

$$|g(\mathbf{y_{1j}}) - g(\mathbf{y_{2j}})| \leq 2.\gamma|\mathbf{y_{1j}} - \mathbf{y_{2j}}|$$

[4]The set $\Omega(\mathbf{Q},\, \hat{\mathbf{x}}_\mathbf{0} + \sum_{\mathbf{i=0}}^{\infty} (\mathbf{I} - \mathbf{Q})\boldsymbol{\Phi}(\mathbf{0},\, \mathbf{i+1})\mathbf{g}(\mathbf{y_{ki}}))$ has exactly one element. For details, see note 3.

by (2.19). Then

$$|\sum_{i=0}^{\infty}(\mathbf{I}-\mathbf{Q})\mathbf{\Phi}(\mathbf{0},\,\mathbf{i}+\mathbf{1})\big(g(\mathbf{y_{1i}})-g(\mathbf{y_{2i}})\big)|\leq 2.\gamma\|\mathbf{y_1}-\mathbf{y_2}\|.|\sum_{i=0}^{\infty}(\mathbf{I}-\mathbf{Q})\mathbf{\Phi}(\mathbf{0},\,\mathbf{i}+\mathbf{1})|\leq \tag{2.34}$$

$$\leq\frac{2.\|\mathbf{y_1}-\mathbf{y_2}\|}{2.3.L_W.K.|\sum_{i=0}^{\infty}(\mathbf{I}-\mathbf{Q})\mathbf{\Phi}(\mathbf{0},\,\mathbf{i}+\mathbf{1})|}.|\sum_{i=0}^{\infty}(\mathbf{I}-\mathbf{Q})\mathbf{\Phi}(\mathbf{0},\,\mathbf{i}+\mathbf{1})|\leq \tag{2.35}$$

$$\leq\frac{\|\mathbf{y_1}-\mathbf{y_2}\|}{3.L_W.K}. \tag{2.36}$$

With respect to (2.33) we can write:

$$|\sum_{i=0}^{\infty}(\mathbf{I}-\mathbf{Q})\mathbf{\Phi}(\mathbf{0},\,\mathbf{i}+\mathbf{1})\big(g(\mathbf{y_{1i}})-g(\mathbf{y_{2i}})\big)+\mathbf{t_1}-\mathbf{t_2}|\leq\frac{\|\mathbf{y_1}-\mathbf{y_2}\|}{3.K}. \tag{2.37}$$

By (2.22):

$$\begin{aligned}&2.K(1+e^{-1})(1-e^{-1})^{-1}.\gamma\|\mathbf{y_1}-\mathbf{y_2}\|\leq\\ &\leq K.2.K(1+e^{-1})(1-e^{-1})^{-1}\frac{\|\mathbf{y_1}-\mathbf{y_2}\|}{10.K(1+e^{-1})(1-e^{-1})^{-1}}\leq\\ &\leq\frac{\|\mathbf{y_1}-\mathbf{y_2}\|}{5}.\end{aligned} \tag{2.38}$$

From (2.37) and (2.38) with respect to (2.26) we obtain for each $j\in Z_0^+$:

$$|\mathbf{w_j}|\leq K.\frac{\|\mathbf{y_1}-\mathbf{y_2}\|}{3.K}+\frac{\|\mathbf{y_1}-\mathbf{y_2}\|}{5}\leq\frac{2}{3}.\|\mathbf{y_1}-\mathbf{y_2}\|$$

The last inequality holds true for each j from Z_0^+ and therefore

$$\|\mathbf{z_1}-\mathbf{z_2}\|=\|\mathbf{w}\|=\sup_{j\in Z_0^+}|\mathbf{w_j}|\leq\frac{2}{3}.\|\mathbf{y_1}-\mathbf{y_2}\|.$$

Denote by $\tilde{\mathbf{z}}$ the unique fixed point[5] of the mapping π. Then $\tilde{\mathbf{z}}_\mathbf{j}$ is a solution of (2.25) satisfying $\|\tilde{\mathbf{z}}\|\leq\delta$. Moreover it is unique since any such solution must be in θ and a fixed point of the maping π.

We have shown that for each $\hat{\mathbf{x}}_\mathbf{0}\in X$, $|\hat{\mathbf{x}}_\mathbf{0}|\leq\sigma$ there exists a unique bounded solution $\tilde{\mathbf{x}}_\mathbf{j}+\tilde{\mathbf{s}}_\mathbf{j}$ ($\tilde{\mathbf{x}}_\mathbf{j}\in X$, $\tilde{\mathbf{s}}_\mathbf{j}\in S$) of (2.18) which satisfies $\tilde{\mathbf{x}}_\mathbf{0}=\hat{\mathbf{x}}_\mathbf{0}$. ■

[5]The existence of this unique fixed point follows from the Banach theorem.

3 Two-sectoral model of economy

In this chapter we present a very simple two-sectoral model of economy. The model may appear somewhat simplistic. However, this simplicity is due to our effort to reduce technicalities to minimum. The similar approach can be used for a wide group of problems regarding the maximization problems in macroeconomics.

3.1 Introduction of the model

We will define the problem in discrete time, in infinite time horizon. The periods are denoted by the elements of the set $\{0, 1, 2, 3, \dots\}$. Let us consider an economy consisting of two sectors and two different capitals invested in the sectors (named as "the first capital" having index 1 and "the second capital" having index 2). The production of each sector is dependent on the level of this sector's capital and the production of the other sector. These assumptions still allows a wide group of possible production functions. We adopted the Cobb-Douglas ones common in the literature on Computable General Equilibrium (CGE) models (for details, see [5]):

$$y_1(t) = d_1 k_1^{\alpha_1}(t) y_2^{\alpha_2}(t), \tag{3.1}$$

$$y_2(t) = d_2 k_2^{\beta_1}(t) y_1^{\beta_2}(t), \tag{3.2}$$

where $y_i(t)$ is the production of the i-th sector ($i \in \{1, 2\}$), $k_j(t)$ is the j-th capital in the economy in period t ($j \in \{1, 2\}$) and $d_1, d_2, \alpha_1, \alpha_2, \beta_1, \beta_2$ are positive constants. Now, we solve the equations (3.1) and (3.2) in terms of $y_1(t)$ and $y_2(t)$:

$$y_1(t) = \left(d_1 d_2^{\alpha_2}\right)^{\frac{1}{1-\alpha_2\beta_2}} k_1^{\frac{\alpha_1}{1-\alpha_2\beta_2}}(t)\, k_2^{\frac{\alpha_2\beta_1}{1-\alpha_2\beta_2}}(t), \tag{3.3}$$

$$y_2(t) = \left(d_1^{\beta_2} d_2\right)^{\frac{1}{1-\alpha_2\beta_2}} k_1^{\frac{\alpha_1\beta_2}{1-\alpha_2\beta_2}}(t)\, k_2^{\frac{\beta_1}{1-\alpha_2\beta_2}}(t). \tag{3.4}$$

Put $c_1 = \left(d_1 d_2^{\alpha_2}\right)^{\frac{1}{1-\alpha_2\beta_2}}$, $c_2 = \left(d_1^{\beta_2} d_2\right)^{\frac{1}{1-\alpha_2\beta_2}}$, $\gamma_1 = \frac{\alpha_1}{1-\alpha_2\beta_2}$, $\gamma_2 = \frac{\alpha_2\beta_1}{1-\alpha_2\beta_2}$, $\sigma_1 = \frac{\alpha_1\beta_2}{1-\alpha_2\beta_2}$ and $\sigma_2 = \frac{\beta_1}{1-\alpha_2\beta_2}$. Then (3.3) and (3.4) can be rewritten as

$$y_1(t) = c_1 k_1^{\gamma_1}(t) k_2^{\gamma_2}(t), \tag{3.5}$$

$$y_2(t) = c_2 k_1^{\sigma_1}(t) k_2^{\sigma_2}(t), \tag{3.6}$$

where c_1, c_2, γ_1, γ_2, σ_1 and σ_2 are appropriate positive constants satisfying $\sigma_1\gamma_2 \neq \sigma_2\gamma_1$ to ensure non-zero denominator $1 - \alpha_2\beta_2$. With respect to known economic theory we allow only cases where $\gamma_1 + \gamma_2 < 1$ and $\sigma_1 + \sigma_2 < 1$ hold simultaneously.

Now, we study a social planner's problem. The social planner is maximizing the sum of discounted utilities of a representative agent in the infinite time horizon,

$$\max \left\{ \sum_{t=0}^{\infty} \beta^t U\big(c\,(t)\big) \right\} \tag{3.7}$$

where β, $0 \leq \beta \leq 1$, is the so-called discount factor, c(t) is the aggregated consumption of the agent at time (period) t and $U : \langle 0, \infty) \longrightarrow \langle 0, \infty)$ is the utility function. Because our model of economy consists of only one type of agent, the solution of the considered problem is the same as solved from agent's point of view.

In economic theory the utility function has to satisfy $U(0) = 0,\ U'(0) = \infty,\ U'(c) > 0$ and $U''(c) < 0$ for all real c. To make the model computationally tractable we assume the following explicit formula of the utility function,

$$U(c) = ac - bc^2 \qquad \text{for all } c \text{ from } (\epsilon,\ K),$$

where a, b, K and ϵ are positive constants ($\epsilon > 0$, $1 \gg \epsilon$) and $K < \frac{a}{2b}$. This assumption does not pose any problems, because we will study the behavior of the model only in a small neighbourhood of the steady state and the value of consumption of this steady state will lie in the interval $(\epsilon,\ K)$.

At each time t ($t = 0, 1, 2, ...$) the representative agent chooses an efficient "rate of consumption" $s(t) > 0$, which we define as

$$s(t) = \frac{c(t)}{y_1(t) + y_2(t)} \qquad t \in \{0, 1, 2, \dots\}. \tag{3.8}$$

We assume that the total production of the economy at time t is equal to the sum of the total investments $i(t)$ and the consumption $c(t)$:

$$y_1(t) + y_2(t) = c(t) + i(t) \qquad t \in \{0, 1, 2, \dots\}. \tag{3.9}$$

From (3.8) and (3.9) we have

$$i(t) = \big(1 - s(t)\big)\big(y_1(t) + y_2(t)\big) \qquad t \in \{0, 1, 2, \dots\}. \tag{3.10}$$

The representative agent splits the total investments $i(t)$ into the investments $i_1(t)$ into the first capital and the investments $i_2(t)$ into the second capital ($i(t) = i_1(t) + i_2(t)$). We allow the representative agent to carry out this splitting by choosing a suitable value of variable $g(t)$ defined as

$$g(t) = \frac{i_1(t)}{i(t)}. \tag{3.11}$$

Then the investments into the second capital can be expressed as

$$i_2(t) = \big(1 - g(t)\big)i(t). \tag{3.12}$$

We admit any real value of $g(t)$ and any nonnegative value of $s(t)$ in order to keep the simplicity of the problem. We allow the agent to consume the existing capital this statement is surprising no more. Also, we do not distinguish the type of products produced by two sectors of economy. We consider that these sectors produce products which can be sold, the representative agent receives money for them and invests into the economy by buying new capital. The execution of these operations is free. So the capitals in economy are driven by the rules,

$$k_1(t+1) = (1 - \delta_1)k_1(t) + i_1 \qquad t \in \{0, 1, 2, \dots\}, \tag{3.13}$$

$$k_2(t+1) = (1 - \delta_2)k_2(t) + i_2 \qquad t \in \{0, 1, 2, \dots\}, \tag{3.14}$$

where δ_1 and δ_2 denote the depreciation rates of the first and the second capital respectively. With regard to (3.10), (3.11) and (3.12) the equalities (3.13) and (3.14) can be rewriten as following,

$$k_1(t+1) = (1 - \delta_1)k_1(t) + g(t)\big(1 - s(t)\big)\big(y_1(t) + y_2(t)\big) \qquad t \in \{0, 1, 2, \dots\},$$
$$k_2(t+1) = (1 - \delta_2)k_2(t) + \big(1 - g(t)\big)\big(1 - s(t)\big)\big(y_1(t) + y_2(t)\big) \qquad t \in \{0, 1, 2, \dots\}.$$

The production of the sectors is determined by (3.5) and (3.6) and thus we obtain the difference equations for both types of capital:

$$k_1(t+1) = (1 - \delta_1)k_1(t) + g(t)\big(1 - s(t)\big)\big(c_1 k_1^{\gamma_1}(t) k_2^{\gamma_2}(t) + c_2 k_1^{\sigma_1}(t) k_2^{\sigma_2}(t)\big), \tag{3.15}$$

$$k_2(t+1) = (1 - \delta_2)k_2(t) + \big(1 - g(t)\big)\big(1 - s(t)\big)\big(c_1 k_1^{\gamma_1}(t) k_2^{\gamma_2}(t) + c_2 k_1^{\sigma_1}(t) k_2^{\sigma_2}(t)\big). \tag{3.16}$$

For a convenient treatment of these equalities we define the functions f_i which determine the amount of i-th capital at time $t+1$ based on the values of all variables $k_1(t)$, $k_2(t)$, $s(t)$ and $g(t)$ one period before. Thus

$$k_1(t+1) = f_1\big(k_1(t),\, k_2(t),\, s(t),\, g(t)\big) \qquad t \in \{0,1,2,\dots\} \tag{3.17}$$

and
$$k_2(t+1) = f_2\big(k_1(t),\, k_2(t),\, s(t),\, g(t)\big) \qquad t \in \{0,1,2,\dots\}. \tag{3.18}$$

The variables $k_1(t)$ and $k_2(t)$ are called predetermined, because their values are driven by the previous state of the system. The behavior of the representative agent from any time t (including the behavior at time t) has no impact on the values of these variables at this time t.

The variables $s(t)$ and $g(t)$ are called non-predetermined, because their current values are not given by the previous states of the system. The agent can choose arbitrary values of these variables and thus control the whole system.

With respect to all assumptions stated above, the representative agent solves the following maximization problem:

$$\max_{\{s(t),\, g(t)\}_{t=0}^{\infty}} \left\{ \sum_{t=0}^{\infty} \beta^t U\Big(s(t)(c_1 k_1^{\gamma_1}(t) k_2^{\gamma_2}(t) + c_2 k_1^{\sigma_1}(t) k_2^{\sigma_2}(t)) \Big) \right\} \tag{3.19}$$

where $U(.)$ is as defined above, the difference equations (3.17) and (3.18) holds for each t from $\{0,1,2,\dots\}$ and the initial amounts of both capitals are given,

$$k_1(0) = k_1^\star, \qquad k_1^\star > 0 \tag{3.20}$$

and
$$k_2(0) = k_2^\star, \qquad k_2^\star > 0. \tag{3.21}$$

In the sequel, we will call this problem as Problem I.

3.2 Analysis of Problem I

Let us consider that for any positive real numbers $k_1^\star$ and $k_2^\star$ there exists a unique sequence $\{\hat{k}_1(t),\, \hat{k}_2(t),\, \hat{s}(t),\, \hat{g}(t)\}_{t=0}^{\infty}$ which maximizes the expression (3.19) and satisfies (3.17), (3.18), (3.20) and (3.21). Assume that the maximized value of expression (3.19) is finite for each positive $k_1^\star$ and $k_2^\star$. Then we can define a mapping $V : R^+ \times R^+ \longrightarrow R$,

$$V(k_1^\star,\, k_2^\star) = \max_{\{s(t),g(t)\}_{t=0}^{\infty}} \left\{ \sum_{t=0}^{\infty} \beta^t U\Big(s(t)(c_1 k_1^{\gamma_1}(t) k_2^{\gamma_2}(t) + c_2 k_1^{\sigma_1}(t) k_2^{\sigma_2}(t)) \Big) \right\} \tag{3.22}$$

$$= \sum_{t=0}^{\infty} \beta^t U\Big(\hat{s}(t)(c_1 \hat{k}_1^{\gamma_1}(t) \hat{k}_2^{\gamma_2}(t) + c_2 \hat{k}_1^{\sigma_1}(t) \hat{k}_2^{\sigma_2}(t)) \Big) \tag{3.23}$$

where the sequence $\{\hat{k}_1(t),\, \hat{k}_2(t),\, \hat{s}(t),\, \hat{g}(t)\}_{t=0}^{\infty}$ satisfies (3.17), (3.18), $\hat{k}_1(0) = k_1^\star$ and $\hat{k}_2(0) = k_2^\star$. To keep the development of the dynamic equations of the model intuitive we assume that V is continuously differentiable. We note that using optimal control theory this assumption can be dropped, cf. e.g. [2].

The function V can be interpretated easily. Indeed, the total utility of the representative agent endowed with the amounts $k_1^\star$ and $k_2^\star$ of capitals at time 0 is expressed as $V(k_1^\star,\, k_2^\star)$. Now, we notice one very important property of the mapping V,

$$V\big(\hat{k}_1(0),\, \hat{k}_2(0)\big) = \max_{\{s(t),g(t)\}_{t=0}^{\infty}} \left\{ \sum_{t=0}^{\infty} \beta^t U(c(t)) \right\} = \max_{\{s(t),g(t)\}_{t=0}^{\infty}} \left\{ U\big(c(0)\big) + \sum_{t=1}^{\infty} \beta^t U\big(c(t)\big) \right\} =$$
$$= U\big(\hat{c}(0)\big) + \beta V\big(\hat{k}_1(1), \hat{k}_2(1)\big)$$

where $\hat{c}(0) = \hat{s}(0)\big(c_1 \hat{k}_1^{\gamma_1}(0) \hat{k}_2^{\gamma_2}(0) + c_2 \hat{k}_1^{\sigma_1}(0) \hat{k}_2^{\sigma_2}(0)\big)$. This chain rule holds at each time t, so we have

$$V\big(\hat{k}_1(t),\, \hat{k}_2(t)\big) = U\big(\hat{c}(t)\big) + \beta V\big(\hat{k}_1(t+1), \hat{k}_2(t+1)\big) =$$
$$= U\big(\hat{c}(t)\big) + \beta V\Big(f_1\big(\hat{k}_1(t),\, \hat{k}_2(t),\, \hat{s}(t),\, \hat{g}(t)\big), f_2\big(\hat{k}_1(t),\, \hat{k}_2(t),\, \hat{s}(t),\, \hat{g}(t)\big) \Big) \tag{3.24}$$

for all t from $\{0,\, 1,\, 2,\, \dots\}$, where $\hat{c}(t) = \hat{s}(t)\big(c_1 \hat{k}_1^{\gamma_1}(t) \hat{k}_2^{\gamma_2}(t) + c_2 \hat{k}_1^{\sigma_1}(t) \hat{k}_2^{\sigma_2}(t)\big)$. The derivation of this chain rule may seem rather artificial, but the rule represents an intuitive fact. The total utility of the representative agent endowed with capital $\hat{k}_1(t)$ and $\hat{k}_2(t)$ in period t equals to the sum of agent's utility generated by his optimal consumption $\hat{c}(t)$ during the period t and discounted total utility at time $t+1$ of the agent endowed with capital $f_1\big(\hat{k}_1(t),\, \hat{k}_2(t),\, \hat{s}(t),\, \hat{g}(t)\big)$ and $f_2\big(\hat{k}_1(t),\, \hat{k}_2(t),\, \hat{s}(t),\, \hat{g}(t)\big)$ coming from the amounts of the capitals and optimal values of the non-predetermined variables at time t.

Once the sequence $\{\hat{k}_1(t),\, \hat{k}_2(t),\, \hat{s}(t),\, \hat{g}(t)\}_{t=0}^{\infty}$ and the mapping V are given, we define the variables $\psi_i(t)$ for $i \in \{1,\, 2\}$ as

$$\psi_i(t) = \frac{\partial V(k_1, k_2)}{\partial k_i}|_{(\hat{k}_1(t),\, \hat{k}_2(t))} \qquad i \in \{1,\, 2\}. \tag{3.25}$$

These variables are usually called "shadow prices" of capital. The variable $\psi_i(t)$ measures the ratio of the marginal change of total utility to its cause, the marginal change of the i-th capital. So the value $\psi_i(t)$ always answers the question: "What is the maximum unit price of the i-th capital, which would be the agent willing to pay for an infinite small amount of the i-th capital in the t-th period?"

3.3 Derivation of necessary conditions

Our aim in this section is twofold, to derive the so-called first order conditions (FOCs) for the solution of Problem I and to bring their intuitive interpretation in the most convenient way.

Firstly, we will deal with the identity (3.24) skipping the hat notation. We derive both sides of this equality by the variable $k_1(t)$ and thus we obtain,

$$\begin{aligned}\frac{\partial V\big(k_1(t),\, k_2(t)\big)}{\partial k_1(t)} =& U'\big(c(t)\big).\frac{\partial c(t)}{\partial k_1(t)} + \beta.\frac{\partial V\big(f_1(t+1),\, f_2(t+1)\big)}{\partial k_1(t+1)}.\frac{\partial f_1(t+1)}{\partial k_1(t)} + \\ &+ \beta.\frac{\partial V\big(f_1(t+1),\, f_2(t+1)\big)}{\partial k_2(t+1)}.\frac{\partial f_2(t+1)}{\partial k_1(t)}\end{aligned}$$

where $f_i(t+1)$ is a short notation of $f_i\big(k_1(t+1),\, k_2(t+1),\, s(t+1),\, g(t+1)\big)$. The latter equality can be with respect to (3.25) written in the form

$$\psi_1(t) = U'\big(c(t)\big).\frac{\partial c(t)}{\partial k_1(t)} + \beta.\left(\psi_1(t+1)\frac{\partial f_1(t+1)}{\partial k_1(t)} + \psi_2(t+1)\frac{\partial f_2(t+1)}{\partial k_1(t)}\right). \tag{3.26}$$

This identity reflects the fact that the shadow price of the marginal amount Δ of the first capital at time t is equal to the marginal utility generated by this capital at time t and the discounted values of both capitals generated at time $t+1$ by the marginal amount Δ of the first capital. Similar equation can be derived from (3.24) by deriving both sides by $k_2(t)$,

$$\psi_2(t) = U'\big(c(t)\big).\frac{\partial c(t)}{\partial k_2(t)} + \beta.\left(\psi_1(t+1)\frac{\partial f_1(t+1)}{\partial k_2(t)} + \psi_2(t+1)\frac{\partial f_2(t+1)}{\partial k_2(t)}\right). \tag{3.27}$$

The representative agent chooses the optimal values of the variables $s(t)$ and $g(t)$ at each period t. He has to set these variables to such levels, where no possible marginal change of them causes an increase of his maximized aggregated utility. Thus by using this rule to the value of $s(t)$ we obtain

$$0 = U'\big(c(t)\big).\frac{\partial c(t)}{\partial s(t)} + \beta\psi_1(t+1)\frac{\partial f_1(t+1)}{\partial s(t)} + \beta\psi_2(t+1)\frac{\partial f_2(t+1)}{\partial s(t)}. \tag{3.28}$$

This means that the change of utility generated by the change of consumption caused by marginal change $\Delta s(t)$ of the variable $s(t)$ and the discounted values of marginal changes of capital expressed in shadow prices caused by the same $\Delta s(t)$ is equal to zero. Now, we use the same rule for $g(t)$,

$$0 = \beta\psi_1(t+1)\frac{\partial f_1(t+1)}{\partial g(t)} + \beta\psi_2(t+1)\frac{\partial f_2(t+1)}{\partial g(t)}$$

from which by (3.17) and (3.18) we can see that,

$$\psi_1(t+1) = \psi_2(t+1) \qquad \text{for all } t \text{ from } \{0,\, 1,\, 2,\, \ldots\ldots\}. \tag{3.29}$$

This identity says that the representative agent will always split the total investments $i(t)$ in the way which will guarantee the same shadow prices of both capitals in the next period. In fact, the agent requires the same marginal production at each time, excluding the time 0, because at this time the initial amounts of capital are given exogenously.
We denote $\psi_1(t+1) = \psi_2(t+1)$ by $\psi(t+1)$ for all $t \in \{0,\, 1,\, 2,\, \ldots\}$. Then (3.28) can be rewritten as follows:

$$0 = U'\big(c(t)\big).\frac{\partial c(t)}{\partial s(t)} + \beta\psi(t+1)\left(\frac{\partial f_1(t+1)}{\partial s(t)} + \frac{\partial f_2(t+1)}{\partial s(t)}\right). \tag{3.30}$$

And hence we can easy evaluate $\psi(t+1)$ as

$$\psi(t+1) = -\frac{U'\big(c(t)\big).\frac{\partial c(t)}{\partial s(t)}}{\beta\left(\frac{\partial f_1(t+1)}{\partial s(t)} + \frac{\partial f_2(t+1)}{\partial s(t)}\right)} \qquad \text{for all } t \in \{0,\, 1,\, 2,\, \ldots\ldots\}. \tag{3.31}$$

With regard to (3.31), for each $t \in \{0,\, 1,\, 2,\, \ldots\}$ from (3.26) we finally obtain:

$$-\frac{U'\big(c(t)\big).\frac{\partial c(t)}{\partial s(t)}}{\beta\left(\frac{\partial f_1(t+1)}{\partial s(t)} + \frac{\partial f_2(t+1)}{\partial s(t)}\right)} = U'\big(c(t+1)\big).\frac{\partial c(t+1)}{\partial k_1(t+1)} -$$

$$-\frac{U'\big(c(t+1)\big).\frac{\partial c(t+1)}{\partial s(t+1)}}{\left(\frac{\partial f_1(t+2)}{\partial s(t+1)}+\frac{\partial f_2(t+2)}{\partial s(t+1)}\right)}\left(\frac{\partial f_1(t+2)}{\partial k_1(t+1)}+\frac{\partial f_2(t+2)}{\partial k_1(t+1)}\right). \tag{3.32}$$

Similarly from (3.27) with regard to (3.31) we can write for each $t \in \{0, 1, 2, \dots\}$,

$$-\frac{U'\big(c(t)\big).\frac{\partial c(t)}{\partial s(t)}}{\beta\left(\frac{\partial f_1(t+1)}{\partial s(t)}+\frac{\partial f_2(t+1)}{\partial s(t)}\right)} = U'\big(c(t+1)\big).\frac{\partial c(t+1)}{\partial k_2(t+1)}-$$

$$-\frac{U'\big(c(t+1)\big).\frac{\partial c(t+1)}{\partial s(t+1)}}{\left(\frac{\partial f_1(t+2)}{\partial s(t+1)}+\frac{\partial f_2(t+2)}{\partial s(t+1)}\right)}\left(\frac{\partial f_1(t+2)}{\partial k_2(t+1)}+\frac{\partial f_2(t+2)}{\partial k_2(t+1)}\right). \tag{3.33}$$

Now, we apply the definitions of $U(.)$, $c(.)$, $f_1(.)$, $f_2(.)$, (3.17) and (3.18) and evaluate all partial derivatives which occur in these conditions to obtain nonlinear FOCs which can be used to determine the set of possible solutions of Problem I. Thus we obtain the first FOC in the form,

$$\frac{a}{\beta}-\frac{2bs}{\beta}(y_1+y_2)=$$

$$=(a-2bs(1))(y_1(1)+y_2(1))\big(1-\delta_1+c_1\gamma_1k_1^{\gamma_1-1}(1)k_2^{\gamma_2}(1)+c_2\sigma_1k_1^{\sigma_1-1}(1)k_2^{\sigma_2}(1)\big) \tag{3.34}$$

where we use the notation "x" for $x(t)$ and "$x(n)$" for $x(t+n)$. The variables $y_1(.)$ and $y_2(.)$ express the production as defined by (3.5) and (3.6). FOC (3.34) should hold true for each t from $\{0, 1, 2, ...\}$. In the same way another FOC appears as

$$\frac{a}{\beta}-\frac{2bs}{\beta}(y_1+y_2)=$$

$$=(a-2bs(1))(y_1(1)+y_2(1))\big(1-\delta_2+c_1\gamma_1k_2^{\gamma_1}(1)k_2^{\gamma_2-1}(1)+c_2\sigma_2k_1^{\sigma_1}(1)k_2^{\sigma_2-1}(1)\big). \tag{3.35}$$

Collecting all conditions which the solution of Problem I has to fulfil, we obtain

$$k_1(1)=(1-\delta_1)k_1+g(1-s)(c_1k_1^{\gamma_1}k_2^{\gamma_2}+c_2k_1^{\sigma_1}k_2^{\sigma_2}), \tag{3.36}$$

$$k_2(1)=(1-\delta_2)k_2+(1-g)(1-s)(c_1k_1^{\gamma_1}k_2^{\gamma_2}+c_2k_1^{\sigma_1}k_2^{\sigma_2}), \tag{3.37}$$

$$\frac{a}{\beta}-\frac{2bs}{\beta}(y_1+y_2)=$$

$$=(a-2bs(1))(y_1(1)+y_2(1))\big(1-\delta_1+c_1\gamma_1k_1^{\gamma_1-1}(1)k_2^{\gamma_2}(1)+c_2\sigma_1k_1^{\sigma_1-1}(1)k_2^{\sigma_2}(1)\big), \tag{3.38}$$

$$\frac{a}{\beta}-\frac{2bs}{\beta}(y_1+y_2)=$$

$$= (a - 2bs(1))(y_1(1) + y_2(1))\big(1 - \delta_2 + c_1\gamma_2 k_1^{\gamma_1}(1)k_2^{\gamma_2 - 1}(1) + c_2\sigma_2 k_1^{\sigma_1}(1)k_2^{\sigma_2 - 1}(1)\big), \quad (3.39)$$

$$k_1(0) = k_1^\star \text{ and } k_2(0) = k_2^\star, \quad (3.40)$$

for all t from $\{0, 1, 2, \dots\}$. Comparing (3.38) and (3.39) we can write

$$0 = c_1 k_1^{\gamma_1 - 1}(1)k_2^{\gamma_2 - 1}(1)\big(\gamma_2 k_1(1) - \gamma_1 k_2(1)\big) + c_2 k_1^{\sigma_1 - 1}(1)k_2^{\sigma_2 - 1}(1)\big(\sigma_2 k_1(1) - \sigma_1 k_2(1)\big) + \delta_1 - \delta_2. \quad (3.41)$$

Substituting the left-hand sides of (3.36) and (3.37) for $k_1(1)$ and $k_2(1)$ respectively we obtain an identity the variables $k_1(t)$, $k_2(t)$, $s(t)$ and $g(t)$ have to satisfy for all $t \in \{0, 1, \dots\}$. By the implicit function theorem (for the verification of the hypotheses of which we refer to Section 3.5) $g(t)$ can be expressed (3.41) as a smooth function of the current values of the remaining three variables, $g(t) = \big(k_1(t),\, k_2(t),\, s(t)\big)$. Therefore, we can reduce the system (3.36), (3.37), (3.38) and (3.39) by eliminating the variable $g(t)$ based on the implicit function $h(.)$ and further deal with the three dimensional nonlinear difference system of the variables $k_1(t)$, $k_2(t)$ and $s(t)$:

$$\begin{aligned}
&k_1(1) = (1 - \delta_1)k_1 + h(k_1,\, k_2,\, s)(1 - s)(c_1 k_1^{\gamma_1} k_2^{\gamma_2} + c_2 k_1^{\sigma_1} k_2^{\sigma_2})\\
&k_2(1) = (1 - \delta_2)k_2 + \big(1 - h(k_1,\, k_2,\, s)\big)(1 - s)(c_1 k_1^{\gamma_1} k_2^{\gamma_2} + c_2 k_1^{\sigma_1} k_2^{\sigma_2})\\
&\tfrac{a}{\beta} - \tfrac{2bs}{\beta}(y_1 + y_2) =\\
&= (a - 2bs(1))(y_1(1) + y_2(1))\big(1 - \delta_1 + c_1\gamma_1 k_1^{\gamma_1 - 1}(1)k_2^{\gamma_2}(1) + c_2\sigma_1 k_1^{\sigma_1 - 1}(1)k_2^{\sigma_2}(1)\big).
\end{aligned} \quad (3.42)$$

3.4 Steady state

By definition, the steady state is such state $(\hat{k}_1,\, \hat{k}_2,\, \hat{s},\, \hat{g})$ of the considered economy which is as solution of Problem I constant over time. In this section we will show the algorithm which we used to locate the steady state of the economy related to Problem I. Mathematica program to carry out this computation is made available in Appendix. The steady state satisfies the equalities $\hat{k}_1 = k_1^\star$, $\hat{k}_2 = k_2^\star$, (3.36), (3.37), (3.38) and (3.39) at each time,

$$\hat{k}_1 = (1 - \delta_1)\hat{k}_1 + \hat{g}(1 - \hat{s})(c_1 \hat{k}_1^{\gamma_1} \hat{k}_2^{\gamma_2} + c_2 \hat{k}_1^{\sigma_1} \hat{k}_2^{\sigma_2}), \quad (3.43)$$

$$\hat{k}_2 = (1 - \delta_2)\hat{k}_2 + (1 - \hat{g})(1 - \hat{s})(c_1 \hat{k}_1^{\gamma_1} \hat{k}_2^{\gamma_2} + c_2 \hat{k}_1^{\sigma_1} \hat{k}_2^{\sigma_2}), \quad (3.44)$$

$$\frac{a}{\beta} - \frac{2b\hat{s}}{\beta}(\hat{y}_1 + \hat{y}_2) = (a - 2b\hat{s})(\hat{y}_1 + \hat{y}_2)\big(1 - \delta_1 + c_1\gamma_1 \hat{k}_1^{\gamma_1 - 1}\hat{k}_2^{\gamma_2} + c_2\sigma_1 \hat{k}_1^{\sigma_1 - 1}\hat{k}_2^{\sigma_2}\big), \quad (3.45)$$

$$\frac{a}{\beta} - \frac{2b\hat{s}}{\beta}(\hat{y}_1 + \hat{y}_2) = (a - 2b\hat{s})(\hat{y}_1 + \hat{y}_2)\big(1 - \delta_2 + c_1\gamma_2\hat{k}_1^{\gamma_1}\hat{k}_2^{\gamma_2 - 1} + c_2\sigma_2\hat{k}_1^{\sigma_1}\hat{k}_2^{\sigma_2 - 1}\big) \tag{3.46}$$

where $\hat{y}_1 = c_1\hat{k}_1^{\gamma_1}\hat{k}_2^{\gamma_2}$ and $\hat{y}_2 = c_2\hat{k}_1^{\sigma_1}\hat{k}_2^{\sigma_2}$.

We will proceed as following. First, we choose the values $\hat{k}_1$, $\hat{k}_2$, $\hat{s}$, $\hat{g}$, γ_1, γ_2, σ_1 and σ_2 satisfying $\hat{k}_1 > 0$, $k_2 > 0$, $\hat{s} > 0$, $\gamma_1 + \gamma_2 < 1$ and $\sigma_1 + \sigma_2 < 1$ arbitrarily. Then we will find appropriate values of c_1, c_2, δ_1 and δ_2 for which, when used as parameters of the economy with the chosen γ_1, γ_2, σ_1 and σ_2, the chosen state $(\hat{k}_1, \hat{k}_2, \hat{s}, \hat{g})$ is steady. To solve this problem, we just need to solve the system of linear equations which comes from (3.43), (3.44), (3.45) and (3.46). For details, see the Mathematica code in Appendix.

3.5 Linearized difference system

Having computed the steady state of the system we now apply the theory of Chapter 2 to the dynamics of the system (3.42) in its neighborhood. To this end we first compute its linearization. Since a direct computation of the lienarization appears to be cumersome we compute it by linearizing the steps leading to (3.42). This technique is presented in the next section. We express the behavior around the steady state in terms of relative deviations $\{\Delta k_1(t), \Delta k_2(t), \Delta s(t), \Delta g(t)\}_{t=0}^{\infty}$ from the steady state,

$$\begin{aligned} k_1(t) &= \big(1 + \Delta k_1(t)\big)\hat{k}_1 + \text{h.o.t.}, \\ k_2(t) &= \big(1 + \Delta k_2(t)\big)\hat{k}_2 + \text{h.o.t.}, \\ s(t) &= \big(1 + \Delta s(t)\big)\hat{s} + \text{h.o.t.}, \\ g(t) &= \big(1 + \Delta g(t)\big)\hat{g} + \text{h.o.t.}, \end{aligned}$$

(h.o.t. standing for higher order terms).

The linearization of equations (3.43), (3.44), (3.45) and (3.43) will take the form of a 4-dimensional linear difference system,

$$M.\begin{pmatrix} \Delta k_1(t+1) \\ \Delta k_2(t+1) \\ \Delta s(t+1) \\ \Delta g(t+1) \end{pmatrix} = N.\begin{pmatrix} \Delta k_1(t) \\ \Delta k_2(t) \\ \Delta s(t) \\ \Delta g(t) \end{pmatrix}, \qquad t \in \{0, 1, 2, \dots\} \tag{3.47}$$

where M, N are square matrices with elements $\{m_{ij}\}$ and $\{n_{ij}\}$ respectively.

The elements of the matrix M are:
$\mathbf{m_{11}} = \hat{k}_1$, $\mathbf{m_{12}} = 0$, $\mathbf{m_{13}} = 0$, $\mathbf{m_{14}} = 0$, $\mathbf{m_{21}} = 0$, $\mathbf{m_{22}} = \hat{k}_2$, $\mathbf{m_{23}} = 0$, $\mathbf{m_{24}} = 0$, $\mathbf{m_{31}} = 2b\hat{s}\big(1 - \delta_1 + c_1\gamma_1\hat{k}_1^{\gamma_1-1}\hat{k}_2^{\gamma_2} + c_2\sigma_1\hat{k}_1^{\sigma_1-1}\hat{k}_2^{\sigma_2}\big)(\gamma_1\hat{y}_1 + \sigma_1\hat{y}_2) - \big(a - 2b\hat{s}(\hat{y}_1 + \hat{y}_2)\big)\big(c_1\gamma_1(\gamma_1 - 1)\hat{k}_1^{\gamma_1-1}\hat{k}_2^{\gamma_2} + c_2\sigma_1(\sigma_1 - 1)\hat{k}_1^{\sigma_1-1}\hat{k}_2^{\sigma_2}\big)$, $\mathbf{m_{32}} = 2b\hat{s}\big(1 - \delta_1 + c_1\gamma_1\hat{k}_1^{\gamma_1-1}\hat{k}_2^{\gamma_2} + c_2\sigma_1\hat{k}_1^{\sigma_1-1}\hat{k}_2^{\sigma_2}\big)(\gamma_2\hat{y}_1 + \sigma_2\hat{y}_2) - \big(a - 2b\hat{s}(\hat{y}_1 + \hat{y}_2)\big)\big(c_1\gamma_1\hat{k}_1^{\gamma_1-1}\gamma_2\hat{k}_2^{\gamma_2} + c_2\sigma_1\hat{k}_1^{\sigma_1-1}\sigma_2\hat{k}_2^{\sigma_2}\big)$, $\mathbf{m_{33}} = 2b\hat{s}(\hat{y}_1 + \hat{y}_2)\big(1 - \delta_1 + c_1\gamma_1\hat{k}_1^{\gamma_1-1}\hat{k}_2^{\gamma_2} + c_2\sigma_1\hat{k}_1^{\sigma_1-1}\hat{k}_2^{\sigma_2}\big)$, $\mathbf{m_{34}} = 0$, $\mathbf{m_{41}} = 2b\hat{s}\big(1 - \delta_2 + c_1\gamma_2\hat{k}_1^{\gamma_1}\hat{k}_2^{\gamma_2-1} + c_2\sigma_2\hat{k}_1^{\sigma_1}\hat{k}_2^{\sigma_2-1}\big)(\gamma_1\hat{y}_1 + \sigma_1\hat{y}_2) - \big(a - 2b\hat{s}(\hat{y}_1 + \hat{y}_2)\big)(c_1\gamma_1\gamma_2\hat{k}_1^{\gamma_1}\hat{k}_2^{\gamma_2-1} + c_2\sigma_1\sigma_2\hat{k}_1^{\sigma_1}\hat{k}_2^{\sigma_2-1}$, $\mathbf{m_{42}} = 2b\hat{s}\big(1 - \delta_2 + c_1\gamma_2\hat{k}_1^{\gamma_1}\hat{k}_2^{\gamma_2-1} + c_2\sigma_2\hat{k}_1^{\sigma_1}\hat{k}_2^{\sigma_2-1}\big)(\gamma_2\hat{y}_1 + \sigma_2\hat{y}_2) - \big(a - 2b\hat{s}(\hat{y}_1 + \hat{y}_2)\big)\big(c_1\gamma_2(\gamma_2 - 1)\hat{k}_1^{\gamma_1}\hat{k}_2^{\gamma_2-1} + c_2\sigma_2(\sigma_2 - 1)\hat{k}_1^{\sigma_1}\hat{k}_2^{\sigma_2-1}$, $\mathbf{m_{43}} = 2b\hat{s}(\hat{y}_1 + \hat{y}_2)\big(1 - \delta_2 + c_1\gamma_2\hat{k}_1^{\gamma_1}\hat{k}_2^{\gamma_2-1} + c_2\sigma_2\hat{k}_1^{\sigma_1}\hat{k}_2^{\sigma_2-1}\big)$, $\mathbf{m_{44}} = 0$.

The elements of N are:
$\mathbf{n_{11}} = (1-\delta_1)\hat{k}_1 + \hat{g}(1-\hat{s})(c_1\gamma_1\hat{k}_1^{\gamma_1}\hat{k}_1^{\gamma_2} + c_2\sigma_1\hat{k}_1^{\sigma_1}\hat{k}_2^{\sigma_2})$, $\mathbf{n_{12}} = \hat{g}(1-\hat{s})(c_1\gamma_2\hat{k}_1^{\gamma_1}\hat{k}_2^{\gamma_2} + c_2\sigma_2\hat{k}_1^{\sigma_1}\hat{k}_2^{\sigma_2})$, $\mathbf{n_{13}} = -\hat{g}(\hat{y}_1 + \hat{y}_2)\hat{s}$, $\mathbf{n_{14}} = (1 - \hat{s})(\hat{y}_1 + \hat{y}_2)\hat{g}$, $\mathbf{n_{21}} = (1 - \hat{g})(1 - \hat{s})(c_1\gamma_1\hat{k}_1^{\gamma_1}\hat{k}_2^{\gamma_2} + c_2\sigma_1\hat{k}_1^{\sigma_1}\hat{k}_2^{\sigma_2})$, $\mathbf{n_{22}} = (1 - \delta_2)\hat{k}_2 + (1 - \hat{g})(1 - \hat{s})(c_1\gamma_2\hat{k}_1^{\gamma_1}\hat{k}_2^{\gamma_2} + c_2\sigma_2\hat{k}_1^{\sigma_1}\hat{k}_2^{\sigma_2})$, $\mathbf{n_{23}} = (\hat{g} - 1)(\hat{y}_1 + \hat{y}_2)\hat{s}$, $\mathbf{n_{24}} = \hat{g}(\hat{s} - 1)(\hat{y}_1 + \hat{y}_2)$, $\mathbf{n_{41}} = 2b\hat{s}(\gamma_1\hat{y}_1 + \sigma_1\hat{y}_2)/\beta$, $\mathbf{n_{42}} = 2b\hat{s}(\gamma_2\hat{y}_1 + \sigma_2\hat{y}_2)/\beta$, $\mathbf{n_{43}} = 2b\hat{s}(\hat{y}_1 + \hat{y}_2)/\beta$, $\mathbf{n_{44}} = 0$.

As expected from Section 3.3 the fourth column of the matrix M is zero and we can eliminate $\Delta g(t)$ and $\Delta g(t + 1)$. To this end we carry out equivalent operations on both sides of (3.47) to obtain a zero row in the matrix on the left side. Then, the corresponding equation of the system after the transformation reads

$$a_1\Delta k_1(t) + a_2\Delta k_2(t) + a_3\Delta s(t) + a_4\Delta g(t) = 0. \tag{3.48}$$

This equality is the linearization of (3.41) and hence the linearization of the implicit function $h(.)$ is:

$$a_1\Delta k_1(t) + a_2\Delta k_2(t) + a_3\Delta s(t) + a_4\Delta h\big(k_1(t),\, k_2(t),\, s(t)\big) = 0. \tag{3.49}$$

where $\Delta\big(k_1(t),\, k_2(t),\, s(t)\big)$ is the relative deviation from the steady state value $h(\hat{k}_1,\, \hat{k}_2,\, \hat{s}) = \hat{g}$. The fourth column of the matrix on the left side is still zero and hence we can successfully eliminate the deviation $\Delta g(t)$ from the system by the implicit function $h(.)$

represented by equality (3.48). We reduced the system by dropping the "zero row on both sides", i.e. the equality (3.48), to the form

$$M^{\star}\begin{pmatrix}\Delta k_1(t+1)\\ \Delta k_2(t+1)\\ \Delta s(t+1)\end{pmatrix} = N^{\star}\begin{pmatrix}\Delta k_1(t)\\ \Delta k_2(t)\\ \Delta s(t)\end{pmatrix} \qquad t\in\{0,\,1,\,2,\,\dots\}.$$

Assume that $M^{\star}$ is regular (we will verify this assumption in Section 3.6 numerically). Then, the dynamics of (3.42) is well-defined and we can write the linearization of (3.42) as

$$\begin{pmatrix}\Delta k_1(t+1)\\ \Delta k_2(t+1)\\ \Delta s(t+1)\end{pmatrix} = (M^{\star})^{-1}N^{\star}\begin{pmatrix}\Delta k_1(t)\\ \Delta k_2(t)\\ \Delta s(t)\end{pmatrix} \qquad t\in\{0,\,1,\,2,\,\dots\}. \tag{3.50}$$

To apply the theory of Chapter 2, we will investigate the eigenvalues of the matrix $(M^{\star})^{-1}N^{\star}$ of the linearized system by carrying out some numerical experiments. We present some results of them with their comments in the sequel.

Note that M^{*} is the steady-state value of the matrix the non-singularity of which is required for the application of the implicit function theorem in Section 3.3. Because of continuity, once it is regular at the steady state, it remains so in it sufficiently small neighborhood.

3.6 Numerical solution

We will study the way, how the representative agent chooses such values of non-predetermined variables ensuring convergence of the solution to the steady state, provided the predetermined variables are sufficiently near their steady state values.

We set the parameters $a = 10$, $b = 1$, $\beta = 0.95$, $\gamma_1 = 0.25$, $\gamma_2 = 0.55$, $\sigma_1 = 0.45$, $\sigma_2 = 0.35$ and the steady state $\hat{k}_1 = 1.1$, $\hat{k}_1 = 1.2$, $\hat{s} = 0.7$, $\hat{g} = 0.5$ and then we compute the missing parametres $c_1 = 0.0650844$, $c_2 = 0.151376$, $\delta_1 = 0.0330144$, $\delta_2 = 0.0302632$ of the economy according the mentioned algorithm. Hence, the matrices M and N are:

$$M = \begin{pmatrix} 1.1 & 0 & 0 & 0 \\ 0 & 1.2 & 0 & 0 \\ 0.626 & -0.175 & 0.357 & 0 \\ -0.156 & 0.602 & 0.357 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 1.078 & 0.015 & -0.085 & -0.036 \\ 0.014 & 1.179 & -0.085 & -0.036 \\ 0.139 & 0.147 & 0.357 & 0 \\ 0.139 & 0.147 & 0.357 & 0 \end{pmatrix}.$$

And finally, we set the initial amounts of capital to values $k_1^\star = (1 + \Delta k_1(0))\hat{k}_1$ and $k_2^\star = (1 + \Delta k_2(0))\hat{k}_2$, where

$$\Delta k_1(0) = 0.01 \qquad \text{and} \qquad \Delta k_2(0) = 0. \tag{3.51}$$

The eigenvalues λ_i and the related eigenvectors $\mathbf{v_i}$ of the matrix $(M^\star)^{-1}N^\star$ are,

$$\lambda_1 = 1.2314 \quad \mathbf{v_1} = \begin{pmatrix} -0.2829 \\ -0.2848 \\ 0.9159 \end{pmatrix}, \qquad \lambda_2 = 0.8548 \quad \mathbf{v_2} = \begin{pmatrix} 0.4249 \\ 0.4258 \\ 0.7999 \end{pmatrix}$$

$$\text{and} \qquad \lambda_3 = 3.6 \times 10^{-17} \quad \mathbf{v_3} = \begin{pmatrix} 0.7371 \\ -0.6757 \\ -0.0092 \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} \Delta k_1(t) \\ \Delta k_2(t) \\ \Delta s(t) \end{pmatrix} = l_1 \lambda_1^t \mathbf{v_1} + l_2 \lambda_2^t \mathbf{v_2} + l_3 \lambda_3^t \mathbf{v_3},$$

where l_1, l_2 and l_3 are appropriate real numbers. The stable solutions of (3.47) are those, where $l_1 = 0$.

Now, we apply theorem 2.6 to system (3.50). In the language of theorem 2.6 we have $\mathbf{B_j} = (\mathbf{M}^\star)^{-\mathbf{1}}\mathbf{N}^\star$, $\mathbf{Q_j} = \mathbf{Q}$ and $\mathbf{h_j} = \mathbf{0}$ for all nonnegative j. The range of the matrix $\mathbf{Q}$ is the sum of the generalized eigenspaces of $(\mathbf{M}^\star)^{-1}\mathbf{N}^\star$ corresponding to the eigenvalues inside the unit circle, e.g. the space generated by the vectors $\mathbf{v_2}$ and $\mathbf{v_3}$. The vectors $\mathbf{v_2}$ and $\mathbf{v_3}$ are linearly independent and hence $l = 2$. In terms of theorem 2.6 the space X is generated by the vectors $(1,\, 0,\, 0)^T$ and $(0,\, 1,\, 0)^T$ and the space S by the vector $(0,\, 0,\, 1)^T$.

The dimension of the space X is $k = 2$ and hence $k = l$. One can see that there exists no nontrivial vector $s^\star \in S$ which for some bounded solution (denoted in theorem 2.6 by $\mathbf{y_j}$) of (3.50) satisfies $\mathbf{s}^\star = \mathbf{y_0}$, because the solution $\mathbf{y_j}$ has to satisfy $\mathbf{y_0} = l_2\mathbf{v_2} + l_3\mathbf{v_3}$. Theorem 2.6 says that there exists a unique solution $\{(\Delta k_1(t),\, \Delta k_2(t),\, \Delta s(t))\}_{t=0}^{\infty}$ of (3.50) which satisfies $\Delta k_1(0) = \mu_1$ and $\Delta k_2(0) = \mu_2$ for any real μ_1 and μ_2. Under these circumstances theorem 2.7 from Section 2.2 states that there exists a unique bounded solution of the FOC's (3.36), (3.37), (3.38) and (3.39) satisfying (3.40) for $k_1^\star$ and $k_2^\star$ sufficiently close to $\hat{k}_1$ and $\hat{k}_2$ respectively.

In the case stated by (3.51), we computed $l_2 = 0.01127$ and $l_3 = 0.00710$ and hence the unique stable solution of (3.47) which satisfies (3.51) is:

$$\begin{pmatrix} \Delta k_1(t) \\ \Delta k_2(t) \\ \Delta s(t) \end{pmatrix} = l_2\lambda_2^t\mathbf{v_2} + l_3\lambda_3^t\mathbf{v_3},$$

The value of $\Delta g(t)$ can be computed from (3.48) for all t from $\{0, 1, 2, \dots\}$.
Figure 1 shows the traces of the variables $\Delta k_1(t)$ and $\Delta k_2(t)$ of the obtained solution over time. Figure 2 exhibits the traces of the variables $\Delta s(t)$ and $\Delta g(t)$ of the same solution

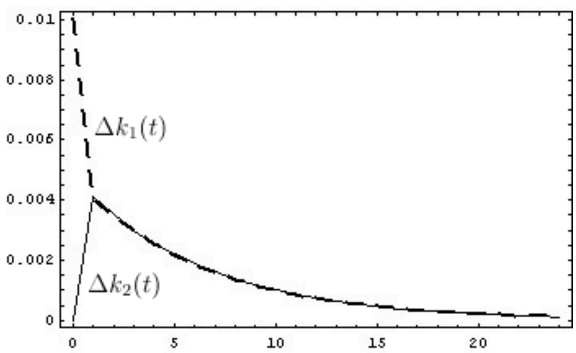

Figure 1: The dynamics of the relative deviations $\Delta k_1(t)$ and $\Delta k_2(t)$ over time.

over time.

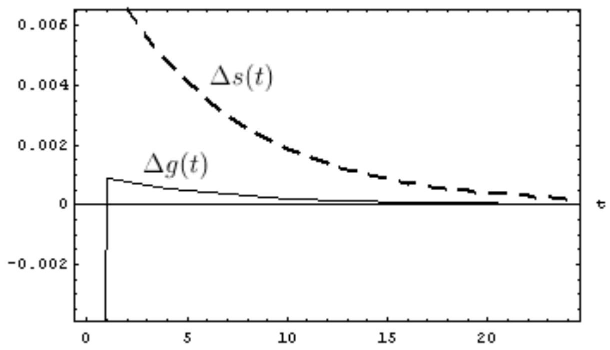

Figure 2: The dynamics of the relative deviations $\Delta s(t)$ and $\Delta g(t)$ over time.

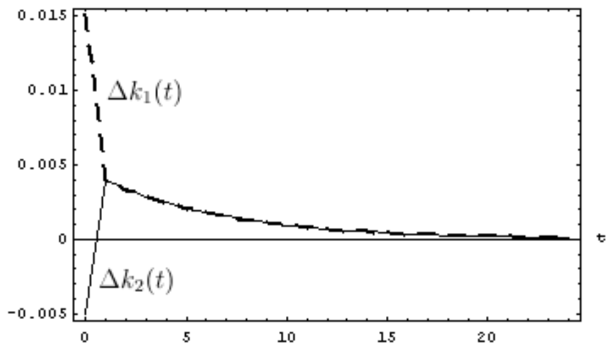

Figure 3: The dynamics of the relative deviations $\Delta k_1(t)$ and $\Delta k_2(t)$ over time.

Figures 3 and 4 show the stable solution of (3.47) satisfying $\Delta k_1(0) = 0.015$ and $\Delta k_2(0) = -0.005$.

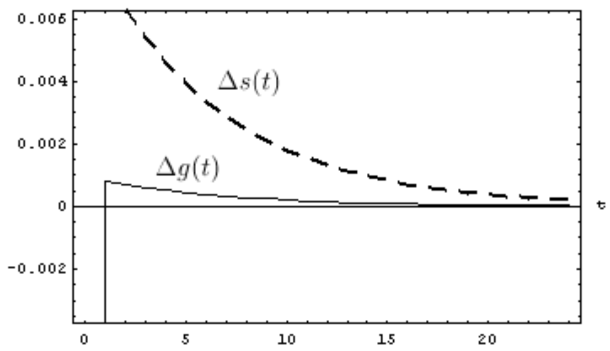

Figure 4: The dynamics of the relative deviations $\Delta s(t)$ and $\Delta g(t)$ over time.

3.7 Comments on the numerical experiments

To begin we recall the results obtained by deriving the FOCs of Problem I. By (3.29) the shadow prices of both capitals are equal at each time excluding 0. The representative agent chooses at each time t the level of the non-predetermined variable $g(t)$ which lead to the same marginal total utility, i.e. shadow price of both capitals. This give us very important knowledge that there is a relation between the amounts of capitals at any time different from 0. We note that the non-predetermined variable $g(t)$ stands for for the splitting of the total investments, whereas the total investments at each time are determined by the agent by the non-predetermined variable $s(t)$. The amount of total investments is determined by the equation (3.28), which creates an equilibrium between the marginal increase (resp. decrease) of current utility and the marginal decrease (resp. increase) of total future utility generated by the same marginal increase (resp. decrease) of $s(t)$. Hence (3.31) expresses the shadow prices of both capitals simultaneously in each period starting from time 1.

Now, let us discuss the numerical experiments from the previous section. We start with the one where the initial amounts of capitals were set by (3.51). The initial amount of the first capital was higher than in steady state and the initial amount of the second capital was at the same level as in the steady state of the whole economy. Let us study the development of the economy after this initial deviation. The total amount of capital in economy is above its steady state value. The production functions (3.5) and (3.6) are concave, so the shadow price of the first capital at time 0 is lower than the shadow price of the second capital. The representative agent therefore chooses $g(0)$ at the level much more lower (see Figure 2) than the steady state level to invest more into the second capital and less into the second capital. This results in the similar relative deviations of the capitals from the steady state in the next period. This rule is applied at each time and the ratio between these relative deviations is the same in any period excluding time 0 whatever the optimal levels of the other variables are. This ratio can represent the relation between the amounts of capital mentioned above. The choice of the optimal value $s(0)$ is determined by the fact, that the total amount of capital at time 0 is over its steady state. This redundant capital does not bring marginally the same value in the future than the amount of capital in the steady state. Therefore, the representative agent starts to "consume" this capital. The optimal way how to do this, is not to consume all redundant capital at time 0. Figure 1 shows the optimal way. One can see that the consumption rate $s(t)$ is above its steady state value during this process. All relative deviations converge to zero.

Similar situation occurs in case $\Delta k_1(0) = 0.015$ and $\Delta k_2(0) = -0.005$ represented by Figure 3 and 4. This simulation shows that different signs of the initial deviations of the capitals pose neither any problem nor qualitative different behavior of the modelled economy. We try to provide a general explanation employing the eigenvectors of (3.50) related to the eigenvalues inside the unit circle, because we looked only for the solution of (3.50) which lies in the space generated by these eigenvectors. These eigenvalues and eigenvectors are:

$$\lambda_2 = 0.8548, \;\; \mathbf{v_2} = \begin{pmatrix} 0.4249 \\ 0.4258 \\ 0.7999 \end{pmatrix} \text{ and } \lambda_3 = 3.6 \times 10^{-17}, \;\; \mathbf{v_3} = \begin{pmatrix} 0.7371 \\ -0.6757 \\ -0.0092 \end{pmatrix}.$$

The eigenvalue λ_3 is very close to zero and the question which could arise is: Is it only an occurence or does this value represents the so-called "machine zero"? The answer can be found in the following consideration.

The eigenvector $\mathbf{v_3}$ causes the adjustment of the deviations $\Delta k_1(t)$ and $\Delta k_2(t)$ at time 1 as one can see in Figure 1 and Figure 3. We presented the economical interpretation of this adjustment saying that the deviations of the capitals have to keep some relation over time satisfying the same shadow prices of the capitals, i.e. the same marginal total utility of the capitals. This relation is determined by the eigenvector $\mathbf{v_3}$ and hence the fact, that the eigenvalue λ_3 is equal to zero is surprising no more. The stable solution of system (3.50) satisfying any initial deviations $\Delta k_1(0)$ and $\Delta k_2(0)$ lies in the subspace created by the vector $\mathbf{v_2}$ starting from the period 1. The interpretation of this eigenvector $\mathbf{v_2}$ which creates the stable path of the whole system is following. This vector lies in the positive ortant, i.e. $\mathbf{v_2} > \mathbf{0}$. When the total amount of capitals in economy is above (resp. below) its amount as in the steady state, the consumption rate is also above (resp. below) the amount as in the steady state. In this way the redundant (resp. missing) capital is "consumed" (resp. "saved") in order to reach the steady state determined by the FOCs.

4 Conclusion

The theory of rational expectations is well developed and therefore the techniques used for solving rational expectations models like the linearization of the FOCs around the steady state are often used without any further explanation. In Section 2 we showed why this linearization is adeguate from mathematical point of view and keeps the qualitative properties of the original nonlinear system around the steady state. We wanted to study the dynamics of the multidimensional difference model and therefore in Section 3 we considered the two-sectoral model of economy with two predetermined and two non-predetermined variables.

In the considered model we allowed the representative agent to manage fully the amounts of both capitals. This property of the model created the degeneration of the FOCs system represented by the same shadow prices of both capitals in each period after time 0. Nevertheless, the model could be analysed and a very intuitive results have been obtained. We believe that a modification of the utility function could lead to a regular system of the FOCs. After this change, the utility function would be dependent on the non-predetermined variable $g(t)$ as well. Then the dynamics of the considered model could be even more interesting.

5 Appendix

We provide Mathematica program used to carry out the calculations in Section 3.6 :

```
k1 = 1.1; k2 = 1.2; a = 10; b = 1; gama1 = 0.25; sigma2 = 0.35;
beta = 0.95; s = 0.7; g = 0.5; cons1 = 0.8; cons2 = 0.8;
gama2 = cons1 - gama1; sigma1 = cons2 - sigma2;

AuxilliaryMatrix={{-g*(1-s)*k1^gama1*k2^gama2, -g*(1-s)*k1^sigma1*k2^sigma2,
k1, 0}, {-(1-g)*(1-s)*k1^gama1*k2^gama2, -(1-g)*(1-s)* k1^sigma1*k2^sigma2,
0, k2}, {gama1*k1^(gama1-1)*k2^gama2, sigma1*k1^(sigma1-1)*k2^sigma2,
-1, 0}, {gama2*k1^gama1*k2^(gama2-1),
sigma2*k1^sigma1*k2^(sigma2-1), 0, -1}};
AuxilliaryVector = {{0}, {0}, {1/beta - 1}, {1/beta - 1}};

c1 = LinearSolve[AuxilliaryMatrix, AuxilliaryVector][[1]][[1]];
c2 = LinearSolve[AuxilliaryMatrix, AuxilliaryVector][[2]][[1]];
delta1 = LinearSolve[AuxilliaryMatrix, AuxilliaryVector][[3]][[1]];
delta2 = LinearSolve[AuxilliaryMatrix, AuxilliaryVector][[4]][[1]];


y1 = c1 k1^gama1 k2^gama2;
y2 = c2 k1^sigma1 k2^sigma2;


Expression1=c1 gama1 k1^(gama1-1) k2^gama2+c2 sigma1 k1^(sigma1-1) k2^sigma2;
Expression2=a - 2 b s (y1+y2);
Expression3=c1 gama2 k1^gama1 k2^(gama2-1)+c2 sigma2 k1^sigma1 k2^(sigma2-1);


MatrixM = {{k1, 0, 0, 0}, {0, k2, 0, 0}, {2 b s(1-delta1+Expression1)
(gama1 y1 + sigma1 y2) - Expression2 (c1 gama1 (gama1-1) k1^(gama1-1) k2^gama2
+ c2 sigma1 (sigma1 - 1) k1^(sigma1 - 1) k2^sigma2),
2 b s (1 - delta1 + Expression1) (gama2 y1 + sigma2 y2) - Expression2 (c1
gama1 k1^(gama1 - 1) gama2 k2^gama2 + c2 sigma1 k1^(sigma1 - 1)
sigma2 k2^sigma2), 2 b s (y1 + y2) (1 - delta1 + Expression1), 0},
```

```
{2 b s (1 - delta2 + Expression3) (gama1 y1 + sigma1 y2) - Expression2
(c1 gama1 gama2 k1^gama1 k2^(gama2 - 1) + c2 sigma1 sigma2 k1^sigma1
k2^(sigma2 - 1)), 2 b s (1 - delta2 + Expression3) (gama2 y1 + sigma2 y2) -
Expression2(c1 gama2(gama2 - 1) k1^gama1 k2^(gama2 - 1) + c2 sigma2 (sigma2
- 1) k1^sigma1 k2^(sigma2 - 1)), 2 b s (y1 + y2) (1 - delta2 + Expression3),
0}};
MatrixN = {{(1 - delta1) k1 + g (1 - s) (c1 gama1 k1^gama1 k2^gama2
+ c2 sigma1 k1^sigma1 k2^sigma2), g (1 - s) (c1 gama2 k1^gama1 k2^gama2
+ c2 sigma2 k1^sigma1 k2^sigma2), -g (y1 + y2) s, (1 - s) (y1 + y2) g},
{(1 - g) (1 - s) (c1 gama1 k1^gama1 k2^gama2 + c2 sigma1 k1^sigma1 k2^sigma2),
(1 - delta2) k2 + (1 - g) (1 - s) (c1 gama2 k1^gama1 k2^gama2
+ c2 sigma2 k1^sigma1 k2^sigma2), -(1 - g) (y1 + y2) s, -(1 - s) (y1 + y2)
g}, {2 b s (gama1 y1 + sigma1 y2)/ beta, 2 b s (gama2 y1 + sigma2 y2)/beta,
2 b s (y1 + y2)/beta, 0}, {2 b s (gama1 y1 + sigma1 y2)/beta, 2 b s (gama2
y1 + sigma2 y2)/beta, 2 b s (y1 + y2)/beta, 0}};

aux1 = Simplify[-MatrixM[[3]][[1]]/MatrixM[[1]][[1]]];
aux2 = Simplify[-MatrixM[[4]][[1]]/MatrixM[[1]][[1]]];
MatrixM[[3]] = Simplify[MatrixM[[3]] + aux1 MatrixM[[1]]];
MatrixN[[3]] = Simplify[MatrixN[[3]] + aux1 MatrixN[[1]]];
MatrixM[[4]] = Simplify[MatrixM[[4]] + aux2 MatrixM[[1]]];
MatrixN[[4]] = Simplify[MatrixN[[4]] + aux2 MatrixN[[1]]];
aux3 = Simplify[-MatrixM[[3]][[2]]/MatrixM[[2]][[2]]];
aux4 = Simplify[-MatrixM[[4]][[2]]/MatrixM[[2]][[2]]];
MatrixM[[3]] = Simplify[MatrixM[[3]] + aux3 MatrixM[[2]]];
MatrixN[[3]] = Simplify[MatrixN[[3]] + aux3 MatrixN[[2]]];
MatrixM[[4]] = Simplify[MatrixM[[4]] + aux4 MatrixM[[2]]];
MatrixN[[4]] = Simplify[MatrixN[[4]] + aux4 MatrixN[[2]]];
aux5 = Simplify[-MatrixM[[4]][[3]]/MatrixM[[3]][[3]]];
MatrixM[[4]] = Simplify[MatrixM[[4]] + aux5 MatrixM[[3]]];
MatrixN[[4]] = Simplify[MatrixN[[4]] + aux5 MatrixN[[3]]];
```

```
aux6 = Simplify[1/MatrixN[[4]][[4]]];

deltag = {-MatrixN[[4]][[1]] aux6, -MatrixN[[4]][[2]] aux6,
-MatrixN[[4]][[3]] aux6, -1};
MatrixN[[1]] = MatrixN[[1]] + MatrixN[[1]][[4]] deltag;
MatrixN[[2]] = MatrixN[[2]] + MatrixN[[2]][[4]] deltag;
MatrixN[[3]] = MatrixN[[3]] + MatrixN[[3]][[4]] deltag;
<< LinearAlgebra`MatrixManipulation`
MatrixM = Submatrix[MatrixM, {1, 1}, {3, 3}];
MatrixN = Submatrix[MatrixN, {1, 1}, {3, 3}];
Print["The eigenvalues of (M^*)^(-1)N^* are:  ",
Eigenvalues[Inverse[MatrixM].MatrixN]]
Print["And the corresponding eigenvectors are:  ",
Eigenvectors[Inverse[MatrixM].MatrixN]]
lambda1 = Eigenvalues[Inverse[MatrixM].MatrixN][[2]];
lambda2 = Eigenvalues[Inverse[MatrixM].MatrixN][[3]];
deltak1 = 0.015;
deltak2 = -0.005;
AuxMat = {{Eigenvectors[Inverse[MatrixM].MatrixN][[2]][[1]],
Eigenvectors[Inverse[MatrixM].MatrixN][[3]][[1]]},
{Eigenvectors[ Inverse[MatrixM].MatrixN][[2]][[2]],
Eigenvectors[Inverse[MatrixM].MatrixN][[3]][[2]]},
{Eigenvectors[ Inverse[MatrixM].MatrixN][[2]][[3]],
Eigenvectors[Inverse[MatrixM].MatrixN][[3]][[3]]}};
constant1 = LinearSolve[ SubMatrix[ AuxMat,
{1, 1}, {2, 2}], {{deltak1}, {deltak2}}][[1]][[1]];
constant2 = LinearSolve[ SubMatrix[ AuxMat,
{1, 1}, {2, 2}], {{deltak1}, {deltak2}}][[2]][[1]];
```

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