Univerzita Komenského v Bratislave, Fakulta matematiky, fyziky a informatiky

Short time oscillations of exchange rates

Diploma Thesis

Bratislava 2007

Tomáš Bokes

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DIPLOMA THESIS

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FAKULTA MATEMATIKY, FYZIKY A INFORMATIKY
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Ekonomická a finančná matematika

Supervisor
Prof. RNDr. Pavol Brunovský, DrSc.

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DIPLOMOVÁ PRÁCA

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Zadanie

Všeobecné:
Analyzovať vlastnosti modelu krátkodobých odchýlok výmenného kurzu s diskrétnym časom (t.j. diferenčnej rovnice vyššieho rádu).

Hlavné:
Analyzovať

- podmienku ohraničenosti trajektórií a určiť hodnotu hranice,
- stabilitu modelu v jeho pevných bodoch.

Prehlasujem, že som diplomovú prácu vypracoval samostatne, iba s pomocou literatúry uvedenej v zozname, konzultácií s vedúcim diplomovej práce a vedomostí získaných počas štúdia.

. . . .

V Bratislave, 30. apríla 2007 Tomáš Bokes

Týmto sa chcem poďakovať Prof. RNDr. Pavlovi Brunovskému, DrSc., svojmu školiteľovi, za všestrannú odbornú pomoc, množstvo cenných pripomienok a rád, ako aj za ochotu a podporu prejavenú pri vedení práce.

Abstract

Abstrakt

BOKES, Tomáš: Short time oscillations of exchange rates [diplomová práca]. Univerzita Komenského v Bratislave. Fakulta matematiky, fyziky a informatiky; Katedra aplikovanej matematiky a štatistiky. Školiteľ: prof. RNDr. Pavol Brunovský, DrSc. Stupeň odbornej kvalifikácie: Magister odboru matematika, špecializácia ekonomická a finančná matematika. Bratislava: FMFI UK, 2007. 33 s.

In this paper we discuss and analyze deviations of a foreign exchange rate from its short term equilibrium. The model was introduced as a delayed differential equation in the thesis Erdélyi (2003) [9] and further analyzed in later works of Brunovský, Erdélyi, Walther (2004) [6], [7] and was transformed into difference equation in theses Boďová (2004) [3] and Szolgayová (2006) [14].

Scope of this paper is to analyze the discrete time model directly derived from the differential equation by replacing the time derivative by time difference. Although an additional factor forcing solutions to be bounded was included in discrete time model in earlier papers, we analyze properties of the model without this element, but with smaller step.

We find a boundary for the initial values and condition for parameters of the model that keep trajectories in the boundary. The functional analysis and center manifold theory is used to analyze stability of the only fixed point $\hat{x} = 0$.

Key words: difference model, exchange rate, discretization

V tejto práci sa zaoberáme odchýlkami výmenného kurzu z jeho krátkodobej rovnováhy. Tento model bol uvedený ako diferenciálna rovnica s oneskorením v diplomovej práci Erdélyi (2003) [9], hlbšie analyzovaný v neskorších prácach Brunovský, Erdélyi, Walther (2004) [6], [7] a jeho transformovaná verzia na diferenčnú rovnicu bola uvedená v diplomových prácach Boďová (2004) [3] a Szolgayová (2006) [14].

Cieľom tejto práce je analyzovať model s diskrétnym časom priamo odvodený z diferenciálnej rovnice nahradením časovej derivácie diferenciou. V predchádzajúcich prácach boli autori nútení obmeniť model s diskrétnym časom pridaním dodatočného ohraničujúceho činiteľa, my sa zaoberáme analýzou modelu bez tohto faktoru, ale s kratším krokom.

Určíme hranicu pre počiatočné hodnoty a podmienku pre parametre, pre ktorú trajektória ostáva ohraničená touto hranicou. Pomocou funkcionálnej analýzy a teórie centrálnej variety analyzujeme stabilitu jediného pevného bodu $\hat{x} = 0$.

Kľúčové slová: diferenčný model, výmenný kurz, diskretizácia

Preface

The real foreign exchange rate is one of the basic indicators of stability and welfare of the country compared to others. Movements of the value of exchange rate became an important element for agents interested in export - import transactions. This is a reason for the need to predict this feature of world economy. Knowledge of the future values of foreign exchange rate would be also very useful for various types of *agents* speculating on the currency market. Many models, more or less precise, were developed during years. Most of them were based on the macroeconomic fundamentals determining the long term state of economy.

In the long term, the exchange rate is believed to be determined by economic fundamentals, expectations and other macroeconomic influences. The general model of real exchange rate is given by

$$S(t) = f(\theta(t)) + x(t), \qquad (p.1)$$

where $f(\theta(t))$ holds for the *natural exchange rate*, if state of economic fundamentals is $\theta(t)$. None of the input factors, except for the expectations, changes in the short term horizon, the long term value S^* is a short term equilibrium. The term $x(t)$, representing expectations of agents, is a deviation shifting the value from equilibrium S^*. This value was modeled in many ways. The model we are presenting is one of them.

We are not introducing this model, because some of its features were already analyzed in earlier papers. Our aim is to perform further analysis to obtain more precise evidence of the behavior of model. We introduce a modification of the model which can be understood as a bridge between the continuous time and discrete time model.

Contents

Introduction

The real foreign exchange rate is a powerful feature of the world economy. Only extremely high long term changes in the value of exchange rate can significantly influence the country wealth and economy. But short term movements, although not high, are still relevant for the individual merchants trying to profit from their international portfolio. If they were able to predict the future value of these short term deviations, hedging of their portfolio would be much easier. There is a possibility to explain the foreign exchange rate employing some economic features, but applying such methods can cause a lack of information. Even if it is impossible to predict the real value exactly, including the expectation of agents trading the currency brings an important improvement of the model.

The model that we are going to talk about is based on decisions of the agents, their expectations and the impact on the value of real foreign exchange rate in the short term. This model was introduced in thesis Erdélyi (2003) [9] and further analysis was done in works Brunovský, Erdélyi, Walther (2004) [6], [7] and theses Boďová (2004) [3] and Szolgayová (2006) [14].

In Erdélyi (2003) [9] the model with continuous time was introduced in form of a nonlinear retarded differential equation:

$$\dot{x}(t) = a(x(t) - x(t-1)) - b|x(t)|x(t), \tag{i.1}$$

where $a, b > 0$ are real parameters. Now we discuss both terms at the right-hand side.

- The first term $a(x(t)-x(t-1)) \approx a(S(t)-S(t-1))$ - agents see the trend of the value of real exchange rate and they set up their expectations according to it.

  If the exchange rate is increasing (decreasing), the foreign currency appreciates (depreciates). Agents expect that the exchange rate will further rise (fall) so they purchase (sell) foreign currency, what implies further appreciation (depreciation).

- The second term $-b|x(t)|x(t)$ - agents know that large deviation is unsupportable, so they expect the trend will turn back.

  As long as the absolute value of deviation from the equilibrium $|x(t)|$ increases (i.e. foreign currency appreciates if the deviation is positive and depreciates if negative), there is a growing number of agents realizing that the trend will turn back and they start to counteract (i.e. sell the foreign currency if $x(t) > 0$ and purchase if $x(t) < 0$). This forces the value of the real exchange rate back to the equilibrium. (This explains the $-bx(t)$.)

The second order of the term is caused by a lack of precise knowledge of the precise exchange rate equilibrium. This leads to the assumption that the higher deviation is, the more agents realize the foreign exchange rate will be pushed back to the equilibrium. (The second term is weighted by absolute value of the $x(t)$.)

When the behavior of agents is impacted by these two types of expectations, it makes us believe that the trajectory of retarded difference equation $(i.1)$ oscillates around zero. Numerical simulations give us evidence of this property of our model (as shown in Figure i.1).

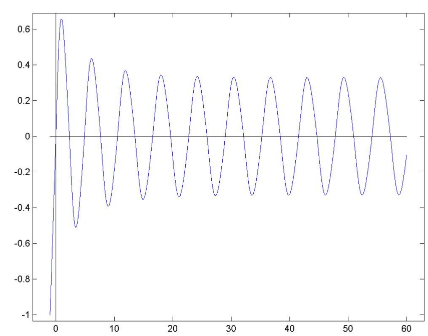

Figure i.1: Solution of the model

This model was analyzed in the mentioned thesis and papers. First the stability of the only fixed point $\hat{x} = 0$ was analyzed. It was found unstable for $a > 1$ and applying the center manifold theory and Lyapunov's second method it was proven asymptotically stable for $a < 1$. There is still no sufficient information about behavior of the system for $a = 1$ to prove or disprove stability. Boundedness of the solution was also shown for the initial values inside the boundary and later also the limit behavior of the trajectories with general initial values was established.

Later in papers Brunovský, Erdélyi, Walther (2004) [6], [7] further analysis was carried out. Since there is no explicit solution for the model, behavior of the trajectories was analyzed also by numerical simulations. Constant parameters a and b were replaced by various stochastic processes ($AR(p)$, $MA(q)$, $ARMA(p,q)$) what brought the model closer to the reality. The time derivative of deviation $\dot{x}(t)$ was also changed by the white noise to decrease the predictability of future values.

In theses Boďová (2004) [3] and Szolgayová (2006) [14], a model with discrete time is analyzed. The equation becomes a difference equation and agents make their decisions simultaneously in discrete time moments.

$$x_{n+1} = x_n + a(x_n - x_{n-1}) - bx_n|x_n|, \tag{i.2}$$

While analyzing the new model a problem of divergence of the trajectories occurred. [As we will show, this is caused by a large ratio between the step size and parameter a.] This problem was solved by implementing a new factor in the model, which holds the value in reasonable boundary (it is done by replacing the constant parameter a with parameter dependent on x_n, i.e. $a(x_n) = a(M - |x_n|)^+$):

$$x_{n+1} = x_n + a(x_n - x_{n-1})(M - |x_n|)^+ - bx_n|x_n|, \tag{i.3}$$

where $(z)^+ = \max(z, 0)$ and $M > 0$ is a constant denoting the number of agents relevant for the model. To get rid of M we can simply rewrite (i.3), by substituting $y_n = Mx_n$ and $\tilde{a} = Ma$, $\tilde{b} = Mb$. Finally we have

$$y_{n+1} = y_n + \tilde{a}(y_n - y_{n-1})(1 - |x_n|)^+ - \tilde{b}y_n|y_n|. \tag{i.4}$$

This model was analyzed in the similar manner as the model with continuous time.

In this paper we analyze the relation between the model with continuous time and the model with discrete time without the additional element $(M - |x_n|)^+$.

In the first part we determine the boundary of the trajectories with bounded initial values. Solving the problem of mathematical programming, we find the condition for the ratio of parameter a and the time step for which the trajectories do not diverge.

The second part analyzes the stability of the model. For the origin, the only fixed point, we prove the same behavior as in the model with continuous time. We analyze the eigenvalues of the characteristic equation of the model. Using the center manifold theorem we prove asymptotic stability of the origin.

In the third part, numerical simulations of the trajectories of model are presented. We include examples of results obtained in the first and the second part. Also numerical simulations for the model with $a = 1$ are carried out.

1 Discrete time model analysis

In this part we examine difference version of our model. Its recurrent rule is

$$x(t+\Delta t) = x(t) + \Big[\ a\big(x(t) - x(t-1)\big) - b|x(t)|x(t)\ \Big]\Delta t. \tag{1.1}$$

It is obtained by replacing the time derivative by time difference in the model with continuous time (i.1),

$$\dot{x}(t) = \frac{dx(t)}{dt} = \frac{x(t+\Delta t) - x(t)}{\Delta t}. \tag{1.2}$$

1.1 Boundedness of trajectories

We begin the analysis of boundedness of the trajectories of difference equation (1.1) that have their initial values in range $[-\mathcal{A}, \mathcal{A}]$ by computing $\mathcal{A}$, the upper bound of the solutions.

Let us have a function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x, y) = x + \Big(a(x-y) - b|x|x\Big)\Delta t. \tag{1.3}$$

To determine the value of $\mathcal{A}$ we have to solve following problem:

$$\max\left\{\ f(x,y)\Big|\ |x|, |y| \le \mathcal{A}\ \right\} \le \mathcal{A}. \tag{1.4}$$

First, we solve mathematical programming problem (i.e. the left side of inequality (1.4)).

We are searching for a maximum of f in the compact set

$$\mathcal{K} = [-\mathcal{A}, \mathcal{A}]^2 \subset \mathbb{R}^2, \tag{1.5}$$

and function $f(x, y)$ is continuous in both arguments, so we are sure about existence of the extreme that we are searching for. We have

$$\frac{\partial f}{\partial x} = 1 + \Big(a - 2b|x|\Big)\Delta t \tag{1.6}$$

$$\frac{\partial f}{\partial y} = -a\Delta t \tag{1.7}$$

We can easily see, there is no $(\hat{x}, \hat{y})$, such that $\nabla f(\hat{x}, \hat{y}) = 0$.

Remark 1.1. *Note that the derivative of function $g(z) = |z|$ is equal to function* $h(z) = sgn(z) = \begin{cases} 1 & \text{if} \quad z > 0 \\ 0 & \text{if} \quad z = 0 \\ -1 & \text{if} \quad z < 0 \end{cases}$ *except of* 0*. The global extreme of function $f(x, y)$ is not at 0 (because $f(-\varepsilon, \hat{y}) \leq f(0, \hat{y}) \leq f(\varepsilon, \hat{y})$, when $0 < \varepsilon << 1 \Rightarrow \varepsilon^2 << \varepsilon$), so we can use $g'(z) = h(z)$ for our purpose.*

The preceding facts imply, that there does not exist an inner extreme. Let us search for extremes on $\partial\mathcal{K}$ (i.e. boundary $\mathcal{K}$). Thanks to (1.7) we know, that the maximum has to be on the part of boundary lying on the line $y = -\mathcal{A}$. [Recall that all parameters are positive, so (1.7) is always negative.]

We have to look for candidates on this part of the boundary. By fixing y we obtain a function of one variable $F(x) = f(x, \tilde{y})$. We have

$$F'(x) = \frac{\partial f}{\partial x}\bigg|_{(x,y)=(x,\tilde{y})} = 1 + \Big(a - 2b|x|\Big)\Delta t. \tag{1.8}$$

Let us search for $\hat{x}$ such that $F'(\hat{x}) = 0$. This equation has two solutions

$$\hat{x}_{1,2} = \pm\frac{a + \tau}{2b}, \tag{1.9}$$

where $\tau = \frac{1}{\Delta t}$. The set $|\hat{x}| \leq \mathcal{A}$ being closed implies that the function F attains its maximum also on the boundary. Sequently we have

$$\hat{x} \in \Big\{ -\frac{a+\tau}{2b}, \frac{a+\tau}{2b}, -\mathcal{A}, \mathcal{A}\Big\} \cap [-\mathcal{A}, \mathcal{A}]. \tag{1.10}$$

The value of $\hat{y}$ was already determined by the negativity of $\frac{\partial f}{\partial y}$: all maxima have to satisfy

$$\hat{y} = -\mathcal{A}. \tag{1.11}$$

So we obtained the candidates for extrema $(\hat{x}, \hat{y})$, where $\hat{x}$ and $\hat{y}$ are defined by (1.10) and (1.11) respectively.

To proceed we distinguish two cases.

Case I $\frac{a+\tau}{2b} \geq \mathcal{A} \Rightarrow \hat{x}_{1,2} = \pm\mathcal{A}$.

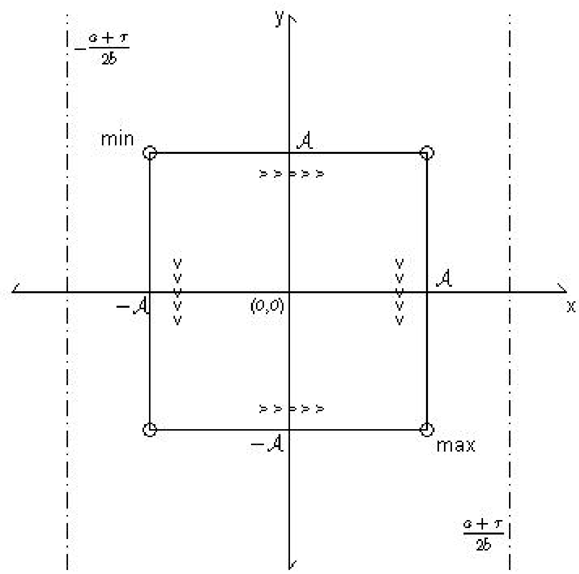

Figure 1.1: Case I

Let us compute the value of $f(x, y)$ in the candidates for extremum:

$$\begin{aligned}
\xi_1 = f(\hat{x}_1, \hat{y}) &= \mathcal{A} + \Big(a(\mathcal{A}+\mathcal{A}) - b\mathcal{A}^2\Big)\frac{1}{\tau} \\
&= \mathcal{A}\Big(1 + \frac{2a}{\tau} - \frac{b}{\tau}\mathcal{A}\Big) \qquad (1.12) \\
\xi_2 = f(\hat{x}_2, \hat{y}) &= -\mathcal{A} + \Big(a(-\mathcal{A}+\mathcal{A}) + b\mathcal{A}^2\Big)\frac{1}{\tau} \\
&= -\mathcal{A}\Big(1 - \frac{b}{\tau}\mathcal{A}\Big) \qquad (1.13)
\end{aligned}$$

The first partial derivative of f with respect to x (1.6) has only positive values on $\mathcal{K}$ (because zero values of this expression lie outside of $\mathcal{K}$). So the point $(\hat{x}_2, \hat{y}) = (-\mathcal{A}, -\mathcal{A})$ is a minmax saddle point (Figure 1.1).

If Case I holds true, then the global maximum is the point $(\hat{x}_1, \hat{y}) = (\mathcal{A}, -\mathcal{A})$. Now we examine the inequality

$$\xi_1 = f(\mathcal{A}, -\mathcal{A}) \leq \mathcal{A}.$$

Let us compute:

$$\begin{aligned} \mathcal{A}\Big(1 + \frac{2a}{\tau} - \frac{b}{\tau}\mathcal{A}\Big) &\leq \mathcal{A} \\ \frac{2a}{\tau} - \frac{b}{\tau}\mathcal{A} &\leq 0 \\ \frac{2a}{b} &\leq \mathcal{A}. \end{aligned} \tag{1.14}$$

Because our aim is to find the optimal boundary, we put $\mathcal{A} = \frac{2a}{b}$. Now we have to find the condition under which also Case I holds true for $\mathcal{A}$:

$$\begin{aligned} \mathcal{A} = \frac{2a}{b} &\leq \frac{a+\tau}{2b} \\ 3a &\leq \tau \end{aligned} \tag{1.15}$$

Recall that $\tau = \frac{1}{\Delta t}$. Thus (1.15) means

$$\Delta t \leq \frac{1}{3a}. \tag{1.16}$$

Hence, in Case I we have $\mathcal{A} = \frac{2a}{b}$ if (1.16) holds true.

Case II Let now $\frac{a+\tau}{2b} < \mathcal{A}$ hence, $\hat{x}_{1,2} = \pm\mathcal{A}$ and $\hat{x}_{3,4} = \pm\frac{a+\tau}{2b}$.
For the additional candidates for extreme, we have

$$\begin{aligned} f(\hat{x}_3, \hat{y}) &= \frac{a+\tau}{2b} + \left(a\Big(\frac{a+\tau}{2b} + \mathcal{A}\Big) - b\Big(\frac{a+\tau}{2b}\Big)^2\right)\frac{1}{\tau} \\ &= \frac{a^2 + 2a\tau + \tau^2}{4b\tau} + \frac{a\mathcal{A}}{\tau} = \frac{(a+\tau)^2}{4b\tau} + \frac{a\mathcal{A}}{\tau} \end{aligned} \tag{1.17}$$

$$\begin{aligned} f(\hat{x}_4, \hat{y}) &= -\frac{a+\tau}{2b} + \left(a\Big(-\frac{a+\tau}{2b} + \mathcal{A}\Big) + b\Big(\frac{a+\tau}{2b}\Big)^2\right)\frac{1}{\tau} \\ &= -\frac{a^2 + 2a\tau + \tau^2}{4b\tau} + \frac{a\mathcal{A}}{\tau} = -\frac{(a+\tau)^2}{4b\tau} + \frac{a\mathcal{A}}{\tau} \end{aligned} \tag{1.18}$$

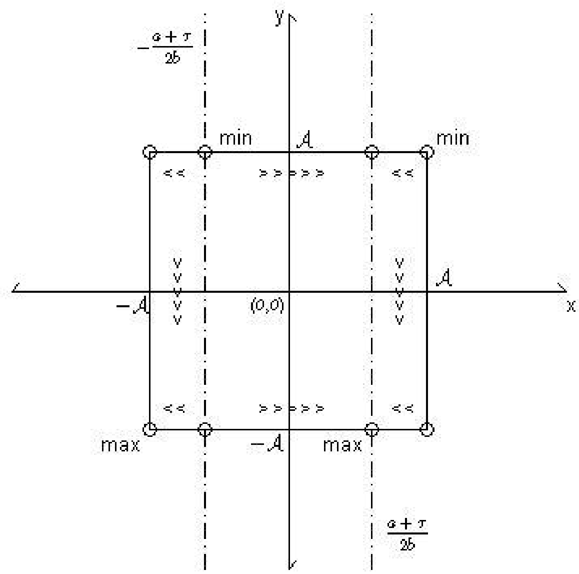

Figure 1.2: Case II

We summarize values of all candidates

$$\begin{aligned}\xi_1 &= f(\hat{x}_1,\hat{y}) = \mathcal{A}\Big(1+\frac{2a}{\tau}-\frac{b}{\tau}\mathcal{A}\Big)\\ \xi_2 &= f(\hat{x}_2,\hat{y}) = -\mathcal{A}\Big(1-\frac{b}{\tau}\mathcal{A}\Big)\\ \xi_3 &= f(\hat{x}_3,\hat{y}) = \frac{(a+\tau)^2}{4b\tau}+\frac{a\mathcal{A}}{\tau}\\ \xi_4 &= f(\hat{x}_4,\hat{y}) = -\frac{(a+\tau)^2}{4b\tau}+\frac{a\mathcal{A}}{\tau}.\end{aligned}$$

Since we know the optimal value of $y=\tilde{y}$ we can fix it and investigate function $F(x)=f(x,\tilde{y})$. We have

$$F''(x)=\frac{\partial^2 f}{\partial x^2}\Big|_{(x,y)=(x,\tilde{y})} = -2b\ sgn(x)\Delta t, \qquad (1.19)$$

F'' is negative (i.e. F is concave) only for x positive. That implies that $(\hat{x}_4,\hat{y})$ cannot be a maximum (it is a minmax saddle point).

Let us now investigate the candidates for extremum on $\partial\mathcal{K}$. For any $\tilde{y}$, it holds

$$x \in \Big[-\mathcal{A}, -\frac{a+\tau}{2b}\Big) \cup \Big(\frac{a+\tau}{2b}, \mathcal{A}\Big] \;\Rightarrow\; \frac{\partial f(x,\tilde{y})}{\partial x} < 0. \tag{1.20}$$

The value of function f increases if $x \searrow -\mathcal{A}$, on the contrary if $x \nearrow \mathcal{A}$ it decreases. So the second maximum is the point $(\hat{x}_2, \hat{y}) = (-\mathcal{A}, -\mathcal{A})$. The point $(\mathcal{A}, -\mathcal{A})$ is a minmax saddle point of f on $\mathcal{K}$.

We have to determine a condition for each maximum to be the global one. To solve problem (1.4) in Case II, we require both inequalities to hold true simultaneously.

For the first of them for ξ_2:

$$\begin{aligned} \xi_2 = -\mathcal{A}\Big(1 - \frac{b}{\tau}\mathcal{A}\Big) &\leq \mathcal{A} \\ \frac{b}{\tau}\mathcal{A} &\leq 2 \\ \mathcal{A} &\leq 2\frac{\tau}{b}. \end{aligned} \tag{1.21}$$

For the second one:

$$\begin{aligned} \xi_3 = \frac{(a+\tau)^2}{4b\tau} + \frac{a\mathcal{A}}{\tau} &\leq \mathcal{A} \\ \frac{(a+\tau)^2}{4b\tau} &\leq \mathcal{A}\Big(1 - \frac{a}{\tau}\Big). \end{aligned} \tag{1.22}$$

If $1 - \frac{a}{\tau} \leq 0 \Leftrightarrow \tau \leq a$ there does not exist any $\mathcal{A} \in \mathbb{R}$ for which inequality (1.22) holds true.

So let $1 - \frac{a}{\tau} > 0 \Leftrightarrow \tau > a$. For $\mathcal{A}$ we have the inequality

$$\frac{(a+\tau)^2}{4b(\tau - a)} \leq \mathcal{A}. \tag{1.23}$$

To satisfy Case II, value $\mathcal{A}$ has to be

$$\mathcal{A} \in \Big[\frac{(a+\tau)^2}{4b(\tau-a)}, 2\frac{\tau}{b}\Big]. \tag{1.24}$$

We search for a condition for the set in (1.24) not to be empty:

$$\begin{aligned} \frac{(a+\tau)^2}{4b(\tau-a)} &\leq 2\frac{\tau}{b} \\ a^2 + 2a\tau + \tau^2 &\leq 8\tau(\tau - a) \\ 0 &\leq 7\tau^2 - 10a\tau - a^2 \end{aligned} \tag{1.25}$$

The solution of the inequality for τ is the set

$$\tau \in \Big(-\infty, a\frac{5-4\sqrt{2}}{7}\Big] \cup \Big[a\frac{5+4\sqrt{2}}{7}, \infty\Big). \tag{1.26}$$

Following from Case I, τ has to satisfy the inequality $3a > \tau$ (because Case II is complementary of Case I).

For Case II, let us put $\mathcal{A} = \frac{(a+\tau)^2}{4b(\tau-a)}$ if $\tau \in (a\frac{5+4\sqrt{2}}{7}, 3a)$, i.e.

$$\Delta t \in (\frac{1}{3a}, \frac{4\sqrt{2}-5}{a}). \tag{1.27}$$

To complete the analysis of boundedness we have to determine the value of the lower bound $\mathcal{B}$, i.e. we have to solve the problem

$$\min \Big\{ \, f(x,y) \Big| \; |x|, |y| \le \mathcal{B} \, \Big\} \ge \mathcal{B}, \tag{1.28}$$

where function f is defined in (1.3).

Since f is odd in both variables, one has

$$\min_{|x|,|y|\le\mathcal{B}} f(x,y) = -\max_{|x|,|y|\le\mathcal{B}} f(x,y) \tag{1.29}$$

so the lower bound has value $-\mathcal{A}$.

Remark 1.2. *While analyzing minimum of f, the optimal value of variable $\hat{y} = \mathcal{A}$ (because the partial derivative of f (1.7) is negative). So values of variable x, that we stated as a saddle points in combination with $\hat{y} = -\mathcal{A}$, become minimums with $\hat{y} = \mathcal{A}$ and vice versa.*

Before we summarize the information obtained, it is useful to introduce following definition.

Definition 1.1. *Let $\tau = \frac{1}{\Delta t}$ be an integer. A set $\mathcal{M}$ will be called an invariant set of the equation Φ in case every its solution satisfying $x(t) \in \mathcal{M}$ for $t = k\Delta t$, $0 \le k \le \tau$ yields $x(t) \in \mathcal{M}$ for all $t = k\Delta t$, $k \in \mathbb{N}$.*

Theorem 1.1. *The set* $|x| < \mathcal{A}$ *with*

$$\mathcal{A}(a,b,\tau) = \begin{cases} \frac{2a}{b} & \textit{if} \quad \frac{a}{\tau} \leq \frac{1}{3} \\ \frac{(a+\tau)^2}{4b(\tau-a)} & \textit{if} \quad \frac{a}{\tau} \in (\frac{1}{3}, 4\sqrt{2}-5] \end{cases} \tag{1.30}$$

is an invariant set of the equation

$$x(t+\Delta t) = x(t) + \Big[\ a\big(x(t) - x(t-1)\big) - b|x(t)|x(t)\ \Big]\Delta t.$$

2 Stability of the zero solution

2.1 Center manifold theory

While analyzing the stability of a nonlinear dynamic system we always need to talk about eigenvalues of the linearization of system in a fixed point. Discrete dynamic system is stable in a fixed point if all eigenvalues are in the unit circle (of the complex plane). If there is at least one eigenvalue lying outside of the unit circle, the system is unstable. If the system has one or more eigenvalues lying on the unit circle, we have to do further analysis to determine the stability of the fixed point. The reduction to center manifold is the solution of this problem.

Let the linearization of dynamic system in the origin defined on $\mathbb{R}^n$ has $m \in \mathbb{N}$ eigenvalues lying on the unit circle. The center manifold theory guarantees the existence of a m-dimensional center manifold tangential in the origin to the eigenspace belonging to eigenvalues lying on the unit circle.

In following theorems we introduce the center manifold theory. More precise theory can be found in Brunovský (1993) [5], Carr (1981) [8], Guckenheimer and Holmes (1983) [10] and Iooss (1979) [11]. Similar technique was also used in thesis of Boďová (2004) [3] and Erdélyi (2003) [9].

Theorem 2.1. *(Center Manifold Theorem)*

Let g be a C^r mapping, $g : \mathbb{R}^n \to \mathbb{R}^n$, with property $g(0) = 0$ and let $A = Dg(0)$. Divide the spectrum of A into three parts, σ_s, σ_c, σ_u with

$$|\lambda| \begin{cases} < 1 & \text{if } \lambda \in \sigma_s, \\ = 1 & \text{if } \lambda \in \sigma_c, \\ > 1 & \text{if } \lambda \in \sigma_u. \end{cases}$$

Let the (generalized) eigenspace of σ_s, σ_c and σ_u be E^s, E^c and E^u, respectively. Then there exists a C^r invariant manifold W^c tangent to E^c at the origin. The manifold W^c is called the center manifold for the mapping of g at 0.

Remark 2.1. *The only reference for the center manifold for discrete dynamical systems we found was* Guckenheimer and Holmes (1983) *[10]. However, in* Guckenheimer and Holmes (1983) *[10] the center manifold is claimed to be C^{r-1} only. Other sources (*Chow and Hale (1983) *[13])* *which establish the center manifold to be C^r are presented only for continuous time systems. Their proof, however, carries over to discrete dynamical systems.*

Let $\sigma_u = \emptyset$. Then, in suitable coordinates, we have

$$\begin{array}{rcl} x_{n+1} & = & Bx_n + \varphi(x_n, y_n) \\ y_{n+1} & = & Cy_n + \psi(x_n, y_n) \end{array} ; \quad (x_n, y_n) \in \mathbb{R}^m \times \mathbb{R}^k, \tag{2.1}$$

where $B \in \mathbb{R}^{m\times m}$ and $C \in \mathbb{R}^{k\times k}$ are matrices whose eigenvalues lie on and in the unit circle respectively, $\varphi(0,0) = 0$ and $\psi(0,0) = 0$, $D\varphi(0,0) = 0$ and $D\psi(0,0) = 0$.

Since the center manifold is tangent to E^c we can locally represent it as a graph

$$W^c = \{(x,y)|y = h(x)\}; \quad h(0) = 0, \tag{2.2}$$

where $h : U \to \mathbb{R}^k$ is defined in some neighborhood $U \subset \mathbb{R}^m$ of the origin. We now consider the projection of the mapping on $y = h(x)$ onto E^c:

$$x_{n+1} = Bx_n + \varphi(x_n, h(x_n)). \tag{2.3}$$

Since $h(x)$ is tangent to $y = 0$, we have for the solution of equation (2.3)

Theorem 2.2. *If the origin $x = 0$ of* (2.3) *is locally asymptotically stable (unstable) then the origin of* (2.2) *is also locally asymptotically stable (unstable).*

2.2 Reduction to center manifold

First we recall difference equation of our model introduced in (1.1)

$$x(t+\Delta t) = x(t) + \Big[\ a\big(x(t) - x(t-1)\big) - b|x(t)|x(t)\ \Big]\Delta t.$$

Now we again substitute $\Delta t = \frac{1}{\tau}$. Realizing that agents are revising their decisions τ times a day (or during another fixed unit of time), we consider τ as integer.

As argued above, our model is a difference equation of order $\tau + 1$, so we rewrite it as a system of $\tau + 1$ equations of first order.

$$y^{n+1} = Ay^n + B(y^n), \tag{2.4}$$

where $y^{n+1}, y^n \in \mathbb{R}^{\tau+1}$, $A \in \mathbb{R}^{(\tau+1)\times(\tau+1)}$ and $B : \mathbb{R}^{\tau+1} \to \mathbb{R}^{\tau+1}$. Rewriting in matrices one has

$$\begin{pmatrix} y_0^{n+1} \\ y_1^{n+1} \\ \vdots \\ y_{\tau-1}^{n+1} \\ y_\tau^{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\frac{a}{\tau} & 0 & 0 & \dots & 1+\frac{a}{\tau} \end{pmatrix} \begin{pmatrix} y_0^n \\ y_1^n \\ \vdots \\ y_{\tau-1}^n \\ y_\tau^n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -\frac{b}{\tau} y_\tau^n |y_\tau^n| \end{pmatrix},$$

where $y_j^{n+i} = x(t + (i + j - \tau)\Delta t)$.

2.2.1 Eigenvalues

The characteristic equation of the system (2.4) is

$$\lambda^{\tau+1} - (1 + \frac{a}{\tau})\lambda^\tau + \frac{a}{\tau} = 0. \tag{2.5}$$

Theorem 2.3. *The equation* (2.5) *has the root* $\lambda_0 = 1$ *for all* a. *The second root* $\lambda_1(a) \in \mathbb{R}^+$ *satisfies*

$$\lambda_1(a) \begin{cases} < 1 & \text{if } a < 1 \\ = 1 & \text{if } a = 1 \\ > 1 & \text{if } a > 1 \end{cases}.$$

The rest of roots lies inside the unit circle (of the complex plane) and there are 2 cases:

- *if* τ *is even,* $\lambda_2 \in \mathbb{R}^-$ *and* $\lambda_j \in \mathbb{C}$, $\Im m(\lambda_j) \neq 0$ *for* $j = 3, \dots, \tau$;
- *if* τ *is odd,* $\lambda_j \in \mathbb{C}$, $\Im m(\lambda_j) \neq 0$ *for* $j = 2, \dots, \tau$.

There is no multiple root if $a \neq 1$ *and if* $a = 1$, $\lambda_0 = \lambda_1 = 1$ *is the only double root.*

Proof First we examine real roots. We define

$$p(\lambda) = \lambda^{\tau+1} - (1 + \frac{a}{\tau})\lambda^\tau + \frac{a}{\tau}.$$

It denotes the characteristic polynomial of system (2.4). By direct inspection we verify that $\lambda_0 = 1$ is its root. To determine the nature of rest of real roots we use the first derivative of $p(\lambda)$:

$$p'(\lambda) = (\tau + 1)\lambda^\tau - (\tau + a)\lambda^{\tau-1}.$$

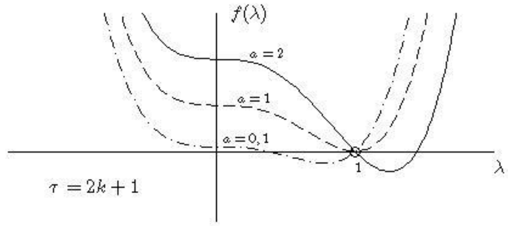

Figure 2.1: Case $\tau = 2k+1$

There are three intervals the monotonicity of $p(\lambda)$ which need to be examined

$p'(\lambda)$	$\big(-\infty, 0\big)$	$\big(0, \frac{\tau+a}{\tau+1}\big)$	$\big(\frac{\tau+a}{\tau+1}, \infty\big)$
τ even	$+$	$-$	$+$
τ odd	$-$	$-$	$+$

Let us search for positive roots. We already have one positive root ($\lambda_0 = 1$) so there is at most one left. Since $p'(1) = 1-a$, $p(0) = \frac{a}{\tau} > 0$ and $\lim\limits_{\lambda\to\infty} p(\lambda) = \infty$, we consider three cases (Figure 2.1-2.2):

- $a < 1$, hence $\frac{\tau+a}{\tau+1} < 1$; together with $p'(1) > 0$ this implies $\lambda_1 \in (0,1)$;
- $a = 1$, hence $\frac{\tau+a}{\tau+1} = 1$; together with $p'(1) = 0$ this implies one is double root so $\lambda_1 = 1$;
- $a > 1$, hence $\frac{\tau+a}{\tau+1} > 1$; together with $p'(1) < 0$ this implies $\lambda_1 > 1$.

If τ is odd we are done with real roots, but if τ is even, we know that $f(-1) = -1-1-\frac{a}{\tau}+\frac{a}{\tau} = -2 < 0$ since $f(0) > 0$ there is a third real root $|\lambda_2| < 1$.

As the second step we prove that all imaginary roots are inside the unit circle if $a \le 1$. We use the goniometric representation of complex number

$$\lambda = \eta(\cos\varphi + i\sin\varphi),$$

where $\eta \in \mathbb{R}_0^+$ and $\varphi \in (-\pi, \pi]$.

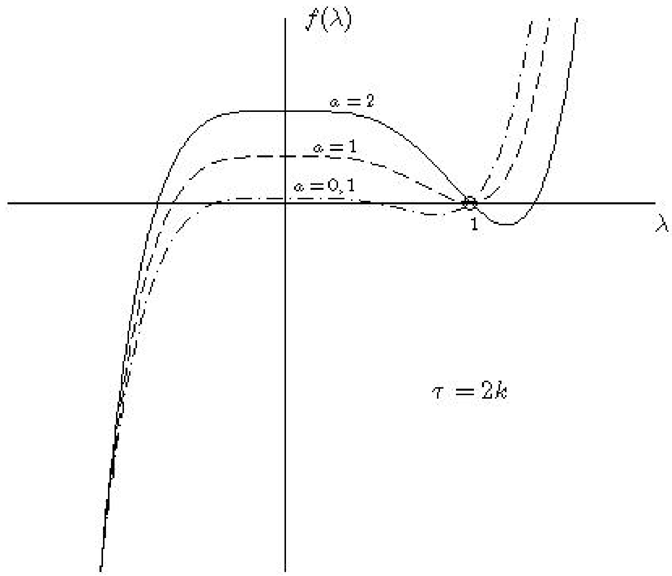

Figure 2.2: Case $\tau = 2k$

By applying Moivre lemma and separating real and imaginary parts we obtain the system of two equations

$$\begin{aligned} 0 &= \eta^{\tau+1}\cos(\tau+1)\varphi - (1+\frac{a}{\tau})\eta^{\tau}\cos\tau\varphi + \frac{a}{\tau} \qquad (2.6)\\ 0 &= \eta^{\tau+1}\sin(\tau+1)\varphi - (1+\frac{a}{\tau})\eta^{\tau}\sin\tau\varphi. \qquad (2.7)\end{aligned}$$

Since $\lambda = 0$ is not a solution of (2.5), we have $\eta \neq 0$ and, in addition, $\sin(\tau+1)\varphi \neq 0$. Indeed $\sin(\tau+1)\varphi = 0$ iff $\varphi = \frac{k}{\tau+1}\pi$, $k = 0, 1 \Rightarrow$ if $k = 0$ the root is not imaginary and if $k = 1$, $\sin(\frac{\tau}{\tau+1})\pi \neq 0$ for $\tau \in \mathbb{N}$ because $\frac{\tau}{\tau+1} \notin \mathbb{N}$. So, from (2.7) we have

$$0 = \eta\sin(\tau+1)\varphi - (1+\frac{a}{\tau})\sin\tau\varphi \;\longrightarrow\; \eta = \left(1+\frac{a}{\tau}\right)\frac{\sin\tau\varphi}{\sin(\tau+1)\varphi}. \qquad (2.8)$$

Substituting (2.8) into (2.6) we obtain

$$\frac{a}{\tau} = \left[\left(1+\frac{a}{\tau}\right)\frac{\sin\tau\varphi}{\sin(\tau+1)\varphi}\right]^{\tau+1}\left(\frac{\sin(\tau+1)\varphi}{\sin\tau\varphi}\cos\tau\varphi - \cos(\tau+1)\varphi\right).$$

If we use the substitution (2.8) again, but in the reverse way, the equation becomes

$$\frac{a}{\tau} = \eta^{\tau+1}\left(\frac{\sin(\tau+1)\varphi}{\sin\tau\varphi}\cos\tau\varphi - \cos(\tau+1)\varphi\right). \tag{2.9}$$

We now simplify the expression in the brackets:

$$\begin{aligned}\frac{\sin(\tau+1)\varphi}{\sin\tau\varphi}\cos\tau\varphi - \cos(\tau+1)\varphi &= \frac{\sin(\tau+1)\varphi\cos\tau\varphi - \cos(\tau+1)\varphi\sin\tau\varphi}{\sin\tau\varphi}\\ &= \frac{\sin[(\tau+1)\varphi - \tau\varphi]}{\sin\tau\varphi} = \frac{\sin\varphi}{\sin\tau\varphi}.\end{aligned} \tag{2.10}$$

Employing (2.10) we obtain a function $\eta = \eta(\varphi)$

$$\eta = \sqrt[\tau+1]{\frac{a}{\tau}\frac{\sin\tau\varphi}{\sin\varphi}}. \tag{2.11}$$

Our aim is to show that $\eta \leq 1$ for $a \leq 1$. Equivalent to this, we can show that $\eta^{\tau+1} < 1$ and we prove that

$$\left|\frac{a}{\tau}\frac{\sin\tau\varphi}{\sin\varphi}\right| < 1, \tag{2.12}$$

we show that for $\varphi \in (-\pi,\pi) - \{0\}$ the inequality is strict. [The expression is equal to one only for $\varphi \to 0$ or for $\varphi \to \pi$, if $\varphi \in (-\pi,\pi]$ as defined and $a = 1$.]

Solving this problem for $a = 1$, guarantees us that the inequality holds true also for all $a < 1$. Moreover, $\sin x$ is odd, so $\frac{\sin\tau\varphi}{\sin\varphi}$ is even. Thanks to this symmetry we are free to analyze only the case $\varphi \in (0,\pi)$, where $\sin\varphi > 0$. The final form of our inequality is

$$\left|\frac{1}{\tau}\sin\tau\varphi\right| < \sin\varphi. \tag{2.13}$$

Expressions on both sides of the inequality are equal to zero at the end-points of interval $(0, \pi)$. Now it suffices if we show that the right-hand side is increasing faster than the left-hand side until the first maximum of the latter and is decreasing faster since the last maximum of the latter [$\sin\varphi$ has only one maximum in $\frac{\pi}{2}$, but $\left|\frac{1}{\tau}\sin\tau\varphi\right|$ has maximum in $\frac{\pi}{2\tau} + \frac{k}{\tau}\pi$ for $k = 0, \dots, \tau - 1$].

Let $\varphi \in \left(0, \frac{\pi}{2\tau}\right) = I_L$ so we can examine the left side of the interval. We want to show that the following inequality holds true

$$0 < \left(\frac{1}{\tau}\sin\tau\varphi\right)' < (\sin\varphi)',$$

i.e.

$$\cos\tau\varphi < \cos\varphi.$$

Since $\cos x$ is decreasing on $(0, \pi)$,

$$\tau\varphi > \varphi$$

i.e.

$$\tau > 1.$$

So this inequality holds true for all $\tau \in \mathbb{N} - \{1\}$, it is enough for our purpose [for $\tau = 1, 2$ there are any imaginary roots of (2.5)].

Now $\varphi \in \left(\pi - \frac{\pi}{2\tau}, \pi\right) = I_R$. While analyzing right-hand side of the interval, there are two cases to be discussed.

In the first case $\tau = 2k + 1$, $k \in \mathbb{N}$. Then $\frac{1}{\tau}\sin\tau\varphi > 0$ for $\varphi \in I_R$ while $(\sin\varphi)' < 0$. Hence (2.13) is equivalent to

$$0 > \left(\frac{1}{\tau}\sin\tau\varphi\right)' > (\sin\varphi)',$$

i.e.

$$\cos\tau\varphi > \cos\varphi.$$

Recalling $\cos x$ is decreasing function on $(0, \pi)$, we have

$$\tau\varphi - 2m\pi < \varphi,$$

where $m \in \mathbb{N}$ is the largest possible to keep $\tau\varphi - 2m\pi \in (0, \pi)$. If $\varphi \in I_R$, $\tau\varphi \in \left((2k+1)\pi - \frac{\pi}{2}, (2k+1)\pi\right)$. Now it is clear that $m = k$ and

$$\begin{aligned} (2k+1)\varphi - 2k\pi &< \varphi \\ \varphi &< \pi, \end{aligned}$$

which is satisfied on all I_R.

In the second case $\tau = 2k$, $k \in \mathbb{N}$, hence $\frac{1}{\tau}\sin\tau\varphi < 0$ for $\varphi \in I_R$.

$$\begin{aligned} 0 > \Big(-\frac{1}{\tau}\sin\tau\varphi\Big)' &> (\sin\varphi)' \\ -\cos\tau\varphi &> \cos\varphi. \end{aligned}$$

Employing $-\cos x = \cos(x+\pi)$ we have

$$\begin{aligned} \cos(\tau\varphi+\pi) &> \cos\varphi \\ \tau\varphi+\pi-2m\pi &< \varphi. \end{aligned}$$

Again we have to determine m. If $\varphi \in I_R$ then $(\tau\varphi+\pi) \in \big(\frac{\pi}{2}+2k\pi, \pi+2k\pi\big)$, hence $m = k$,

$$\begin{aligned} 2k\varphi+\pi-2k\pi &< \varphi \\ \varphi &< \pi, \end{aligned}$$

so we have the same inequality as in the first case.

Thanks to symmetry of an even function, this inequality behaves in the same manner for negative φ. We have proven that for $a = 1$ inequality (2.12) holds true. As we mentioned above, this implies that it holds true also for all $a < 1$.

Solving the previous problem we showed that $|\lambda| = |\eta| \le 1$ ($\lambda \in \mathbb{C}$) if $a < 1$.

In the final part of the proof we show that the eigenvalues are distinct if $a \ne 1$ and one root is double for $a = 1$.

If there existed some double root λ_ω for equation (2.5), λ_ω would be a root of the equation

$$\Big(\lambda^{\tau+1} - (1+\frac{a}{\tau})\lambda^\tau + \frac{a}{\tau}\Big)' = 0$$

i.e.

$$(\tau+1)\lambda^\tau - (\tau+a)\lambda^{\tau-1} = 0$$

as well.

This equation has τ roots. There is a $(\tau-1)$-multiple root $\lambda_k' = 0$ for $k = 1,\ldots,\tau-1$ and the last root is $\lambda_0' = \frac{\tau+a}{\tau+1}$. Any of these roots is not also a root of (2.5) if $a \ne 1$. This implies the characteristic equation has no multiple roots if $a \ne 1$. For $a = 1$ we have $\lambda_0' = 1$ and since $\lambda_0 = 1$ is root of (2.5), $\lambda_0 = 1$ is double root.

This completes the proof of theorem. $\square$

2.2.2 Center manifold

Our aim is to reduce the system (2.4) to the center manifold belonging to eigenvalue $\lambda_0 = 1$. Now we determine form of the nonlinear part of system.

We need to derive matrix of projection mentioned.

Definition 2.1. *The resolvent of operator $\mathcal{T} \in L(\mathbb{R}^n, \mathbb{R}^n)$ is defined as $\mathcal{R}_\lambda^\mathcal{T} = (\lambda I - \mathcal{T})^{-1}$, where $\lambda \in \mathbb{R}$ and I is the identity matrix.*

Definition 2.2. *Let $\mathcal{T} \in L(\mathbb{R}^n, \mathbb{R}^n)$. We define a class $\mathfrak{U}(\mathcal{T})$ of complex functions g such that*

- *the domain $\mathcal{D}(g)$ of g is an open set in the complex plane;*
- *the (complex) derivative of g exists in each point x, $x \in \mathcal{D}(g)$.*

By Taylor (1973) [15] we know that for all $\psi \in \mathfrak{U}(\mathcal{T})$ the following equation holds true

$$\psi(\mathcal{T}) = \frac{1}{2\pi i} \int_C \psi(\zeta) \mathcal{R}_\zeta^\mathcal{T} d\zeta, \tag{2.14}$$

where C is an anticlockwise oriented simple Jordan curve containing all eigenvalues of $\mathcal{T}$ in its interior.

More generally, we have

$$\psi(\mathcal{T})\Big|_E = \frac{1}{2\pi i} \int_\mathfrak{C} \psi(\zeta) \mathcal{R}_\zeta^\mathcal{T} d\zeta, \tag{2.15}$$

if E is an invariant subspace of spectral set σ_E and $\mathfrak{C}$ encircles σ_E, but $\sigma \setminus \sigma_E$ lies outside the curve $\mathfrak{C}$.

According to this theorem we set the curve $\mathfrak{C}$ a circle with $\lambda_0 = 1$ as center that does not contain any of the other eigenvalues. The right-hand expression becomes the matrix of projection we are searching for [recall that A is a linear operator]:

$$\mathcal{P} \stackrel{Def}{=} \frac{1}{2\pi i} \int_\mathfrak{C} \mathcal{R}_\lambda^A d\lambda. \tag{2.16}$$

The nonlinear part of (2.4) can be written as

$$B(y^n) = \Big(- \frac{b}{\tau} y_\tau^n |y_\tau^n| \Big) \mathfrak{e}, \tag{2.17}$$

where $\mathfrak{e} = (0, \dots, 0, 1)^T \in \mathbb{R}^{\tau+1}$.

As we can see, the value $\frac{b}{\tau}y^n_\tau|y^n_\tau|$ is the norm of $B(y^n)$. We are free to omit the constant part with respect to λ (i.e. the norm multiplied by -1) and calculate the projection of (2.17):

$$\tilde{\mathfrak{e}} = \mathcal{P}\mathfrak{e} = \frac{1}{2\pi i}\int_{\mathfrak{C}}(\lambda I - A)^{-1}\mathfrak{e}d\lambda = \frac{1}{2\pi i}\int_{\mathfrak{C}} d(\lambda)d\lambda, \tag{2.18}$$

where d is the solution of

$$(\lambda I - A)\mathbf{d} = \mathfrak{e}, \tag{2.19}$$

i.e., in components,

$$\begin{pmatrix} \lambda & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 \\ \frac{a}{\tau} & 0 & \dots & \lambda-(1+\frac{a}{\tau}) \end{pmatrix} \begin{pmatrix} d_0 \\ \vdots \\ d_{\tau-1} \\ d_\tau \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{2.20}$$

We have

$$\begin{aligned} \lambda d_i &= d_{i+1} \qquad \text{for } i = 0,\dots,\tau-1 \\ \lambda d_\tau - (1+\frac{a}{\tau})d_\tau + \frac{a}{\tau}d_0 &= 1, \end{aligned}$$

hence

$$d_i = \frac{\lambda^i}{\lambda^{\tau+1} - (1+\frac{a}{\tau})\lambda^\tau + \frac{a}{\tau}} \qquad \forall i = 0,\dots,\tau. \tag{2.21}$$

It is useful to rewrite (2.21) in the form

$$d_i = \frac{\lambda^i}{(\lambda-1)(\lambda^\tau - \frac{a}{\tau}\sum_{j=0}^{\tau-1}\lambda^j)} \qquad \forall i = 0,\dots,\tau. \tag{2.22}$$

Now we can find the projection of $\mathfrak{e}$. For each $k = 0,\dots,\tau$ we have

$$\tilde{\mathfrak{e}}_k = \frac{1}{2\pi i}\int_{\mathfrak{C}} d_k(\lambda)d\lambda = \frac{1}{2\pi i}\int_{\mathfrak{C}} \frac{\varphi_k(\lambda)}{\lambda-1}d\lambda, \tag{2.23}$$

where

$$\varphi_k(\lambda) = \frac{\lambda^k}{\lambda^\tau - \frac{a}{\tau}\sum_{j=0}^{\tau-1}\lambda^j}. \tag{2.24}$$

The function $\varphi_k(\lambda)$ being holomorphic on the interior of $\mathfrak{C}$, according to Cauchy formula for the circle, we have

$$\tilde{\mathfrak{e}}_k = \varphi_k(1) = \frac{1}{1-a}. \tag{2.25}$$

Thus, the projection of nonlinear part of (2.4) to E has form

$$\mathcal{P}B(y) = -\frac{b}{\tau(1-a)} y_\tau^n |y_\tau^n| \mathfrak{v}_0, \tag{2.26}$$

where $\mathfrak{v}_0$ is the eigenvector of eigenvalue $\lambda_0 = 1$ [i.e. $\mathfrak{v}_0 = (1, \ldots, 1)^T$].

According to the center manifold theory, the center manifold is defined as

$$\mathfrak{y} \stackrel{Def}{=} \{u\mathfrak{v} + h(u) |\ u \in \mathbb{R}\}, \tag{2.27}$$

where $\mathfrak{v}$ is the eigenvector belonging to eigenvalue lying on the unit circle, function $h : \mathbb{R} \to \mathbb{R}^{\tau+1}$ and $h(u)$ is $o(u)$ in all components.

Remark 2.2. *Recall that for $f, g : \mathcal{D}(f) \subset \mathbb{R} \to \mathbb{R}$, $\mathcal{D}(f)$ is an open set, we set $f(x) = o(g(x))$ for $x \to \bar{x}$, where $g : \mathcal{D}(g) \subset \mathbb{R} \to \mathbb{R}$ if*

$$\lim_{x \to \bar{x}} \left| \frac{f(x)}{g(x)} \right| = 0.$$

If we follow the center manifold flow $\mathfrak{y}$ ($\Rightarrow y = y(u)$), we find that the norm of $B(y)$ equals to

$$\|B(y(u))\| = \frac{b}{\tau} |u + o(u)|(u + o(u)). \tag{2.28}$$

The projection of (2.4) to the eigenspace of eigenvalue λ_0 is

$$u^{n+1}\mathfrak{v}_0 = u^n\mathfrak{v}_0 - \frac{b}{\tau(1-a)} |u^n + o(u^n)|(u^n + o(u^n))\mathfrak{v}_0. \tag{2.29}$$

Simplifying and changing the notation in (2.29) we obtain

$$u_{n+1} = u_n - \frac{b}{\tau(1-a)} |u_n + o(u_n)|(u_n + o(u_n)). \tag{2.30}$$

2.3 Stability

Theorem 2.4. *If $b > 0$ and $\tau \in \mathbb{N}$ then the fixed point $\hat{y} = (0, \ldots, 0) \in \mathbb{R}^{\tau+1}$ of (2.4) is*

- *asymptotically stable if $a < 1$;*
- *unstable if $a > 1$.*

Proof According to the Center manifold theorem 2.2 we analyze the stability of $\hat{y}$ on the center manifold flow. We prove the statement in the theorem for system (2.30) and $\hat{u} = 0$. The center manifold theory implies the same behavior for system mentioned in Theorem 2.4.

Remark 2.2 implies that there exists $\delta > 0$ so that $sgn(u_n + o(u_n)) = sgn(u_n)$, for all $|u_n| < \delta$ and we have

$$u_{n+1} = u_n - \frac{b}{\tau(1-a)}(u_n + o(u_n))^2 sgn(u_n). \tag{2.31}$$

We omit elements of higher order in the square

$$\begin{aligned} u_{n+1} &= u_n - \frac{b}{\tau(1-a)} u_n^2 sgn(u_n) + o(u_n^2) \\ &= u_n - \frac{b}{\tau(1-a)} u_n |u_n| + o(u_n^2). \end{aligned} \tag{2.32}$$

For $|u_n| < \varepsilon$ and sufficient small $\varepsilon > 0$ it holds true

$$o(u_n^2) < \frac{b}{2\tau(1-a)} u_n^2. \tag{2.33}$$

Substituting (2.33) into (2.32) we have

$$\begin{aligned} |u_{n+1}| &< |u_n| - \frac{b}{\tau(1-a)} u_n^2 + \frac{b}{2\tau(1-a)} u_n^2 \\ &< |u_n| \Big(1 - \frac{b}{2\tau(1-a)} |u_n| \Big). \end{aligned} \tag{2.34}$$

To prove the theorem we need to consider two cases. First let $a < 1$.

We have to solve the following problem

$$0 < 1 - \frac{b}{2\tau(1-a)} |u_n| < 1. \tag{2.35}$$

We have $(1-a) > 0$ and recall that $b, \tau > 0$. This implies that the right-hand inequality holds true. The left-hand inequality holds true if $|u_n| < \frac{2\tau(1-a)}{b}$.

If $|u_n| < \min\big\{\delta, \varepsilon, \frac{2\tau(1-a)}{b}\big\}$, following inequality holds true

$$|u_{n+1}| < |u_n|. \tag{2.36}$$

This implies that $\hat{u} = 0$ is asymptotically stable point of the system (2.30) and the center manifold theory guarantees that the first part of theorem holds true (i.e. $\hat{y}$ is asymptotically stable for $a < 1$).

On the other hand if $a > 1$, we have $\lambda_1 > 1$ as we have proved above in part 2.2.1. This implies instability of the origin. $\square$

3 Numerical simulations

In this section we present numerical simulations of the model. All simulations are performed in mathematical software *Matlab 7.0*.

Remark 3.1. *The value of parameter b does not affect the solution of model directly. By substituting $\tilde{x}(t) = bx(t)$ into* (1.1) *and multiplying both sides of equation by b we have*

$$\tilde{x}(t + \Delta t) = \tilde{x}(t) + \Big[\ a\big(\tilde{x}(t) - \tilde{x}(t-1)\big) - |\tilde{x}(t)|\tilde{x}(t)\ \Big]\Delta t. \tag{3.1}$$

The parameter b sets the scale of $x(t)$. Increase in b decreases $x(t)$ and vice versa.

According to Remark 3.1 the value of parameter b is held constant in the presented sample. Without loss of generality we set $b = 1$ in all examples of trajectories of the model.

3.1 Boundedness

For a certain ratio of parameters a and $\tau = \frac{1}{\Delta t}$, there exists a boundary for trajectories having their initial values inside this boundary (Theorem 1.1).

We set the value of parameter $a > 1$ in all examples. The simulations indicate instability of the model at its only fixed point $\hat{x} = 0$ and the solution oscillates around zero. The model with $a < 1$ is asymptotically stable so we expect that trajectory of such model does not exceed the initial boundary (we do not present any example of such model in this section). The trajectory of model with $a = 1$ is analyzed in Section 3.3.

We set the initial values of trajectories in the examples as points lying on curve

$$x(t) = t, \quad t = \frac{k}{\tau}, \text{ for } k \in \{-\tau, \ldots, 0\}. \tag{3.2}$$

Figure 3.1 presents trajectories of the model with ratio of parameters $\frac{a}{\tau} = 0.3$. [The first trajectory is a solution of the model with $\tau = 9$ and the second one has $\tau = 20$.] The ratio is less than $\frac{1}{3}$ so the boundary $\mathcal{A} = \frac{2a}{b}$. [We have $\mathcal{A} = 5.4$ and $\mathcal{A} = 12$ in the first and the second case, respectively.]

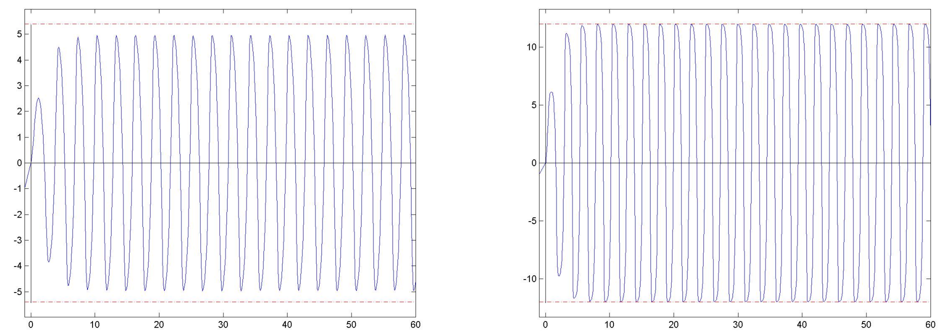

Figure 3.1: Ratio $\frac{a}{\tau} = 0.3$

Both solutions of model presented in Figure 3.2 have ratio of the parameters $\frac{a}{\tau} = 0.6$. [The value of parameter τ is the same as before.] Hence $\frac{a}{\tau} \in (\frac{1}{3}, 4\sqrt{2} - 5)$ and the boundary $\mathcal{A} = \frac{(a+\tau)^2}{4b(\tau-a)}$. [We have $\mathcal{A} = 14.4$ and $\mathcal{A} = 32$ in the first and the second case, respectively.] The trajectory of both models exceeds the lower boundary $\tilde{\mathcal{A}} = \frac{2a}{b}$. [$\tilde{\mathcal{A}} = 10.8$ and $\tilde{\mathcal{A}} = 24$ for the first and the second trajectory, respectively.]

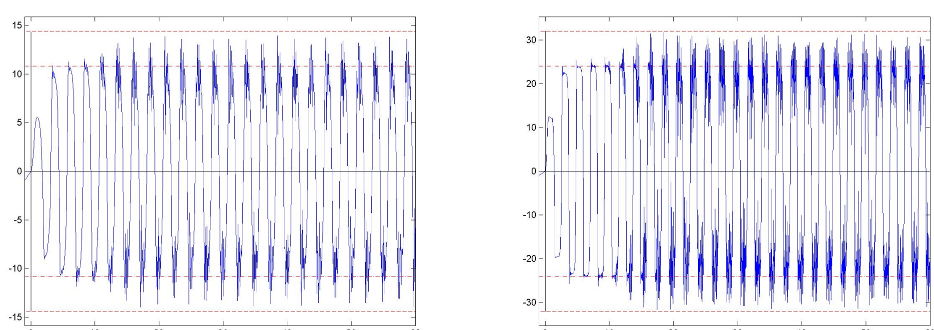

Figure 3.2: Ratio $\frac{a}{\tau} = 0.6$

3.2 Stability

Stability of the model in its fixed point $\hat{x} = 0$ depends only on the value of parameter a (Theorem 2.4).

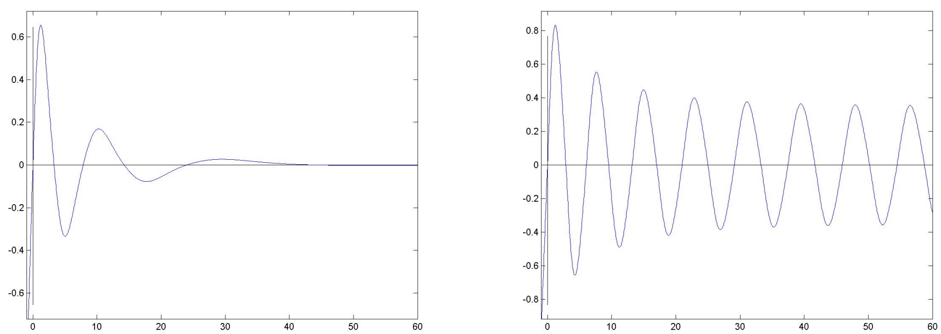

Figure 3.3: Stability; $\tau = 20$

The first trajectory in Figure 3.3 is the solution of model with $a = 0.9$, i.e. the model is asymptotically stable in its fixed point. On the other hand the second trajectory belongs to the model with $a = 1.1$ and the solution is unstable.

3.3 Trajectories in case $a = 1$

For $a = 1$, the linearization of model at its fixed point has two eigenvalues lying on the unit circle. Also the numerical simulations confirm that the stability of such system is not very transparent.

We set the initial values for all solutions in this section as

$$x(t) = \sin(t), \quad t = \frac{k}{\tau}, \text{ for } k \in \{-\tau, \dots, 0\}. \tag{3.3}$$

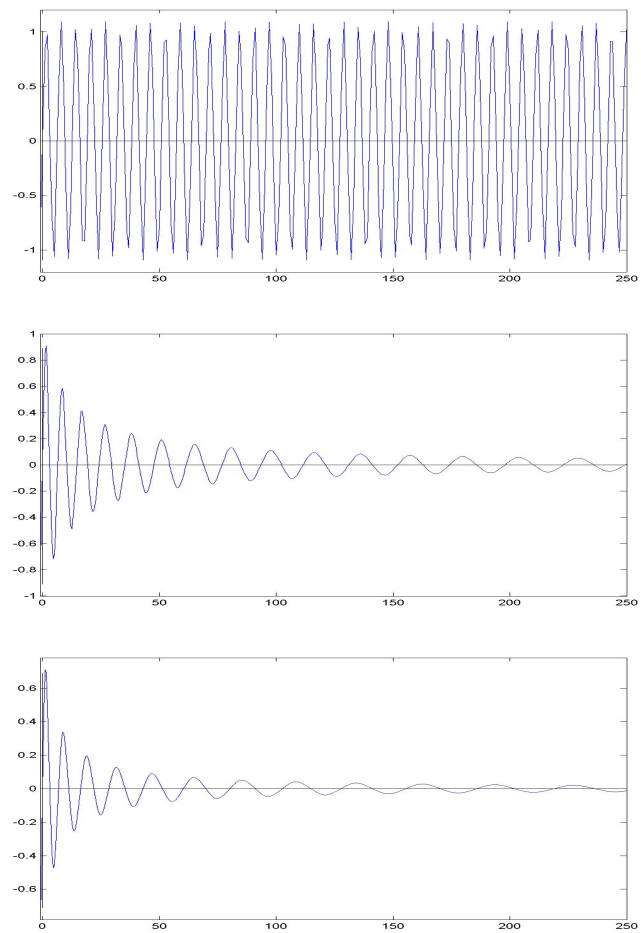

Figure 3.4: Trajectories; $a = 1$

The first trajectory in Figure 3.4 is the solution of model with the value of parameter $\tau = 1$. According to the simulations, amplitudes of the trajectories of rest of solutions seem to be decreasing in t. In fact, there is no evidence for such statement. The model of the second and the third trajectory has the value of parameter $\tau = 2$ and $\tau = 20$, respectively.

Conclusions

In this paper we analyze the model for deviations from the short term equilibrium of foreign exchange rate. The original model, introduced in ERDÉLYI (2003) [9], has a form of delayed differential equation - reflects the situation where decisions are made continuously. We modify the model into difference equation - where decisions are made in certain time moments. Although there were earlier works (BOĎOVÁ (2004) [3] and SZOLGAYOVÁ (2006) [14]) analyzing the discrete version of model, this approach needed an extra factor to be included into model to keep the boundedness of the trajectories. To eliminate this factor we use smaller step instead.

Our aim was to analyze particular properties of the model with continuous time that were causing difficulties in the discretization. First we set the values of parameters and boundary for the trajectories with bounded initial values (by the same boundary).

We found that eigenvalues of the linearization of the model are not outside the unit circle (of complex plane) for $a < 1$. Since there is always one eigenvalue lying on the unit circle, we had to make further analysis of stability of the system. The system (belonging to the model) is asymptotically stable for $a < 1$ and is unstable for $a > 1$.

As we already stated, the model presented in this paper can be understood as a bridge between the model with continuous and discrete time. The property of boundedness of trajectories and their limits (for bounded and unbounded initial values, respectively) observed in the continuous time model is vanishing in the discretization. According to numerical simulations and analysis of our model there exists a range, increasing with τ, where the property holds true. Although we decrease the step to keep the property of boundedness, other properties, as stability, are not affected by the discretization.

One can say that the model is not a sufficient image of the real world behavior of foreign exchange rate. We must agree that the basic version of model is too simple to be good approximation of the reality, but there are various ways to improve the model by employing some stochastic feature into it (e.g. $a \to a(t)$, $b \to b(t)$, etc.).

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