FAKULTA MATEMATIKY, FYZIKY A INFORMATIKY
UNIVERZITY KOMENSKÉHO V BRATISLAVE

DIPLOMOVÁ PRÁCA

Bratislava 2008

Miloš Gabriš

FAKULTA MATEMATIKY, FYZIKY A INFORMATIKY
UNIVERZITY KOMENSKÉHO V BRATISLAVE

Ekonomická a finančná matematika

RIEŠENIE VEĽKÝCH SÚSTAV LINEÁRNYCH ROVNÍC MODERNÝMI METÓDAMI

Diplomová práca

Diplomant : Miloš Gabriš
Vedúci diplomovej práce : doc. RNDr. Milan Hamala, CSc. Bratislava 2008

Čestne prehlasujem, že som túto prácu vypracoval samostatne, len s použitím literatúry uvedenej v zozname.

V Bratislave 26. apríla 2008

Miloš Gabriš

Poďakovanie

Aj touto cestou by som sa chcel poďakovať svojmu vedúcemu diplomovej práce doc. RNDr. Milanovi Hamalovi, CSc. za jeho odborné vedenie, poskytovanie cenných rád a za množstvo času, ktorý mi venoval pri vypracovávaní diplomovej práce.
Taktiež, veľké ďakujem, patrí mojim rodičom za podporu a trpezlivosť počas štúdia.

Abstrakt

V tejto práci by sme radi čitateľa oboznámili s metódami pre riešenie veľkých riedkych sústav lineárnych rovníc.

1. kapitola oboznamuje čitateľa o vhodných prístupoch pri riešení riedkych sústav, t.j. aké sú výhody riedkej matice voči hustej matici, ako sa to dajú preniesť tieto výhody do efektívnosti algoritmu, ktorý má riešiť našu sústavu a uvedime si algoritmy pre tvorbu vzájomne konjungovaných vektorov.

2. kapitola nás prevedie metódami na riešenie ekvivalentnej sústavy lineárnych rovníc, a to, OrthoDir a GCR, GMRes metódou, LSQR a SymmLQ metódami.

3. kapitola obsahuje popis k dosiahnutím výsledkom z numerického experimentu, kde sme testovali 6 metód pre stovky vygenerovaných úloh $\boldsymbol{Ax} = \boldsymbol{b}$ s kladne definitnou maticou $\boldsymbol{A}$.

Obsah

Kapitola 1

Úvod - motivácia

Našou snahou je riešiť sústavu lineárnych rovníc

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{1.1}$$

kde $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ (tzv. matica koeficientov) je veľká, regulárna, a najmä, riedka matica, a $\boldsymbol{b} \in \mathbb{R}^n$ (tzv. vektor pravých strán).

Prečo požadujeme riedku maticu? Aké sú to riedke matice?

1.1 Riedke matice

Ide o triedu matíc, ktorých väčšina prvkov je rovná nule . To nám umožňuje

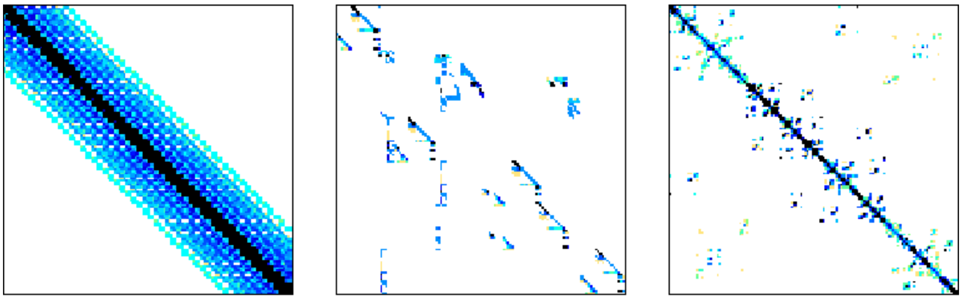
- *úspornejšie uloženie v pamäti počítača*, a tiež
- *menší počet operácií*, napr. pri maticovo-vektorovom násobení.

Len pre ilustráciu spomeňme najjednoduchší, a snáď aj najprirodzenejší, spôsob uloženia riedkych matíc. Miesto potreby dvojrozmerného poľa, ktoré vlastne kopíruje štruktúru matice, vystačíme si s tromi jednorozmernými poľami, povedzme $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{s}$ a dvomi hodnotami m, n. Pole $\boldsymbol{s}$ bude obsahovať samotné nenulové hodnoty pôvodnej matice, polia $\boldsymbol{i}, \boldsymbol{j}$ potom ich riadkové, resp. ich stĺpcové indexy. Teda, napr. $k-$ta nenulová hodnota s_k má umiestnenie v pôvodnej matici (i_k, j_k). To vyžaduje rovnakú veľkosť týchto polí. Hodnoty m, n slúžia na zachytenie rozmerov matice.
Z uvedeného ďalej plynie, že jedna nenulová hodnota je popísaná tromi hodnotami (i_k, j_k, s_k) a teda má zmysel uvažovať tento spôsob uloženia matice, len ak počet nenulových hodnôt neprevýši tretinu všetkých prvkov matice.

Čo sa týka menšieho počtu operácií, pouvažujme nad týmto:
Majme vektor $\boldsymbol{u}$ dĺžky n s počtom nenúl $k \ll n$. Ďalej označme vektor $\boldsymbol{v} = (1, 2, 3, \ldots, n)^T$. Teraz sa pokúsme vyčísliť počet aritmetických operácií, potrebných na výpočet ich skalárneho súčinu, t.j. $z = \boldsymbol{u}^T\boldsymbol{v} = \sum_{i=1}^{n} u_i v_i$. Nie je príliš ťažké si uvedomiť, že vo všeobecnosti je potreba n násobení $(u_i v_i)$ a $n-1$ sčítaní $(i = 1, \ldots, n)$, teda spolu $2n-1$ operácií.

Ak vezmeme do úvahy riedkosť vektora $\boldsymbol{u}$, ušetríme $n-k$ násobení a $n-k-1$ sčítaní (spolu $2(n-k)-1$ operácií).
Bez újmy na všeobecnosti (BÚNV) môžeme predpokladať, že riedka matica rozmerov $m \times n$, ktorá má rovnaké relatívne zastúpenie nenulových hodnôt ako vektor $\boldsymbol{u}$ (t.j. k/n), pozostáva z m riadkov s riedkosťou ako vektor $\boldsymbol{u}$, a teda počet aritmetických operácií, ktorých sa môžeme vyvarovať v prípade výpočtu $\boldsymbol{Av}$ je $[2(n-k)-1]m$.
Ak má matica $\boldsymbol{A}$ rozmery, povedzme $m = n = 10^5$ a riedkosť 5%, (t.j. $k/n = 0.05$), bude ušetrených operácií 999 900 000 $\approx$ 1mld.

V praxi nie je ničím výnimočným stretnúť sa s riedkymi maticami. Na obr.1.1 sú niektoré z nich. Ide o štvorcové matice.



Obrázok 1.1: Riedke matice v praxi

(1) $n = 36\,057,\ 335\,552$ nenúl, riedkosť $\doteq 0.026\%$, el. obvody

(2) $n = 497,\ 1\,721$ nenúl, riedkosť $\doteq 0.69\%$, chem. proces

(3) $n = 14\,822,\ 715\,804$ nenúl, riedkosť $\doteq 0.32\%$, dyn. kvapalín

Tieto i mnohé ďalšie riedke matice spolu s ich základnými vlastnosťami (ako symetria, regularita a pod.) môžete nájsť na *http://www.cise.ufl.edu/research/sparse/matrices.*

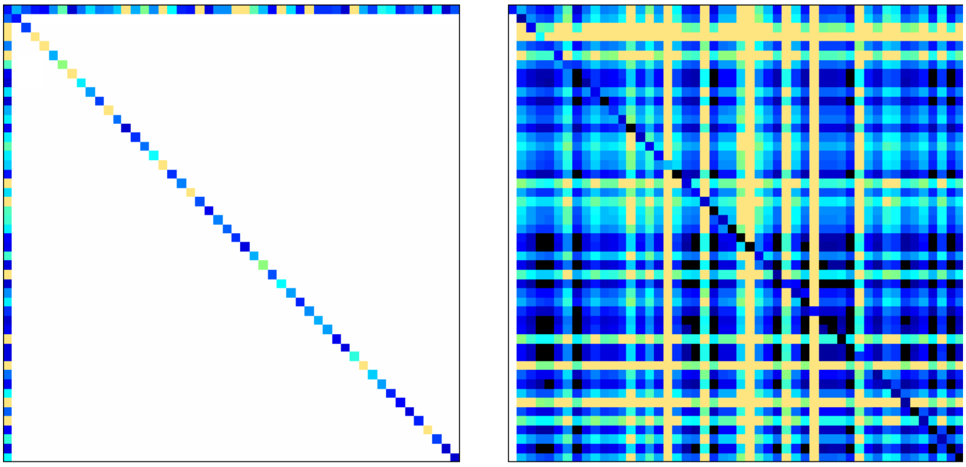
Keďže metódy, ktorými sa chceme zaoberať sú vo všeobecnosti výpočtovo zložitejšie ako najznámejšia Gaussova eliminačná metóda (GEM), príp. iné metódy, nemalo by zmysel vôbec nad nimi uvažovať. Ale riedkosť matice (a teda jej už spomínané výhody), ako jedna z požadovaných vlastností matice koeficientov systému, spôsobí ohromnú prevahu práve týchto metód.

1.2 Nevhodné metódy a iný prístup

Požiadavky, ktoré kladieme na metódy riešiace problém (1.1) sú

- *nevnášanie nových nenulových hodnôt do matice* $\boldsymbol{A}$
- *(teoreticky) presné riešenie po konečnom počte krokov*

To vylučuje také metódy ako GEM a Choleského rozklad, ktoré modifikujú maticu $\boldsymbol{A}$ a tiež Jacobiho metódu a SOR metódu, pretože nepočítajú presné riešenie (ani teoreticky, t.j. s presnou aritmetikou). Čo presne znamená modifikácia matice $\boldsymbol{A}$ v prípade metódy GEM si ukážeme na malom príklade.



(1) (2)

Obrázok 1.2: Vplyv GEM na štruktúru riedkej matice

(1) $n = 50$, 148 nenúl, 1 980 Bytov[1]

(2) $n = 50$, 2451 nenúl, 20 000 Bytov[2]

Ako môžeme vidieť, už prvá fáza GEM-u, kedy sa nuluje prvý stĺpec matice $\boldsymbol{A}$ pomocou jej prvého riadku spôsobí, že matica $\boldsymbol{A}$ stratí riedkosť, ktorá je necelých 6% a stane sa plnou maticou.

Postup, ktorý si zvolíme, aby sme sa vyvarovali nežiadúcim efektom, je generovanie postupnosti aproximačných riešení $\boldsymbol{x}_i$ rekurentným vzťahom:

$$\boldsymbol{x}_{i+1} = \boldsymbol{x}_i + \boldsymbol{c}_i \tag{1.2}$$

kde $\boldsymbol{x}_j$ je j-ta aproximácia riešenia sústavy (1.1), $\boldsymbol{c}_j$ je j-ta korekcia a $\boldsymbol{x}_1$ je ľubovoľné.

[1] v MATLAB-e, uložená ako riedka matica
[2] v MATLAB-e, uložená ako plná matica, t.j. ako dvojrozmerné pole

Korekciu vyberieme tak, aby sme

- *zmenšili normu chyby aproximácie* a/alebo
- *redukovali dimenziu podpriestoru, v ktorom chyba aproximačného riešenia leží*

Do ďalších úvah zahrnieme aj tzv. *ekvivalentný systém*

$$\boldsymbol{G}\boldsymbol{x} = \boldsymbol{h}\,, \quad \boldsymbol{G} \succ 0 \tag{1.3}$$

k systému $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, kde $\boldsymbol{G}$ je kladne definitná ($\boldsymbol{G} \succ 0$). Ekvivalencia systémov spočíva v tom, že oba majú rovnaké riešenie $\boldsymbol{x}^*$. Teda $\boldsymbol{A}\boldsymbol{x}^* = \boldsymbol{b}$ a tiež $\boldsymbol{G}\boldsymbol{x}^* = \boldsymbol{h}$. (Z toho vyplývajú aj vzťahy: $\boldsymbol{x}^* = \boldsymbol{A}^{-1}\boldsymbol{b} = \boldsymbol{G}^{-1}\boldsymbol{h}$ a $\boldsymbol{h} = \boldsymbol{G}\boldsymbol{A}^{-1}\boldsymbol{b}$). Jedným zo spôsobov ako riešiť pôvodnú sústavu lineárnych rovníc je, najprv, skonštruovanie príslušného ekvivalentného systému k pôvodnému systému a potom jeho vyriešenie.
Napr. systém normálnych rovníc $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{b}$, môže taktiež poslúžiť ako ekvivalentný systém ($\boldsymbol{G} = \boldsymbol{A}^T\boldsymbol{A}$, $\boldsymbol{h} = \boldsymbol{A}^T\boldsymbol{b}$). Nevýhodou tohto systému však je, že číslo podmienenosti $\mathrm{c}(\boldsymbol{A}^T\boldsymbol{A})$ matice $\boldsymbol{A}^T\boldsymbol{A}$ je druhou mocninou čísla podmienenosti $\mathrm{c}(\boldsymbol{A})$ matice $\boldsymbol{A}$, t.j. $\mathrm{c}(\boldsymbol{A}^T\boldsymbol{A}) = \mathrm{c}^2(\boldsymbol{A})$. To môže spôsobiť nemalé numerické ťažkosti, ak $\mathrm{c}(\boldsymbol{A})$ je veľké.
Teraz treba ešte rozhodnúť, z kade budeme vyberať korekciu $\boldsymbol{c}_i$ v (1.2), resp. akú dimenziu má podpriestor, v ktorom leží. Pri malej snahe si ľahko uvedomíme, že ak $\boldsymbol{x}_1$ je ľubovoľná (nastrelená) hodnota prvej aproximácie $\boldsymbol{x}^*$, $\boldsymbol{c}_1 \in \mathbb{R}^n$ (t.j. leží v (pod)priestore dimenzie n) a volíme túto korekciu podľa prvého prístupu uvedeného vyššie; aby sme minimalizovali normu chyby aproximácie; potom $\boldsymbol{x}_2$ musí byť rovné $\boldsymbol{x}^*$. Vyzerá to síce jednoducho, avšak by sme sa potýkali s riešením rovnako obtiažnej sústavy rovníc ako je pôvodná sústava. Preto sa obmedzíme na metódy, v ktorých $\boldsymbol{c}_i$ leží v podpriestore dimenzie 1. Čiže môžeme písať, že $\boldsymbol{c}_i \in z\boldsymbol{p}_i$, kde $\boldsymbol{p}_i$, $\boldsymbol{p}_i \neq 0_n$ je bázovým (vytvárajúcim) vektorom tohto podpriestoru a $z \in \mathbb{R}$ je parameter. $\boldsymbol{c}_i$ je potom vyjadrená cez vzťah $\boldsymbol{c}_i = z_i\boldsymbol{p}_i$, pre nejaké vhodný skalár $z = z_i$. Ak to dáme do súvisu s rovnicou (1.2), dostaneme :

$$\boldsymbol{x}_{i+1} = \boldsymbol{x}_i + z_i\boldsymbol{p}_i$$

resp.

$$\boldsymbol{x} \equiv \boldsymbol{x}(z) = \boldsymbol{x}_i + z\boldsymbol{p}_i \tag{1.4}$$

(Ide o parametrické vyjadrenie afínneho podpriestoru dimenzie 1 (priamky), v ktorom leží nová aproximácia $\boldsymbol{x}_{i+1}$ (vektor $\boldsymbol{p}_i$ sa tiež nazýva smerový vektor priamky)).

Aby sme sa mohli dostať v teórii ďalej, treba nám ešte zvoliť normu, ktorú pre jej vhodné vlastnosti budeme využívať a tiež prijať nejaké pomocné označenia a vzťahy.
Aj keď sú všetky normy na $\mathbb{R}^n$ navzájom ekvivalentné, nami zvolená tzv. *eliptická norma* má predsa len výsosnejšie postavenie. Takto je definovaná :

$$\|x\|_E \equiv \sqrt{x^T \boldsymbol{E} x} \tag{1.5}$$

kde $\boldsymbol{E} \succ 0$. O akú konkrétnu maticu $\boldsymbol{E}$ pôjde sa dozvieme v nasledujúcom texte.
Označme

$$\boldsymbol{e} \equiv \boldsymbol{e}(z) \equiv \boldsymbol{x}(z) - \boldsymbol{x}^* \tag{1.6}$$

tzv. *vektor chyby* (error vector), ktorý má len teoretický význam, nakoľko $\boldsymbol{x}^*$ nepoznáme.

$$\boldsymbol{r} \equiv \boldsymbol{r}(z) \equiv \boldsymbol{A}\boldsymbol{x}(z) - \boldsymbol{b}$$

tzv. *vektor rezíduí* (residual vector), ktorého význam, naopak, je čisto praktický, keďže platí :

$$\boldsymbol{r} = \boldsymbol{A}\boldsymbol{e}\,, \text{ resp } \quad \boldsymbol{e} = \boldsymbol{A}^{-1}\boldsymbol{r}$$

Pre i-tu aproximáciu $\boldsymbol{x}_i$ použijeme nasledovné označenie i-teho vektora chyby a i-teho vektora rezíduí

$$\boldsymbol{e}_i = \boldsymbol{x}_i - \boldsymbol{x}^* \text{ a } \quad \boldsymbol{r}_i = \boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{b}.$$

Obdobné označenia máme aj pre ekvivalentný systém

$$\begin{aligned} \boldsymbol{f} &\equiv \boldsymbol{f}(z) \equiv \boldsymbol{G}\boldsymbol{x}(z) - \boldsymbol{h} \\ \boldsymbol{f} &= \boldsymbol{G}\boldsymbol{e}\,, \text{ resp } \ \boldsymbol{e} = \boldsymbol{G}^{-1}\boldsymbol{f} \\ \boldsymbol{f}_i &= \boldsymbol{G}\boldsymbol{x}_i - \boldsymbol{h} \end{aligned} \tag{1.7}$$

1.2.1 Redukcia normy chyby

Pod redukciou normy chyby, resp. štvorca normy chyby, rozumieme voľbu $\boldsymbol{c}_i$, resp. z_i takú, aby

$$z_i \equiv arg \min_{z \in \mathbb{R}} \|\boldsymbol{e}\|_E^2 \,.$$

Keď to rozpíšeme a označíme si

$$\boldsymbol{E}_* = \boldsymbol{G}^{-1}\boldsymbol{E}\boldsymbol{G}^{-1}$$

dostávame :

$$\begin{aligned} \|\boldsymbol{e}\|_E^2 &\stackrel{(1.7)}{=} \left\|\boldsymbol{G}^{-1}\boldsymbol{f}\right\|_E^2 \stackrel{(1.5)}{=} \|\boldsymbol{f}\|_{E_*}^2 = \|\boldsymbol{G}\boldsymbol{x} - \boldsymbol{h}\|_{E_*}^2 = \\ &= \|\boldsymbol{G}(\boldsymbol{x}_i + \boldsymbol{p}_i z) - \boldsymbol{h}\|_{E_*}^2 = \|(\boldsymbol{G}\boldsymbol{x}_i - \boldsymbol{h}) + \boldsymbol{G}\boldsymbol{p}_i z\|_{E_*}^2 = \\ &= \|\boldsymbol{f}_i + \boldsymbol{G}\boldsymbol{p}_i z\|_{E_*}^2 \stackrel{(1.5)}{=} \left(\boldsymbol{f}_i^T + \boldsymbol{p}_i^T \boldsymbol{G} z\right) \boldsymbol{G}^{-1}\boldsymbol{E}\boldsymbol{G}^{-1} \left(\boldsymbol{f}_i + \boldsymbol{p}_i \boldsymbol{G} z\right) = \\ &= \|\boldsymbol{f}_i\|_{E_*}^2 + 2\left(\boldsymbol{p}_i^T \boldsymbol{E}\boldsymbol{G}^{-1}\boldsymbol{f}_i\right) z + \left(\boldsymbol{p}_i^T \boldsymbol{E}\boldsymbol{p}_i\right) z^2 \end{aligned}$$

Pozn.: Pri odvádzaní (1.8) sme využili

- $\boldsymbol{G} = \boldsymbol{G}^T \Rightarrow \boldsymbol{G}^{-1} = \left(\boldsymbol{G}^{-1}\right)^T \triangleq \boldsymbol{G}^{-T}$
- $\boldsymbol{G} \succ 0$ $(\Rightarrow \boldsymbol{G}^{-1} \succ 0)$, $\boldsymbol{E} \succ 0 \Rightarrow \boldsymbol{E}_* = \boldsymbol{G}^{-1}\boldsymbol{E}\boldsymbol{G}^{-1} \succ 0$

Jedná sa o nájdenie minima konvexnej kvadratickej funkcie jednej premennej z. Na to nám poslúži podmienka prvého rádu (***F***irst ***O***rder ***C***ondition).

FOC :

$$\left.\frac{\mathrm{d}\,\|\boldsymbol{e}\|_E^2}{\mathrm{d}z}\right|_{z=z_i} = 0$$

$$2z_i\left(\boldsymbol{p}_i^T\boldsymbol{E}\boldsymbol{p}_i\right) + 2\left(\boldsymbol{p}_i^T\boldsymbol{E}\boldsymbol{G}^{-1}\boldsymbol{f}_i\right) = 0$$

$$\Downarrow$$

$$z_i = -\left(\boldsymbol{p}_i^T\boldsymbol{E}\boldsymbol{p}_i\right)^{-1}\left(\boldsymbol{p}_i^T\boldsymbol{E}\boldsymbol{G}^{-1}\boldsymbol{f}_i\right) \tag{1.8}$$

Tu už môžeme uvažovať o vhodnom výbere matice $\boldsymbol{E}$. Aby sme, čo možno najviac, zjednodušili výraz $\boldsymbol{E}\boldsymbol{G}^{-1}$, zvolíme $\boldsymbol{E} = \boldsymbol{G}$. Potom

$$z_i = -\left(\boldsymbol{p}_i^T\boldsymbol{G}\boldsymbol{p}_i\right)^{-1}\left(\boldsymbol{p}_i^T\boldsymbol{f}_i\right) = -\frac{\left(\boldsymbol{p}_i^T\boldsymbol{f}_i\right)}{d_i} \tag{1.9}$$

Samozrejme

$$d_i \equiv \boldsymbol{p}_i^T\boldsymbol{G}\boldsymbol{p}_i \neq 0 \tag{1.10}$$

lebo $\boldsymbol{p}_i \neq 0_n$ a $\boldsymbol{G} \succ 0$.
Ak dosadíme z (1.9) hodnotu z_i do (1.2), získame iteračný vzťah :

$$\boldsymbol{x}_{i+1} = \boldsymbol{x}_i - \boldsymbol{p}_i\,\frac{\boldsymbol{p}_i^T\boldsymbol{f}_i}{d_i} \tag{1.11}$$

Vynásobením maticou $\boldsymbol{G}$ zľava

$$\boldsymbol{G}\boldsymbol{x}_{i+1} = \boldsymbol{G}\boldsymbol{x}_i - \boldsymbol{G}\boldsymbol{p}_i\,\frac{\boldsymbol{p}_i^T\boldsymbol{f}_i}{d_i}$$

a ďalej odčítaním, od oboch strán rovnice, vektora pravých strán

$$\boldsymbol{G}\boldsymbol{x}_{i+1} - \boldsymbol{h} = \boldsymbol{G}\boldsymbol{x}_i - \boldsymbol{h} - \boldsymbol{G}\boldsymbol{p}_i\,\frac{\boldsymbol{p}_i^T\boldsymbol{f}_i}{d_i}$$

dostávame

$$\boldsymbol{f}_{i+1} = \boldsymbol{f}_i - \boldsymbol{G}\boldsymbol{p}_i\,\frac{\boldsymbol{p}_i^T\boldsymbol{f}_i}{d_i}\,. \tag{1.12}$$

Všimnime si, že ak by v (1.11) výraz $\boldsymbol{p}_i^T \boldsymbol{f}_i$ bol nulový ($\boldsymbol{p}_i^T \boldsymbol{f}_i = 0$), potom $\boldsymbol{x}_{i+1} = \boldsymbol{x}_i$. Čiže nová aproximácia $\boldsymbol{x}_{i+1}$ by nám nepriniesla pokles hodnoty účelovej funkcie $\|\boldsymbol{e}\|_G$. A preto všetky „inteligentné“ metódy tento výber $\boldsymbol{p}_i$ vylučujú. Ďalej teda predpokladajme, naviac ku nenulovosti $\boldsymbol{p}_i$, že $\boldsymbol{p}_i^T \boldsymbol{f}_i \neq 0$. Následne, z týchto dvoch predpokladov o $\boldsymbol{p}_i$, vyplýva

$$z_i = -\left(\boldsymbol{p}_i^T \boldsymbol{G} \boldsymbol{p}_i\right)^{-1} \left(\boldsymbol{p}_i^T \boldsymbol{f}_i\right) = -\frac{\left(\boldsymbol{p}_i^T \boldsymbol{f}_i\right)}{d_i} \neq 0. \tag{1.13}$$

a tiež

$$\boldsymbol{x}_{i+1} \overset{(1.13)}{=} \boldsymbol{x}_i - \underbrace{\boldsymbol{p}_i \frac{\boldsymbol{p}_i^T \boldsymbol{f}_i}{d_i}}_{\neq 0_n} \neq \boldsymbol{x}_i \tag{1.14}$$

Pozrime sa, teraz, na niektoré vlastnosti $\boldsymbol{x}_{i+1}$ a $\boldsymbol{f}_{i+1}$:

V1: Jednoznačnosť a existencia $\boldsymbol{x}_{i+1}$ (na to postačuje $\boldsymbol{G} \succ 0$ a $\boldsymbol{p}_i \neq 0_n$, aby $d_i \neq 0$)

V2:

$$\|\boldsymbol{e}_{i+1}\|_G = \|\boldsymbol{x}_{i+1} - \boldsymbol{x}^*\|_G < \|\boldsymbol{x}_i - \boldsymbol{x}^*\|_G = \|\boldsymbol{e}_i\|_G \tag{1.15}$$

Toto má veľmi jednoduché zdôvodnenie. Obe hodnoty $\boldsymbol{x}_{i+1}$ aj $\boldsymbol{x}_i$ ležia na tej istej priamke $\boldsymbol{x}(z) = \boldsymbol{x}_i + z\boldsymbol{p}_i$, pričom $\boldsymbol{x}_{i+1} \neq \boldsymbol{x}_i$ (1.14). Ale len $\boldsymbol{x}_{i+1}$, ako jediné, minimalizuje $\|\boldsymbol{e}(z)\|_G^2 = \|\boldsymbol{x}(z) - \boldsymbol{x}^*\|_G^2$. Z toho, teda, $\|\boldsymbol{x}_{i+1} - \boldsymbol{x}^*\|_G^2 < \|\boldsymbol{x}_i - \boldsymbol{x}^*\|_G^2$. Teraz aplikujme na túto nerovnosť monotónne rastúcu funkciu $y = \sqrt{x}$ a vlastnosť V2 (1.15) ihneď vyplynie.

V3:

$$\boldsymbol{p}_i^T \boldsymbol{f}_{i+1} = 0\ , \tag{1.16}$$

tzv. *Galerkinova podmienka.* Stačí vynásobiť (1.12) zľava vektorom $\boldsymbol{p}_i^T$:

$$\boldsymbol{p}_i^T \boldsymbol{f}_{i+1} = \boldsymbol{p}_i^T \left(\boldsymbol{f}_i - \underbrace{\boldsymbol{p}_i^T \boldsymbol{G} \boldsymbol{p}_i}_{d_i} \frac{\boldsymbol{p}_i^T \boldsymbol{f}_i}{d_i} \right) = \boldsymbol{p}_i^T \boldsymbol{f}_i - \boldsymbol{p}_i^T \boldsymbol{f}_i = 0.$$

Slovami povedané: $(i+1)$ - vé rezíduum ekvivalentného systému leží v ortogonálnom doplnku vektora $\boldsymbol{p}_i$.

Aké $\boldsymbol{p}_i$ voliť? Spomeňme zopár možností:

- stĺpce matice $\boldsymbol{I}_n$, $\boldsymbol{A}$ alebo $\boldsymbol{A}^T$ vo všeobecnosti brané cyklicky (v rámci jednej matice),
- $\boldsymbol{p}_i = -\boldsymbol{f}_i$ (smer najprudšieho spádu funkcie $F(\boldsymbol{x}) = \frac{1}{2}\,\boldsymbol{x}^T \boldsymbol{G} \boldsymbol{x} + \boldsymbol{h}^T \boldsymbol{x}$, viď str. 14–15)

Hoci norma $\|e\|_G$ monotónne klesá, nie je garantované, že klesá k nule, ani, že pôjde o rýchlu konvergenciu.
Príklad 1
Máme riešiť sústavu 2 rovníc o 2 neznámych $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, kde

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \left[\boldsymbol{x}^* = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]$$

Nech náš ekvivalentný systém bude systémom normálnych rovníc $\boldsymbol{A}^T \boldsymbol{A} \boldsymbol{x} = \boldsymbol{A}^T \boldsymbol{b}$, t.j.

$$\boldsymbol{G} = \boldsymbol{A}^T \boldsymbol{A} = \begin{pmatrix} 10 & -10 \\ -10 & 20 \end{pmatrix}, \quad \boldsymbol{h} = \boldsymbol{A}^T \boldsymbol{b} = \begin{pmatrix} 0 \\ 10 \end{pmatrix}$$

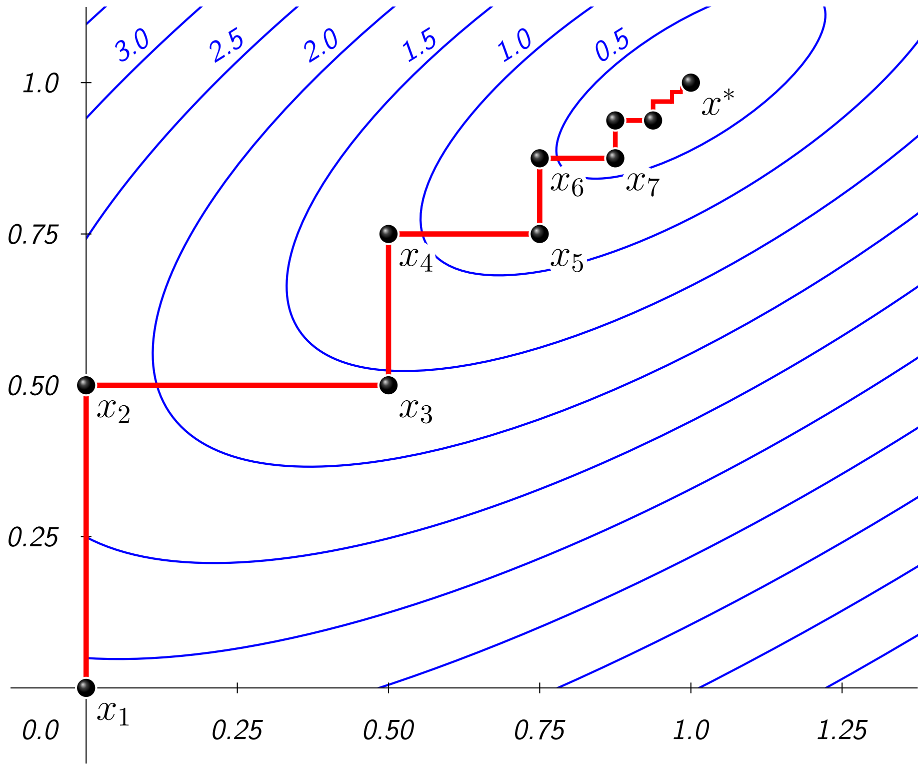
Zvoľme

$$\boldsymbol{x}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

a v iteračnom procese (1.11) definujme smery

$$\boldsymbol{p}_i = -\boldsymbol{f}_i = (\boldsymbol{h} - \boldsymbol{G}\boldsymbol{x}_i)$$

Riešme úlohu pomocou vzťahu (1.11). Pozrime sa na obrázok (1.3).



Obrázok 1.3: Priebeh iterácií

Elipsy na ňom predstavujú body rovnakej vzdialenosti od $\boldsymbol{x}^*$ v $\boldsymbol{G}-$norme. Sú to, vlastne, „vrstevnice“ funkcie

$$F(\boldsymbol{x}) = \frac{1}{2}\,\boldsymbol{x}^T\boldsymbol{G}\boldsymbol{x} + \boldsymbol{h}^T\boldsymbol{x}.$$

Prečo?
Chceme minimalizovať „štvorec“ normy $\|\boldsymbol{e}\|_G^2$, resp. nájsť $\hat{\boldsymbol{x}}$, ktoré minimalizuje túto normu. Ale

$$\begin{aligned}
\|\boldsymbol{e}\|_G^2 &= (\boldsymbol{x} - \boldsymbol{x}^*)^T\,\boldsymbol{G}\,(\boldsymbol{x} - \boldsymbol{x}^*) \\
&= \boldsymbol{x}^T\boldsymbol{G}\boldsymbol{x} - 2\,\underbrace{(\boldsymbol{x}^*)^T\,\boldsymbol{G}}_{\boldsymbol{h}}\,\boldsymbol{x} + \underbrace{(\boldsymbol{x}^*)^T\,\boldsymbol{G}\boldsymbol{x}^*}_{\text{const}} \\
&= 2\underbrace{\left(\frac{1}{2}\,\boldsymbol{x}^T\boldsymbol{G}\boldsymbol{x} - \boldsymbol{h}^T\boldsymbol{x}\right)}_{F(\boldsymbol{x})} + \text{const} \\
&= 2F(\boldsymbol{x}) + \text{const}.
\end{aligned}$$

Ak chceme zistiť polohu minima, $\hat{\boldsymbol{x}}$, funkcie $F(\boldsymbol{x})$, ktorá je rýdzokonvexná, stačí, aby bola splnená podmienka prvého rádu (FOC) :

$$\nabla F(\hat{\boldsymbol{x}}) = 0_n \;\Leftrightarrow\; \boldsymbol{G}\hat{\boldsymbol{x}} = \boldsymbol{h}.$$

A teda z jednoznačnosti riešenia ekvivalentného systému (1.3) $\boldsymbol{G}\boldsymbol{x} = \boldsymbol{h}$ máme

$$\boldsymbol{G}\hat{\boldsymbol{x}} = \boldsymbol{h} \stackrel{\exists!}{\Rightarrow} \boldsymbol{x}^* = \hat{\boldsymbol{x}}.$$

Riešenie sústavy $\boldsymbol{G}\boldsymbol{x} = \boldsymbol{h}$ je, teda, ekvivalentné nájdeniu minima funkcie $F(\boldsymbol{x})$.
Jednou z metód minimalizácie funkcie je aj tzv. *Cauchyho metóda najväčšieho spádu*, kedy sa smer optimalizácie v každej iterácii volí, práve, záporný gradient funkcie $F(\boldsymbol{x})$, (t.j. $-\nabla F(\boldsymbol{x}) = -\boldsymbol{f} = \boldsymbol{h} - \boldsymbol{G}\boldsymbol{x}$).
Z obrázku (1.3) vidíme, že postupnosť $\boldsymbol{x}_i$ vytvára „schody“ od $\boldsymbol{x}_1 = (0,0)^T$ k $\boldsymbol{x}^* = (1,1)^T$, pričom každý schod je o polovicu nižší a užší než ten predchádzajúci. Aj sedliackym rozumom si už vieme dať dohromady, že presné riešenie nezískame nikdy. Iba sa k nemu blížime. Ako to vidí numerika sa pozrieme do tabuľky 1.1, kde okrem nášho výberu $\boldsymbol{p}_i$ uvidíme aj správanie sa ostatných spomínaných možností voľby $\boldsymbol{p}_i$. Údajmi sú hodnoty normy $\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_G$.

	Výber $\boldsymbol{p}_i$			
Iterácia	$\boldsymbol{I}_{\bullet i}$	$\boldsymbol{A}_{\bullet i}$	$\boldsymbol{A}^T_{\bullet i}$	$-\boldsymbol{f}_i$
2	2.2361	2.2627	1.1767	1.5811
3	1.5811	2.0239	0.9791	1.1180
5	0.7906	1.6191	0.6778	0.5590
10	0.1398	0.9268	0.2703	0.0988
20	0.0044	0.3037	0.0430	0.0031
30	0.0001	0.0995	0.0068	0.0001
50	1.3E-07	0.0107	0.0002	9.4E-08
100	3.9E-15	4.0E-05	1.8E-08	2.8E-15

Tabuľka 1.1:

Presnosť sa zdá byť relatívne dobrá, avšak za akú cenu. 50-násobný počet iterácií voči dimenzii úlohy. A ako to bude pre nejaký vyšší rozmer úlohy?

Príklad 2 ($n = 100$)
V tomto prípade (tabuľka 1.2), okrem nelichotivého počtu iterácií, sme nedosiahli ani uspokojivú presnosť riešenia.
Treba sa lepšie zamyslieť nad výberom vektorov $\boldsymbol{p}_i$. Tu sa otvára priestor pre využitie vlastnosti V3 (1.16); Galerkinovej podmienky.

	Výber $\boldsymbol{p}_i$			
Iterácia	$\boldsymbol{I}_{\bullet i}$	$\boldsymbol{A}_{\bullet i}$	$\boldsymbol{A}^T_{\bullet i}$	$-\boldsymbol{f}_i$
2	96.564	97.615	94.926	31.540
3	95.465	96.813	90.771	24.331
5	94.452	94.198	90.667	17.537
10	91.049	93.155	81.085	10.850
20	87.887	89.762	74.624	6.3222
30	79.656	79.218	67.365	4.3467
50	62.797	71.142	62.231	2.5517
100	40.867	52.269	44.676	1.2919
500	9.2190	23.811	25.785	0.3150
1 000	6.0958	17.667	19.779	0.2138
2 000	3.6676	13.314	14.386	0.1430
3 000	2.4460	11.124	11.836	0.1035
5 000	1.2814	8.8584	9.4352	0.0606
10 000	0.5405	6.9612	6.8210	0.0194
20 000	0.2543	6.0092	4.3726	0.0022
30 000	0.1827	5.5725	3.4126	0.0004
50 000	0.1317	5.0208	2.7191	0.0003
100 000	0.0769	4.2751	2.0565	0.0002

Tabuľka 1.2:

1.2.2 Redukcia dimenzie podpriestoru rezíduí

Tento prístup voľby $\boldsymbol{p}_i$ je „sofistikovanejší“ ako tie predošlé. V čom spočíva?
Označme najprv, kvôli lepšej manipulácii a zjednodušeniu

$$\mathbb{P}_i \equiv \left(\ \boldsymbol{p}_1 \ \middle| \ \boldsymbol{p}_2 \ \middle| \ \cdots \ \middle| \ \boldsymbol{p}_i \ \right) \in \mathbb{R}^{n\times i}$$

Ak teraz budeme pre $\boldsymbol{p}_j$, $j = 1, 2, \ldots, i$, požadovať, aby boli navzájom lineárne nezávislé (LN) a súčasne, aby $\boldsymbol{p}_j^T\boldsymbol{f}_{i+1} = 0$ (v novom označení $\mathrm{h}(\mathbb{P}_i) = i$ a $\mathbb{P}_i^T\boldsymbol{f}_{i+1} = 0_n$) dosiahneme, že $\boldsymbol{f}_{i+1}$ bude ležať v podpriestore dimenzie $n-i$, t.j. s pribúdajúcimi vektormi $\boldsymbol{p}_i$ (a teda rastom i) klesá dimenzia k nule. A čo sa stane, ak $i = n$? Potom $\boldsymbol{f}_{n+1} = 0_n$, pretože jediný vektor, ktorý leží v podpriestore s dimenziou $n - i = n - n = 0$ je práve 0_n. A teda

$$0_n = \boldsymbol{f}_{n+1} = \boldsymbol{G}\boldsymbol{x}_{n+1} - \boldsymbol{h} \Rightarrow \boldsymbol{G}\boldsymbol{x}_{n+1} = \boldsymbol{h} \overset{\exists!}{\Rightarrow} \boldsymbol{x}_{n+1} = \boldsymbol{x}^*$$

To nám garantuje (teoreticky) maximálne n iterácií na získanie presného riešenia. Ale ako budovať maticu $\mathbb{P}_i$ s danými vlastnosťami? Pomôže nám vzťah (1.12) pre $\boldsymbol{f}_{i+1}$:

$$\boldsymbol{f}_{i+1} = \boldsymbol{f}_i - \boldsymbol{G}\boldsymbol{p}_i \frac{\boldsymbol{p}_i^T\boldsymbol{f}_i}{d_i}$$

Vynásobme ho $\boldsymbol{p}_j^T$ zľava, $j \leq i$

$$\boldsymbol{p}_j^T\boldsymbol{f}_{i+1} = \boldsymbol{p}_j^T\boldsymbol{f}_i - \boldsymbol{p}_j^T\boldsymbol{G}\boldsymbol{p}_i \frac{\boldsymbol{p}_i^T\boldsymbol{f}_i}{d_i}$$

Poďme postupne: ($d_i > 0$)

- $i = 1,\ j = 1 \qquad \boldsymbol{p}_1^T \boldsymbol{f}_2 = 0$ (Galerkin)
- $i = 2,\ j = 2 \qquad \boldsymbol{p}_2^T \boldsymbol{f}_3 = 0$ (Galerkin)

  $j = 1 \qquad \boldsymbol{p}_1^T \boldsymbol{f}_3 = \underbrace{\boldsymbol{p}_1^T \boldsymbol{f}_2}_{=0} - \boldsymbol{p}_1^T \boldsymbol{G} \boldsymbol{p}_2 \dfrac{\boldsymbol{p}_2^T \boldsymbol{f}_2}{d_2} = 0$, ak $\boldsymbol{p}_1^T \boldsymbol{G} \boldsymbol{p}_2 = 0$

Prvý výraz na pravej strane je nulový, čo vyplýva z Galerkinovej podmienky (V3). Aby sme anulovali aj druhý výraz položíme $\boldsymbol{p}_1^T \boldsymbol{G} \boldsymbol{p}_2 = 0$. Inými slovami, vektor $\boldsymbol{p}_2$ zvolíme tak, aby daná bilineárna forma bola nulová.

- $i = 3,\ j = 3 \qquad \boldsymbol{p}_3^T \boldsymbol{f}_4 = 0$ (Galerkin)

  $j = 2 \qquad \boldsymbol{p}_2^T \boldsymbol{f}_4 = \underbrace{\boldsymbol{p}_2^T \boldsymbol{f}_3}_{=0} - \boldsymbol{p}_2^T \boldsymbol{G} \boldsymbol{p}_3 \dfrac{\boldsymbol{p}_3^T \boldsymbol{f}_3}{d_3} = 0$, ak $\boldsymbol{p}_2^T \boldsymbol{G} \boldsymbol{p}_3 = 0$

  $j = 1 \qquad \boldsymbol{p}_1^T \boldsymbol{f}_4 = \underbrace{\boldsymbol{p}_1^T \boldsymbol{f}_3}_{=0} - \boldsymbol{p}_1^T \boldsymbol{G} \boldsymbol{p}_3 \dfrac{\boldsymbol{p}_3^T \boldsymbol{f}_3}{d_3} = 0$, ak $\boldsymbol{p}_1^T \boldsymbol{G} \boldsymbol{p}_3 = 0$

Zhrňme to takto: ak $\mathbb{P}_{i-1}^T \boldsymbol{f}_i = 0_{i-1}$ a $\mathbb{P}_{i-1}^T \boldsymbol{G} \boldsymbol{p}_i = 0_{i-1}$, potom

$$\mathbb{P}_i^T \boldsymbol{f}_{i+1} = 0_i.$$

Že to platí ukážeme veľmi jednoducho. Píšme:

$$\mathbb{P}_i^T \boldsymbol{f}_{i+1} = \left(\ \mathbb{P}_{i-1} \ \middle| \ \boldsymbol{p}_i \ \right)^T \left(\boldsymbol{f}_i - \boldsymbol{G} \boldsymbol{p}_i \frac{\boldsymbol{p}_i^T \boldsymbol{f}_i}{d_i} \right) =$$

$$= \left(\frac{\mathbb{P}_{i-1}^T \boldsymbol{f}_i - \mathbb{P}_{i-1}^T \boldsymbol{G} \boldsymbol{p}_i \dfrac{\boldsymbol{p}_i^T \boldsymbol{f}_i}{d_i}}{\boldsymbol{p}_i^T \boldsymbol{f}_i - \boldsymbol{p}_i^T \boldsymbol{G} \boldsymbol{p}_i \dfrac{\boldsymbol{p}_i^T \boldsymbol{f}_i}{d_i}} \right) = \left(\frac{0_{i-1}}{0} \right) = 0_i.$$

Pozn.: $\boldsymbol{G} \succ 0$ a platí $\boldsymbol{p}_j^T \boldsymbol{G} \boldsymbol{p}_k = 0,\ \ j \neq k$ potom $\boldsymbol{p}_i$ sa nazývajú konjungované vektory vzhľadom ku $\boldsymbol{G}$.

Zostáva nám zodpovedať otázku, či takto získané $\boldsymbol{p}_i$ spĺňajú lineárnu nezávislosť (LN), t.j. je $\mathrm{h}\left(\mathbb{P}_i\right) = i$?
Vytvorme si pomocnú maticu

$$\mathbb{D}_i = \mathbb{P}_i^T \boldsymbol{G} \mathbb{P}_i = \begin{pmatrix} \boldsymbol{p}_1^T \boldsymbol{G} \boldsymbol{p}_1 & \boldsymbol{p}_1^T \boldsymbol{G} \boldsymbol{p}_2 & \cdots & \boldsymbol{p}_1^T \boldsymbol{G} \boldsymbol{p}_i \\ \boldsymbol{p}_2^T \boldsymbol{G} \boldsymbol{p}_1 & \boldsymbol{p}_2^T \boldsymbol{G} \boldsymbol{p}_2 & \cdots & \boldsymbol{p}_2^T \boldsymbol{G} \boldsymbol{p}_i \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{p}_i^T \boldsymbol{G} \boldsymbol{p}_1 & \boldsymbol{p}_2^T \boldsymbol{G} \boldsymbol{p}_i & \cdots & \boldsymbol{p}_i^T \boldsymbol{G} \boldsymbol{p}_i \end{pmatrix}_{i \times i} \tag{1.17}$$

Potom

$$\mathbb{D}_i = \begin{pmatrix} \boldsymbol{p}_1^T \boldsymbol{G} \boldsymbol{p}_1 & 0 & \cdots & 0 \\ 0 & \boldsymbol{p}_2^T \boldsymbol{G} \boldsymbol{p}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{p}_i^T \boldsymbol{G} \boldsymbol{p}_i \end{pmatrix} \overset{(1.10)}{=} \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_i \end{pmatrix}$$

Ide o diagonálnu maticu. Platí:

$$\det(\mathbb{D}_i) = \prod_{j=1}^{i} \boldsymbol{p}_j^T \boldsymbol{G} \boldsymbol{p}_j = \prod_{j=1}^{i} d_j.$$

A teda je regulárna, keďže $d_i \neq 0\,, \forall i$. Následne, poľahky, vyplýva, že ak $\mathbb{D}_i$ je regulárna a $\boldsymbol{G} \succ 0$, potom musia byť stĺpce matice $\mathbb{P}_i$ LN, t.j. $\mathrm{h}(\mathbb{P}_i) = i$.

1.2.3 Konštrukcia konjungovaných vektorov

Od problému, ako rátať maticu $\mathbb{P}_i$, sme postúpili k úlohe, ako konštruovať vzájomne konjungované vektory $\boldsymbol{p}_i$ vzhľadom k matici $\boldsymbol{G}$. Uvedieme si 2 zovšeobecnené algoritmy (a to : *Gram-Schmidtov* a *Arnoldiho*).

Najprv však odvodíme alternatívne vyjadrenie $\boldsymbol{f}_{i+1}$:

$$\begin{aligned}
\boldsymbol{f}_{i+1} &= \boldsymbol{f}_i - \boldsymbol{G}\boldsymbol{p}_i \frac{\boldsymbol{p}_i^T \boldsymbol{f}_i}{d_i} \\
&= \left(\boldsymbol{I}_n - \boldsymbol{G}\boldsymbol{p}_i \frac{\boldsymbol{p}_i^T}{d_i} \right) \boldsymbol{f}_i \\
&= \left(\boldsymbol{I}_n - \boldsymbol{G}\boldsymbol{p}_i \frac{\boldsymbol{p}_i^T}{d_i} \right) \left(\boldsymbol{I}_n - \boldsymbol{G}\boldsymbol{p}_{i-1} \frac{\boldsymbol{p}_{i-1}^T}{d_{i-1}} \right) \boldsymbol{f}_{i-1} \\
&= \left(\boldsymbol{I}_n - \boldsymbol{G}\boldsymbol{p}_i \frac{\boldsymbol{p}_i^T}{d_i} \right) \left(\boldsymbol{I}_n - \boldsymbol{G}\boldsymbol{p}_{i-1} \frac{\boldsymbol{p}_{i-1}^T}{d_{i-1}} \right) \cdots \left(\boldsymbol{I}_n - \boldsymbol{G}\boldsymbol{p}_1 \frac{\boldsymbol{p}_1^T}{d_1} \right) \boldsymbol{f}_1
\end{aligned}$$

Ak si zadefinujeme maticu $\mathbb{Q}_i$:

$$\mathbb{Q}_i = \begin{cases} \boldsymbol{I}_n, & \mathrm{i} = 0; \\ \prod_{j=1}^{i} \left(\boldsymbol{I}_n - \boldsymbol{G}\boldsymbol{p}_j \frac{\boldsymbol{p}_j^T}{d_j} \right), & \mathrm{i} > 0. \end{cases} \tag{1.18}$$

tak

$$\boldsymbol{f}_{i+1} = \mathbb{Q}_i \boldsymbol{f}_1 \tag{1.19}$$

To však nie je všetko. Pokúsme sa uplatniť výhod, vyplývajúcich zo symetrie matice $\boldsymbol{G}$ a k nej konjungovaných vektorov $\boldsymbol{p}_i$, na vzťah pre $\mathbb{Q}_i$:

$$\begin{aligned}
\mathbb{Q}_i &\overset{(1.18)}{=} \left(\boldsymbol{I}_n - \boldsymbol{G}\boldsymbol{p}_i \frac{\boldsymbol{p}_i^T}{d_i} \right) \left(\boldsymbol{I}_n - \boldsymbol{G}\boldsymbol{p}_{i-1} \frac{\boldsymbol{p}_{i-1}^T}{d_{i-1}} \right) \mathbb{Q}_{i-2} \\
&= \left(\boldsymbol{I}_n - \boldsymbol{G}\boldsymbol{p}_i \frac{\boldsymbol{p}_i^T}{d_i} - \boldsymbol{G}\boldsymbol{p}_{i-1} \frac{\boldsymbol{p}_{i-1}^T}{d_{i-1}} + \frac{\boldsymbol{G}\boldsymbol{p}_i}{d_i} \underbrace{\left(\boldsymbol{p}_i^T \boldsymbol{G} \boldsymbol{p}_{i-1} \right)}_{=0} \frac{\boldsymbol{p}_{i-1}^T}{d_{i-1}} \right) \mathbb{Q}_{i-2} \\
&= \left(\boldsymbol{I}_n - \boldsymbol{G}\boldsymbol{p}_i \frac{\boldsymbol{p}_i^T}{d_i} - \boldsymbol{G}\boldsymbol{p}_{i-1} \frac{\boldsymbol{p}_{i-1}^T}{d_{i-1}} \right) \left(\boldsymbol{I}_n - \boldsymbol{G}\boldsymbol{p}_{i-2} \frac{\boldsymbol{p}_{i-2}^T}{d_{i-2}} \right) \mathbb{Q}_{i-3} \\
&\vdots \\
&= \left(\boldsymbol{I}_n - \boldsymbol{G} \sum_{j=1}^{i} \frac{\boldsymbol{p}_j \boldsymbol{p}_j^T}{d_j} \right) \mathbb{Q}_0
\end{aligned}$$

Dostali sme 2. výraz pre $\mathbb{Q}_i$:

$$\mathbb{Q}_i = \begin{cases} \boldsymbol{I}_n, & \text{i} = 0; \\ \boldsymbol{I}_n - \boldsymbol{G}\displaystyle\sum_{j=1}^{i} \frac{\boldsymbol{p}_j \boldsymbol{p}_j^T}{d_j}, & \text{i} > 0. \end{cases} \tag{1.20}$$

Užitočné vlastnosti matíc $\mathbb{Q}_i$:

W1:

$$\mathbb{P}_i^T \mathbb{Q}_i = 0_{i\times n}\,, \text{resp. } \mathbb{Q}_i^T \mathbb{P}_i = 0_{n\times i} \tag{1.21}$$

Dôkaz:

$$\mathbb{P}_i^T \left(\boldsymbol{I}_n - \boldsymbol{G}\mathbb{P}_i \mathbb{D}_i^{-1} \mathbb{P}_i^T\right) = \mathbb{P}_i^T - \left(\mathbb{P}_i^T \boldsymbol{G} \mathbb{P}_i\right) \mathbb{D}_i^{-1} \mathbb{P}_i^T = \mathbb{P}_i^T - \mathbb{P}_i^T = 0_{i\times n}$$

W2:

$$\mathbb{Q}_i \boldsymbol{G} \mathbb{P}_i = 0_{n\times i}\,, \text{resp. } \mathbb{P}_i^T \boldsymbol{G} \mathbb{Q}_i^T = 0_{i\times n} \tag{1.22}$$

Dôkaz:

$$\left(\boldsymbol{I}_n - \boldsymbol{G}\mathbb{P}_i \mathbb{D}_i^{-1} \mathbb{P}_i^T\right) \boldsymbol{G}\mathbb{P}_i = \boldsymbol{G}\mathbb{P}_i - \boldsymbol{G}\mathbb{P}_i \mathbb{D}_i^{-1} \left(\mathbb{P}_i^T \boldsymbol{G}\mathbb{P}_i\right) = \boldsymbol{G}\mathbb{P}_i - \boldsymbol{G}\mathbb{P}_i = 0_{n\times i}$$

W3:

$$\mathbb{Q}_j \mathbb{Q}_i = \mathbb{Q}_i \mathbb{Q}_j = \mathbb{Q}_i\,,\; j \leq i \tag{1.23}$$

Dôkaz: Najprv nech $j = i$:

$$\begin{aligned} \mathbb{Q}_j \mathbb{Q}_i &= \mathbb{Q}_i^2 = \left(\boldsymbol{I}_n - \boldsymbol{G}\mathbb{P}_i \mathbb{D}_i^{-1} \mathbb{P}_i^T\right)\left(\boldsymbol{I}_n - \boldsymbol{G}\mathbb{P}_i \mathbb{D}_i^{-1} \mathbb{P}_i^T\right) \\ &= \Big(\boldsymbol{I}_n - 2\boldsymbol{G}\mathbb{P}_i \mathbb{D}_i^{-1} \mathbb{P}_i^T + \boldsymbol{G}\mathbb{P}_i \mathbb{D}_i^{-1} \underbrace{\mathbb{P}_i^T \boldsymbol{G}\mathbb{P}_i \mathbb{D}_i^{-1}}_{\boldsymbol{I}_n} \mathbb{P}_i^T\Big) = \mathbb{Q}_i \end{aligned}$$

Pre $j < i$:

$$\begin{aligned} \mathbb{Q}_i \mathbb{Q}_j &\overset{(1.18)}{=} \Bigg[\Pi_{k=j+1}^{i} \left(\boldsymbol{I}_n - \boldsymbol{G}\boldsymbol{p}_k \frac{\boldsymbol{p}_k^T}{d_k}\right) \underbrace{\mathbb{Q}_j \Bigg] \mathbb{Q}_j} \overset{(1.)}{=} \mathbb{Q}_i \\ &= \quad \mathbb{Q}_j \mathbb{Q}_i = \left(\mathbb{Q}_i^T \mathbb{Q}_j^T\right)^T = \left(\Bigg[\Pi_{k=j+1}^{i} \left(\boldsymbol{I}_n - \boldsymbol{p}_k \frac{\boldsymbol{p}_k^T \boldsymbol{G}}{d_k}\right) \underbrace{\mathbb{Q}_j^T \Bigg] \mathbb{Q}_j^T}\right)^T = \mathbb{Q}_i \end{aligned}$$

W4:

$$\mathbb{Q}_j \boldsymbol{G}\boldsymbol{p}_i = \boldsymbol{G}\boldsymbol{p}_i\,,\; j \leq i \tag{1.24}$$

Dôkaz: Nech $\boldsymbol{s}_i = (0, 0, \ldots, 0, 1)^T \in \mathbb{R}^{i\times 1}$ je posledný stĺpec matice $\boldsymbol{I}_i$

$$\begin{aligned} \mathbb{Q}_j \boldsymbol{G}\boldsymbol{p}_i &= \mathbb{Q}_j \boldsymbol{G} \left(\mathbb{P}_i \boldsymbol{s}_i\right) = \left(\mathbb{Q}_j \boldsymbol{G}\mathbb{P}_i\right) \boldsymbol{s}_i \\ \mathbb{Q}_j \boldsymbol{G}\mathbb{P}_i &= \left(\boldsymbol{I}_n - \boldsymbol{G}\mathbb{P}_j \mathbb{D}_j^{-1} \mathbb{P}_j^T\right) \boldsymbol{G}\mathbb{P}_i = \boldsymbol{G}\mathbb{P}_i - \boldsymbol{G}\mathbb{P}_j \mathbb{D}_j^{-1} \left(\mathbb{P}_j^T \boldsymbol{G}\mathbb{P}_i\right) \\ &= \boldsymbol{G}\mathbb{P}_i - \boldsymbol{G}\mathbb{P}_j \mathbb{D}_j^{-1} \left(\mathbb{D}_j \;\middle|\; 0_{j\times(i-j)}\;\right) = \boldsymbol{G}\mathbb{P}_i - \left(\boldsymbol{G}\mathbb{P}_j \;\middle|\; 0_{j\times(i-j)}\;\right) \\ \mathbb{Q}_j \boldsymbol{G}\boldsymbol{p}_i &= \left(\mathbb{Q}_j \boldsymbol{G}\mathbb{P}_i\right) \boldsymbol{s}_i = \left(\boldsymbol{G}\mathbb{P}_i\right) \boldsymbol{s}_i - \left(\boldsymbol{G}\mathbb{P}_j \;\middle|\; 0_{j\times(i-j)}\;\right) \boldsymbol{s}_i = \boldsymbol{G}\boldsymbol{p}_i \end{aligned}$$

Pričom sme si pomohli vyjadrením matice $\mathbb{P}_j^T \boldsymbol{G}\mathbb{P}_i$ nasledovne:

$$\mathbb{P}_j^T \boldsymbol{G}\mathbb{P}_i = \left(\; \mathbb{P}_j^T \boldsymbol{G}\mathbb{P}_j \;\middle|\; \mathbb{P}_j^T \boldsymbol{G}\boldsymbol{p}_{j+1} \;\middle|\; \cdots \;\middle|\; \mathbb{P}_j^T \boldsymbol{G}\boldsymbol{p}_i \;\right) = \left(\; \mathbb{D}_j \;\middle|\; 0_j \;\middle|\; \cdots \;\middle|\; 0_j \;\right)_{j\times i}$$

W5:

$$\mathbb{Q}_i = \begin{cases} \boldsymbol{I}_n, & \text{i} = 0; \\ \boldsymbol{I}_n - \boldsymbol{G}\mathbb{P}_i\mathbb{D}_i^{-1}\mathbb{P}_i^T, & \text{i} > 0. \end{cases} \tag{1.25}$$

Dôkaz: Kľúčovým prvkom odvodenia „maticovej verzie“ pre $\mathbb{Q}_i$ je previesť sumu $\sum_{j=1}^{i} \frac{\boldsymbol{p}_j\boldsymbol{p}_j^T}{d_j}$ na maticový zápis. Všimnime si, že platí :

$$\mathbb{P}_i\mathbb{P}_i^T = \left(\boldsymbol{p}_1 \mid \boldsymbol{p}_2 \mid \cdots \mid \boldsymbol{p}_i \right) \begin{pmatrix} \boldsymbol{p}_1^T \\ \boldsymbol{p}_2^T \\ \vdots \\ \boldsymbol{p}_i^T \end{pmatrix} = \boldsymbol{p}_1\boldsymbol{p}_1^T + \boldsymbol{p}_2\boldsymbol{p}_2^T + \cdots + \boldsymbol{p}_i\boldsymbol{p}_i^T$$

$$\mathbb{D}_i^{-1} = \begin{pmatrix} d_1^{-1} & & \\ & \ddots & \\ & & d_i^{-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{d_1} & & \\ & \ddots & \\ & & \frac{1}{d_i} \end{pmatrix}$$

A čo je $\mathbb{P}_i\mathbb{D}_i^{-1}\mathbb{P}_i^T$?

$$\mathbb{P}_i\mathbb{D}_i^{-1}\mathbb{P}_i^T = \left(\boldsymbol{p}_1 \mid \boldsymbol{p}_2 \mid \cdots \mid \boldsymbol{p}_i \right) \begin{pmatrix} \frac{1}{d_1} & & \\ & \ddots & \\ & & \frac{1}{d_i} \end{pmatrix} \begin{pmatrix} \boldsymbol{p}_1^T \\ \boldsymbol{p}_2^T \\ \vdots \\ \boldsymbol{p}_i^T \end{pmatrix}$$

$$= \frac{\boldsymbol{p}_1\boldsymbol{p}_1^T}{d_1} + \frac{\boldsymbol{p}_2\boldsymbol{p}_2^T}{d_2} + \cdots + \frac{\boldsymbol{p}_i\boldsymbol{p}_i^T}{d_i} = \sum_{j=1}^{i} \frac{\boldsymbol{p}_j\boldsymbol{p}_j^T}{d_j}$$

W6: Ak vo (**??**) $\mathbb{P}_i$ zameníme za $\mathbb{P}_i\mathbb{M}_i$, kde $\mathbb{M}_i$ je nejaká regulárna matica, $\mathbb{Q}_i$ sa nezmení, t.j.

$$\boldsymbol{I}_n - \boldsymbol{G}\left(\mathbb{P}_i\mathbb{M}_i\right)\left[\left(\mathbb{P}_i\mathbb{M}_i\right)^T \boldsymbol{G}\left(\mathbb{P}_i\mathbb{M}_i\right)\right]^{-1}\left(\mathbb{P}_i\mathbb{M}_i\right)^T = \mathbb{Q}_i \tag{1.26}$$

Dôkaz: Aplikovaním známeho pravidla

$$(\boldsymbol{XYZ})^{-1} = \boldsymbol{Z}^{-1}\boldsymbol{Y}^{-1}\boldsymbol{X}^{-1}$$

dostávame

$$\begin{aligned} & \boldsymbol{I}_n - \boldsymbol{G}\left(\mathbb{P}_i\mathbb{M}_i\right)\left[\left(\mathbb{P}_i\mathbb{M}_i\right)^T \boldsymbol{G}\left(\mathbb{P}_i\mathbb{M}_i\right)\right]^{-1}\left(\mathbb{P}_i\mathbb{M}_i\right)^T \\ = \; & \boldsymbol{I}_n - \boldsymbol{G}\mathbb{P}_i\mathbb{M}_i\left[\mathbb{M}_i^T\left(\mathbb{P}_i^T\boldsymbol{G}\mathbb{P}_i\right)\mathbb{M}_i\right]^{-1}\mathbb{M}_i^T\mathbb{P}_i^T \\ = \; & \boldsymbol{I}_n - \boldsymbol{G}\mathbb{P}_i\underbrace{\mathbb{M}_i\mathbb{M}_i^{-1}}\mathbb{D}_i^{-1}\underbrace{\mathbb{M}_i^{-T}\mathbb{M}_i^T}\mathbb{P}_i^T \\ = \; & \boldsymbol{I}_n - \boldsymbol{G}\mathbb{P}_i\mathbb{D}_i^{-1}\mathbb{P}_i^T \\ = \; & \mathbb{Q}_i \end{aligned}$$

Pozn.: Vlastnosť (1.23) pre $j = i$ ($\mathbb{Q}_i^2 = \mathbb{Q}_i$) sa nazýva *idempotentnosť* matice. Matica $\mathbb{Q}_i$ je tzv. všeobecný (neortogonálny) projektor (oblique projector). (Keby, naviac. $\mathbb{Q}_i = \mathbb{Q}_i^T$, išlo by o ortogonálny projektor (orthogonal projector)).

Vráťme sa teraz už k tomu, ako generovať postupnosť vzájomne konjungovaných vektorov $\{\boldsymbol{p}_j\}$. Ide o to, pridať nový vektor $\boldsymbol{p}_{i+1}$ k „starým“, aby platilo $\mathbb{P}_i^T \boldsymbol{G} \boldsymbol{p}_{i+1} = 0_i$. Na to sa hodí vlastnosť W2 (1.22). Vynásobením tohto vzťahu

$$\mathbb{P}_i^T \boldsymbol{G} \mathbb{Q}_i^T = 0_{i \times n}$$

sprava ľubovoľným vektorom $\boldsymbol{w}_{i+1}$ dostávame

$$\mathbb{P}_i^T \boldsymbol{G} \mathbb{Q}_i^T \boldsymbol{w}_{i+1} = 0_i$$

Potom stačí definovať vektor $\boldsymbol{p}_{i+1}$ vzťahom

$$\boldsymbol{p}_{i+1} = \mathbb{Q}_i^T \boldsymbol{w}_{i+1} \tag{1.27}$$

a dostávame požadovanú vlastnosť

$$\mathbb{P}_i^T \boldsymbol{G} \boldsymbol{p}_{i+1} = 0_i.$$

1.3 Algoritmy pre tvorbu vzájomne konjungovaných vektorov

Zovšeobecnený Gram-Schmidtov algoritmus (*GGSA* = ***G**eneralised **G**ram-**S**chmidt **A**lgorithm*) si popíšeme ako prvý.
Je daná matica $\boldsymbol{G} \succ 0$, $\boldsymbol{G} = \boldsymbol{G}^T \in \mathbb{R}^{n \times n}$ a je daný systém lineárne nezávislých vektorov $\{\boldsymbol{w}_i\} \in \mathbb{R}^n$ $(i = 1, 2, \ldots, n)$. G – konjungované vektory $\{\boldsymbol{p}_i\}$ sa vytvárajú takto :

$\mathbb{Q}_0 = \boldsymbol{I}_n$, $\boldsymbol{p}_1 = \boldsymbol{w}_1$, $d_1 = \boldsymbol{p}_1^T \boldsymbol{G} \boldsymbol{p}_1$

for $i = 1, \ldots, n$

$$\mathbb{Q}_i = \mathbb{Q}_{i-1} - \boldsymbol{G} \boldsymbol{p}_i \left(\frac{\boldsymbol{p}_i}{d_i} \right)^T$$

$$\boldsymbol{p}_{i+1} = \mathbb{Q}_i^T \boldsymbol{w}_{i+1}$$

$$d_{i+1} = \boldsymbol{p}_{i+1}^T \boldsymbol{G} \boldsymbol{p}_{i+1}$$

end for.

Pozn.: Ak $\boldsymbol{G} = \boldsymbol{I}_n \Rightarrow$ *GGSA* generuje I – konjungované, t.j. ortogonálne, vektory $\boldsymbol{p}_i$.

Aký je vzťah medzi $\boldsymbol{w}_j$ a $\boldsymbol{p}_j$? Píšme

$$\begin{aligned}\boldsymbol{p}_1 &= \mathbb{Q}_0^T\boldsymbol{w}_1 = \boldsymbol{I}_n\boldsymbol{w}_1 = 1.\boldsymbol{w}_1\,, \qquad U_1 \equiv 1\,(\text{skalár})\\ \mathbb{P}_1 &= \mathbb{W}_1 U_1 \Rightarrow \mathbb{W}_1 = \mathbb{P}_1 U_1^{-1}\end{aligned}$$

$$\boldsymbol{p}_2 = \mathbb{Q}_1^T\boldsymbol{w}_2 = \left(\boldsymbol{I}_n - \mathbb{P}_1\mathbb{P}_1^T\mathbb{D}_1^{-1}\boldsymbol{G}\right)\boldsymbol{w}_2 \overset{(W5)}{\underset{\substack{\uparrow\\ \mathbb{P}_1 \to \mathbb{P}_1 U_1^{-1} = \mathbb{W}_1}}{=}} \left(\boldsymbol{I}_n - \mathbb{W}_1\mathbb{W}_1^T\left(\mathbb{W}_1^T\boldsymbol{G}\mathbb{W}_1\right)^{-1}\boldsymbol{G}\right)\boldsymbol{w}_2 =$$

$$= \boldsymbol{w}_2 - \mathbb{W}_1\underbrace{\left[\mathbb{W}_1^T\left(\mathbb{W}_1^T\boldsymbol{G}\mathbb{W}_1\right)^{-1}\boldsymbol{G}\boldsymbol{w}_2\right]}_{=-u_2}\,, \quad \text{ak} \quad \mathbb{W}_1^T\boldsymbol{G}\mathbb{W}_1 \quad \text{je regulárna}$$

$$\mathbb{P}_2 = \left(\mathbb{P}_1 \mid \boldsymbol{p}_2\right) = \left(\mathbb{W}_1 U_1 \mid \mathbb{W}_1 u_2 + \boldsymbol{w}_2\right) = \left(\underbrace{\left(\mathbb{W}_1 \mid \boldsymbol{w}_2\right)}_{\mathbb{W}_2}\begin{pmatrix}U_1\\ \hline 0\end{pmatrix} \;\middle|\; \left(\mathbb{W}_1 \mid \boldsymbol{w}_2\right)\begin{pmatrix}u_2\\ \hline 1\end{pmatrix}\right) =$$

$$= \mathbb{W}_2\boldsymbol{U}_2\,, \quad \text{kde} \quad \boldsymbol{U}_2 = \left(\begin{array}{c|c}\boldsymbol{U}_1 & u_2\\ \hline 0 & 1\end{array}\right)_{2\times 2}$$

$\Rightarrow$ $\mathbb{W}_2 = \mathbb{P}_2\boldsymbol{U}_2^{-1}$, keďže $\boldsymbol{U}_2$ je horná trojuholníková matica s *jednotkami* na hlavnej diagonále, a teda je regulárna

---

$$\boldsymbol{p}_3 = \mathbb{Q}_2^T\boldsymbol{w}_3 = \left(\boldsymbol{I}_n - \mathbb{P}_2\mathbb{P}_2^T\mathbb{D}_2^{-1}\boldsymbol{G}\right)\boldsymbol{w}_3 \overset{(W5)}{\underset{\substack{\uparrow\\ \mathbb{P}_2 \to \mathbb{P}_2 \boldsymbol{U}_2^{-1} = \mathbb{W}_2}}{=}} \left(\boldsymbol{I}_n - \mathbb{W}_2\mathbb{W}_2^T\left(\mathbb{W}_2^T\boldsymbol{G}\mathbb{W}_2\right)^{-1}\boldsymbol{G}\right)\boldsymbol{w}_3 =$$

$$= \boldsymbol{w}_3 - \mathbb{W}_2\underbrace{\left[\mathbb{W}_2^T\left(\mathbb{W}_2^T\boldsymbol{G}\mathbb{W}_2\right)^{-1}\boldsymbol{G}\boldsymbol{w}_3\right]}_{=-\boldsymbol{u}_3}\,, \quad \text{ak} \quad \mathbb{W}_2^T\boldsymbol{G}\mathbb{W}_2 \quad \text{je regulárna}$$

$$\mathbb{P}_3 = \left(\mathbb{P}_2 \mid \boldsymbol{p}_3\right) = \left(\mathbb{W}_2 \boldsymbol{U}_2 \mid \mathbb{W}_2 \boldsymbol{u}_3 + \boldsymbol{w}_3\right) = \left(\underbrace{\left(\mathbb{W}_2 \mid \boldsymbol{w}_3\right)}_{\mathbb{W}_3}\begin{pmatrix}\boldsymbol{U}_2\\ \hline 0_2^T\end{pmatrix} \;\middle|\; \left(\mathbb{W}_2 \mid \boldsymbol{w}_3\right)\begin{pmatrix}\boldsymbol{u}_3\\ \hline 1\end{pmatrix}\right) =$$

$$= \mathbb{W}_3\boldsymbol{U}_3\,, \quad \text{kde} \quad \boldsymbol{U}_3 = \left(\begin{array}{c|c}\boldsymbol{U}_2 & \boldsymbol{u}_3\\ \hline 0_2^T & 1\end{array}\right)_{3\times 3} \quad \text{je opäť regulárna}$$

$\vdots$ samozrejme , ak $\mathbb{W}_k^T\boldsymbol{G}\mathbb{W}_k$ je regulárne pre $3 < k < j$

$$\boldsymbol{p}_{j+1} = \mathbb{Q}_j^T \boldsymbol{w}_{j+1} = \Big(\boldsymbol{I}_n - \mathbb{P}_j \mathbb{P}_j^T \mathbb{D}_j^{-1} \boldsymbol{G}\Big) \boldsymbol{w}_{j+1} \overset{(W5)}{=} \Big(\boldsymbol{I}_n - \mathbb{W}_j \mathbb{W}_j^T \left(\mathbb{W}_j^T \boldsymbol{G} \mathbb{W}_j\right)^{-1} \boldsymbol{G}\Big) \boldsymbol{w}_{j+1} =$$

$$\uparrow$$

$$\mathbb{P}_j \to \mathbb{P}_j \boldsymbol{U}_j^{-1} = \mathbb{W}_j$$

$$= \boldsymbol{w}_{j+1} - \mathbb{W}_j \underbrace{\left[\mathbb{W}_j^T \left(\mathbb{W}_j^T \boldsymbol{G} \mathbb{W}_j\right)^{-1} \boldsymbol{G} \boldsymbol{w}_{j+1}\right]}_{=-\boldsymbol{u}_{j+1}}, \quad \text{ak} \quad \mathbb{W}_j^T \boldsymbol{G} \mathbb{W}_j \quad \text{je regulárna}$$

$$\mathbb{P}_{j+1} = \left(\mathbb{P}_j \mid \boldsymbol{p}_{j+1}\right) = \left(\underbrace{\left(\mathbb{W}_j \mid \boldsymbol{w}_{j+1}\right)}_{\mathbb{W}_{j+1}} \begin{pmatrix} \boldsymbol{U}_j \\ \hline 0_j^T \end{pmatrix} \;\middle|\; \left(\mathbb{W}_j \mid \boldsymbol{w}_{j+1}\right) \begin{pmatrix} \boldsymbol{u}_{j+1} \\ \hline 1 \end{pmatrix}\right) =$$

$$= \mathbb{W}_{j+1} \boldsymbol{U}_{j+1}, \quad \text{kde} \quad \boldsymbol{U}_{j+1} = \begin{pmatrix} \boldsymbol{U}_j & \boldsymbol{u}_{j+1} \\ \hline 0_j^T & 1 \end{pmatrix}_{(j+1)\times(j+1)}$$

je horná trojuholníková matica s *jednotkami* na hlavnej diagonále, takže regulárna

---

Treba poznamenať, že regularita $\mathbb{W}_j^T \boldsymbol{G} \mathbb{W}_j$, vo všeobecnosti, vôbec neznamená aj regularitu $\mathbb{W}_k^T \boldsymbol{G} \mathbb{W}_k$, pre $k < j$. Avšak máme $\boldsymbol{G}$ kladne definitnú a tiež $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n$ sú LN vektory. Potom $\mathbb{W}_1, \mathbb{W}_2, \ldots, \mathbb{W}_n$ majú plnú stĺpcovú hodnosť a platí:

$$\mathbb{W}_k^T \boldsymbol{G} \mathbb{W}_k \succ 0, \quad k = 1, 2, \ldots, n, \qquad \text{keďže} \quad \boldsymbol{G} \succ 0.$$

Takže LN vektorov $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n$ je nielen nutnou, ale i *postačujúcou* podmienkou pre „bezproblémové“ generovanie G– konjungovaných vektorov $\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n$ pomocou *GGSA* algoritmu.

Pozn.: Voľba pre $\mathbb{W}_n$ tu môže byť napr. $\boldsymbol{I}_n$, alebo $\boldsymbol{A}$ či $\boldsymbol{A}^T$.

Ilustrácia *GGSA*: vráťme sa k príkladu 1 a 2. V príklade 1 sme riešili systém ekvivalentných rovníc $\boldsymbol{G}\boldsymbol{x} = \boldsymbol{h}$ s

$$\boldsymbol{G} = \boldsymbol{A}^T \boldsymbol{A} = \begin{pmatrix} 10 & -10 \\ -10 & 20 \end{pmatrix}, \quad \boldsymbol{h} = \boldsymbol{A}^T \boldsymbol{b} = \begin{pmatrix} 0 \\ 10 \end{pmatrix}$$

Za $\mathbb{W}_2$ zvoľme $\boldsymbol{I}_2$. Ako bude vyzerať $\mathbb{P}_2$? Rátajme :

$$\boldsymbol{p}_1 = \boldsymbol{w}_1 = (1\,,\,0)^T$$

$$\mathbb{Q}_1 = \boldsymbol{I}_2 - \boldsymbol{G}\boldsymbol{p}_1 \left(\frac{\boldsymbol{p}_1}{d_1}\right)^T, \quad d_1 \neq 0 \quad (\text{to vieme, lebo} \quad \boldsymbol{G} \succ 0\,, \;\; \boldsymbol{w}_1 \neq 0_2)\,,$$

$$d_1 = \boldsymbol{p}_1^T \boldsymbol{G} \boldsymbol{p}_1 = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 10 & -10 \\ -10 & 20 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 & -10 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 10 \quad (= g_{11})$$

$$\mathbb{Q}_1^T = \boldsymbol{I}_2 - \boldsymbol{p}_1 \frac{\boldsymbol{p}_1^T \boldsymbol{G}}{d_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{\begin{pmatrix} 10 & -10 \end{pmatrix}}{10} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\boldsymbol{p}_2 = \mathbb{Q}_1^T \boldsymbol{w}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbb{P}_2 = \begin{pmatrix} \boldsymbol{p}_1 \mid \boldsymbol{p}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Overme si ich konjungovanosť :

$$\mathbb{P}_2^T \boldsymbol{G} \mathbb{P}_2 = \mathbb{D}_2 = \begin{pmatrix} \boldsymbol{p}_1^T \boldsymbol{G} \boldsymbol{p}_1 & \boldsymbol{p}_1^T \boldsymbol{G} \boldsymbol{p}_2 \\ \boldsymbol{p}_2^T \boldsymbol{G} \boldsymbol{p}_1 & \boldsymbol{p}_2^T \boldsymbol{G} \boldsymbol{p}_2 \end{pmatrix} = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$$

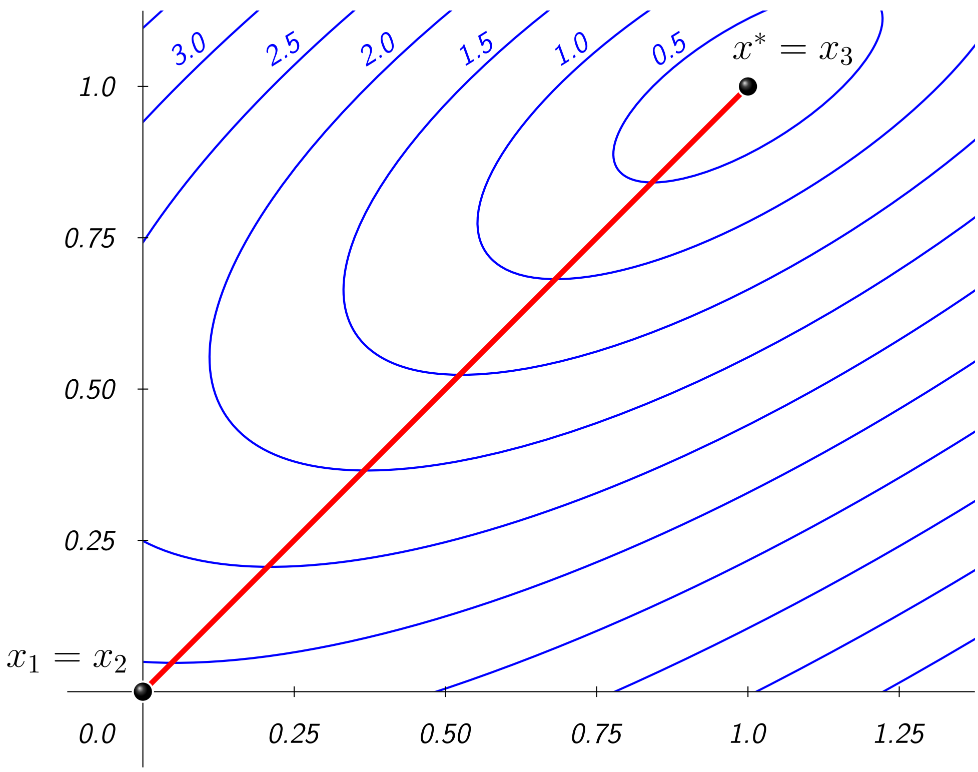
Ešte skúsme napísať maticu „prechodu“ od $\mathbb{W}_2$ k $\mathbb{P}_2$ (t.j. maticu $\boldsymbol{U}_2$) :

$$\mathbb{P}_2 = \mathbb{W}_2 \boldsymbol{U}_2 \xrightarrow{\mathbb{W}_2 = \boldsymbol{I}_2} \boldsymbol{U}_2 = \mathbb{P}_2$$

Dopočítajme, pomocou získanej matice $\mathbb{P}_2$, resp. vektorov $\boldsymbol{p}_1$ a $\boldsymbol{p}_2$, ďalšie aproximácie $\boldsymbol{x}_{i+1}$, $i\ = 2, 3$. Máme:

$$\begin{aligned}
\boldsymbol{x}_1 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ \boldsymbol{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \boldsymbol{f}_1 = \boldsymbol{G}\boldsymbol{x}_1 - \boldsymbol{h} = -\boldsymbol{h} = \begin{pmatrix} 0 \\ -10 \end{pmatrix} \\
\boldsymbol{x}_2 &= \ \boldsymbol{x}_1 - \boldsymbol{p}_1 \frac{\boldsymbol{p}_1^T \boldsymbol{f}_1}{d_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{0}{10} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \boldsymbol{x}_1 \\
\boldsymbol{f}_2 &= \boldsymbol{f}_1, \ \boldsymbol{p}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
d_2 &= \boldsymbol{p}_2^T \boldsymbol{G} \boldsymbol{p}_2 = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 10 & -10 \\ -10 & 20 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 10 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 10 \\
\boldsymbol{x}_3 &= \ \boldsymbol{x}_2 - \boldsymbol{p}_2 \frac{\boldsymbol{p}_2^T \boldsymbol{f}_2}{d_2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{-10}{10} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \boldsymbol{x}^*
\end{aligned}$$

Ako môžeme vidieť, stalo sa nám, že $\boldsymbol{p}_1^T \boldsymbol{f}_1 = 0$. Či už kvôli „nevhodnej“ voľbe matice $\mathbb{W}_2$ alebo výberu počiatočnej aproximácie $\boldsymbol{x}_1$. Metódy na riešenie sústavy lineárnych rovníc, $\boldsymbol{G}\boldsymbol{x} = \boldsymbol{h}$, ktorými sa budeme zaoberať ale toto nepripúšťajú.
Na nasledujúcom obrázku (1.4) je znázornený priebeh iterácií.



Obrázok 1.4: Priebeh iterácií - použitie *GGSA*

Skúsme taktiež nahliadnuť do nasledujúcej tabuľky 1.3. Ide o tabuľku 1.2 k príkladu 2, ktorá je doplnená o voľbu vzájomne konjungovaných vektorov $\boldsymbol{p}_i$ ku matici G. Pripomeňme, že rozmer úlohy je 100. Nielenže tento výber vektorov $\boldsymbol{p}_i$, ako bolo spomínané, garantuje (teoreticky) maximálny počet iteácií n (100), ale i dosiahnutá presnosť riešenia je veľmi dobrá ($1.8E$–12).

	Výber $\boldsymbol{p}_i$				
Iterácia	$\boldsymbol{I}_{\bullet i}$	$\boldsymbol{A}_{\bullet i}$	$\boldsymbol{A}^T_{\bullet i}$	$-\boldsymbol{f}_i$	$\boldsymbol{p}_j^T\boldsymbol{G}\boldsymbol{p}_k$
2	96.564	97.615	94.926	31.540	96.564
3	95.465	96.813	90.771	24.331	95.456
5	94.452	94.198	90.667	17.537	94.407
10	91.049	93.155	81.085	10.850	90.551
20	87.887	89.762	74.624	6.3222	87.027
30	79.656	79.218	67.365	4.3467	76.514
50	62.797	71.142	62.231	2.5517	51.612
100	40.867	52.269	44.676	1.2919	1.8E-12
500	9.2190	23.811	25.785	0.3150	———
1 000	6.0958	17.667	19.779	0.2138	
2 000	3.6676	13.314	14.386	0.1430	
3 000	2.4460	11.124	11.836	0.1035	
5 000	1.2814	8.8584	9.4352	0.0606	
10 000	0.5405	6.9612	6.8210	0.0194	
20 000	0.2543	6.0092	4.3726	0.0022	
30 000	0.1827	5.5725	3.4126	0.0004	
50 000	0.1317	5.0208	2.7191	0.0003	
100 000	0.0769	4.2751	2.0565	0.0002	

Tabuľka 1.3:

Špeciálnym prípadom *GGSA* je *Zovšeobecnený Arnoldiho* algoritmus (*GAA*). Tu nie sú vektory $\boldsymbol{w}_i$, $i = 1, 2, \ldots, n$ vopred nami zadané, ako pri *GGSA*, ale ich volíme nasledovným spôsobom :

$$\begin{aligned} \boldsymbol{w}_1 &\in \mathbb{R}^n \\ \boldsymbol{w}_{i+1} &= \boldsymbol{B}\boldsymbol{p}_i h_{i+1,i}^{-1} \quad i = 1, 2, \ldots\ , \end{aligned} \tag{1.28}$$

kde $\boldsymbol{B} \in \mathbb{R}^{n\times n}$ je nejaká vhodná regulárna matica a $h_{i+1,i}$ je tzv. normalizačný skalár.
Je daná matica $\boldsymbol{G} \succ 0$, $\boldsymbol{G} = \boldsymbol{G}^T \in \mathbb{R}^{n\times n}$, regulárna matica $\boldsymbol{B} \in \mathbb{R}^n$ a je zvolený vektor $\boldsymbol{w}_1 \neq 0_n$ (vhodne normalizovaný). G – konjungované vektory $\{\boldsymbol{p}_i\}$ sa pomocou *GAA* vytvárajú takto :

$\mathbb{Q}_0 = \boldsymbol{I}_n$, $\boldsymbol{p}_1 = \boldsymbol{w}_1$, $d_1 = \boldsymbol{p}_1^T\boldsymbol{G}\boldsymbol{p}_1$

for $i = 1, \ldots, n$

$$\mathbb{Q}_i = \mathbb{Q}_{i-1} - \boldsymbol{G}\boldsymbol{p}_i \left(\frac{\boldsymbol{p}_i}{d_i}\right)^T$$

$$\boldsymbol{p}_{i+1} = \mathbb{Q}_i^T \left(\boldsymbol{B}\boldsymbol{p}_i h_{i+1,i}^{-1}\right), \quad h_{i+1,i} = \|\boldsymbol{p}_{i+1}\|$$

$$d_{i+1} = \boldsymbol{p}_{i+1}^T\boldsymbol{G}\boldsymbol{p}_{i+1}$$

end for.

Pozn.: GAA = ***G**eneralised **A**rnoldi **A**lgorithm*

Preskúmajme nejaké vzťahy vektorov $\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_i$ získané týmto algoritmom. Vychádzajme z definícií (1.27) a (1.28)

$$\boldsymbol{p}_{i+1} = \mathbb{Q}_i^T \boldsymbol{B}\boldsymbol{p}_i h_{i+1,i}^{-1}$$

a ďalej ju upravujme takto (pre $1 \leq j < i$):

$$\boldsymbol{p}_j h_{j+1,j} = \mathbb{Q}_j^T \boldsymbol{B}\boldsymbol{p}_j \overset{(1.25)}{=} \left(\boldsymbol{I}_n - \mathbb{P}_j \mathbb{D}_j^{-1} \mathbb{P}_j^T \boldsymbol{G}\right) \boldsymbol{B}\boldsymbol{p}_j = \boldsymbol{B}\boldsymbol{p}_j - \mathbb{P}_j \underbrace{\left(\mathbb{D}_j^{-1} \mathbb{P}_j^T \boldsymbol{G}\boldsymbol{B}\boldsymbol{p}_j\right)}_{\triangleq \boldsymbol{h}_j}$$

$$\boldsymbol{B}\boldsymbol{p}_j = \mathbb{P}_j \boldsymbol{h}_j + \boldsymbol{p}_{j+1} h_{j+1,j} = \underbrace{\left(\mathbb{P}_j \mid \boldsymbol{p}_{j+1} \right)}_{\mathbb{P}_{j+1}} \left(\frac{\boldsymbol{h}_j}{h_{j+1,j}} \right) = \mathbb{P}_{j+1} \boldsymbol{h}_j^* , \tag{1.29}$$

kde

$$\boldsymbol{h}_j^* = \left(\boldsymbol{h}_j \mid h_{j+1,j}\right)^T = \left(h_{1j} \quad h_{2j} \quad \cdots \quad h_{jj} \mid h_{j+1,j}\right)^T$$

Čo bude $\boldsymbol{B}\mathbb{P}_i$?

$$\boldsymbol{B}\mathbb{P}_i = \boldsymbol{B}\left(\boldsymbol{p}_1 \mid \boldsymbol{p}_2 \mid \cdots \mid \boldsymbol{p}_i\right) = \left(\boldsymbol{B}\boldsymbol{p}_1 \mid \boldsymbol{B}\boldsymbol{p}_2 \mid \cdots \mid \boldsymbol{B}\boldsymbol{p}_i\right) \overset{(1.29)}{=} \left(\mathbb{P}_2 \boldsymbol{h}_1^* \mid \mathbb{P}_3 \boldsymbol{h}_2^* \mid \cdots \mid \mathbb{P}_{i+1} \boldsymbol{h}_i^*\right)$$

$$\boldsymbol{B}\mathbb{P}_i = \left(\underbrace{\left(\mathbb{P}_2 \mid \boldsymbol{p}_3 \mid \cdots \mid \boldsymbol{p}_{i+1}\right)}_{\mathbb{P}_{i+1}} \begin{pmatrix} \boldsymbol{h}_1^* \\ 0 \\ \vdots \\ 0 \end{pmatrix} \,\middle|\, \left(\mathbb{P}_3 \mid \boldsymbol{p}_4 \mid \cdots \mid \boldsymbol{p}_{i+1}\right) \begin{pmatrix} \boldsymbol{h}_2^* \\ 0 \\ \vdots \\ 0 \end{pmatrix} \,\middle|\, \cdots \,\middle|\, \mathbb{P}_{i+1} \boldsymbol{h}_{i+1}^* \right)$$

Dostávame dôležitý vzťah

$$\boldsymbol{B}\mathbb{P}_i = \mathbb{P}_{i+1} \boldsymbol{H}_{i+1}^* , \tag{1.30}$$

kde

$$\boldsymbol{H}_{i+1}^* = \begin{pmatrix} h_{1,1} & h_{1,2} & \cdots & h_{1,i} \\ h_{2,1} & h_{2,2} & \ddots & \vdots \\ 0 & h_{3,2} & \ddots & h_{i-1,i} \\ \vdots & \ddots & \ddots & h_{i,i} \\ 0 & \cdots & 0 & h_{i+1,i} \end{pmatrix}_{(i+1)\times i}$$

Matica $\boldsymbol{H}_{i+1}^*$ je typu hornej *Hessenbergovej* matice. Môžeme ju ďalej písať takto :

$$\boldsymbol{H}_{i+1}^* = \left(\begin{array}{c|c} \boldsymbol{H}_i^* & \boldsymbol{h}_i \\ \hline 0_i^T & h_{i+1,i} \end{array} \right)_{(i+1)\times i} = \left(\frac{\boldsymbol{H}_i}{0_i^T \mid h_{i+1,i}} \right) \tag{1.31}$$

a

$$\boldsymbol{H}_i = \left(\boldsymbol{H}_i^* \mid \boldsymbol{h}_i\right) = \begin{pmatrix} h_{1,1} & h_{1,2} & h_{1,3} & \cdots & h_{1,i} \\ h_{2,1} & h_{2,2} & h_{2,3} & \cdots & h_{2,i} \\ 0 & h_{3,2} & h_{3,3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & h_{i-1,i} \\ 0 & \cdots & 0 & h_{i,i-1} & h_{i,i} \end{pmatrix}_{i\times i}$$

Kapitola 2

Metódy na riešenie lineárnych sústav

Teraz, keď už máme vzťahy pre výpočet $\boldsymbol{x}_{i+1}$ (1.11) a $\boldsymbol{p}_{i+1}$ (1.27), môžeme sformulovať „univerzálny" algoritmus iteračnej metódy riešenia ekvivalentného systému rovníc $\boldsymbol{Gx} = \boldsymbol{h}$ (1.3).
Je daná ekvivalentná sústava rovníc, t.j. matica koeficientov $\boldsymbol{G} \succ 0$, $\boldsymbol{G} \in \mathbb{R}^{n\times n}$ a vektor pravých strán $\boldsymbol{h} \in \mathbb{R}^n$. Nech matica $\boldsymbol{B} \in \mathbb{R}^n$ v GAA je rovná súčinu $\boldsymbol{KG}$ ($\boldsymbol{B} = \boldsymbol{KG}$), pre nejakú ľubovolnú regulárnu maticu $\boldsymbol{K} \in \mathbb{R}^n$. Potom štruktúra algoritmu je nasledovaná.

$\mathbb{Q}_0 = \boldsymbol{I}_n$, $\boldsymbol{f}_1 = \boldsymbol{Gx}_1 - \boldsymbol{h}$, $\boldsymbol{p}_1 = \boldsymbol{Kf}_1$, $d_1 = \boldsymbol{p}_1^T\boldsymbol{Gp}_1$

for $i = 1, \ldots, n$

$$\boldsymbol{x}_{i+1} = \boldsymbol{x}_i - \boldsymbol{p}_i \frac{\boldsymbol{p}_i^T\boldsymbol{f}_i}{d_i}$$

$$\boldsymbol{f}_{i+1} = \boldsymbol{f}_i - \boldsymbol{Gp}_i \frac{\boldsymbol{p}_i^T\boldsymbol{f}_i}{d_i}$$

if $\left\|\boldsymbol{f}_{i+1}\right\|_\infty = 0$ *then*

$\boldsymbol{x}^* = \boldsymbol{x}_{i+1}$, STOP

end if

$$\mathbb{Q}_i = \mathbb{Q}_{i-1} - \boldsymbol{Gp}_i \left(\frac{\boldsymbol{p}_i}{d_i}\right)^T$$

(1) $\boldsymbol{p}_{i+1} = \mathbb{Q}_i^T\boldsymbol{KGp}_i h_{i+1,i}^{-1}$ (prípad GAA)

(2) $\boldsymbol{p}_{i+1} = \mathbb{Q}_i^T\boldsymbol{Kf}_{i+1}$ (prípad $GGSA$)

$d_{i+1} = \boldsymbol{p}_{i+1}^T\boldsymbol{Gp}_{i+1}$

end for.

Pozn.: Volí sa buď krok (1) alebo (2) pre výpočet vektora $\boldsymbol{p}_{i+1}$.

Ako možno vidieť $\boldsymbol{p}_1$ nie je ľubovoľný vektor (na rozdiel od $\boldsymbol{x}_1$). Prečo?
Nech $\boldsymbol{B} = \boldsymbol{KG}$ (prípad GAA). Počítajme týmto algoritmom postupnosť aproximácií riešení $\{\boldsymbol{x}_i\}$. Za predpokladu, že $\boldsymbol{p}_{s+1} = 0_n$ (a teda $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_s \neq 0_n$) vieme zabezpečiť pre ľubovoľný výber parametrov $\boldsymbol{G}$ a $\boldsymbol{K}$, aby posledná dosiahnutá aproximácia $\boldsymbol{x}_{s+1}$ bola riešením ekvivalent-

nej sústavy $\boldsymbol{G}\boldsymbol{x} = \boldsymbol{h}$, resp. aby $\boldsymbol{f}_{s+1} = 0_n$? Čo sa núka je vhodný výber $\boldsymbol{p}_1$.
Z rovnice (1.30)

$$\boldsymbol{B}\mathbb{P}_i = \mathbb{P}_{i+1}\boldsymbol{H}^*_{i+1}$$

vyplýva pre $i = s$ a za predpokladu $\boldsymbol{p}_{s+1} = 0_n$ rovnosť:

$$\boldsymbol{B}\mathbb{P}_s = \mathbb{P}_{s+1}\boldsymbol{H}^*_{s+1} \overset{(1.31)}{=} \mathbb{P}_s\boldsymbol{H}_s + \underbrace{\boldsymbol{p}_{s+1}}_{0_n}\left(0_s^T \mid h_{s+1,s}\right) = \mathbb{P}_s\boldsymbol{H}_s \tag{2.1}$$

Dosaďme za $\boldsymbol{B} = \boldsymbol{K}\boldsymbol{G}$ a nech $\boldsymbol{s}_1$ označuje prvý stĺpec matice $\boldsymbol{I}_s$. Potom postupnými úpravami (ktoré sú v každom riadku vyznačené vpravo) dostávame :

$$\begin{aligned}
\boldsymbol{K}\boldsymbol{G}\mathbb{P}_s &= \mathbb{P}_s\boldsymbol{H}_s && \big/\, \boldsymbol{K}^{-1} \text{ zľava} \\
\boldsymbol{G}\mathbb{P}_s &= \boldsymbol{K}^{-1}\mathbb{P}_s\boldsymbol{H}_s && \big/\, \boldsymbol{H}_s^{-1} \text{ sprava} \\
\boldsymbol{G}\mathbb{P}_s\boldsymbol{H}_s^{-1} &= \boldsymbol{K}^{-1}\mathbb{P}_s && \big/\, \mathbb{Q}_s \text{ zľava} \\
\underbrace{\left(\mathbb{Q}_s\boldsymbol{G}\mathbb{P}_s\right)}_{=0_{n\times s}}\boldsymbol{H}_s^{-1} &= \mathbb{Q}_s\boldsymbol{K}^{-1}\mathbb{P}_s && \big/\, \boldsymbol{s}_1 \text{ sprava} \\
0_n &= \mathbb{Q}_s\left(\boldsymbol{K}^{-1}\boldsymbol{p}_1\right)
\end{aligned}$$

a tiež chceme, aby platilo :

$$0_n = \boldsymbol{f}_{s+1} = \mathbb{Q}_s\boldsymbol{f}_1$$

Porovnaním posledných dvoch rovníc nám vyjde voľba pre $\boldsymbol{f}_1$:

$$\boldsymbol{f}_1 = \boldsymbol{K}^{-1}\boldsymbol{p}_1\,, \quad \text{resp.} \quad \boldsymbol{p}_1 = \boldsymbol{K}\boldsymbol{f}_1$$

V prípade ($GGSA$), kedy sa volí $\boldsymbol{p}_{i+1} = \mathbb{Q}_i^T\boldsymbol{K}\boldsymbol{f}_{i+1}$, nám pri $\boldsymbol{p}_{s+1} = 0_n$ pomôže garantovať $\boldsymbol{f}_{s+1} = 0_n$ vhodný výber matice $\boldsymbol{K}$. Lebo

$$\begin{aligned}
0_n &= \boldsymbol{p}_{s+1} = \mathbb{Q}_s^T\boldsymbol{K}\boldsymbol{f}_{s+1} && \big/\, \boldsymbol{f}_1^T \text{ zľava} \\
0 &= \boldsymbol{f}_1^T\boldsymbol{p}_{s+1} = \underbrace{\boldsymbol{f}_1^T\mathbb{Q}_s^T}\boldsymbol{K}\boldsymbol{f}_{s+1} \overset{(1.19)}{=} \boldsymbol{f}_{s+1}^T\boldsymbol{K}\boldsymbol{f}_{s+1}
\end{aligned}$$

Ak $\boldsymbol{K}$ bude spĺňať, podobne ako kladne definitná matica,

$$\forall\, \boldsymbol{x} \neq 0_n \quad \boldsymbol{x}^T\boldsymbol{K}\boldsymbol{x} > 0$$

potom

$$\boldsymbol{f}_{s+1} = 0_n$$

Pozn.: $\boldsymbol{K}$ nemusí byť, nutne, symetrická.

Niektoré z jej vlastností sú podobné ako pre kladne definitné matice, napr.

- $\boldsymbol{K}$ je regulárna
- $\forall\, \boldsymbol{x} \neq 0_n \quad \boldsymbol{x}^T\boldsymbol{K}^{-1}\boldsymbol{x} > 0$

  Dôkaz:

  $$\boldsymbol{x}^T\boldsymbol{K}\boldsymbol{x} = \boldsymbol{x}^T\boldsymbol{K}^T\boldsymbol{x} = \boldsymbol{x}^T\boldsymbol{K}^T\boldsymbol{K}^{-1}\boldsymbol{K}\boldsymbol{x} = \boldsymbol{y}^T\boldsymbol{K}^{-1}\boldsymbol{y} > 0$$

  Pričom iné mocniny $\boldsymbol{K}$ nemusia byť!
- $\forall\, \boldsymbol{x} \neq 0_n \quad \boldsymbol{M}^T\boldsymbol{K}\boldsymbol{M} > 0$, ak $\boldsymbol{M}$ má plnú stĺpcovú hodnosť.

2.1 Dlhé rekurencie (Long recurrences)

2.1.1 *OrthoDir* a *GCR*

Pokúsme sa pomocou nášho „univerzálneho" algoritmu vyrátať riešenie systému normálnych rovníc

$$\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{b}\,,$$

t.j.

$$\boldsymbol{G} = \boldsymbol{A}^T\boldsymbol{A}\,,\;\; \boldsymbol{h} = \boldsymbol{A}^T\boldsymbol{b} \;\text{ a nech }\; \boldsymbol{K} = \boldsymbol{A}^{-T}.$$

Čo dostaneme?

Prípad *GAA* ($h_{i+1,i} \equiv 1\,,\ \forall\, i$) – metóda *OrthoDir*

$$\begin{aligned}
\boldsymbol{x}_1 &\in \mathbb{R}^n, \quad \boldsymbol{r}_1 = \boldsymbol{A}\boldsymbol{x}_1 - \boldsymbol{b}, \quad \boldsymbol{p}_1 = \boldsymbol{K}\boldsymbol{f}_1 = \boldsymbol{r}_1 \\
\boldsymbol{x}_{i+1} &= \boldsymbol{x}_i + \boldsymbol{p}_i\,\frac{\boldsymbol{p}_i^T\boldsymbol{A}^T\boldsymbol{r}_i}{\boldsymbol{p}_i^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{p}_i} \\
\boldsymbol{p}_{i+1} &= \boldsymbol{A}\boldsymbol{p}_i - \sum_{j=1}^{i}\boldsymbol{p}_j\,\frac{\boldsymbol{p}_j^T\boldsymbol{A}^T\boldsymbol{A}^2\boldsymbol{p}_i}{\boldsymbol{p}_j^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{p}_j}
\end{aligned} \tag{2.2}$$

Pozn.:

- *OrthoDir* = ***Ortho***gonal ***Dir***ection
- V každom kroku sa minimalizuje $\|\boldsymbol{e}\|_G \overset{(1.7)}{=} \left\|\boldsymbol{G}^{-1}\boldsymbol{f}\right\|_G \overset{(1.5)}{=} \|\boldsymbol{f}\|_{G^{-1}} = \|\boldsymbol{r}\|$

Prípad *GGSA* – metóda *GCR*

Predpokladajme, že $\boldsymbol{A}$ spĺňa

$$\forall\, \boldsymbol{x} \neq 0_n \quad \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} > 0,$$

pričom nemusí byť symetrická.
Potom

$$\begin{aligned}
\boldsymbol{x}_1 &\in \mathbb{R}^n, \quad \boldsymbol{r}_1 = \boldsymbol{A}\boldsymbol{x}_1 - \boldsymbol{b}, \quad \boldsymbol{p}_1 = \boldsymbol{r}_1 \\
\boldsymbol{x}_{i+1} &= \boldsymbol{x}_i + \boldsymbol{p}_i\,\frac{\boldsymbol{p}_i^T\boldsymbol{A}^T\boldsymbol{r}_i}{\boldsymbol{p}_i^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{p}_i} \\
\boldsymbol{p}_{i+1} &= \boldsymbol{r}_{i+1} - \sum_{j=1}^{i}\boldsymbol{p}_j\,\frac{\boldsymbol{p}_j^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{r}_{i+1}}{\boldsymbol{p}_j^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{p}_j}
\end{aligned}$$

Pozn.:

- *GCR* = ***G***eneralised ***C***onjugeted ***R***esiduals
- V každom kroku sa minimalizuje $\|\boldsymbol{r}\|$

Všimnime si, že na vyčíslenie $\boldsymbol{p}_{i+1}$ v oboch prípadoch, nám treba poznať celú maticu $\mathbb{P}_i$. A navyše so zvyšujúcim sa i nám lineárne rastie pamäť, potrebná na jej uloženie. Tiež počet operácií (a teda času) na zrátanie danej sumy, stúpa. Inými slovami: tieto metódy využívajú plný tvar projekčnej matice $\mathbb{Q}_i$ (1.20).
To sa dá riešiť, viacmenej, takýmito spôsobmi :

- *reštartovanie* – vypočíta sa prvých m iterácií a potom sa opäť začne od znovu s $\boldsymbol{x}_1 = \boldsymbol{x}_{m+1}$ a zvyšok získanej informácie sa „zmaže“ (čo môže byť nežiadúce)
- *skrátenie* – namiesto celej formuly pre $\boldsymbol{p}_{i+1}$ sa použije iba posledných $k \leq i$ členov matice $\mathbb{P}_i$, t.j. $\boldsymbol{p}_{i+1-k}, \ldots, \boldsymbol{p}_i$ a tak pre *OrthoDir* potom máme

$$\boldsymbol{p}_{i+1} = \boldsymbol{A}\boldsymbol{p}_i - \sum_{i+1-k}^{i} \boldsymbol{p}_j \frac{\boldsymbol{p}_j^T \boldsymbol{A}^T \boldsymbol{A}^2 \boldsymbol{p}_i}{\boldsymbol{p}_j^T \boldsymbol{A}^T \boldsymbol{A} \boldsymbol{p}_j}$$

- *vhodný výber* $\boldsymbol{K}$ – môžeme anulovať takmer všetky členy v sume (niečo ako skrátenie, dokonca pre malé k) a dostaneme tzv. krátku rekurenciu (viď odsek 2.2).

2.1.2 *GMRes* metóda

GMRes (**G**eneralised **M**inimal **Res**iduals) využíva *OAA* s $\boldsymbol{B} = \boldsymbol{A}$ a $\boldsymbol{p}_1 = \boldsymbol{r}_1 / \|\boldsymbol{r}_1\|$ na generovanie postupnosti ortonormálnych vektorov $\{\boldsymbol{p}_j\}$. S ich pomocou sa snažíme v každom kroku minimalizovať normu $\|\boldsymbol{r}\|$. Štruktúra *GMRes* sa líši od „univerzálneho“ algoritmu, a preto nemôžeme použiť rekurentný vzťah pre $\boldsymbol{x}_{i+1}$. Treba aproximácie počítať inou technikou.
Budeme hľadať vhodné $\boldsymbol{x}_{i+1}$ z podpriestoru :

$$\boldsymbol{x}(\boldsymbol{z}) \equiv \boldsymbol{x}_1 + \mathbb{P}_i \boldsymbol{z}\,, \quad \boldsymbol{z} \in \mathbb{R}^i \tag{2.3}$$

Potom

$$\begin{aligned} \boldsymbol{r}(\boldsymbol{z}) &= \boldsymbol{A}\boldsymbol{x}(\boldsymbol{z}) - \boldsymbol{b} = (\boldsymbol{A}\boldsymbol{x}_1 - \boldsymbol{b}) + \boldsymbol{A}\mathbb{P}_i \boldsymbol{z} = \boldsymbol{r}_1 + \boldsymbol{A}\mathbb{P}_i \boldsymbol{z} \\ \boldsymbol{r}(\boldsymbol{z}) &= \boldsymbol{p}_1 \|\boldsymbol{r}_1\| + \boldsymbol{A}\mathbb{P}_i \boldsymbol{z} = \mathbb{P}_{i+1} \boldsymbol{s}_1 \|\boldsymbol{r}_1\| + \boldsymbol{A}\mathbb{P}_i \boldsymbol{z}, \end{aligned}$$

kde $\boldsymbol{s}_1$ je prvý stĺpec matice $\boldsymbol{I}_{i+1}$.
Využijeme rovnosť (1.30)

$$\boldsymbol{B}\mathbb{P}_i = \mathbb{P}_{i+1} \boldsymbol{H}_{i+1}^* \overset{\boldsymbol{B}=\boldsymbol{A}}{\Longrightarrow} \boldsymbol{A}\mathbb{P}_i = \mathbb{P}_{i+1} \boldsymbol{H}_{i+1}^*$$

a dosaďme do $\boldsymbol{r}(\boldsymbol{z})$:

$$\boldsymbol{r}(\boldsymbol{z}) = \mathbb{P}_{i+1} \boldsymbol{s}_1 \|\boldsymbol{r}_1\| + \mathbb{P}_{i+1} \boldsymbol{H}_{i+1}^* \boldsymbol{z} = \mathbb{P}_{i+1} \left(\boldsymbol{H}_{i+1}^* \boldsymbol{z} + \boldsymbol{s}_1 \|\boldsymbol{r}_1\|\right)$$

Norma $\boldsymbol{r}(\boldsymbol{z})$, ktorú chceme minimalizovať bude

$$\|\boldsymbol{r}(\boldsymbol{z})\| = \left\|\mathbb{P}_{i+1} \left(\boldsymbol{H}_{i+1}^* \boldsymbol{z} + \boldsymbol{s}_1 \|\boldsymbol{r}_1\|\right)\right\| = \left\|\boldsymbol{H}_{i+1}^* \boldsymbol{z} + \boldsymbol{s}_1 \|\boldsymbol{r}_1\|\right\| ,$$

pričom druhá rovnosť platí, pretože matica $\mathbb{P}_i$ má ortonormálne stĺpce.
Taktiež

$$\|\boldsymbol{r}(\boldsymbol{z})\| = \left\|\boldsymbol{Q} \left(\boldsymbol{H}_{i+1}^* \boldsymbol{z} + \boldsymbol{s}_1 \|\boldsymbol{r}_1\|\right)\right\|$$

pre ľubovoľnú ortogonálnu maticu $\boldsymbol{Q}$, takže i pre $\boldsymbol{Q} = \boldsymbol{Q}_{i+1}$, ktorá spĺňa :

$$\boldsymbol{Q}_{i+1}\boldsymbol{H}^*_{i+1} = \left(\frac{\boldsymbol{U}_i}{0_i^T}\right) ,$$

kde $\boldsymbol{U}_i$ je horná trojuholníková matica. Ako dospejeme k matici $\boldsymbol{U}_i$ a samotnej $\boldsymbol{Q}_{i+1}$? (viď ďalej).
Za predpokladu, že máme už takú maticu $\boldsymbol{Q}_{i+1}$ a definujeme si jej prvý stĺpec

$$\boldsymbol{Q}_{i+1}\boldsymbol{s}_1 \equiv \left(\frac{\boldsymbol{a}_i}{\gamma_{i+1}}\right) ,$$

rátajme :

$$\|\boldsymbol{r}(\boldsymbol{z})\| = \left\|\boldsymbol{Q}_{i+1}\left(\boldsymbol{H}^*_{i+1}\boldsymbol{z} + \boldsymbol{s}_1\|\boldsymbol{r}_1\|\right)\right\| = \left\|\left(\frac{\boldsymbol{U}_i}{0_i^T}\right)\boldsymbol{z} + \left(\frac{\boldsymbol{a}_i}{\gamma_{i+1}}\right)\|\boldsymbol{r}_1\|\right\|$$

následne

$$\|\boldsymbol{r}(\boldsymbol{z})\|^2 = \|\boldsymbol{U}_i\boldsymbol{z} + \boldsymbol{a}_i\|\boldsymbol{r}_1\|\|^2 + (\gamma_{i+1}\|\boldsymbol{r}_1\|)^2$$

Keďže $\boldsymbol{U}_i$ je regulárna (lebo $\boldsymbol{H}^*_{i+1}$ má plnú stĺpcovú hodnosť), z toho vyplýva, že $\|\boldsymbol{r}(\boldsymbol{z})\|$ je minimalizovaná, ak

$$\boldsymbol{z} = -\boldsymbol{U}_i^{-1}\boldsymbol{a}_i\|\boldsymbol{r}_1\| .$$

S týmto výberom $\boldsymbol{z}$ potom máme

$$\|\boldsymbol{r}_{i+1}\| = |\gamma_{i+1}|\,\|\boldsymbol{r}_1\|$$

a z rovnice (2.3)

$$\boldsymbol{x}_{i+1} = \boldsymbol{x}_1 - \mathbb{P}_i\boldsymbol{U}_i^{-1}\boldsymbol{a}_i\|\boldsymbol{r}_1\| . \tag{2.4}$$

Už len získať $\boldsymbol{U}_i^{-1}$ a $\boldsymbol{a}_i$. Začnime najprv zisťovaním, čo je $\boldsymbol{U}_i$ a $\boldsymbol{Q}_{i+1}$.
Nech $i = 1$ $\left(\boldsymbol{Q}_{i+1} = \boldsymbol{Q}_2\,, \boldsymbol{U}_i = \boldsymbol{U}_1\,, \boldsymbol{H}^*_{i+1} = \boldsymbol{H}^*_2\right)$

$$\boldsymbol{Q}_2\boldsymbol{H}^*_2 = \begin{pmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{pmatrix}\begin{pmatrix} h_{1,1} \\ h_{2,1} \end{pmatrix} = \left(\frac{\boldsymbol{U}_1}{0}\right)$$

$\boldsymbol{Q}_2$ je tzv. matica rotácií v rovine, kde c_1 – je kosínus a s_1 – sínus príslušného uhla. Keďže ortogonálne matice nemenia dĺžku vektorov, vieme, čo bude $\boldsymbol{U}_1$. A síce, platí :

$$\|\boldsymbol{Q}_2\boldsymbol{H}^*_2\| = \left\|\begin{pmatrix} h_{1,1} \\ h_{2,1} \end{pmatrix}\right\| = \left\|\left(\frac{\boldsymbol{U}_1}{0}\right)\right\|$$

$$\Downarrow$$

$$\sqrt{h_{1,1}^2 + h_{2,1}^2} = \sqrt{\boldsymbol{U}_1^2 + 0^2} = |\boldsymbol{U}_1|$$

Zvoľme za $\boldsymbol{U}_1$ kladnú hodnotu, t.j. $\boldsymbol{U}_1 = \sqrt{h_{1,1}^2 + h_{2,1}^2}$ $\left(\text{norma vektora } \left(h_{1,1}\,, h_{2,1}\right)^T\right)$.

Dopočítajme, teraz, maticu $\boldsymbol{Q}_2$:

$$\left.\begin{array}{rcll} h_{1,1}\, c_1 - h_{2,1}\, s_1 & = & \boldsymbol{U}_1 & /\, h_{1,1} \\ h_{2,1}\, c_1 + h_{1,1}\, s_1 & = & 0 & /\, h_{2,1} \end{array}\right] \oplus$$

$$\left(h_{1,1}^2 + h_{2,1}^2\right) c_1 = h_{1,1}\, \boldsymbol{U}_1$$

$$\Downarrow$$

$$c_1 = \frac{h_{1,1}\boldsymbol{U}_1}{h_{1,1}^2 + h_{2,1}^2} = \frac{h_{1,1}}{\sqrt{h_{1,1}^2 + h_{2,1}^2}}, \quad s_1 = -\frac{h_{2,1}}{h_{1,1}}\, c_1 = -\frac{h_{2,1}}{\sqrt{h_{1,1}^2 + h_{2,1}^2}}$$

Teraz predpokladjame, že poznáme $\boldsymbol{Q}_i$ a $\boldsymbol{U}_{i-1}$ $\Big($takže $\boldsymbol{Q}_i \boldsymbol{H}_i^* = \left(\boldsymbol{U}_{i-1}^{-T} \mid 0_i\right)^T\Big)$. Ako upravíme na hornú trojuholníkovú maticu, $\boldsymbol{H}_{i+1}^*$?
Matica $\boldsymbol{H}_{i+1}^*$ má „peknú“ štruktúru, ktorá nám pomôže.

$$\boldsymbol{H}_{i+1}^* = \left(\begin{array}{c|c} \boldsymbol{H}_i^* & \boldsymbol{h}_i \\ \hline 0_i^T & h_{i+1,i} \end{array}\right)$$

vynásobme ju zľava maticou, ktorá je opäť ortogonálna :

$$\left(\begin{array}{c|c} \boldsymbol{Q}_i & 0_i \\ \hline 0_i^T & 1 \end{array}\right) \left(\begin{array}{c|c} \boldsymbol{H}_i^* & \boldsymbol{h}_i \\ \hline 0_i^T & h_{i+1,i} \end{array}\right) = \left(\begin{array}{c|c} \boldsymbol{Q}_i \boldsymbol{H}_i^* & \boldsymbol{Q}_i h_i \\ \hline 0_{i-1}^T & h_{i+1,i} \end{array}\right) = \left(\begin{array}{c|c} \boldsymbol{U}_{i-1} & \boldsymbol{u}_i \\ 0_{i-1}^T & \omega_i \\ 0_{i-1}^T & h_{i+1,i} \end{array}\right)_{(i+1)\times i},$$

ak definujeme

$$\boldsymbol{Q}_i h_i \equiv \left(\begin{array}{c} \boldsymbol{u}_i \\ \hline \omega_i \end{array}\right). \tag{2.5}$$

Ešte nám chýba anulácia posledného riadku. Aj tu použijeme maticu otočenia (v rovine) na posledné dva riadky, pričom prvých $(i-1)$ riadkov ponecháme bez zmeny, t.j. vynásobíme práve získanú maticu, špeciálnou (ortogonálnou) maticou, zľava :

$$\left(\begin{array}{c|c|c} \boldsymbol{I}_{i-1} & 0_{i-1} & 0_{i-1} \\ 0_{i-1}^T & c_i & -s_i \\ 0_{i-1}^T & s_i & c_i \end{array}\right) \left(\begin{array}{c|c} \boldsymbol{U}_{i-1} & \boldsymbol{u}_i \\ 0_{i-1}^T & \omega_i \\ 0_{i-1}^T & h_{i+1,i} \end{array}\right) = \left(\begin{array}{c|c} \boldsymbol{U}_{i-1} & \boldsymbol{u}_i \\ 0_{i-1}^T & \omega_i^* \\ 0_{i-1}^T & 0 \end{array}\right),$$

kde

$$\begin{aligned} \omega_i^* &= \left\| \begin{pmatrix} \omega_i \\ h_{i+1,i} \end{pmatrix} \right\| = \sqrt{\omega_i^2 + h_{i+1,i}^2} \\ \omega_i &\overset{(2.5)}{=} \left(\boldsymbol{s}_i^T \boldsymbol{Q}_i\right) \boldsymbol{h}_i = \boldsymbol{q}_i^T \boldsymbol{h}_i \\ c_i &= \frac{\omega_i}{\omega_i^*}, \quad s_i = -\frac{h_{i+1,i}}{\omega_i^*} \end{aligned}$$

Pozn.:

- $\omega_1^* = \boldsymbol{U}_1$
- $\boldsymbol{q}_i^T$ označme posledný $(i$ – ty) riadok matice $\boldsymbol{Q}_i$
- $\boldsymbol{Q}_{-i}$ označme „zvyšok“ matice $\boldsymbol{Q}_i$ (prvých $(i-1)$ riadkov)

Pre $\boldsymbol{U}_i$ a $\boldsymbol{Q}_{i+1}$ máme :

$$\boldsymbol{U}_i = \left(\begin{array}{c|c} \boldsymbol{U}_{i-1} & \boldsymbol{u}_i \\ \hline 0_{i-1}^T & \omega_i^* \end{array}\right) \tag{2.6}$$

a

$$\boldsymbol{Q}_{i+1} = \left(\begin{array}{c|c|c} \boldsymbol{I}_{i-1} & 0_{i-1} & 0_{i-1} \\ 0_{i-1}^T & c_i & -s_i \\ 0_{i-1}^T & s_i & c_i \end{array}\right) \left(\begin{array}{c|c} \boldsymbol{Q}_{-i} & 0_{i-1} \\ \boldsymbol{q}_i^T & 0 \\ 0_{i-1}^T & 1 \end{array}\right) = \left(\begin{array}{c|c} \boldsymbol{Q}_{-i} & 0_{i-1} \\ c_i\boldsymbol{q}_i^T & -s_i \\ s_i\boldsymbol{q}_i^T & c_i \end{array}\right)_{(i+1)\times(i+1)},$$

pričom sme vyjadrili $\boldsymbol{Q}_i$ ako

$$\boldsymbol{Q}_i = \left(\begin{array}{c} \boldsymbol{Q}_{-i} \\ \hline \boldsymbol{q}_i^T \end{array}\right).$$

Ak sa pozrieme lepšie na štruktúru $\boldsymbol{Q}_{i+1}$ určite nám neunikne, že ide o *dolnú* Hessenbergovu maticu.
K vyjadreniu $\boldsymbol{U}_i^{-1}$ a $\boldsymbol{a}_i$ už nie je ďaleko.

$$\boldsymbol{U}_i^{-1} = \left(\begin{array}{c|c} \boldsymbol{U}_{i-1} & \boldsymbol{u}_i \\ \hline 0_{i-1}^T & \omega_i^* \end{array}\right)^{-1}$$

Použijeme štandardný postup :

$$\left(\begin{array}{c|c||c|c} \boldsymbol{U}_{i-1} & \boldsymbol{u}_i & \boldsymbol{I}_{i-1} & 0_{i-1} \\ \hline 0_{i-1}^T & \omega_i^* & 0_{i-1}^T & 1 \end{array}\right) \begin{array}{l} /\boldsymbol{U}_{i-1}^{-1} \text{ zľava} \\ /\omega_i^{*-1} \end{array} \sim \left(\begin{array}{c|c||c|c} \boldsymbol{I}_{i-1} & \boldsymbol{U}_{i-1}^{-1}\boldsymbol{u}_i & \boldsymbol{U}_{i-1}^{-1} & 0_{i-1} \\ \hline 0_{i-1}^T & 1 & 0_{i-1}^T & \omega_i^{*-1} \end{array}\right) \begin{array}{l} \oplus \\ /-\boldsymbol{U}_{i-1}^{-1}\boldsymbol{u}_i \end{array}$$

$$\sim \left(\begin{array}{c|c||c|c} \boldsymbol{I}_{i-1} & 0_{i-1} & \boldsymbol{U}_{i-1}^{-1} & -\boldsymbol{U}_{i-1}^{-1}\boldsymbol{u}_i\,\omega_i^{*-1} \\ \hline 0_{i-1}^T & 1 & 0_{i-1}^T & \omega_i^{*-1} \end{array}\right)$$

$$\Rightarrow \boldsymbol{U}_i^{-1} = \left(\begin{array}{c|c} \boldsymbol{U}_{i-1}^{-1} & -\boldsymbol{U}_{i-1}^{-1}\boldsymbol{u}_i\,\omega_i^{*-1} \\ \hline 0_{i-1}^T & \omega_i^{*-1} \end{array}\right) \tag{2.7}$$

A čo $\boldsymbol{a}_i$?
Nech $\boldsymbol{i}_+ =$ prvý stĺpec $\boldsymbol{I}_{i+1}$ a $\boldsymbol{i} =$ prvý stĺpec $\boldsymbol{I}_i$.

$$\left(\begin{array}{c} \boldsymbol{a}_i \\ \hline \gamma_{i+1} \end{array}\right) = \boldsymbol{Q}_{i+1}\boldsymbol{i}_+ = \left(\begin{array}{c|c} \boldsymbol{Q}_{-i} & 0_{i-1} \\ c_i\boldsymbol{q}_i^T & -s_i \\ s_i\boldsymbol{q}_i^T & c_i \end{array}\right) \boldsymbol{i}_+ = \left(\begin{array}{c} \boldsymbol{Q}_{-i} \\ c_i\boldsymbol{q}_i^T \\ s_i\boldsymbol{q}_i^T \end{array}\right) \boldsymbol{i} = \left(\begin{array}{c} \boldsymbol{Q}_{-i}\boldsymbol{i} \\ c_i\left(\boldsymbol{q}_i^T\boldsymbol{i}\right) \\ s_i\left(\boldsymbol{q}_i^T\boldsymbol{i}\right) \end{array}\right) = \left(\begin{array}{c} \boldsymbol{a}_{i-1} \\ c_i\gamma_i \\ s_i\gamma_i \end{array}\right) \tag{2.8}$$

$$\Rightarrow \boldsymbol{a}_i = \left(\begin{array}{c} \boldsymbol{a}_{i-1} \\ \hline c_i\gamma_i \end{array}\right) \tag{2.9}$$

Označme si $\boldsymbol{Y}_i = \mathbb{P}_i\boldsymbol{U}_i^{-1}$ a počítajme

$$\boldsymbol{Y}_i = \left(\mathbb{P}_{i-1} \mid \boldsymbol{p}_i\right) \left(\begin{array}{c|c} \boldsymbol{U}_{i-1}^{-1} & -\boldsymbol{U}_{i-1}^{-1}\boldsymbol{u}_i\,\omega_i^{*-1} \\ \hline 0_{i-1}^T & \omega_i^{*-1} \end{array}\right) = \left(\mathbb{P}_{i-1}\boldsymbol{U}_{i-1}^{-1} \mid -\mathbb{P}_{i-1}\boldsymbol{U}_{i-1}^{-1}\boldsymbol{u}_i\,\omega_i^{*-1} + \boldsymbol{p}_i\omega_i^{*-1}\right) =$$

$$= \left(\boldsymbol{Y}_{i-1} \mid (\boldsymbol{p}_i - \boldsymbol{Y}_{i-1}\boldsymbol{u}_i)\,\omega_i^{*-1}\right)$$

$$\Rightarrow \boldsymbol{y}_i = (\boldsymbol{p}_i - \boldsymbol{Y}_{i-1}\boldsymbol{u}_i)\,\omega_i^{*-1}$$

a dosadzujme do (2.4)

$$\begin{aligned}
\boldsymbol{x}_{i+1} &= \boldsymbol{x}_1 - \mathbb{P}_i \boldsymbol{U}_i^{-1} \boldsymbol{a}_i \|\boldsymbol{r}_1\| = \boldsymbol{x}_1 - \boldsymbol{Y}_i \boldsymbol{a}_i \|\boldsymbol{r}_1\| \\
&= \boldsymbol{x}_1 - \left(\boldsymbol{Y}_{i-1} \mid \boldsymbol{y}_i\right) \left(\frac{\boldsymbol{a}_{i-1}}{c_i\gamma_i}\right) \|\boldsymbol{r}_1\| = \\
&= \underbrace{\boldsymbol{x}_1 - \boldsymbol{Y}_{i-1}\boldsymbol{a}_{i-1} \|\boldsymbol{r}_1\|}_{\boldsymbol{x}_i} - \boldsymbol{y}_i c_i \gamma_i \|\boldsymbol{r}_1\| = \\
&= \boldsymbol{x}_i - (\boldsymbol{p}_i - \boldsymbol{Y}_{i-1}\boldsymbol{u}_i) \frac{c_i\gamma_i}{\omega_i^*} \|\boldsymbol{r}_1\|
\end{aligned}$$

Pozn.:

- Z (2.8) $\gamma_{i+1} = s_i\gamma_i = \prod_{j=1}^{i} s_j$
- Ak by $\boldsymbol{A} = \boldsymbol{A}^T$ a $\boldsymbol{p}_{s+1} = 0_n$:

$$\begin{aligned}
\boldsymbol{A}\mathbb{P}_s &= \mathbb{P}_{s+1}\boldsymbol{H}_{s+1}^* = \mathbb{P}_s\boldsymbol{H}_s \quad /\mathbb{P}_s^T \\
\mathbb{P}_s^T\boldsymbol{A}\mathbb{P}_s &= \mathbb{P}_s^T\mathbb{P}_s\boldsymbol{H}_s = \boldsymbol{H}_s
\end{aligned}$$

Keďže $\boldsymbol{H}_s$ je horná trojuholníková aj symetrická zároveň, potom je *trojdiagonálna*. A všetky $\boldsymbol{H}_i \quad 1 \le i \le s$ sú taktiež trojdiagonálne, lebo sú jej hlavné podmatice. Táto vlastnosť matice nám uľahčí množstvo výpočtov.

2.2 Krátke rekurencie (Short recurrences)

Ako sme naznačili v predchádzajúcom odseku 2.1 (dlhé rekurencie), problémy s riešením rozsiahlych riedkych sústav, nám hrozia, hlavne s požiadavkou na pamäť, ktorá s rastom i (počtom iterácií) narastá. S tým sa tiež spája množstvo operácií, potrebných na vyčíslenie matice $\mathbb{Q}_i^T$. Tá sa vo všeobecnosti, neukladá, ale vyčísluje sa pomocou uložených vektorov $\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_i$:

$$\mathbb{Q}_i^T = \boldsymbol{I}_n - \sum_{j=1}^{i} \boldsymbol{p}_j \frac{\boldsymbol{p}_j^T\boldsymbol{G}}{d_j} .$$

Vlastne chceme vypočítať vektor $\mathbb{Q}_i^T\boldsymbol{w}_{\boldsymbol{i+1}}$, a teda, treba nám počítať v každej iterácii (pre každé i) sumu

$$\sum_{j=1}^{i} \boldsymbol{p}_j \frac{\boldsymbol{p}_j^T\boldsymbol{G}\boldsymbol{w}_{i+1}}{d_j} .$$

Načrtli sme už aj nejaké možné „kvázi“ postupy, ako obísť ten ohromný počet maticovo–vektorových násobení, ale snáď najlepšie je pokúsiť sa zvoliť „vhodnejší“ parameter $\boldsymbol{K}$.
Zauvažujme takto : majme rovnicu (1.24) (jedna z vlastností $\mathbb{Q}_i$ (W4))

$$\mathbb{Q}_j\boldsymbol{G}\boldsymbol{p}_i = \boldsymbol{G}\boldsymbol{p}_i , \quad j < i$$

upravme si indexáciu matice $\mathbb{Q}$ (kvôli lepšej prehľadnosti) : $j \to j-1$

$$\begin{aligned}
\mathbb{Q}_{j-1}\boldsymbol{G}\boldsymbol{p}_i &= \boldsymbol{G}\boldsymbol{p}_i , \quad j \le i \quad /\boldsymbol{w}_j^T \text{ zľava} \\
\underbrace{\boldsymbol{w}_j^T\mathbb{Q}_{j-1}}_{\boldsymbol{p}_j^T}\boldsymbol{G}\boldsymbol{p}_i &= \boldsymbol{w}_j^T\boldsymbol{G}\boldsymbol{p}_i = 0 , \quad j < i
\end{aligned}$$

Teraz zvoľme za $\boldsymbol{w}_j$ tzv. *Lanczosov* výber :

$$\boldsymbol{w}_j = \boldsymbol{KGp}_{j-1}$$

a dosaďme :

$$\boldsymbol{p}_j^T \boldsymbol{Gp}_i = \boldsymbol{p}_{j-1}^T \boldsymbol{GK}^T \boldsymbol{Gp}_i = 0 \;, \quad j < i \;,$$

resp. $(j-1 \rightarrow j)$

$$\boldsymbol{p}_j^T \boldsymbol{GK}^T \boldsymbol{Gp}_i = 0 \;, \quad j \leq i-2 \;.$$

Vráťme sa k sume :

$$\sum_{j=1}^{i} \boldsymbol{p}_j \frac{\boldsymbol{p}_j^T \boldsymbol{Gw}_{i+1}}{d_j} = \sum_{j=1}^{i} \boldsymbol{p}_j \frac{\boldsymbol{p}_j \boldsymbol{GKGp}_{i+1}}{d_j} = \boldsymbol{p}_i \frac{\boldsymbol{p}_i \boldsymbol{GKGp}_i}{d_i} + \boldsymbol{p}_{i-1} \frac{\boldsymbol{p}_{i-1} \boldsymbol{GKGp}_i}{d_{i-1}} \;,$$

ak $\boldsymbol{K} = \boldsymbol{K}^T$. Potom pre $\boldsymbol{p}_{i+1}$ platí (3–členná rekurencia) :

$$\boldsymbol{p}_{i+1} = \mathbb{Q}_i^T \boldsymbol{KGp}_i = \boldsymbol{KGp}_i - \boldsymbol{p}_i \frac{\boldsymbol{p}_i^T \boldsymbol{GKGp}_i}{d_i} - \boldsymbol{p}_{i-1} \frac{\boldsymbol{p}_{i-1}^T \boldsymbol{GKGp}_i}{d_{i-1}} \;. \tag{2.10}$$

Keďže, v praxi sa vyskytujú rôzne numerické obtiaže, skúsme ich trochu zmierniť „vložením“ do našich úvah škálovací skalár $h_{i+1,i}$.

$$\begin{aligned} \boldsymbol{p}_{i+1} &= \mathbb{Q}_i^T \boldsymbol{KGp}_i h_{i+1,i}^{-1} \quad / \boldsymbol{p}_{i+1}^T \boldsymbol{G} \text{ zľava} \\ d_{i+1} &= \boldsymbol{p}_{i+1}^T \boldsymbol{Gp}_{i+1} = \boldsymbol{p}_{i+1}^T \boldsymbol{G} \mathbb{Q}_i^T \boldsymbol{KGp}_i h_{i+1,i}^{-1} \end{aligned}$$

$$\begin{aligned} d_{i+1} &= \left(\boldsymbol{p}_{i+1}^T \boldsymbol{GKGp}_i - \underbrace{(\boldsymbol{p}_{i+1}^T \boldsymbol{Gp}_i)}_{=0} \frac{\boldsymbol{p}_i^T \boldsymbol{GKGp}_i}{d_i} - \underbrace{(\boldsymbol{p}_{i+1}^T \boldsymbol{Gp}_{i-1})}_{=0} \frac{\boldsymbol{p}_{i-1}^T \boldsymbol{GKGp}_i}{d_{i-1}} \right) h_{i+1,i}^{-1} = \\ &= \boldsymbol{p}_{i+1}^T \boldsymbol{GKGp}_i h_{i+1,i}^{-1} \end{aligned}$$

$$\Rightarrow \; \boldsymbol{p}_{i+1}^T \boldsymbol{GKGp}_i = h_{i+1,i} \, d_{i+1}$$

$i+1 \rightarrow i$:

$$\boldsymbol{p}_{i-1}^T \boldsymbol{GKGp}_i = \boldsymbol{p}_i^T \boldsymbol{GKGp}_{i-1} = h_{i,i-1} \, d_i$$

následne

$$\boldsymbol{p}_{i+1} = \left(\boldsymbol{KGp}_i - \boldsymbol{p}_i \frac{\boldsymbol{p}_i^T \boldsymbol{GKGp}_i}{d_i} - \boldsymbol{p}_{i-1} \frac{d_i}{d_{i-1}} \, h_{i,i-1} \right) h_{i+1,i}^{-1} \tag{2.11}$$

Pre výber $\boldsymbol{w}_j = \boldsymbol{K}\boldsymbol{f}_j$ (tzv. *HS*[1] výber) môžeme urobiť podobné úvahy.

$$\begin{aligned}
\boldsymbol{p}_j^T \boldsymbol{f}_{i+1} &\overset{(1.19)}{=} 0\,, \quad j \le i \quad \text{(Galerkinova podmienka)} \\
\boldsymbol{f}_{i+1} &= \mathbb{Q}_i \boldsymbol{f}_1 \overset{(3.)}{=} \mathbb{Q}_{j-1}(\mathbb{Q}_i \boldsymbol{f}_1) = \mathbb{Q}_{j-1}\boldsymbol{f}_{i+1}\,, \quad j \le i+1 \quad \Big/ \boldsymbol{w}_j^T \ \ \text{zľava} \qquad (2.12) \\
\boldsymbol{w}_j^T \boldsymbol{f}_{i+1} &= \underbrace{\boldsymbol{w}_j^T \mathbb{Q}_{j-1}}_{\boldsymbol{p}_j^T} \boldsymbol{f}_{i+1} = \boldsymbol{p}_j^T \boldsymbol{f}_{i+1} = 0\,, \quad j \le i \\
&\Downarrow \ \ \boldsymbol{w}_j = \boldsymbol{K}\boldsymbol{f}_j \\
\boldsymbol{f}_j^T \boldsymbol{K}^T \boldsymbol{f}_{i+1} &= 0\,, \quad j \le i \qquad (2.13) \\
\boldsymbol{f}_{j+1} &\overset{(1.12)}{=} \boldsymbol{f}_j - \boldsymbol{G}\boldsymbol{p}_j \frac{\boldsymbol{p}_j^T \boldsymbol{f}_j}{d_j} \quad \Big/ ()^T \\
\boldsymbol{f}_{j+1}^T &= \boldsymbol{f}_j^T - \frac{\boldsymbol{p}_j^T \boldsymbol{f}_j}{d_j} \boldsymbol{p}_j^T \boldsymbol{G} \quad \Big/ \boldsymbol{K}^T \boldsymbol{f}_{i+1} \ \ \text{sprava} \\
\boldsymbol{f}_{j+1}^T \boldsymbol{K}^T \boldsymbol{f}_{i+1} &= \boldsymbol{f}_j^T \boldsymbol{K}^T \boldsymbol{f}_{i+1} - \frac{\boldsymbol{p}_j^T \boldsymbol{f}_j}{d_j} \boldsymbol{p}_j^T \boldsymbol{G}\boldsymbol{K}^T \boldsymbol{f}_{i+1}
\end{aligned}$$

Nech $j \le i-1$ a $\boldsymbol{p}_j^T \boldsymbol{f}_j \ne 0$, potom výraz na ľavej strane a prvý výraz na pravej strane, v predchádzajúcej rovnici, je nulový (z (2.13)) a platí

$$\frac{\boldsymbol{p}_j^T \boldsymbol{f}_j}{d_j} \left(\boldsymbol{p}_j^T \boldsymbol{G}\boldsymbol{K}^T \boldsymbol{f}_{i+1}\right) = 0\,,$$

resp.

$$\boldsymbol{p}_j^T \boldsymbol{G}\boldsymbol{K}^T \boldsymbol{f}_{i+1} = 0\,, \quad j \le i-1\,.$$

Suma

$$\sum_{j=1}^{i} \boldsymbol{p}_j \frac{\boldsymbol{p}_j \boldsymbol{G}\boldsymbol{K}\boldsymbol{f}_{i+1}}{d_j}$$

sa zredukuje iba na jeden člen

$$\boldsymbol{p}_i \frac{\boldsymbol{p}_i \boldsymbol{G}\boldsymbol{K}\boldsymbol{f}_{i+1}}{d_i}\,,$$

ak $\boldsymbol{K} = \boldsymbol{K}^T$. A pre $\boldsymbol{p}_{i+1}$ máme (2–člennú rekurenciu)

$$\boldsymbol{p}_{i+1} = \mathbb{Q}_i^T \boldsymbol{K}\boldsymbol{f}_{i+1} = \boldsymbol{K}\boldsymbol{f}_{i+1} - \boldsymbol{p}_i \frac{\boldsymbol{p}_i \boldsymbol{G}\boldsymbol{K}\boldsymbol{f}_{i+1}}{d_i}\,.$$

Opäť skúsme použiť škálovanie :

$$\begin{aligned}
\boldsymbol{p}_{i+1} &= \mathbb{Q}_i^T \boldsymbol{K}\boldsymbol{f}_{i+1} h_{i+1,i}^{-1} \quad \Big/ \boldsymbol{f}_{i+1}^T \ \ \text{zľava} \\
\boldsymbol{p}_{i+1}^T \boldsymbol{f}_{i+1} &= \boldsymbol{f}_{i+1}^T \boldsymbol{p}_{i+1} = \underbrace{\boldsymbol{f}_{i+1}^T \mathbb{Q}_i^T}_{} \boldsymbol{K}\boldsymbol{f}_{i+1} h_{i+1,i}^{-1} \overset{(2.12)}{=} \boldsymbol{f}_{i+1}^T \boldsymbol{K}\boldsymbol{f}_{i+1} h_{i+1,i}^{-1} = o_{i+1} h_{i+1,i}^{-1} \qquad (2.14) \\
o_{i+1} &\equiv \boldsymbol{f}_{i+1}^T \boldsymbol{K}\boldsymbol{f}_{i+1} \overset{(1.12)}{=} \underbrace{\boldsymbol{f}_i^T \boldsymbol{K}\boldsymbol{f}_{i+1}}_{\overset{(2.13)}{=} 0_n} - \frac{\boldsymbol{p}_i^T \boldsymbol{f}_i}{d_i}\left(\boldsymbol{p}_i^T \boldsymbol{G}\boldsymbol{K}\boldsymbol{f}_{i+1}\right) \overset{(2.14)}{=} -\frac{o_i\, h_{i,i-1}^{-1}}{d_i}\left(\boldsymbol{p}_i^T \boldsymbol{G}\boldsymbol{K}\boldsymbol{f}_{i+1}\right)
\end{aligned}$$

[1] *HS = Hestensen – Stiefel*

$$\Rightarrow\ \boldsymbol{p}_i^T \boldsymbol{G}\boldsymbol{K}\boldsymbol{f}_{i+1} = -\frac{o_{i+1}}{o_i}\, h_{i,i-1}\, d_i$$

$$\Downarrow$$

$$\boldsymbol{p}_{i+1} = \mathbb{Q}_i^T \boldsymbol{K}\boldsymbol{f}_{i+1} h_{i+1,i}^{-1} = \left(\boldsymbol{K}\boldsymbol{f}_{i+1} + \boldsymbol{p}_i\, \frac{\boldsymbol{o}_{i+1}}{o_i}\, h_{i,i-1}\right) h_{i+1,i}^{-1} \tag{2.15}$$

Ako v prípade dlhých rekurencií, uveďme si všeobecný algoritmus pre krátke formuly.
Je daná matica $\boldsymbol{G} \succ 0$, $\boldsymbol{G} = \boldsymbol{G}^T$, $\boldsymbol{G} \in \mathbb{R}^{n\times n}$ a vektor pravých strán $\boldsymbol{h} \in \mathbb{R}^n$. Ďalej máme zvolenú nejakú regulárnu maticu $\boldsymbol{K} \in \mathbb{R}^n$, $\boldsymbol{K} = \boldsymbol{K}^T$. Potom štruktúra algoritmu je nasledovaná.

$\mathbb{Q}_0 = \boldsymbol{I}_n$, $\boldsymbol{f}_1 = \boldsymbol{G}\boldsymbol{x}_1 - \boldsymbol{h}$, $\boldsymbol{p}_1 = \boldsymbol{K}\boldsymbol{f}_1$, $d_1 = \boldsymbol{p}_1^T \boldsymbol{G}\boldsymbol{p}_1$

for $i = 1, \ldots, n$

(1) $\boldsymbol{x}_{i+1} = \boldsymbol{x}_i - \boldsymbol{p}_i\, \dfrac{\boldsymbol{p}_i^T \boldsymbol{f}_i}{d_i}$

(2) $\boldsymbol{x}_{i+1} = \boldsymbol{x}_i - \boldsymbol{p}_i\, \dfrac{o_i\, h_{i,i-1}^{-1}}{d_i}$

(1) $\boldsymbol{f}_{i+1} = \boldsymbol{f}_i - \boldsymbol{G}\boldsymbol{p}_i\, \dfrac{\boldsymbol{p}_i^T \boldsymbol{f}_i}{d_i}$

(2) $\boldsymbol{f}_{i+1} = \boldsymbol{f}_i - \boldsymbol{G}\boldsymbol{p}_i\, \dfrac{o_i\, h_{i,i-1}^{-1}}{d_i}$

if $\left\|\boldsymbol{f}_{i+1}\right\|_\infty = 0$ *then*

$\boldsymbol{x}^* = \boldsymbol{x}_{i+1}$, STOP

end if

(1) $\boldsymbol{p}_{i+1} = \left(\boldsymbol{K}\boldsymbol{G}\boldsymbol{p}_i + \boldsymbol{p}_i\, \dfrac{\boldsymbol{p}_i^T \boldsymbol{G}\boldsymbol{K}\boldsymbol{G}\boldsymbol{p}_i}{d_i} + \boldsymbol{p}_{i-1} d_i\, h_{i,i-1}\right) h_{i+1,i}^{-1}$

(2) $\boldsymbol{p}_{i+1} = \left(\boldsymbol{K}\boldsymbol{f}_{i+1} + \boldsymbol{p}_i\, \dfrac{\boldsymbol{o}_{i+1}}{o_i}\, h_{i,i-1}\right) h_{i+1,i}^{-1}$

$d_{i+1} = \boldsymbol{p}_{i+1}^T \boldsymbol{G}\boldsymbol{p}_{i+1}$

end for.

Pozn.: volia sa buď kroky (1) (v prípade Lanczosovho výberu $\boldsymbol{w}_j = \boldsymbol{K}\boldsymbol{G}\boldsymbol{p}_{j-1}$ – metódy *Lanczosovho* typu), alebo kroky (2) (HS voľba : $\boldsymbol{w}_j = \boldsymbol{K}\boldsymbol{f}_j$ – metódy *HS* typu). Pre (2) sme použili vyjadrenie $\boldsymbol{p}_i^T \boldsymbol{f}_i = o_i\, h_{i,i-1}^{-1}$ z (2.14).

Na tomto mieste treba podotknúť, že všetky algoritmy (vrátane už uvádzaných) čerpáme z knihy [1], t.j. v tejto práci budú uvádzané bez zásahu autora, resp. odvodené podľa uvedenej literatúry.

Ku konkrétnym metódam sa dostaneme výberom vhodných parametrov a ich dosadením do všeobecného algoritmu pre tú, či onú voľbu $\boldsymbol{w}_j$.
V knihe [1] je viacero metód, ktorými sa autori zaoberajú, pričom my sme sa obmedzili iba na niektoré z nich.

Sú to tieto (uvádzame názvy prebraté z [1]):

- *HS* výber :

$$\left.\begin{array}{lll} CG & - & \boldsymbol{C}onjugate\ \boldsymbol{G}radient \\ CR & - & \boldsymbol{C}onjugate\ \boldsymbol{R}esiduals \end{array}\right\} \boldsymbol{A} = \boldsymbol{A}^T$$
$$\begin{array}{lll} CGNR & - & \boldsymbol{CG}\ \boldsymbol{N}ormal\ \boldsymbol{R}esiduals \end{array}$$

- *Lanczov* výber :

$$\left.\begin{array}{lll} MCR & - & \boldsymbol{M}odified\ \boldsymbol{CR} \\ OD & - & \boldsymbol{O}rthogonal\ \boldsymbol{D}irections \\ StOD & - & \boldsymbol{St}abilised\ \boldsymbol{OD} \\ SymmLQ & - & \boldsymbol{Symm}etric\ \boldsymbol{LQ} \end{array}\right\} \boldsymbol{A} = \boldsymbol{A}^T$$
$$\begin{array}{lll} LSQR & - & \boldsymbol{L}east\text{–}\boldsymbol{S}quares\ \boldsymbol{QR} \end{array}$$

Ešte uveďme, že všetky metódy, okrem *CGNR* a *LSQR*, vyžadujú symetriu matice $\boldsymbol{A}$ $\big($t.j. $\boldsymbol{A} = \boldsymbol{A}^T\big)$. Pre nasledujúce algoritmy používame všeobecný algoritmus pre krátke rekurencie (s konkrétnymi parametrami), uvedený na strane 37 a volíme $\mathrm{h}_{i+1,i} = h_{i,i-1} = 1$.

2.2.1 Metódy HS typu

A) Metóda konjungovaných gradientov (CG = $\boldsymbol{C}$onjugate $\boldsymbol{G}$radient method)

Parametre :

$$\boldsymbol{A} = \boldsymbol{A}^T, \quad \boldsymbol{G} = \boldsymbol{A}, \quad \boldsymbol{h} = \boldsymbol{b}, \quad \boldsymbol{K} = \boldsymbol{I}_n$$

Počiatočné hodnoty :

$$\boldsymbol{x}_1 \in \mathbb{R}^n, \quad \boldsymbol{p}_1 = \boldsymbol{K}\boldsymbol{f}_1 = \boldsymbol{r}_1 = \boldsymbol{A}\boldsymbol{x}_1 - \boldsymbol{b},$$

Iterácie :

$$\begin{aligned} \boldsymbol{x}_{i+1} &= \boldsymbol{x}_i - \boldsymbol{p}_i \,\frac{\boldsymbol{r}_i^T\boldsymbol{r}_i}{\boldsymbol{p}_i^T\boldsymbol{p}_i} \\ \boldsymbol{r}_{i+1} &= \boldsymbol{r}_i - \boldsymbol{A}\boldsymbol{p}_i \,\frac{\boldsymbol{r}_i^T\boldsymbol{r}_i}{\boldsymbol{p}_i^T\boldsymbol{p}_i} \\ \boldsymbol{p}_{i+1} &= \boldsymbol{r}_{i+1} + \boldsymbol{p}_i \,\frac{\boldsymbol{r}_{i+1}^T\boldsymbol{r}_{i+1}}{\boldsymbol{r}_i^T\boldsymbol{r}_i} \end{aligned}$$

Pozn. Ak matica $\boldsymbol{A}$ je indefinitná, algoritmus môže zlyhať (vplyvom straty kladnej definitnosti matice $\boldsymbol{G} = \boldsymbol{A}$, lebo môže platiť, že $\boldsymbol{p}_i^T\boldsymbol{G}\boldsymbol{p}_i = 0$, i keď $\boldsymbol{p}_i \neq 0_n$). Treba testovať v algoritme podmienku nenulovosti d_i.

B) Metóda konjungovaných rezíduí (CR = $\boldsymbol{C}$onjugate $\boldsymbol{R}$esiduals method)

Parametre :

$$\boldsymbol{A} = \boldsymbol{A}^T, \quad \boldsymbol{G} = \boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{A}^2, \quad \boldsymbol{h} = \boldsymbol{A}^T\boldsymbol{b} = \boldsymbol{A}\boldsymbol{b}, \quad \boldsymbol{K} = \boldsymbol{A}^{-T} = \boldsymbol{A}^{-1}$$

Počiatočné hodnoty :

$$\boldsymbol{x}_1 \in \mathbb{R}^n, \quad \boldsymbol{r}_1 = \boldsymbol{A}\boldsymbol{x}_1 - \boldsymbol{b}, \quad \boldsymbol{f}_1 = \boldsymbol{G}\boldsymbol{x}_1 - \boldsymbol{h}, \quad \boldsymbol{p}_1 = \boldsymbol{K}\boldsymbol{f}_1 = \boldsymbol{r}_1, \quad \boldsymbol{q}_1 = \boldsymbol{A}\boldsymbol{p}_1$$

Iterácie :

$$\begin{aligned}
\boldsymbol{x}_{i+1} &= \boldsymbol{x}_i - \boldsymbol{p}_i \, \frac{\boldsymbol{r}_i^T \boldsymbol{A} \boldsymbol{r}_i}{\boldsymbol{q}_i^T \boldsymbol{q}_i} \\
\boldsymbol{r}_{i+1} &= \boldsymbol{r}_i - \boldsymbol{q}_i \, \frac{\boldsymbol{r}_i^T \boldsymbol{A} \boldsymbol{r}_i}{\boldsymbol{q}_i^T \boldsymbol{q}_i} \\
\boldsymbol{p}_{i+1} &= \boldsymbol{r}_{i+1} + \boldsymbol{p}_i \, \frac{\boldsymbol{r}_{i+1}^T \boldsymbol{A} \boldsymbol{r}_{i+1}}{\boldsymbol{r}_i^T \boldsymbol{A} \boldsymbol{r}_i} \\
\boldsymbol{q}_{i+1} &= \boldsymbol{A} \boldsymbol{r}_{i+1} + \boldsymbol{q}_i \, \frac{\boldsymbol{r}_{i+1}^T \boldsymbol{A} \boldsymbol{r}_{i+1}}{\boldsymbol{r}_i^T \boldsymbol{A} \boldsymbol{r}_i}
\end{aligned}$$

Postupnosť $\{\boldsymbol{q}_i\} = \{\boldsymbol{A}\boldsymbol{p}_i\}$ generujeme, aby sme sa vyvarovali maticovo – vektorovému násobeniu $\big($namiesto $\boldsymbol{p}_i^T \boldsymbol{A}^2 \boldsymbol{p}_i$ rátame $\boldsymbol{q}_i^T \boldsymbol{q}_i\big)$.

C) Metóda normálnych konjungovaných rezíduí (*CGNR* = *CG* *N*ormal *R*esiduals method)

Parametre :

$$\boldsymbol{G} = \boldsymbol{A}^T \boldsymbol{A}, \quad \boldsymbol{h} = \boldsymbol{A}^T \boldsymbol{b}, \quad \boldsymbol{K} = \boldsymbol{I}_n$$

Počiatočné hodnoty :

$$\boldsymbol{x}_1 \in \mathbb{R}^n, \quad \boldsymbol{r}_1 = \boldsymbol{A}\boldsymbol{x}_1 - \boldsymbol{b}, \quad \boldsymbol{f}_1 = \boldsymbol{G}\boldsymbol{x}_1 - \boldsymbol{h}, \quad \boldsymbol{p}_1 = \boldsymbol{K}\boldsymbol{f}_1 = \boldsymbol{A}^T \boldsymbol{r}_1$$

Iterácie :

$$\begin{aligned}
\boldsymbol{x}_{i+1} &= \boldsymbol{x}_i - \boldsymbol{p}_i \, \frac{\boldsymbol{r}_i^T \boldsymbol{A}\boldsymbol{A}^T \boldsymbol{r}_i}{\boldsymbol{p}_i^T \boldsymbol{A}^T \boldsymbol{A} \boldsymbol{p}_i} \\
\boldsymbol{r}_{i+1} &= \boldsymbol{r}_i - \boldsymbol{A}\boldsymbol{p}_i \, \frac{\boldsymbol{r}_i^T \boldsymbol{A}\boldsymbol{A}^T \boldsymbol{r}_i}{\boldsymbol{p}_i^T \boldsymbol{A}^T \boldsymbol{A} \boldsymbol{p}_i} \\
\boldsymbol{p}_{i+1} &= \boldsymbol{A}^T \boldsymbol{r}_{i+1} + \boldsymbol{p}_i \, \frac{\boldsymbol{r}_{i+1}^T \boldsymbol{A}\boldsymbol{A}^T \boldsymbol{r}_{i+1}}{\boldsymbol{r}_i^T \boldsymbol{A}\boldsymbol{A}^T \boldsymbol{r}_i}
\end{aligned}$$

Poznamenajme, že vyčíslenie výrazu $\boldsymbol{p}_i^T \boldsymbol{A}^T \boldsymbol{A} \boldsymbol{p}_i$ $\big($resp., $\boldsymbol{r}_i^T \boldsymbol{A}\boldsymbol{A}^T \boldsymbol{r}_i\big)$ obdobne, ako pri *CR* metóde, vyžaduje 2 kroky : najprv výpočet pomocného vektora $\boldsymbol{q}_i = \boldsymbol{A}\boldsymbol{p}_i$ a potom výpočet daného výrazu ako skalárny súčin $\boldsymbol{q}_i^T \boldsymbol{q}_i$ $\Big($resp., $\boldsymbol{A}^T \boldsymbol{r}_i$ a následne $\big(\boldsymbol{A}^T \boldsymbol{r}_i\big)^T \big(\boldsymbol{A}^T \boldsymbol{r}_i\big)\Big)$.

2.2.2 Metódy Lanczosovho typu

A) Modifikovaná metóda konjugovaných rezíduí (*MCR* = *M*odified *C*onjugate *R*esiduals)

Parametre :

$$\boldsymbol{A} = \boldsymbol{A}^T, \quad \boldsymbol{G} = \boldsymbol{A}^T \boldsymbol{A} = \boldsymbol{A}^2, \quad \boldsymbol{h} = \boldsymbol{A}^T \boldsymbol{b} = \boldsymbol{A}\boldsymbol{b}, \quad \boldsymbol{K} = \boldsymbol{A}^{-T} = \boldsymbol{A}^{-1}$$

Počiatočné hodnoty :

$$\boldsymbol{x}_1 \in \mathbb{R}^n, \quad \boldsymbol{r}_1 = \boldsymbol{A}\boldsymbol{x}_1 - \boldsymbol{b}, \quad \boldsymbol{f}_1 = \boldsymbol{G}\boldsymbol{x}_1 - \boldsymbol{h}, \quad \boldsymbol{p}_1 = \boldsymbol{K}\boldsymbol{f}_1 = \boldsymbol{r}_1, \quad \boldsymbol{q}_1 = \boldsymbol{A}\boldsymbol{p}_1$$

Iterácie :

$$
\begin{aligned}
\boldsymbol{x}_{i+1} &= \boldsymbol{x}_i - \boldsymbol{p}_i \, \frac{\boldsymbol{q}_i^T \boldsymbol{r}_i}{\boldsymbol{q}_i^T \boldsymbol{q}_i} \\
\boldsymbol{r}_{i+1} &= \boldsymbol{r}_i - \boldsymbol{q}_i \, \frac{\boldsymbol{q}_i^T \boldsymbol{r}_i}{\boldsymbol{q}_i^T \boldsymbol{q}_i} \\
\boldsymbol{p}_{i+1} &= \boldsymbol{q}_i - \boldsymbol{p}_i \, \frac{\boldsymbol{q}_i^T \boldsymbol{A} \boldsymbol{q}_i}{\boldsymbol{q}_i^T \boldsymbol{q}_i} - \boldsymbol{p}_{i-1} \, \frac{\boldsymbol{q}_i^T \boldsymbol{q}_i}{\boldsymbol{q}_{i-1}^T \boldsymbol{q}_{i-1}} \;, \qquad i \geq 2 \\
\boldsymbol{q}_{i+1} &= \boldsymbol{A}\boldsymbol{q}_i - \boldsymbol{q}_i \, \frac{\boldsymbol{q}_i^T \boldsymbol{A} \boldsymbol{q}_i}{\boldsymbol{q}_i^T \boldsymbol{q}_i} - \boldsymbol{q}_{i-1} \, \frac{\boldsymbol{q}_i^T \boldsymbol{q}_i}{\boldsymbol{q}_{i-1}^T \boldsymbol{q}_{i-1}} \;, \qquad i \geq 2
\end{aligned}
$$

Opäť tu používame postupnosť $\{\boldsymbol{q}_i\} = \{\boldsymbol{A}\boldsymbol{p}_i\}$, čím si zjednodušíme násobenie $\boldsymbol{p}_i^T \boldsymbol{A}^3 \boldsymbol{p}_i$ na $\boldsymbol{q}_i^T \boldsymbol{A} \boldsymbol{q}_i$.

B) Metóda ortogonálnych smerov (*OD* = ***O**rthogonal **D**irections method*)

Parametre :

$$\boldsymbol{A} = \boldsymbol{A}^T, \quad \boldsymbol{G} = \boldsymbol{I}_n, \quad \boldsymbol{h} = \boldsymbol{A}^{-1}\boldsymbol{b}, \quad \boldsymbol{K} = \boldsymbol{A}$$

Počiatočné hodnoty :

$$\boldsymbol{x}_1 \in \mathbb{R}^n, \quad \boldsymbol{r}_1 = \boldsymbol{A}\boldsymbol{x}_1 - \boldsymbol{b}, \quad \boldsymbol{f}_1 = \boldsymbol{G}\boldsymbol{x}_1 - \boldsymbol{h}, \quad \boldsymbol{p}_1 = \boldsymbol{K}^2 \boldsymbol{f}_1 = \boldsymbol{A}\boldsymbol{r}_1$$

Iterácie :

$$
\begin{aligned}
\boldsymbol{x}_2 &= \boldsymbol{x}_1 - \boldsymbol{p}_1 \, \frac{\boldsymbol{r}_1^T \boldsymbol{r}_1}{\boldsymbol{p}_1^T \boldsymbol{p}_1} \\
\boldsymbol{r}_2 &= \boldsymbol{r}_1 - \boldsymbol{A}\boldsymbol{p}_1 \, \frac{\boldsymbol{r}_1^T \boldsymbol{r}_1}{\boldsymbol{p}_1^T \boldsymbol{p}_1} \\
\boldsymbol{x}_{i+1} &= \boldsymbol{x}_i - \boldsymbol{p}_i \, \frac{\boldsymbol{p}_{i-1}^T \boldsymbol{r}_i}{\boldsymbol{p}_i^T \boldsymbol{p}_i} \;, \qquad i \geq 2 \\
\boldsymbol{r}_{i+1} &= \boldsymbol{r}_i - \boldsymbol{A}\boldsymbol{p}_i \, \frac{\boldsymbol{p}_{i-1}^T \boldsymbol{r}_i}{\boldsymbol{p}_i^T \boldsymbol{p}_i} \;, \qquad i \geq 2 \\
\boldsymbol{p}_{i+1} &= \boldsymbol{A}\boldsymbol{p}_i - \boldsymbol{p}_i \, \frac{\boldsymbol{p}_i^T \boldsymbol{A} \boldsymbol{p}_i}{\boldsymbol{p}_i^T \boldsymbol{p}_i} - \boldsymbol{p}_{i-1} \, \frac{\boldsymbol{p}_i^T \boldsymbol{p}_i}{\boldsymbol{p}_{i-1}^T \boldsymbol{p}_{i-1}}
\end{aligned}
$$

Pozn. Vynásobme vzťah (2.10) vektorom $\boldsymbol{f}_{i+1}$ zľava a dostaneme

$$\boldsymbol{p}_{i+1}^T \boldsymbol{f}_{i+1} = \boldsymbol{p}_i^T \boldsymbol{G}\boldsymbol{K} \boldsymbol{f}_{i+1},$$

pričom sme využili Galerkinovu podmienku, čím sme anulovali zvyšné členy. Teraz, keďže $\boldsymbol{G} = \boldsymbol{I}_n$, $\boldsymbol{K} = \boldsymbol{A}$, a teda $\boldsymbol{r}_{i+1} = \boldsymbol{A}\boldsymbol{f}_{i+1}$, a ak $i \to i-1$ máme

$$\boldsymbol{p}_i^T \boldsymbol{f}_i = \boldsymbol{p}_{i-1}^T \boldsymbol{r}_{i+1}.$$

A práve táto rovnica sa objavuje v iteráciách pre $\boldsymbol{x}_i$ a $\boldsymbol{r}_i$.
V praxi sa ukazuje byť tento algoritmus numericky nestabilný.
Preto vznikla jeho stabilizovanejšia podoba (*StOD*)

C) Stabilizovaná metóda ortogonálnych smerov (*StOD* = *St*abilised *O*rthogonal *D*irections method)

Parametre :

$$\boldsymbol{A} = \boldsymbol{A}^T, \quad \boldsymbol{G} = \boldsymbol{I}_n, \quad \boldsymbol{h} = \boldsymbol{A}^{-1}\boldsymbol{b}, \quad \boldsymbol{K} = \boldsymbol{A}$$

Počiatočné hodnoty :

$$\boldsymbol{x}_1 \in \mathbb{R}^n, \quad \boldsymbol{r}_1 = \boldsymbol{A}\boldsymbol{x}_1 - \boldsymbol{b}, \quad \boldsymbol{v}_1 = \boldsymbol{r}_1, \quad \boldsymbol{p}_1 = \boldsymbol{A}\boldsymbol{v}_1$$

Iterácie :

$$\begin{aligned}
\boldsymbol{x}_{i+1} &= \boldsymbol{x}_i - \boldsymbol{p}_i \frac{\boldsymbol{v}_i^T \boldsymbol{r}_i}{\boldsymbol{p}_i^T \boldsymbol{p}_i}, \\
\boldsymbol{r}_{i+1} &= \boldsymbol{r}_i - \boldsymbol{A}\boldsymbol{p}_i \frac{\boldsymbol{v}_i^T \boldsymbol{r}_i}{\boldsymbol{p}_i^T \boldsymbol{p}_i}, \\
\boldsymbol{p}_{i+1} &= \boldsymbol{A}\boldsymbol{p}_i - \boldsymbol{p}_i \frac{\boldsymbol{p}_i^T \boldsymbol{A}\boldsymbol{p}_i}{\boldsymbol{p}_i^T \boldsymbol{p}_i} - \boldsymbol{p}_{i-1} \frac{\boldsymbol{p}_i^T \boldsymbol{p}_i}{\boldsymbol{p}_{i-1}^T \boldsymbol{p}_{i-1}}
\end{aligned}$$

Tu počítame výraz $\boldsymbol{p}_i^T \boldsymbol{f}_i$ nepriamo – pomocou postupnosti $\{\boldsymbol{v}_i\} = \left\{\boldsymbol{A}^{-1}\boldsymbol{p}_i\right\}$.

D) Symetrická metóda využívajúca LQ rozklad (*SymmLQ* = *Symm*etric *O*rthogonal *LQ*)

Parametre :

$$\boldsymbol{A} = \boldsymbol{A}^T, \quad \boldsymbol{G} = \boldsymbol{I}_n, \quad \boldsymbol{K} = \boldsymbol{A}$$

Počiatočné hodnoty :

$$\boldsymbol{x}_1 \in \mathbb{R}^n, \quad \boldsymbol{r}_1 = \boldsymbol{A}\boldsymbol{x}_1 - \boldsymbol{b}, \quad \boldsymbol{p}_1 = \boldsymbol{r}_1 / \|\boldsymbol{r}_1\|$$

Tento algoritmus rieši rovnicu $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ (pôvodný systém (1.1)), kde $\boldsymbol{A}$ je symetrická a vo všeobecnosti indefinitná. Generuje postupnosť ortonormálnych vektorov $\{\boldsymbol{y}_i\} \in \mathbb{R}^n$ a v každom kroku minimalizuje Euklidovskú normu vektora chyby $\boldsymbol{e}(z)$ nad podpriestorom :

$$\boldsymbol{x}(z) = \boldsymbol{x}_i + z\boldsymbol{y}_i \tag{2.16}$$

Dosadením do vzťahu (1.6)

$$\boldsymbol{e}(z) = \boldsymbol{x}(z) - \boldsymbol{x}^*$$

výrazu pre $\boldsymbol{x}(z)$ (2.16) dostávame :

$$\begin{aligned}
\boldsymbol{e}(z) &= \boldsymbol{x}_i + z\boldsymbol{y}_i - \boldsymbol{x}^* \\
&= (\boldsymbol{x}_i - \boldsymbol{x}^*) + z\boldsymbol{y}_i \\
&= \boldsymbol{e}_i + z\boldsymbol{y}_i
\end{aligned} \tag{2.17}$$

Ďalej platí, že $\boldsymbol{y}_i^T \boldsymbol{y}_i = 1$ (lebo sú ortonormálne) a hodnotu $z = z_i$, ktorá minimalizuje $\|\boldsymbol{e}(z)\|^2$ zistíme takto :

$$\begin{aligned}
\|\boldsymbol{e}(z)\|^2 &\overset{(2.17)}{=} \|\boldsymbol{e}_i + z\boldsymbol{y}_i\|^2 = (\boldsymbol{e}_i + z\boldsymbol{y}_i)^T (\boldsymbol{e}_i + z\boldsymbol{y}_i) \\
&= \|\boldsymbol{e}_i\|^2 + 2\left(\boldsymbol{y}_i^T \boldsymbol{e}_i\right) z + \underbrace{\left(\boldsymbol{y}_i^T \boldsymbol{y}_i\right)}_{=1} z^2 \\
&= z^2 + 2\left(\boldsymbol{y}_i^T \boldsymbol{e}_i\right) z + \|\boldsymbol{e}_i\|^2
\end{aligned}$$

Ide o rýdzokonvexnú funkciu, a preto pre nájdenie minima stačí, aby bola splnená podmienka prvého rádu (*FOC*)

FOC :

$$\left.\frac{\mathrm{d}\,\|\boldsymbol{e}\|_E^2}{\mathrm{d}z}\right|_{z=z_i} = 0$$

$$2z_i + 2\left(\boldsymbol{y}_i^T\boldsymbol{e}_i\right) = 0$$

$$z_i = -\boldsymbol{y}_i^T\boldsymbol{e}_i \tag{2.18}$$

Dosaďme tento výsledok, spätne, do rovníc (2.16) a (2.17) za skalár z :

$$\boldsymbol{x}_{i+1} \equiv \boldsymbol{x}(z_i) = \boldsymbol{x}_i - \boldsymbol{y}_i\left(\boldsymbol{y}_i^T\boldsymbol{e}_i\right) \tag{2.19}$$

a

$$\begin{aligned}\boldsymbol{e}_{i+1} \equiv \boldsymbol{e}(z_i) &= \boldsymbol{e}_i - \boldsymbol{y}_i\left(\boldsymbol{y}_i^T\boldsymbol{e}_i\right)\\ &= \left(\boldsymbol{I}_n - \boldsymbol{y}_i\boldsymbol{y}_i^T\right)\boldsymbol{e}_i\end{aligned} \tag{2.20}$$

Získaný výsledok pre $\boldsymbol{e}_{i+1}$ je rekurentným vzťahom, a preto ho rozpíšme ďalej :

$$\begin{aligned}\boldsymbol{e}_{i+1} &= \left(\boldsymbol{I}_n - \boldsymbol{y}_i\boldsymbol{y}_i^T\right)\boldsymbol{e}_i\\ &= \left(\boldsymbol{I}_n - \boldsymbol{y}_i\boldsymbol{y}_i^T\right)\left(\boldsymbol{I}_n - \boldsymbol{y}_{i-1}\boldsymbol{y}_{i-1}^T\right)\boldsymbol{e}_i\\ &= \left(\boldsymbol{I}_n - \boldsymbol{y}_i\boldsymbol{y}_i^T\right)\left(\boldsymbol{I}_n - \boldsymbol{y}_{i-1}\boldsymbol{y}_{i-1}^T\right)\cdots\left(\boldsymbol{I}_n - \boldsymbol{y}_1\boldsymbol{y}_1^T\right)\boldsymbol{e}_1\\ &= \left(\boldsymbol{I}_n - \sum_{j=1}^{i}\boldsymbol{y}_j\boldsymbol{y}_j^T\right)\boldsymbol{e}_1\end{aligned} \tag{2.21}$$

Označme si

$$\mathbb{Y}_i = \left(\ \boldsymbol{y}_1 \mid \boldsymbol{y}_2 \mid \cdots \mid \boldsymbol{y}_i\ \right)_{n\times i}$$

a môžeme (2.21) vyjadriť rovnicou

$$\boldsymbol{e}_{i+1} = \left(\boldsymbol{I}_n - \sum_{j=1}^{i}\boldsymbol{y}_j\boldsymbol{y}_j^T\right)\boldsymbol{e}_1 = \left(\boldsymbol{I}_n - \mathbb{Y}_i\mathbb{Y}_i^T\right)\boldsymbol{e}_1$$

Všimnime si, že ak $i = n$, tak $\mathbb{Y}_n$ je štvorcová ortogonálna matica, a teda $\boldsymbol{I}_n - \mathbb{Y}_n\mathbb{Y}_n^T = 0_{n\times n}$.
To garantuje (teoreticky) maximálne n iterácií.
V praxi sa však potýkame s úlohou počítať $\boldsymbol{x}_{i+1}$ pomocou vzťahu (2.19), ale $\boldsymbol{e}_i$ je pre nás neznáma veličina.
Tento problém budeme riešiť generovaním pomocnej postupnosti ortonormálnych vektorov $\{\boldsymbol{p}_i\}$, prostredníctvom *GAA* algoritmu, pričom za maticu $\boldsymbol{B}$ volíme maticu $\boldsymbol{A}$.
Z rovnice (1.30) s $\boldsymbol{B} = \boldsymbol{A}$ máme :

$$\boldsymbol{A}\mathbb{P}_i = \mathbb{P}_{i+1}\boldsymbol{H}_{i+1}^*, \tag{2.22}$$

kde $\boldsymbol{H}_{i+1}^* \in \mathbb{R}^{(i+1)\times i}$ je horná *Hessenbergova* matica (viď. strana 26).
Podobne, ako pri *GMRes* metóde (strana 30), označme $\boldsymbol{Q}_{i+1}$ ortogonálnu maticu takú, pre ktorú platí, že

$$\boldsymbol{Q}_{i+1}\boldsymbol{H}_{i+1}^* = \left(\frac{\boldsymbol{U}_i}{0_i^T}\right),$$

kde $\boldsymbol{U}_i \in \mathbb{R}^{i \times i}$ je horná trojuholníková matica. Maticu $\mathbb{Y}_i$ a vektor $\boldsymbol{c}_{i+1}$ definujme takto :

$$\left(\begin{array}{c|c} \mathbb{Y}_i & \boldsymbol{c}_{i+1} \end{array} \right) = \mathbb{P}_{i+1}\boldsymbol{Q}_{i+1}^T$$

Potom z konštrukcie $\mathbb{Y}_i$ vyplýva, že

$$\mathbb{Y}_i^T \mathbb{Y}_i = \boldsymbol{I}_i,$$

t.j. $\mathbb{Y}_i$ má ortonormálne stĺpce. Upravme rovnicu (2.22) :

$$\begin{aligned} \boldsymbol{A}\mathbb{P}_i &= \boldsymbol{P}_{i+1}\boldsymbol{H}_{i+1}^* = \underbrace{\mathbb{P}_{i+1}\boldsymbol{Q}_{i+1}^T}\underbrace{\boldsymbol{Q}_{i+1}\boldsymbol{H}_{i+1}^*} \\ &= \left(\begin{array}{c|c} \mathbb{Y}_i & \boldsymbol{c}_{i+1} \end{array} \right) \left(\frac{\boldsymbol{U}_i}{0_i^T} \right) \\ &= \mathbb{Y}_i \boldsymbol{U}_i \end{aligned} \tag{2.23}$$

Vynásobením tejto rovnice maticou $\boldsymbol{A}^{-1}$ zľava a následne maticou $\boldsymbol{U}_i^{-1}$ sprava dostaneme :

$$\boldsymbol{A}^{-1}\mathbb{Y}_i = \mathbb{P}_i \boldsymbol{U}_i^{-1} \tag{2.24}$$

Teraz aplikujme vzťah medzi i–tym vektorom rezíduí $\boldsymbol{r}_i$ a i– tym vektorom chyby $\boldsymbol{e}_i$ pôvodného systému lineárnych rovníc $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ (1.1), t.j. vzťah

$$\boldsymbol{e}_i = \boldsymbol{A}^{-1}\boldsymbol{r}_i$$

a tiež fakt, že

$$\boldsymbol{A}^{-1} = \boldsymbol{A}^{-T}, \qquad \text{keďže} \qquad \boldsymbol{A} = \boldsymbol{A}^T.$$

Vynásobme (2.24) vektorom $\boldsymbol{r}_i^T$ zľava

$$\begin{aligned} \underbrace{\boldsymbol{r}_i^T \boldsymbol{A}^{-1}} \mathbb{Y}_i &= \boldsymbol{r}_i^T \mathbb{P}_i \boldsymbol{U}_i^{-1} \\ \boldsymbol{e}_i^T \mathbb{Y}_i &= \boldsymbol{r}_i^T \mathbb{P}_i \boldsymbol{U}_i^{-1} \end{aligned}$$

Ak $\boldsymbol{s}_i$ bude označovať i–ty stĺpec matice I_i, potom

$$\boldsymbol{e}_i^T \boldsymbol{y}_i = \boldsymbol{e}_i^T \mathbb{Y}_i \boldsymbol{s}_i = \boldsymbol{r}_i^T \mathbb{P}_i \boldsymbol{U}_i^{-1} \boldsymbol{s}_i.$$

Dosaďme tento vzťah do (2.19) a dostaneme

$$\boldsymbol{x}_{i+1} = \boldsymbol{x}_i - \boldsymbol{y}_i \boldsymbol{r}_i^T \mathbb{P}_i \boldsymbol{U}_i^{-1} \boldsymbol{s}_i,$$

kde všetky hodnoty na pravej strane sú známe.
Ako počítať maticu $\mathbb{P}_i \boldsymbol{U}_i^{-1}$?
Podrobný postup sme uviedli na strane 33, v sekcii *GMRes* metóda. Pričom sú tu vysvetlené všetky premenné a spôsoby ich výpočtu.
Z tohto dôvodu tento postup znovu neuvádzame a prejdeme ďalej.
Nech $\mathbb{T}_i = \left(\begin{array}{c|c|c|c} \boldsymbol{t}_1 & \boldsymbol{t}_2 & \cdots & \boldsymbol{t}_i \end{array} \right) \equiv \mathbb{P}_i \boldsymbol{U}_i^{-1}$ (POZOR v predošlom texte, na ktorý vás upozorňujeme, je zobraté $\boldsymbol{Y}_i \equiv \mathbb{P}_i \boldsymbol{U}_i^{-1}$). Potom platí

$$\boldsymbol{t}_i = \frac{\boldsymbol{p}_i - \mathbb{T}_{i-1}\boldsymbol{u}_i}{\omega_i^*}$$

Lenže $\boldsymbol{u}_i$ má všetky zložky nulové, okrem dvoch posledných. Označme ich u_{i-1} a u_i. To vyplýva z toho, že matica $\boldsymbol{H}_{i+1}^*$ je trojdiagonálna, lebo platí

$$\mathbb{P}_i^T \boldsymbol{A} \mathbb{P}_i = \mathbb{P}_i^T \mathbb{P}_{i+1} \boldsymbol{H}_{i+1}^* = \boldsymbol{H}_i$$

a $\boldsymbol{A} = \boldsymbol{A}^T$. Preto máme

$$\boldsymbol{t}_i = \frac{\boldsymbol{p}_i - \boldsymbol{t}_{i-1}u_i - \boldsymbol{t}_{i-2}u_{i-1}}{\omega_i^*}$$

Nakoniec píšme z (2.23)

$$\begin{aligned} \boldsymbol{y}_i &= \mathbb{Y}_i \boldsymbol{s}_i = \boldsymbol{A} \underbrace{\mathbb{P}_i \boldsymbol{U}_i^{-1} \boldsymbol{s}_i} \\ &= \boldsymbol{A}\boldsymbol{t}_i. \end{aligned}$$

A tak sme odvodili vzťah pre $\boldsymbol{x}_{i+1}$ a $\boldsymbol{r}_{i+1}$

$$\begin{aligned} \boldsymbol{x}_{i+1} &= \boldsymbol{x}_i - \boldsymbol{A}\boldsymbol{t}_i \left(\boldsymbol{r}_i^T \boldsymbol{t}_i\right) \\ \boldsymbol{r}_{i+1} &= \boldsymbol{r}_i - \boldsymbol{A}^2 \boldsymbol{t}_i \left(\boldsymbol{r}_i^T \boldsymbol{t}_i\right) \end{aligned}$$

E) Metóda najmenších štvorcov využívajúca QR rozklad (*LSQR* = *L*east–*S*quares *QR* method)

Parametre :

$$\boldsymbol{G} = \begin{pmatrix} \boldsymbol{I}_n & 0_{n\times n} \\ 0_{n\times n} & \boldsymbol{I}_n \end{pmatrix}, \quad \boldsymbol{K} = \begin{pmatrix} 0_{n\times n} & \boldsymbol{A}^T \\ \boldsymbol{A} & 0_{n\times n} \end{pmatrix}, \quad \boldsymbol{p}_i = \begin{pmatrix} 0_n \\ \boldsymbol{s}_1 \end{pmatrix}$$

Počiatočné hodnoty :

$$\boldsymbol{x}_1 \in \mathbb{R}^n, \quad \boldsymbol{r}_1 = \boldsymbol{A}\boldsymbol{x}_1 - \boldsymbol{b}, \quad \boldsymbol{s}_1 = \boldsymbol{r}_1 / \|\boldsymbol{r}_1\|$$

LSQR konštruuje postupnosť vektorov $\{\boldsymbol{w}_j\} \in \mathbb{R}^n$ navzájom konjungovaných vzhľadom ku matici $\boldsymbol{A}^T\boldsymbol{A}$ a potom ich použije na riešenie systému normálnych rovníc

$$\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{b}.$$

Postupnosť aproximácií riešenia tejto sústavy $\{\boldsymbol{x}_i\}$ je počítaná pomocou rovnice (2.2), ktorú používajú, napr. metódy *GCR* a *OrthoDir*

$$\boldsymbol{x}_{i+1} = \boldsymbol{x}_i + \boldsymbol{p}_i \frac{\boldsymbol{p}_i^T \boldsymbol{A}^T \boldsymbol{r}_i}{\boldsymbol{p}_i^T \boldsymbol{A}^T \boldsymbol{A} \boldsymbol{p}_i}.$$

Ak v tejto rovnici zameníme $\boldsymbol{w}_j$ za $\boldsymbol{p}_i$ a i za j dostaneme

$$\boldsymbol{x}_{j+1} = \boldsymbol{x}_j + \boldsymbol{w}_j \frac{\boldsymbol{w}_j^T \boldsymbol{A}^T \boldsymbol{r}_j}{\boldsymbol{w}_j^T \boldsymbol{A}^T \boldsymbol{A} \boldsymbol{w}_j}. \tag{2.25}$$

Vektory $\boldsymbol{w}_j$ budeme počítať pomocou postupnosti ortonormálnych vektorov $\{\boldsymbol{v}_i\}$.
Dosadením parametrov a počiatočných hodnôt do vzťahu (2.11) zistíme, že vektory $\boldsymbol{p}_j$ majú tvar

$$\begin{pmatrix} \boldsymbol{v}_j \\ 0_n \end{pmatrix}$$

pre j párne a

$$\begin{pmatrix} 0_n \\ \boldsymbol{s}_j \end{pmatrix}$$

pre j nepárne, kde

$$\begin{aligned} \boldsymbol{v}_{j+1} &= \left(\boldsymbol{A}^T\boldsymbol{s}_j - \boldsymbol{v}_{j-1}\beta_j\right)/\alpha_{j+1} \quad j \quad \text{nepárne} && (2.26)\\ \boldsymbol{s}_{j+1} &= \left(\boldsymbol{A}\boldsymbol{v}_j - \boldsymbol{s}_{j-1}\alpha_j\right)/\beta_{j+1} \quad j \quad \text{párne} && (2.27) \end{aligned}$$

a α_{j+1} a β_{j+1} sú vyberané tak, aby $\|\boldsymbol{v}_{j+1}\| = \|\boldsymbol{s}_{j+1}\| = 1$. Obe postupnosti sa stávajú týmto ortonormálnymi.

Nech teraz

$$\mathbb{V}_i = \left(\boldsymbol{v}_2 \mid \boldsymbol{v}_4 \mid \cdots \mid \boldsymbol{v}_{2i} \right)_{n\times i}$$

a

$$\mathbb{S}_{i+1} = \left(\boldsymbol{s}_1 \mid \boldsymbol{s}_3 \mid \cdots \mid \boldsymbol{s}_{2i+1} \right)_{n\times (i+1)}$$

Upravme rovnicu (2.27) na

$$\boldsymbol{A}\boldsymbol{v}_j = \alpha_j\boldsymbol{s}_{j-1} + \beta_{j+1}\boldsymbol{s}_{j+1}. \tag{2.28}$$

Čo bude $\boldsymbol{A}\mathbb{V}_i$?

$$\begin{aligned} \boldsymbol{A}\mathbb{V}_i &= \boldsymbol{A}\left(\boldsymbol{v}_2 \mid \boldsymbol{v}_4 \mid \cdots \mid \boldsymbol{v}_{2i} \right)\\ &= \left(\boldsymbol{A}\boldsymbol{v}_2 \mid \boldsymbol{A}\boldsymbol{v}_4 \mid \cdots \mid \boldsymbol{A}\boldsymbol{v}_{2i} \right)\\ &\overset{(2.28)}{=} \left(\alpha_2\boldsymbol{s}_1 + \beta_3\boldsymbol{s}_3 \mid \alpha_4\boldsymbol{s}_3 + \beta_5\boldsymbol{s}_5 \mid \cdots \mid \alpha_{2i}\boldsymbol{s}_{2i-1} + \beta_{2i+1}\boldsymbol{s}_{2i+1} \right)\\ &= \left(\mathbb{S}_{i+1}\begin{pmatrix}\alpha_2\\ \beta_3\\ 0\\ 0\\ \vdots\\ 0\end{pmatrix} \middle| \ \mathbb{S}_{i+1}\begin{pmatrix}0\\ \alpha_4\\ \beta_5\\ 0\\ \vdots\\ 0\end{pmatrix} \middle| \cdots \middle| \ \mathbb{S}_{i+1}\begin{pmatrix}0\\ 0\\ \vdots\\ 0\\ \alpha_{2i}\\ \beta_{2i+1}\end{pmatrix} \right)\\ &= \mathbb{S}_{i+1}\boldsymbol{H}^*_{i+1}, \end{aligned} \tag{2.29}$$

kde

$$\boldsymbol{H}^*_{i+1} = \begin{pmatrix} \alpha_2 & 0 & \cdots & 0\\ \beta_3 & \alpha_4 & \ddots & 0\\ 0 & \beta_5 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \alpha_{2i}\\ 0 & \cdots & 0 & \beta_{2i+1} \end{pmatrix}_{(i+1)\times i}$$

je horná *Hessenbergova* matica.

Ak $\boldsymbol{Q}_{i+1}$ je ortogonálna matica, ktorá spĺňa

$$\boldsymbol{Q}_{i+1}\boldsymbol{H}^*_{i+1} = \left(\frac{\boldsymbol{U}_i}{0_i^T}\right),$$

kde $\boldsymbol{U}_i$ je horná trojuholníková matica, môžeme definovať maticu $\mathbb{Y}_i \in \mathbb{R}^{n\times i}$ a vektor $\boldsymbol{c}_{i+1}$ takto :

$$\left(\mathbb{Y}_i \mid \boldsymbol{c}_{i+1} \right) = \mathbb{S}_{i+1}\boldsymbol{Q}^T_{i+1}$$

Rovnicu (2.29) môžeme teraz napísať :

$$\begin{aligned} \boldsymbol{A}\mathbb{V}_i &= \mathbb{S}_{i+1}\boldsymbol{Q}^T_{i+1}\boldsymbol{Q}_{i+1}\boldsymbol{H}^*_{i+1}\\ &= \mathbb{Y}_i\boldsymbol{U}_i \end{aligned}$$

Ak teraz

$$\mathbb{W}_i = \left(\ \boldsymbol{w}_1 \ \middle| \ \boldsymbol{w}_2 \ \middle| \ \cdots \ \middle| \ \boldsymbol{w}_i \ \right) \equiv \mathbb{V}_i \boldsymbol{U}_i^{-1} \tag{2.30}$$

potom

$$\mathbb{Y}_i = \boldsymbol{A}\mathbb{W}_i$$

a

$$\mathbb{W}_i^T \boldsymbol{A}^T \boldsymbol{A} \mathbb{W}_i = \mathbb{Y}_i^T \mathbb{Y}_i = \boldsymbol{I}_i,$$

t.j.

$$\boldsymbol{w}_j^T \boldsymbol{A}^T \boldsymbol{A} \boldsymbol{w}_j = 1 \ , \quad j = 1, 2, \ldots, i.$$

Toto zjednoduší rovnicu (2.25) na :

$$\boldsymbol{x}_{j+1} = \boldsymbol{x}_j + \left(\boldsymbol{w}_j^T \boldsymbol{A}^T \boldsymbol{r}_j \right) \boldsymbol{w}_j$$

Ako v prípade *SymmLQ*, počítanie matíc $\mathbb{W}_i$ z rovnice (2.30) je popísané v sekcii *GMRes* (strana 30).
Takže tu uvedieme výsledný vzťah pre vektory $\boldsymbol{w}_i$

$$\boldsymbol{w}_i = \frac{\boldsymbol{v}_{2i} - \boldsymbol{w}_{i-1} u_i}{\omega_i^*},$$

kde u_i je jediný nenulový prvok (na poslednej pozícii) vektora $\boldsymbol{u}_i$ matice $\boldsymbol{U}_i$ (viď. (2.6) a stranu 32).

Kapitola 3

Numerické experimenty

V tejto kapitole sa zameriame na výsledky, získané „testovaním“ niektorých uvedených metód. Vybrali sme si tieto metódy

- HS typu : *CG* (conjugate gradient), *CR* (conjugate residuals), *CGNR* (*CG* normal residuals),
- Lanczosovho typu : *SymmLQ* (symmetric LQ), *LSQR* (least–squares QR)

a *GMRes* (generalised minimal residuals).

Na naprogramovanie jednotlivých metód sme použili software MATLAB, ktorý je veľmi vhodný pre prácu s maticami. Testovať sa dajú rôzne regulárne sústavy lineárnych rovníc, avšak my sme sa obmedzili na kladne definitné systémy, t.j. uvažujme len prípad kladne definitnej matice $\boldsymbol{A} \succ 0$. V tomto prípade, podľa teórie, by algoritmy nemali zlyhávať, nakoľko je potom vždy matica $\boldsymbol{G} \succ 0$.
Systém $\boldsymbol{Ax} = \boldsymbol{b}$ (1.1) s kladne definitnou maticou $\boldsymbol{A}$ budeme generovať s vopred známym riešením sústavy, kvôli kontrole správnosti riešenia. Keďže sa ťažšie generujú riedke *regulárne* matice, pomohli sme si vlastnosťami diagonálne dominantných regulárnych matíc. Potom za $\boldsymbol{A}$ zvolíme diagonálne dominantnú, symetrickú a teda kladne definitnú maticu.
Správanie sa algoritmov sme testovali na úlohách rozmeru $n = 100$ až $n = 50\,000$, pričom sme menili aj samotnú „riedkosť“ matíc a tiež „kvalitu“ (normu vektora $\boldsymbol{e}_1 = \boldsymbol{x}_1 - \boldsymbol{x}^*$) štartovacieho bodu, t.j. prvej aproximácie $\boldsymbol{x}_1$
Samotný experiment sme zostavili ako sériu 100 úloh pre danú metódu, daný rozmer úlohy n, danú riedkosť matice A a daný štartovací bod. Z týchto 100 úloh sme ako výsledok (pre každú testovaciu metódu) dostali tieto údaje:

- *Pres_min_e* - najhoršiu dosiahnutú presnosť (zo všetkých úloh) $\|\boldsymbol{x}_s - \boldsymbol{x}^*\|_\infty$, kde $\boldsymbol{x}_s$ je posledná dosiahnutá aproximácia, príslušným algoritmom
- *Pres_min_r* - najhoršiu dosiahnutú presnosť $\|\boldsymbol{Ax}_s - \boldsymbol{b}\|_\infty$
- *Pres_avg_e* - priemernú dosiahnutú presnosť $\|\boldsymbol{x}_s - \boldsymbol{x}^*\|_\infty$
- *Pres_avg_r* - priemernú dosiahnutú presnosť $\|\boldsymbol{Ax}_s - \boldsymbol{b}\|_\infty$
- *Iter_max* - maximalny počet iterácií, ktoré potrebovala metóda na niektorú z úloh
- *Iter_avg* - priemerný počet ietrácií

Všetky výsledky (tabuľky a grafy) sú uvedené v prílohe str. 50–131.
Na grafoch sú priebehy iterácií jednotlivých metód (z priemerných hodnôt) pre $n = 1000$, riedkosť = 5%, $\|\boldsymbol{e}_1\|_\infty = 104$ (str. 50–53). Ďalšie grafy sú pre $n = 1000$, riedkosť = 5% a rôzne vzdialenosti $\boldsymbol{x}_1$ od $\boldsymbol{x}^*$ (str. 54–59). Potom nasledujú tabuľky (60–131).

3.1 Záver

Z množstva údajov, ktoré máme k dispozícii, je zrejmé, že dané metódy sa veľmi dobre osvedčili. Je pravda, že sú medzi nimi kvalitatívne rozdiely (napr. *CG* a *LSQR*), ale ani jedna z nich nezlyhala. Počet iterácií aj pre veľké rozmery úloh je pomerne uspokojivý, hlavne pri menšej riedkosti matíc, pretože pre väčšie rozmery a pri veľkej riedkosti je nárast iterácií mnohonásobný (viď. napr. metóda *CGNR* $n = 50\,000$, tab. CGNR 12, str. 95). Najlepšie vykazujú výsledky metódy *CG* a *CR*.
Všetky naprogramované metódy sa môžu testovať aj na iných regulárnych maticiach $\boldsymbol{A}$ než kladne definitných.

Literatúra

a) Hlavný zdroj použitý pri spracovaní diplomovej práce

[1] Ch. G. Broyden, M. T. Vespucci, *Krylov Solvers for Linear Algebraic Systems*, Elsevier 2004

b) Zdroje použité pri štúdiu danej problematiky

[2] Karim M. Abadir, Jan R. Magnus, *Matrix Algebra*, Cambridge University Press 2005

[3] Y. Saad, *The Lanczos Biorthogonalization Algorithm and Other Oblique Projection Methods for Solving Large Unsymmetric Systems*, SIAM Journal on Numerical Analysis, Vol. 19, No. 3. (Jun., 1982), pp. 485-506.

[4] Y. Saad, *Krylov Subspace Methods for Solving Large Unsymmetric Linear Systems*, Mathematics of Computation, Vol. 37, No. 155. (Jul., 1981), pp. 105-126.

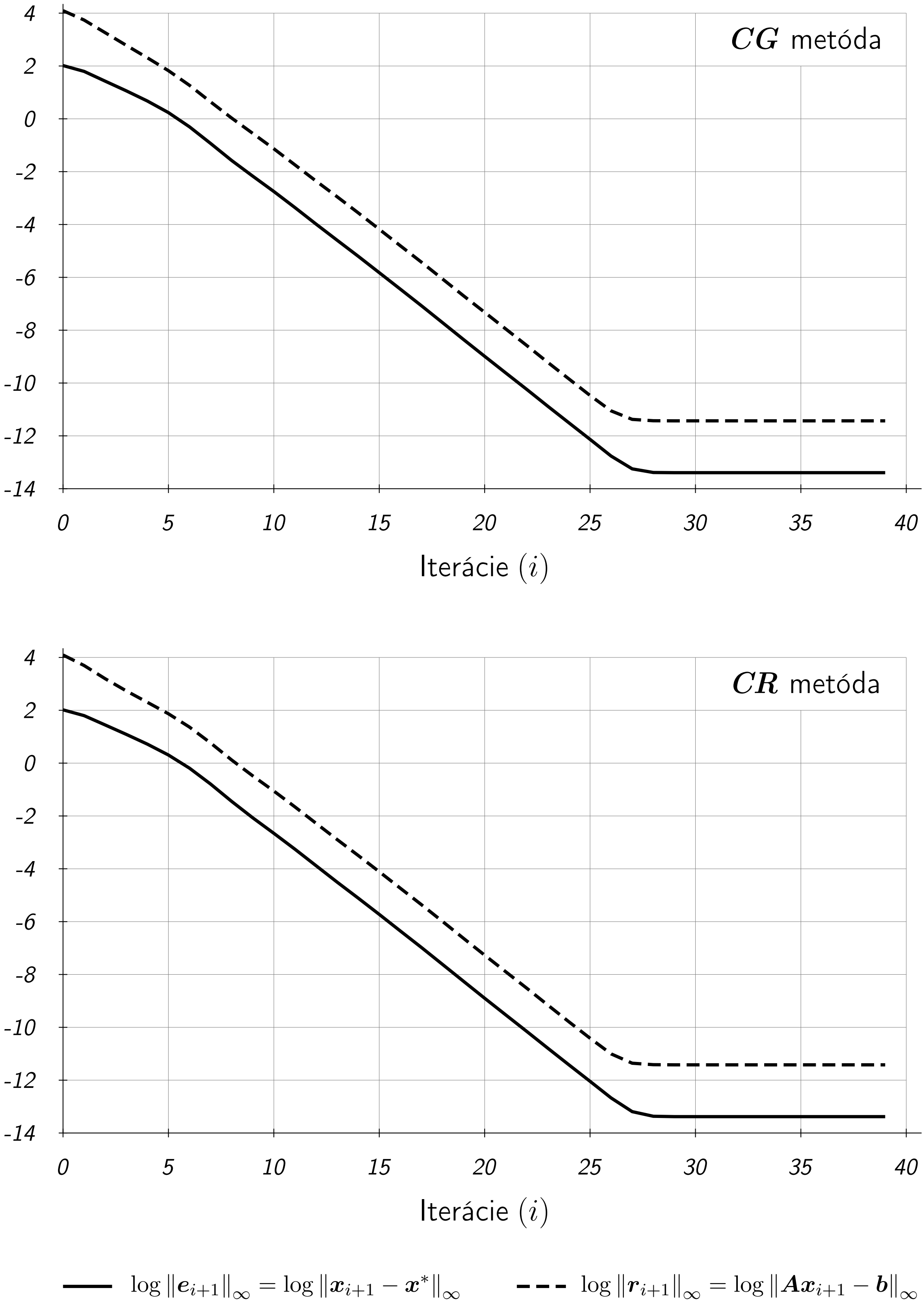
[5] Y. Saad, M. H. Schultz, *GMRES: A Generalized Minimal Residual Algorithm for Solving Nonsymmetric Linear Systems*, SIAM J. Sci. Stat. Comput. Vol. 7, No. 3, July 1986

[6] Y. Saad, M. H. Schultz, *Krylov Subspace Methods for Solving Large Unsymmetric Linear Systems*, Mathematics of Computation, Vol. 37, No. 155. (Jul., 1981), pp. 105-126.

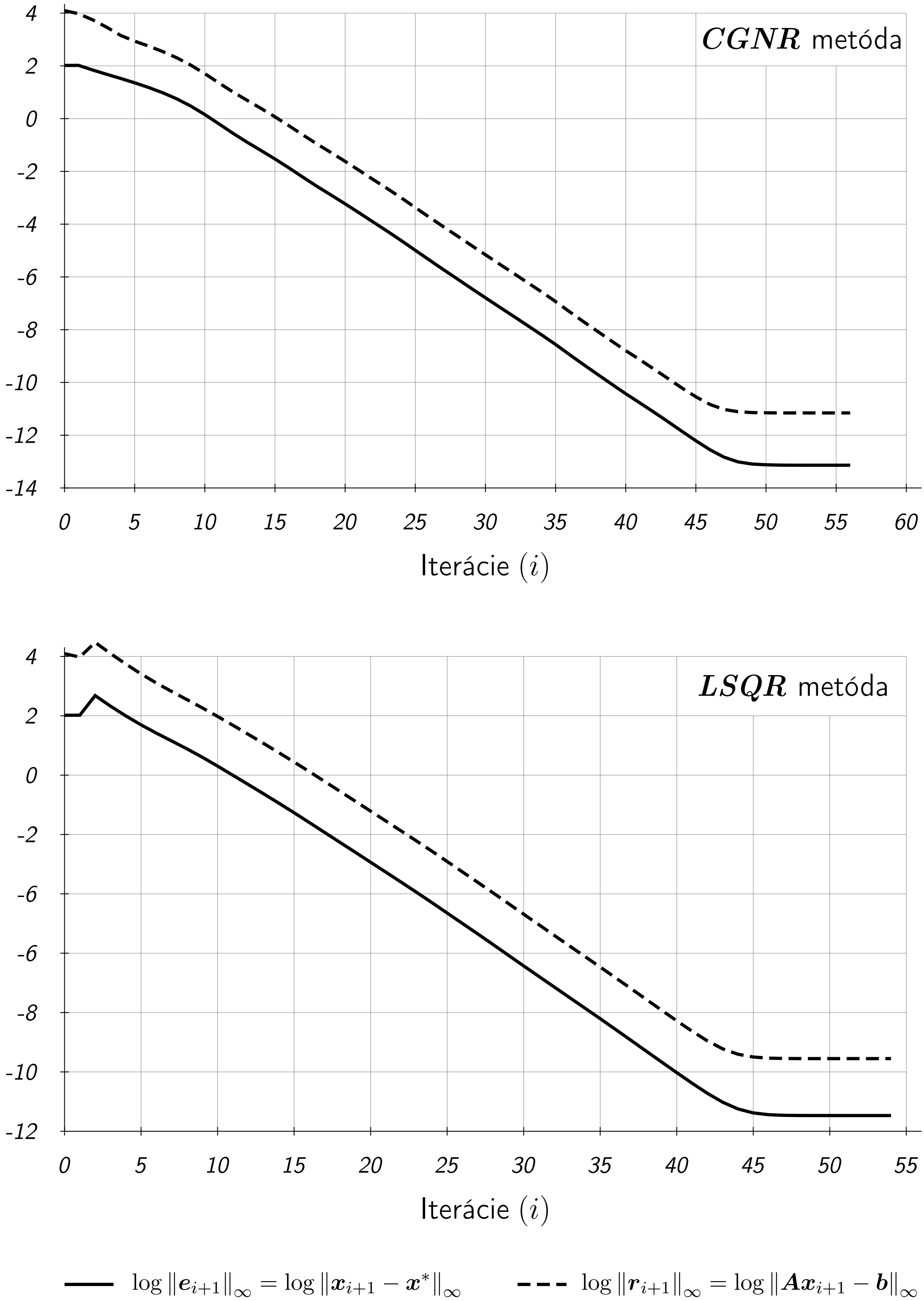
[7] *http://www.cise.ufl.edu/research/sparse/matrices*

Príloha 1

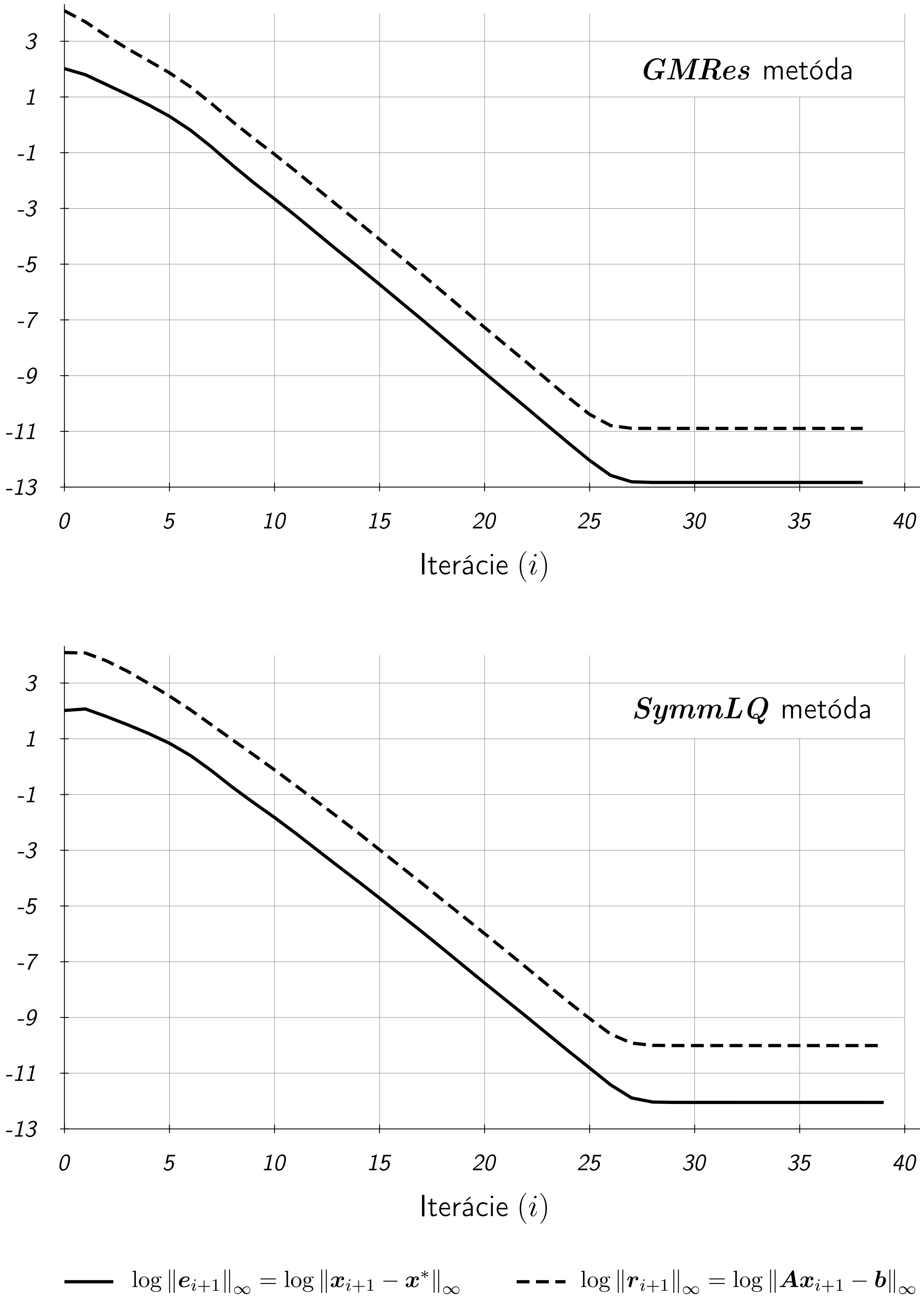
Grafy a tabuľky



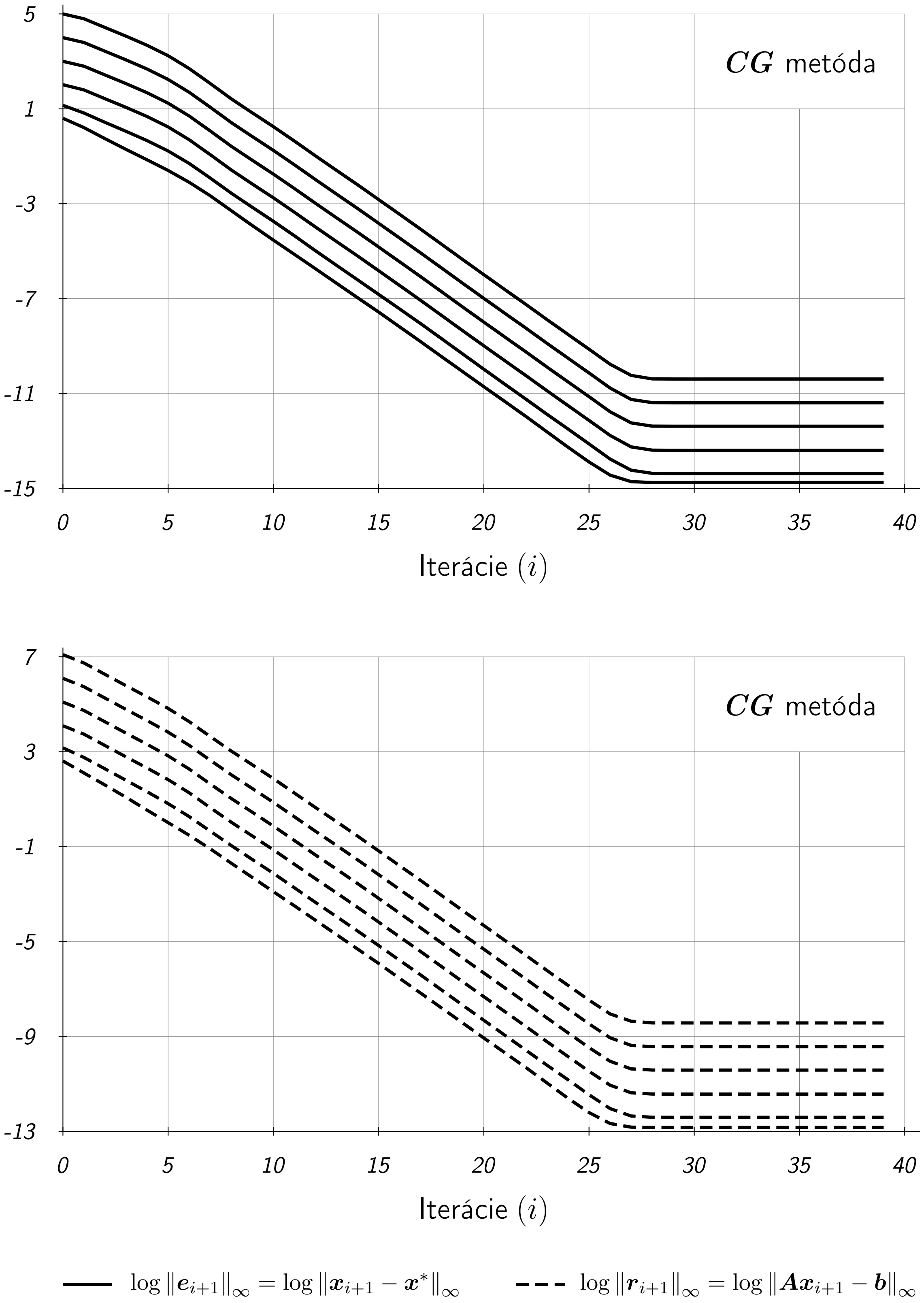
Obrázok 3.1: $n = 1000$, riedkosť $= 5\%$, $\|\boldsymbol{e}_1\|_\infty = 104$, uvažované sú priemerné hodnoty



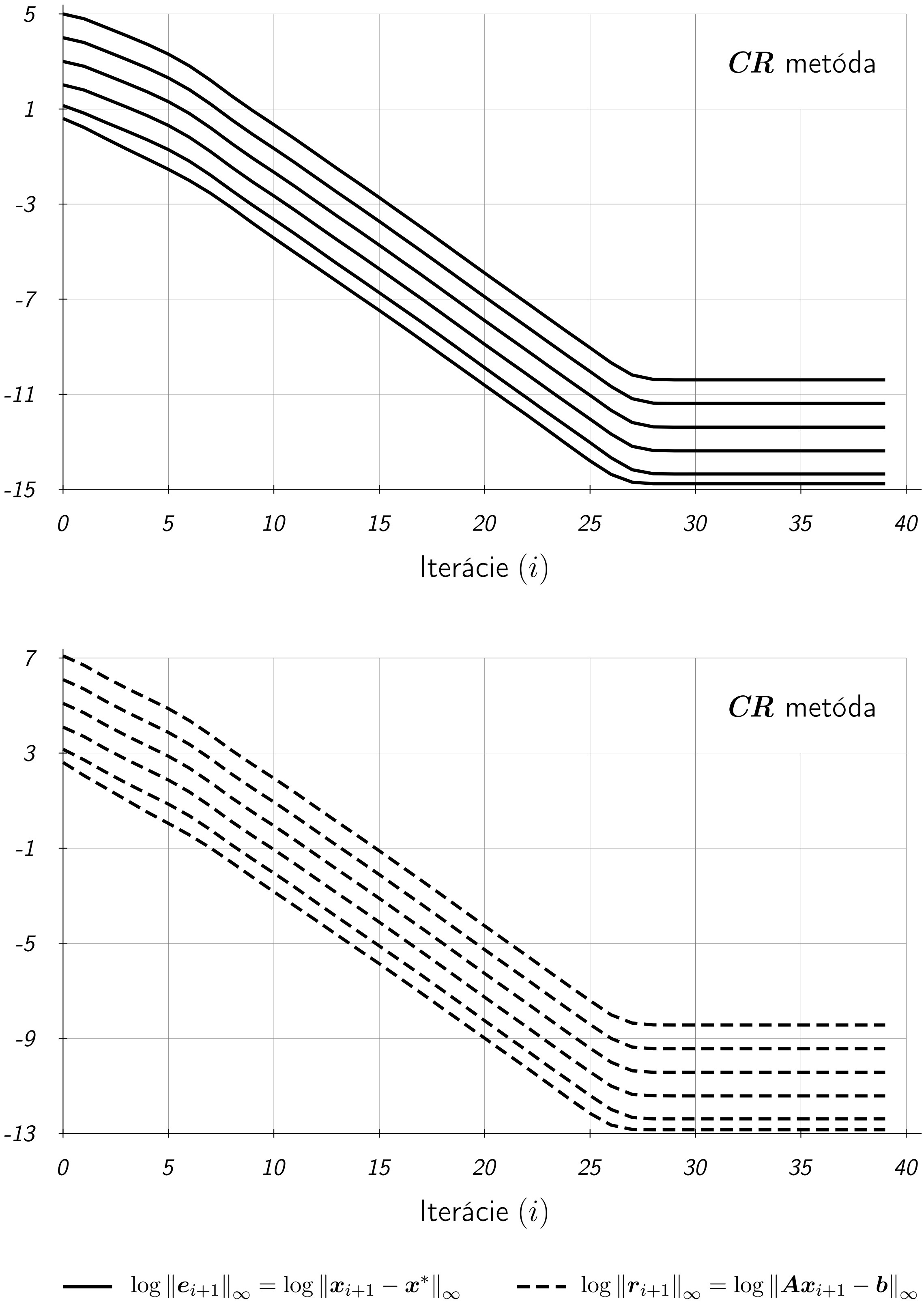
Obrázok 3.2: $n = 1000$, riedkosť $= 5\%$, $\|\boldsymbol{e}_1\|_\infty = 104$, uvažované sú priemerné hodnoty



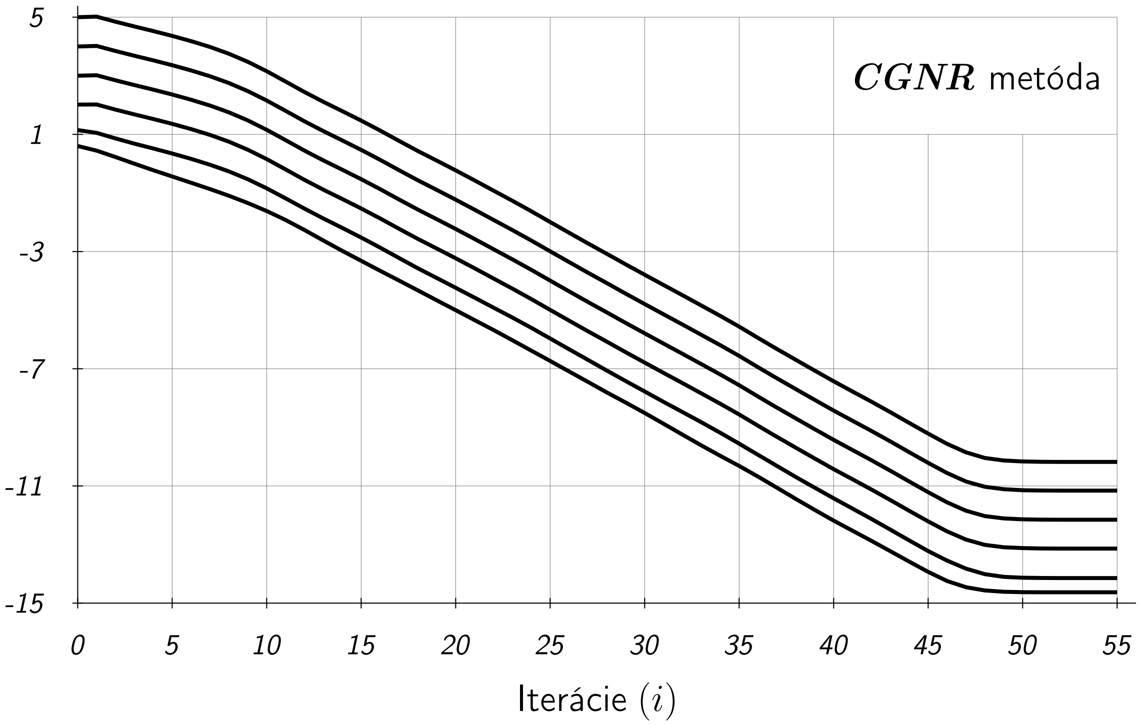
Obrázok 3.3: $n = 1000$, riedkosť $= 5\%$, $\|\boldsymbol{e}_1\|_\infty = 104$, uvažované sú priemerné hodnoty

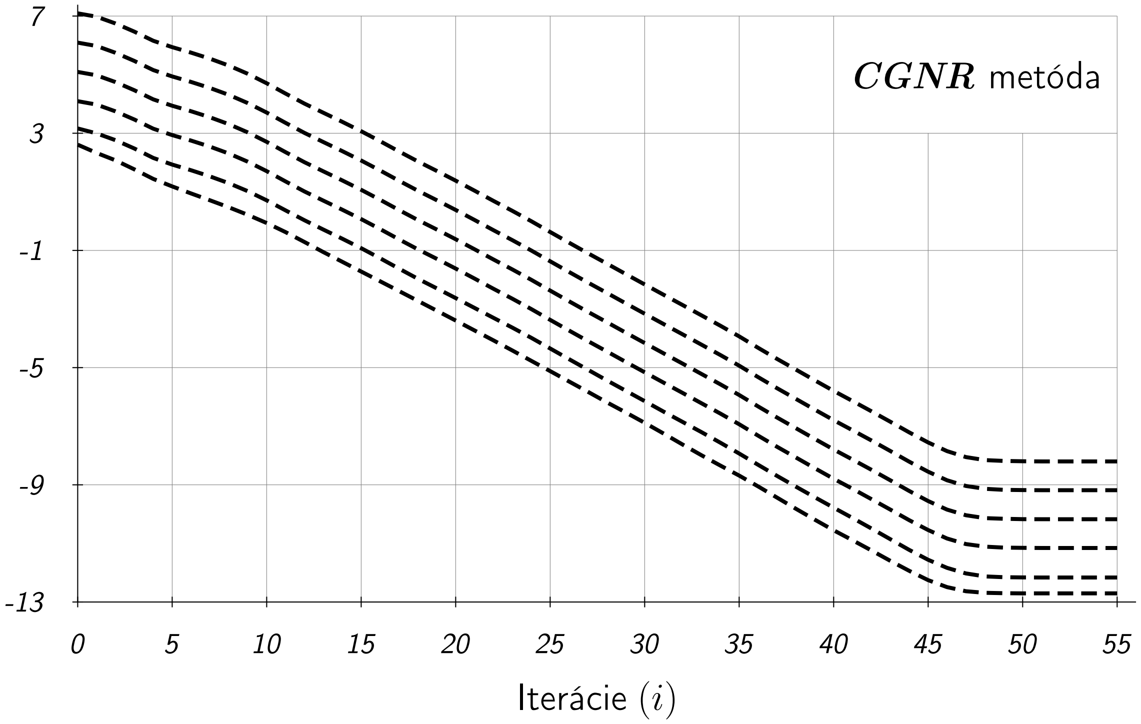


Obrázok 3.4: $n = 1000$, riedkosť $= 5\%$, uvažované sú priemerné hodnoty



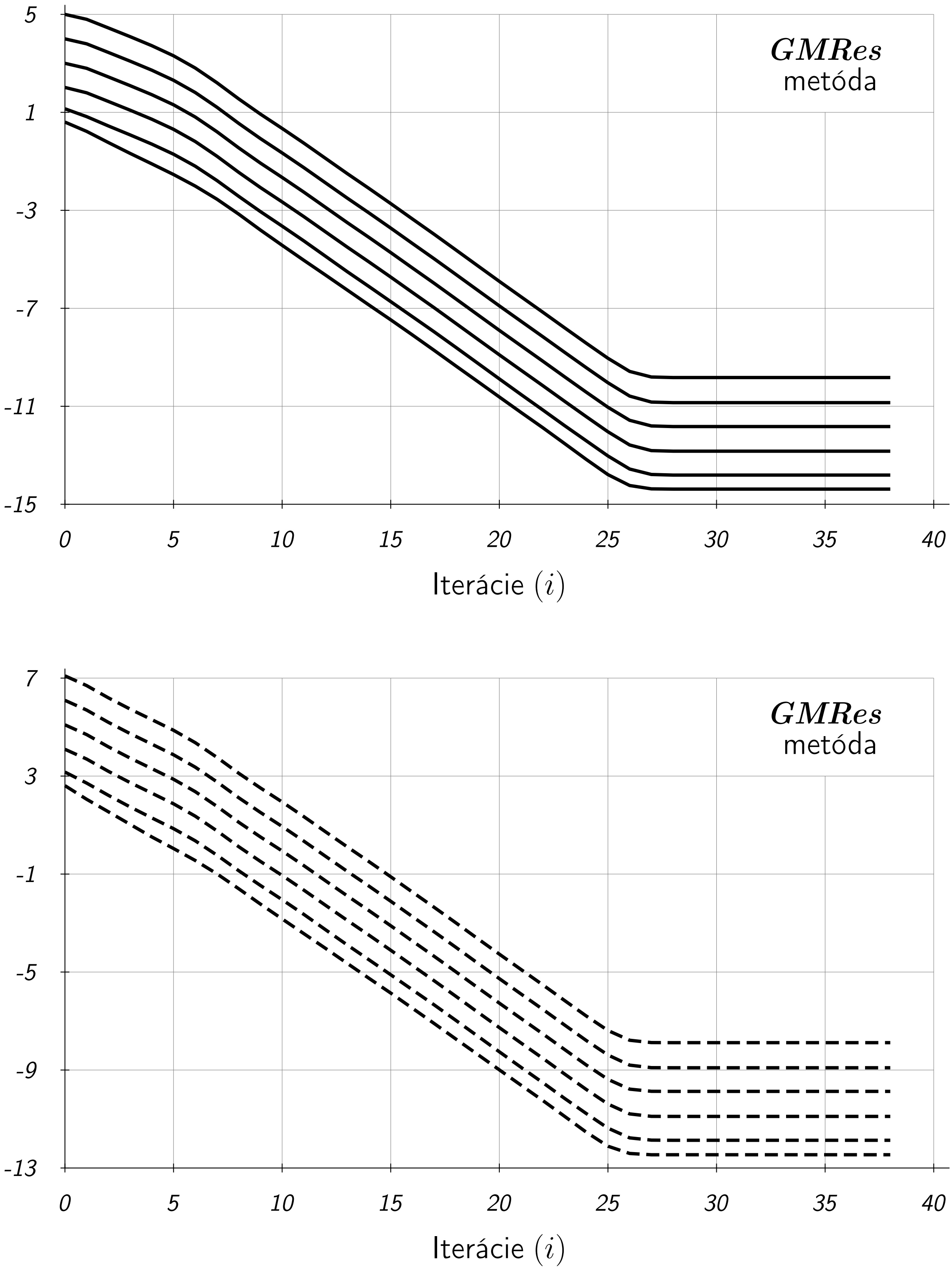
Obrázok 3.5: $n = 1000$, riedkosť $= 5\%$, uvažované sú priemerné hodnoty





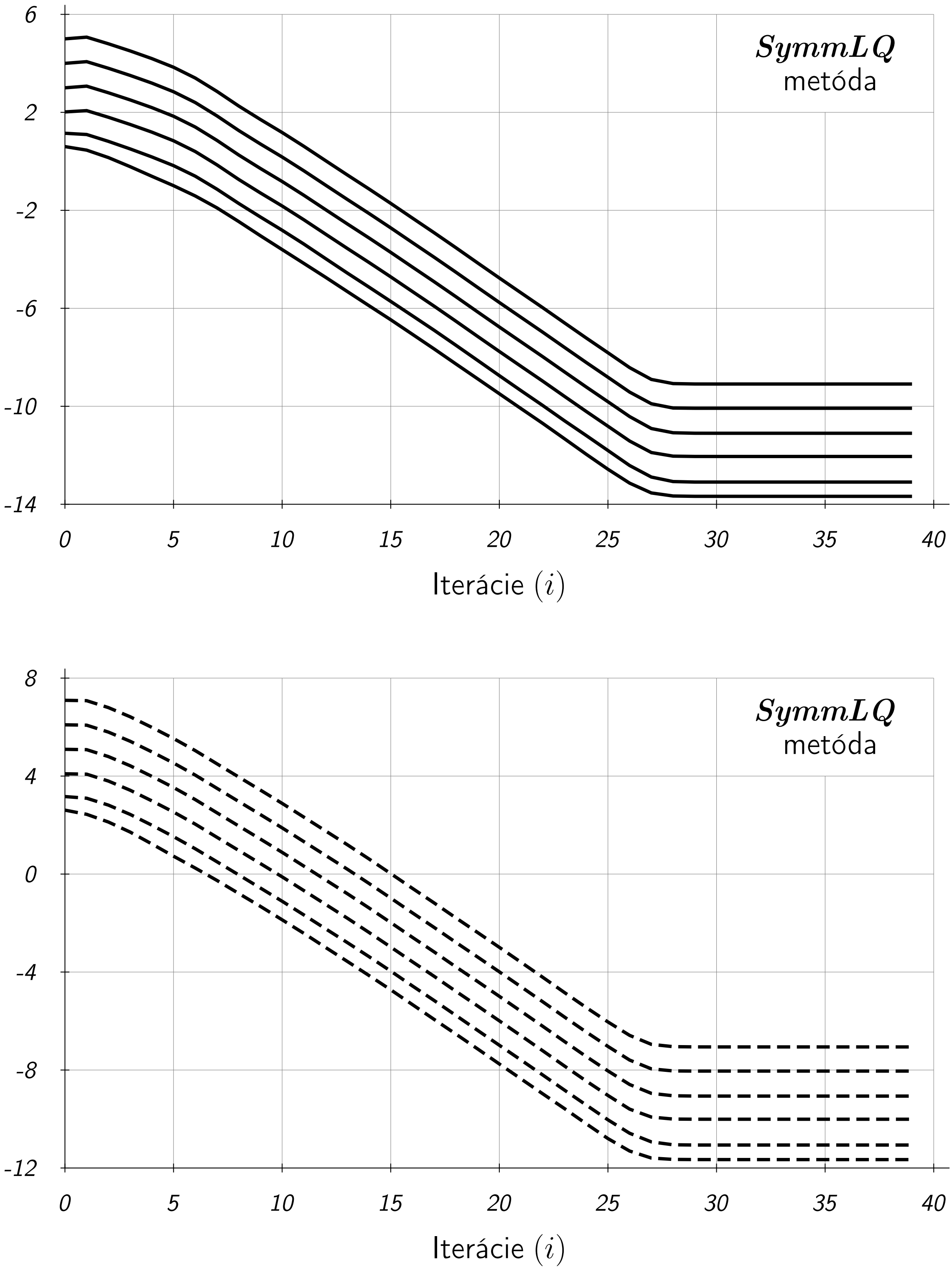
$\log \|\boldsymbol{e}_{i+1}\|_\infty = \log \|\boldsymbol{x}_{i+1} - \boldsymbol{x}^*\|_\infty$ $\quad$ $\log \|\boldsymbol{r}_{i+1}\|_\infty = \log \|\boldsymbol{A}\boldsymbol{x}_{i+1} - \boldsymbol{b}\|_\infty$

Obrázok 3.6: $n = 1000$, riedkosť $= 5\%$, uvažované sú priemerné hodnoty



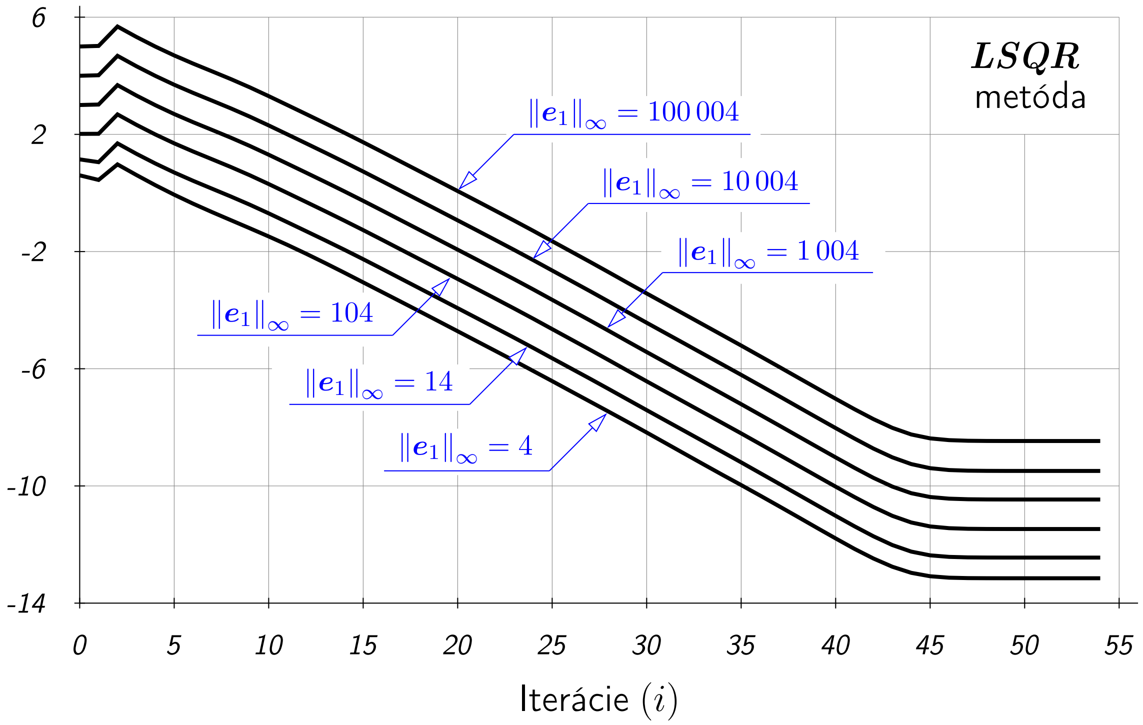
—— $\log \|\boldsymbol{e}_{i+1}\|_\infty = \log \|\boldsymbol{x}_{i+1} - \boldsymbol{x}^*\|_\infty$ - - - $\log \|\boldsymbol{r}_{i+1}\|_\infty = \log \|\boldsymbol{A}\boldsymbol{x}_{i+1} - \boldsymbol{b}\|_\infty$

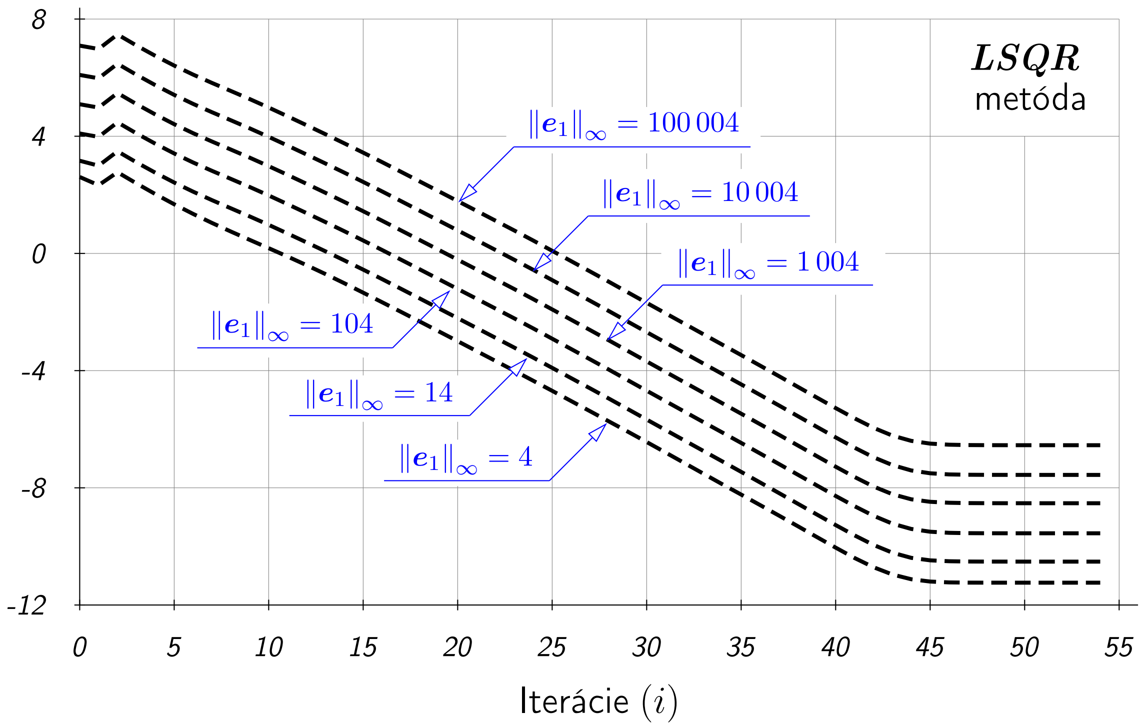
Obrázok 3.7: $n = 1000$, riedkosť $= 5\%$, uvažované sú priemerné hodnoty



— $\log \|\boldsymbol{e}_{i+1}\|_\infty = \log \|\boldsymbol{x}_{i+1} - \boldsymbol{x}^*\|_\infty$ - - - $\log \|\boldsymbol{r}_{i+1}\|_\infty = \log \|\boldsymbol{A}\boldsymbol{x}_{i+1} - \boldsymbol{b}\|_\infty$

Obrázok 3.8: $n = 1000$, riedkosť $= 5\%$, uvažované sú priemerné hodnoty





—— $\log\|\boldsymbol{e}_{i+1}\|_\infty = \log\|\boldsymbol{x}_{i+1} - \boldsymbol{x}^*\|_\infty$ - - - $\log\|\boldsymbol{r}_{i+1}\|_\infty = \log\|\boldsymbol{A}\boldsymbol{x}_{i+1} - \boldsymbol{b}\|_\infty$

Obrázok 3.9: $n = 1000$, riedkosť $= 5\%$, uvažované sú priemerné hodnoty

CG metóda, $n = 100$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
			Riedkosť = 5%			
4	3.11E-15	3.91E-14	1.87E-15	1.69E-14	68	63.30
14	7.22E-15	9.77E-14	3.99E-15	3.66E-14	69	64.27
104	6.24E-14	6.47E-13	3.61E-14	3.04E-13	68	64.09
1 004	6.77E-13	7.99E-12	3.62E-13	3.05E-12	69	64.44
10 004	6.93E-12	6.95E-11	3.64E-12	2.97E-11	70	64.40
100 004	6.83E-11	7.87E-10	3.62E-11	3.00E-10	69	64.11
			Riedkosť = 10%			
4	2.66E-15	5.68E-14	1.62E-15	2.77E-14	55	51.07
14	8.44E-15	2.06E-13	3.45E-15	6.35E-14	56	51.87
104	5.12E-14	1.20E-12	3.08E-14	5.71E-13	57	51.80
1 004	6.38E-13	1.44E-11	3.05E-13	5.57E-12	56	51.72
10 004	4.92E-12	1.22E-10	2.86E-12	5.23E-11	57	51.83
100 004	7.63E-11	1.54E-09	2.95E-11	5.42E-10	56	51.75
			Riedkosť = 15%			
4	2.22E-15	7.11E-14	1.52E-15	3.98E-14	51	46.50
14	5.33E-15	1.85E-13	3.18E-15	8.83E-14	51	47.15
104	6.52E-14	2.03E-12	2.86E-14	8.04E-13	51	47.12
1 004	6.41E-13	1.83E-11	2.75E-13	7.32E-12	51	47.22
10 004	6.21E-12	1.98E-10	2.78E-12	7.45E-11	51	47.11
100 004	7.05E-11	2.31E-09	2.91E-11	7.74E-10	51	47.13
			Riedkosť = 20%			
4	2.44E-15	1.14E-13	1.43E-15	4.76E-14	47	42.90
14	7.11E-15	3.27E-13	3.15E-15	1.11E-13	46	43.32
104	5.81E-14	2.55E-12	2.78E-14	1.01E-12	46	43.41
1 004	4.58E-13	1.87E-11	2.64E-13	9.33E-12	46	43.45
10 004	5.18E-12	2.17E-10	2.69E-12	9.68E-11	47	43.46
100 004	7.21E-11	2.75E-09	2.79E-11	9.85E-10	46	43.40
			Riedkosť = 25%			
4	2.66E-15	9.95E-14	1.51E-15	6.56E-14	50	46.69
14	7.33E-15	4.41E-13	3.15E-15	1.48E-13	51	47.40
104	7.42E-14	4.83E-12	2.86E-14	1.36E-12	51	47.36
1 004	8.86E-13	5.53E-11	3.08E-13	1.53E-11	51	47.30
10 004	5.22E-12	2.86E-10	2.75E-12	1.30E-10	51	47.38
100 004	7.25E-11	4.27E-09	2.92E-11	1.40E-09	51	47.25

Tabuľka *CG 1:*

		***CG** metóda,* $n = 200$				
	Pres_ min		*Pres_ avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
			Riedkosť = 5%			
4	2.66E-15	6.39E-14	1.83E-15	3.29E-14	59	53.90
14	8.44E-15	2.06E-13	3.72E-15	7.38E-14	60	54.83
104	5.77E-14	1.31E-12	3.45E-14	6.70E-13	59	54.85
1 004	5.99E-13	1.57E-11	3.50E-13	6.87E-12	60	54.93
10 004	9.01E-12	2.13E-10	3.43E-12	6.73E-11	59	54.92
100 004	6.15E-11	1.58E-09	3.43E-11	6.63E-10	59	54.86
			Riedkosť = 10%			
4	2.22E-15	8.53E-14	1.59E-15	5.22E-14	48	44.70
14	6.55E-15	2.70E-13	3.46E-15	1.30E-13	49	45.25
104	6.35E-14	3.07E-12	3.13E-14	1.15E-12	49	45.33
1 004	5.79E-13	2.63E-11	3.20E-13	1.18E-11	49	45.21
10 004	8.85E-12	3.74E-10	3.18E-12	1.22E-10	48	45.23
100 004	1.01E-10	5.40E-09	3.05E-11	1.12E-09	49	45.20
			Riedkosť = 15%			
4	2.22E-15	1.42E-13	1.50E-15	7.54E-14	45	41.41
14	5.53E-15	3.48E-13	3.35E-15	1.83E-13	46	42.01
104	5.90E-14	4.22E-12	3.35E-14	1.84E-12	45	41.94
1 004	5.47E-13	3.16E-11	3.17E-13	1.69E-11	46	42.01
10 004	5.64E-12	3.94E-10	2.89E-12	1.57E-10	45	42.15
100 004	7.01E-11	3.47E-09	3.12E-11	1.71E-09	46	42.07
			Riedkosť = 20%			
4	2.22E-15	1.71E-13	1.45E-15	9.46E-14	44	39.73
14	6.66E-15	5.68E-13	3.39E-15	2.44E-13	43	40.04
104	5.89E-14	4.76E-12	3.19E-14	2.33E-12	43	40.01
1 004	6.19E-13	4.69E-11	3.14E-13	2.29E-11	44	40.05
10 004	9.40E-12	8.39E-10	3.27E-12	2.45E-10	43	40.03
100 004	6.92E-11	6.05E-09	3.22E-11	2.36E-09	44	40.06
			Riedkosť = 25%			
4	1.78E-15	1.71E-13	1.42E-15	1.19E-13	42	38.51
14	6.54E-15	6.42E-13	3.46E-15	3.19E-13	43	38.97
104	5.77E-14	6.34E-12	3.31E-14	3.08E-12	42	39.06
1 004	6.28E-13	6.35E-11	3.35E-13	3.09E-11	43	38.93
10 004	7.31E-12	7.99E-10	3.40E-12	3.16E-10	42	39.03
100 004	8.47E-11	8.35E-09	3.11E-11	2.81E-09	44	39.12

Tabuľka *CG 2:*

CG metóda, $n = 300$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|\|_\infty$	$\|\|\boldsymbol{x}_i - \boldsymbol{x}^*\|\|_\infty$	$\|\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|\|_\infty$	$\|\|\boldsymbol{x}_i - \boldsymbol{x}^*\|\|_\infty$	$\|\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|\|_\infty$	*max*	*avg*
			Riedkosť = 6.7%			
4	3.11E-15	9.95E-14	1.69E-15	5.58E-14	49	45.23
14	8.33E-15	3.98E-13	3.69E-15	1.39E-13	49	45.83
104	6.88E-14	3.06E-12	3.50E-14	1.35E-12	49	45.86
1 004	6.86E-13	3.00E-11	3.48E-13	1.32E-11	50	45.88
10 004	7.03E-12	3.22E-10	3.50E-12	1.30E-10	50	45.90
100 004	8.52E-11	3.83E-09	3.43E-11	1.29E-09	50	45.91
			Riedkosť = 10%			
4	2.66E-15	1.71E-13	1.63E-15	7.93E-14	45	41.87
14	7.33E-15	5.12E-13	3.50E-15	1.93E-13	47	42.38
104	6.39E-14	3.87E-12	3.44E-14	1.89E-12	46	42.37
1 004	7.28E-13	4.81E-11	3.32E-13	1.85E-11	46	42.39
10 004	6.14E-12	3.61E-10	3.07E-12	1.65E-10	46	42.50
100 004	7.04E-11	5.16E-09	3.25E-11	1.82E-09	47	42.35
			Riedkosť = 16.7%			
4	2.22E-15	2.27E-13	1.53E-15	1.31E-13	42	38.95
14	6.66E-15	6.54E-13	3.57E-15	3.27E-13	43	39.42
104	8.10E-14	7.76E-12	3.40E-14	3.14E-12	43	39.42
1 004	6.31E-13	5.83E-11	3.50E-13	3.22E-11	43	39.43
10 004	6.31E-12	5.74E-10	3.24E-12	2.93E-10	42	39.48
100 004	6.80E-11	6.98E-09	3.58E-11	3.24E-09	42	39.42
			Riedkosť = 20%			
4	2.22E-15	2.27E-13	1.49E-15	1.49E-13	41	38.24
14	8.17E-15	9.50E-13	3.55E-15	3.84E-13	42	38.66
104	8.39E-14	9.66E-12	3.63E-14	4.00E-12	42	38.70
1 004	1.16E-12	1.50E-10	3.78E-13	4.07E-11	42	38.63
10 004	7.53E-12	9.52E-10	3.32E-12	3.65E-10	42	38.79
100 004	7.47E-11	9.09E-09	3.33E-11	3.71E-09	42	38.77
			Riedkosť = 25%			
4	2.22E-15	2.84E-13	1.62E-15	2.02E-13	44	39.96
14	7.11E-15	1.14E-12	3.82E-15	5.08E-13	43	40.33
104	6.45E-14	9.49E-12	3.45E-14	4.70E-12	43	40.38
1 004	7.19E-13	1.08E-10	3.55E-13	4.83E-11	43	40.40
10 004	7.10E-12	1.03E-09	3.52E-12	4.69E-10	44	40.36
100 004	8.43E-11	1.26E-08	3.61E-11	4.84E-09	43	40.39

Tabuľka *CG 3:*

CG metóda, $n = 500$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
			Riedkosť = 5%			
4	2.66E-15	1.14E-13	1.87E-15	7.92E-14	53	48.46
14	8.22E-15	4.83E-13	3.99E-15	1.90E-13	54	49.32
104	6.48E-14	3.33E-12	3.57E-14	1.69E-12	54	49.43
1 004	5.81E-13	3.54E-11	3.64E-13	1.73E-11	54	49.40
10 004	7.18E-12	4.09E-10	3.62E-12	1.74E-10	54	49.32
100 004	6.51E-11	3.39E-09	3.86E-11	1.83E-09	54	49.25
			Riedkosť = 10%			
4	2.22E-15	2.27E-13	1.62E-15	1.31E-13	43	39.65
14	6.66E-15	6.25E-13	3.69E-15	3.33E-13	44	40.22
104	7.64E-14	7.59E-12	3.67E-14	3.33E-12	44	40.20
1 004	7.23E-13	6.55E-11	3.63E-13	3.29E-11	44	40.20
10 004	6.52E-12	7.10E-10	3.58E-12	3.26E-10	44	40.22
100 004	7.52E-11	7.74E-09	3.67E-11	3.30E-09	43	40.18
			Riedkosť = 15%			
4	3.77E-15	6.25E-13	1.71E-15	2.15E-13	45	41.03
14	7.55E-15	1.08E-12	4.00E-15	5.35E-13	45	41.50
104	6.53E-14	9.01E-12	3.86E-14	5.24E-12	45	41.52
1 004	6.45E-13	9.74E-11	3.90E-13	5.22E-11	45	41.50
10 004	7.28E-12	1.06E-09	3.71E-12	5.01E-10	45	41.53
100 004	7.62E-11	1.23E-08	3.93E-11	5.18E-09	45	41.55
			Riedkosť = 20%			
4	2.22E-15	4.55E-13	1.57E-15	2.64E-13	40	37.45
14	1.29E-14	2.73E-12	4.36E-15	7.99E-13	40	37.86
104	9.46E-14	2.00E-11	4.08E-14	7.33E-12	41	38.02
1 004	1.13E-12	2.33E-10	4.23E-13	7.71E-11	40	37.99
10 004	7.82E-12	1.46E-09	4.06E-12	7.28E-10	40	37.97
100 004	1.13E-10	2.29E-08	4.10E-11	7.50E-09	41	37.96
			Riedkosť = 25%			
4	3.11E-15	7.96E-13	1.86E-15	4.14E-13	49	46.16
14	9.50E-15	2.53E-12	5.06E-15	1.14E-12	50	46.78
104	9.73E-14	2.59E-11	4.84E-14	1.12E-11	49	46.84
1 004	9.76E-13	2.67E-10	4.88E-13	1.14E-10	50	46.83
10 004	1.14E-11	2.97E-09	4.88E-12	1.15E-09	50	46.84
100 004	8.95E-11	2.27E-08	4.84E-11	1.13E-08	49	46.79

Tabuľka *CG 4*:

CG *metóda*, $n = 1\,000$						
	Pres_ min		*Pres_ avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
			Riedkosť = 5%			
4	3.11E-15	3.13E-13	1.78E-15	1.47E-13	43	40.02
14	6.88E-15	7.67E-13	4.27E-15	3.88E-13	44	40.62
104	7.76E-14	8.17E-12	4.04E-14	3.70E-12	44	40.63
1 004	9.94E-13	8.85E-11	4.18E-13	3.81E-11	44	40.57
10 004	9.86E-12	1.08E-09	4.09E-12	3.68E-10	44	40.65
100 004	9.46E-11	1.05E-08	4.08E-11	3.69E-09	43	40.62
			Riedkosť = 10%			
4	2.22E-15	4.55E-13	1.71E-15	2.71E-13	40	37.89
14	7.77E-15	1.45E-12	4.56E-15	7.98E-13	41	38.27
104	7.08E-14	1.29E-11	4.18E-14	7.24E-12	42	38.35
1 004	7.64E-13	1.36E-10	4.31E-13	7.57E-11	41	38.34
10 004	8.30E-12	1.55E-09	4.34E-12	7.56E-10	41	38.36
100 004	8.10E-11	1.41E-08	4.22E-11	7.27E-09	41	38.36
			Riedkosť = 15%			
4	2.66E-15	7.96E-13	1.74E-15	4.28E-13	40	37.36
14	8.66E-15	2.39E-12	4.97E-15	1.30E-12	41	37.69
104	8.99E-14	2.66E-11	4.70E-14	1.23E-11	41	37.71
1 004	8.37E-13	2.37E-10	4.58E-13	1.20E-10	41	37.68
10 004	8.08E-12	2.19E-09	4.41E-12	1.15E-09	41	37.67
100 004	1.16E-10	3.37E-08	4.61E-11	1.21E-08	41	37.78
			Riedkosť = 20%			
4	2.66E-15	1.02E-12	1.75E-15	5.88E-13	40	37.45
14	1.02E-14	3.98E-12	5.48E-15	1.99E-12	40	37.75
104	9.93E-14	3.74E-11	4.99E-14	1.79E-11	40	37.80
1 004	9.91E-13	3.78E-10	5.14E-13	1.81E-10	41	37.72
10 004	1.12E-11	4.36E-09	5.13E-12	1.85E-09	40	37.70
100 004	9.07E-11	3.32E-08	5.12E-11	1.84E-08	40	37.74
			Riedkosť = 25%			
4	3.55E-15	1.71E-12	1.85E-15	8.07E-13	38	35.63
14	1.20E-14	5.68E-12	6.49E-15	2.97E-12	38	35.86
104	1.39E-13	6.87E-11	6.02E-14	2.79E-11	38	35.94
1 004	1.25E-12	5.80E-10	6.01E-13	2.77E-10	38	35.86
10 004	1.22E-11	5.94E-09	6.29E-12	2.88E-09	38	35.89
100 004	1.31E-10	6.03E-08	6.04E-11	2.78E-08	38	35.92

Tabuľka *CG 5:*

CG metóda, $n = 2\,000$						
	Pres_ min		*Pres_ avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
			Riedkosť = 5%			
4	2.66E-15	5.12E-13	1.81E-15	2.96E-13	41	38.29
14	7.11E-15	1.25E-12	4.52E-15	7.78E-13	42	38.86
104	8.14E-14	1.41E-11	4.57E-14	7.78E-12	42	38.86
1 004	1.02E-12	1.90E-10	4.54E-13	7.79E-11	42	38.90
10 004	7.26E-12	1.28E-09	4.45E-12	7.74E-10	42	38.80
100 004	1.04E-10	1.73E-08	4.30E-11	7.40E-09	42	38.83
			Riedkosť = 10%			
4	2.44E-15	7.96E-13	1.81E-15	5.73E-13	39	36.97
14	1.02E-14	3.64E-12	5.33E-15	1.79E-12	40	37.36
104	8.72E-14	3.25E-11	4.94E-14	1.68E-11	39	37.36
1 004	9.35E-13	3.30E-10	4.89E-13	1.63E-10	40	37.40
10 004	1.05E-11	4.14E-09	4.89E-12	1.62E-09	40	37.45
100 004	1.05E-10	4.00E-08	4.73E-11	1.60E-08	40	37.44
			Riedkosť = 15%			
4	2.66E-15	1.36E-12	1.87E-15	9.33E-13	39	36.80
14	1.07E-14	6.14E-12	5.86E-15	3.01E-12	39	37.17
104	1.04E-13	5.98E-11	5.62E-14	2.85E-11	40	37.25
1 004	8.94E-13	4.64E-10	5.61E-13	2.86E-10	39	37.21
10 004	1.05E-11	5.41E-09	5.60E-12	2.85E-09	39	37.22
100 004	8.96E-11	4.99E-08	5.53E-11	2.81E-08	40	37.26
			Riedkosť = 20%			
4	3.11E-15	2.27E-12	2.05E-15	1.38E-12	40	37.35
14	1.20E-14	8.30E-12	7.05E-15	4.99E-12	39	37.61
104	1.56E-13	1.16E-10	6.41E-14	4.55E-11	40	37.67
1 004	1.21E-12	9.36E-10	6.61E-13	4.66E-10	39	37.63
10 004	1.16E-11	8.25E-09	6.77E-12	4.79E-09	40	37.71
100 004	1.08E-10	7.69E-08	6.59E-11	4.68E-08	40	37.65
			Riedkosť = 25%			
4	3.55E-15	3.64E-12	2.42E-15	2.14E-12	38	35.53
14	1.31E-14	1.32E-11	8.72E-15	7.99E-12	38	35.74
104	1.22E-13	1.15E-10	7.90E-14	7.19E-11	38	35.78
1 004	1.70E-12	1.62E-09	8.27E-13	7.71E-10	38	35.80
10 004	1.80E-11	1.70E-08	8.23E-12	7.56E-09	38	35.82
100 004	1.35E-10	1.37E-07	8.08E-11	7.42E-08	38	35.83

Tabuľka *CG 6:*

CG metóda, $n = 3\,000$							
	Pres_min		Pres_avg		Iterácie	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	max	avg
			Riedkosť = 0.15%			
4	5.33E-15	8.53E-14	3.14E-15	3.66E-14	86	78.37
14	1.88E-14	2.84E-13	8.76E-15	1.08E-13	85	79.49
104	2.08E-13	2.81E-12	8.77E-14	1.07E-12	87	79.53
1 004	2.00E-12	3.06E-11	8.54E-13	1.05E-11	84	79.55
10 004	2.30E-11	3.01E-10	8.40E-12	1.03E-10	88	79.69
100 004	1.91E-10	3.12E-09	8.30E-11	1.01E-09	87	79.61
			Riedkosť = 5%			
4	3.33E-15	1.25E-12	1.88E-15	6.01E-13	39	37.33
14	7.99E-15	2.73E-12	5.34E-15	1.78E-12	40	37.81
104	8.75E-14	2.62E-11	5.03E-14	1.70E-11	40	37.83
1 004	7.04E-13	2.60E-10	4.95E-13	1.63E-10	40	37.95
10 004	7.37E-12	2.54E-09	5.01E-12	1.66E-09	40	37.82
100 004	9.04E-11	3.25E-08	5.00E-11	1.68E-08	40	37.86
			Riedkosť = 10%			
4	2.22E-15	1.14E-12	1.89E-15	9.07E-13	39	36.70
14	9.10E-15	4.55E-12	5.70E-15	2.83E-12	39	37.13
104	8.84E-14	4.96E-11	5.40E-14	2.66E-11	39	37.17
1 004	7.92E-13	4.17E-10	5.62E-13	2.73E-10	39	37.15
10 004	9.11E-12	4.79E-09	5.42E-12	2.67E-09	40	37.18
100 004	8.29E-11	4.16E-08	5.33E-11	2.63E-08	39	37.08
			Riedkosť = 15%			
4	3.33E-15	2.96E-12	2.17E-15	1.72E-12	38	35.62
14	1.07E-14	9.32E-12	7.39E-15	6.05E-12	38	35.94
104	1.22E-13	1.07E-10	6.80E-14	5.47E-11	38	35.97
1 004	1.09E-12	9.88E-10	6.90E-13	5.96E-10	38	35.94
10 004	1.15E-11	1.03E-08	6.76E-12	5.66E-09	38	35.99
100 004	1.18E-10	1.06E-07	7.06E-11	5.78E-08	38	35.97

Tabuľka *CG 7:*

CG *metóda*, $n = 5\,000$						
	Pres_ min		*Pres_ avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
Riedkosť = 0.1%						
4	6.66E-15	1.14E-13	3.28E-15	3.90E-14	87	80.50
14	2.26E-14	3.17E-13	9.94E-15	1.29E-13	89	81.63
104	3.25E-13	5.27E-12	1.02E-13	1.31E-12	89	81.54
1 004	2.80E-12	4.06E-11	9.53E-13	1.18E-11	87	81.41
10 004	1.79E-11	2.43E-10	9.18E-12	1.15E-10	87	81.57
100 004	2.66E-10	3.74E-09	1.01E-10	1.27E-09	90	81.49
Riedkosť = 3%						
4	2.66E-15	6.82E-13	1.94E-15	4.73E-13	40	37.86
14	1.07E-14	3.07E-12	5.46E-15	1.39E-12	41	38.31
104	7.83E-14	2.09E-11	5.00E-14	1.25E-11	41	38.37
1 004	8.86E-13	2.39E-10	5.05E-13	1.28E-10	41	38.33
10 004	8.25E-12	2.20E-09	4.88E-12	1.22E-09	41	38.43
100 004	8.77E-11	2.55E-08	5.12E-11	1.29E-08	41	38.36
Riedkosť = 5%						
4	2.66E-15	1.36E-12	1.95E-15	9.49E-13	39	37.02
14	1.01E-14	5.23E-12	5.86E-15	2.82E-12	40	37.51
104	1.14E-13	6.39E-11	5.56E-14	2.69E-11	40	37.41
1 004	1.05E-12	5.37E-10	5.68E-13	2.70E-10	40	37.44
10 004	7.97E-12	3.91E-09	5.36E-12	2.60E-09	39	37.50
100 004	8.88E-11	4.57E-08	5.26E-11	2.57E-08	40	37.58
Riedkosť = 10%						
4	2.66E-15	1.59E-12	2.13E-15	1.48E-12	38	36.30
14	1.15E-14	1.02E-11	7.59E-15	5.77E-12	38	36.70
104	7.59E-14	5.66E-11	6.33E-14	4.73E-11	38	36.90
1 004	7.37E-13	5.88E-10	6.13E-13	4.99E-10	39	37.10
10 004	8.28E-12	7.05E-09	6.75E-12	5.44E-09	39	36.80
100 004	1.34E-10	1.18E-07	7.47E-11	5.37E-08	38	36.90

Tabuľka *CG 8:*

CG metóda, $n = 10\,000$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
Riedkosť = 0.05%						
4	8.66E-15	1.71E-13	3.64E-15	4.75E-14	88	82.96
14	2.82E-14	4.90E-13	1.21E-14	1.62E-13	94	83.81
104	2.62E-13	4.24E-12	1.21E-13	1.61E-12	90	83.86
1 004	4.34E-12	6.47E-11	1.25E-12	1.67E-11	89	83.78
10 004	3.82E-11	6.87E-10	1.25E-11	1.66E-10	89	83.80
100 004	2.73E-10	3.73E-09	1.16E-10	1.52E-09	89	83.95
Riedkosť = 0.5%						
4	3.11E-15	3.13E-13	2.20E-15	1.90E-13	44	41.55
14	1.07E-14	1.14E-12	6.11E-15	5.88E-13	46	42.05
104	1.07E-13	1.08E-11	5.87E-14	5.61E-12	45	42.06
1 004	1.01E-12	1.10E-10	5.85E-13	5.58E-11	45	42.10
10 004	1.19E-11	1.26E-09	6.04E-12	5.73E-10	45	42.10
100 004	1.21E-10	1.20E-08	6.15E-11	5.92E-09	45	42.02
Riedkosť = 1%						
4	2.66E-15	5.12E-13	2.08E-15	3.39E-13	44	40.67
14	1.18E-14	2.50E-12	5.96E-15	1.05E-12	44	41.12
104	8.62E-14	1.51E-11	5.59E-14	9.93E-12	44	41.17
1 004	8.43E-13	1.56E-10	5.55E-13	9.74E-11	44	41.16
10 004	8.86E-12	1.77E-09	5.49E-12	9.68E-10	44	41.15
100 004	1.35E-10	2.74E-08	5.75E-11	1.02E-08	44	41.16
Riedkosť = 2%						
4	2.22E-15	7.96E-13	2.02E-15	6.71E-13	39	37.70
14	9.10E-15	3.01E-12	5.78E-15	1.89E-12	39	37.90
104	6.08E-14	2.08E-11	5.01E-14	1.67E-11	39	38.00
1 004	7.27E-13	2.41E-10	5.46E-13	1.78E-10	39	38.00
10 004	7.10E-12	2.40E-09	5.36E-12	1.77E-09	39	38.00
100 004	7.56E-11	2.19E-08	5.55E-11	1.73E-08	40	38.10

Tabuľka *CG 9:*

CG metóda, $n = 20\,000$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\lVert x_1 - x^* \rVert_\infty$	$\lVert x_i - x^* \rVert_\infty$	$\lVert Ax_i - x^* \rVert_\infty$	$\lVert x_i - x^* \rVert_\infty$	$\lVert Ax_i - x^* \rVert_\infty$	*max*	*avg*
			Riedkosť = 0.025%			
4	8.44E-15	1.21E-13	3.94E-15	5.39E-14	92	85.74
14	4.44E-14	7.11E-13	1.51E-14	2.11E-13	92	86.14
104	2.93E-13	5.27E-12	1.41E-13	1.93E-12	92	86.52
1 004	3.67E-12	5.80E-11	1.52E-12	2.11E-11	92	86.28
10 004	4.26E-11	7.16E-10	1.53E-11	2.07E-10	92	86.48
100 004	3.39E-10	4.78E-09	1.42E-10	1.94E-09	92	86.38
			Riedkosť = 0.15%			
4	3.55E-15	2.27E-13	2.47E-15	1.43E-13	50	47.13
14	1.57E-14	1.31E-12	7.58E-15	4.89E-13	51	47.64
104	1.04E-13	7.80E-12	7.01E-14	4.36E-12	51	47.77
1 004	1.67E-12	1.27E-10	7.00E-13	4.45E-11	52	47.73
10 004	1.47E-11	1.19E-09	7.05E-12	4.46E-10	51	47.79
100 004	1.08E-10	6.87E-09	6.93E-11	4.38E-09	51	47.73
			Riedkosť = 0.25%			
4	3.33E-15	3.69E-13	2.27E-15	2.00E-13	45	42.24
14	1.02E-14	1.11E-12	6.56E-15	6.30E-13	46	42.72
104	1.20E-13	1.38E-11	6.55E-14	6.42E-12	46	42.72
1 004	1.05E-12	1.12E-10	6.51E-13	6.19E-11	45	42.68
10 004	1.46E-11	1.72E-09	6.44E-12	6.22E-10	46	42.74
100 004	1.13E-10	1.23E-08	6.34E-11	6.12E-09	46	42.80
			Riedkosť = 0.375%			
4	2.22E-15	3.41E-13	2.20E-15	2.84E-13	45	42.40
14	1.18E-14	1.79E-12	7.30E-15	1.05E-12	46	43.10
104	1.13E-13	1.54E-11	6.91E-14	9.74E-12	46	43.00
1 004	9.01E-13	1.24E-10	6.23E-13	8.45E-11	46	43.10
10 004	7.43E-12	1.06E-09	5.92E-12	8.40E-10	46	43.30
100 004	7.96E-11	1.13E-08	6.01E-11	8.72E-09	46	43.30

Tabuľka *CG 10:*

CG *metóda,* $n = 30\,000$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
			Riedkosť = 0.015%			
4	8.44E-15	1.28E-13	4.19E-15	5.76E-14	93	86.76
14	4.57E-14	6.96E-13	1.66E-14	2.30E-13	94	87.32
104	5.51E-13	9.21E-12	1.64E-13	2.31E-12	94	87.47
1 004	4.25E-12	6.11E-11	1.61E-12	2.24E-11	94	87.27
10 004	4.88E-11	6.83E-10	1.61E-11	2.24E-10	96	87.32
100 004	5.67E-10	7.45E-09	1.64E-10	2.24E-09	92	87.35
			Riedkosť = 0.05%			
4	4.88E-15	2.98E-13	2.82E-15	1.19E-13	58	54.16
14	2.66E-14	1.51E-12	9.40E-15	4.52E-13	59	54.68
104	1.60E-13	8.26E-12	8.51E-14	3.90E-12	58	54.62
1 004	1.74E-12	8.95E-11	8.59E-13	4.03E-11	59	54.87
10 004	1.59E-11	7.31E-10	8.55E-12	3.98E-10	59	54.83
100 004	2.29E-10	1.31E-08	8.68E-11	4.05E-09	59	54.71
			Riedkosť = 0.1%			
4	4.00E-15	2.56E-13	2.56E-15	1.52E-13	50	47.64
14	1.55E-14	1.03E-12	8.41E-15	5.54E-13	51	48.00
104	1.60E-13	1.27E-11	7.59E-14	4.86E-12	51	48.09
1 004	1.74E-12	1.25E-10	7.35E-13	4.73E-11	51	48.08
10 004	2.66E-11	2.13E-09	8.09E-12	5.20E-10	51	48.05
100 004	1.90E-10	1.38E-08	7.84E-11	5.07E-09	51	48.21
			Riedkosť = 0.15%			
4	3.11E-15	3.13E-13	2.44E-15	2.27E-13	45	42.40
14	9.43E-15	8.97E-13	7.09E-15	7.33E-13	45	43.00
104	1.20E-13	1.15E-11	6.91E-14	6.78E-12	46	43.20
1 004	8.18E-13	7.61E-11	6.56E-13	6.60E-11	45	43.20
10 004	7.83E-12	7.78E-10	6.75E-12	6.67E-10	46	43.00
100 004	9.96E-11	1.02E-08	7.04E-11	6.95E-09	45	42.90

Tabuľka *CG 11:*

CG *metóda*, $n = 50\,000$							
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\|x_1 - x^*\|\|_\infty$	$\|\|x_i - x^*\|\|_\infty$	$\|\|Ax_i - x^*\|\|_\infty$	$\|\|x_i - x^*\|\|_\infty$	$\|\|Ax_i - x^*\|\|_\infty$	*max*	*avg*
			Riedkosť = 0.01%			
4	8.88E-15	1.78E-13	4.75E-15	7.04E-14	95	88.43
14	4.37E-14	6.57E-13	1.98E-14	2.87E-13	96	88.84
104	5.68E-13	9.32E-12	1.94E-13	2.79E-12	96	88.96
1 004	4.02E-12	6.25E-11	1.89E-12	2.75E-11	97	88.86
10 004	4.63E-11	7.54E-10	1.84E-11	2.61E-10	96	89.09
100 004	4.10E-10	6.68E-09	1.98E-10	2.86E-09	94	88.85
			Riedkosť = 0.02%			
4	1.30E-14	2.49E-13	4.16E-15	1.00E-13	85	76.66
14	4.46E-14	9.43E-13	1.37E-14	3.58E-13	84	77.39
104	3.34E-13	1.12E-11	1.32E-13	3.59E-12	85	77.51
1 004	3.89E-12	1.33E-10	1.29E-12	3.48E-11	85	77.44
10 004	4.68E-11	7.49E-10	1.35E-11	3.58E-10	85	77.39
100 004	4.95E-10	1.57E-08	1.33E-10	3.53E-09	86	77.66
			Riedkosť = 0.04%			
4	4.44E-15	2.27E-13	2.94E-15	1.29E-13	59	55.20
14	1.89E-14	1.06E-12	9.57E-15	4.54E-13	59	55.85
104	1.99E-13	1.07E-11	9.88E-14	4.66E-12	61	55.84
1 004	1.52E-12	7.96E-11	9.26E-13	4.39E-11	60	55.97
10 004	1.94E-11	1.07E-09	9.80E-12	4.57E-10	60	55.91
100 004	2.02E-10	9.73E-09	9.20E-11	4.24E-09	61	55.90
			Riedkosť = 0.06%			
4	2.66E-15	1.99E-13	2.38E-15	1.52E-13	51	48.60
14	1.52E-14	1.23E-12	9.00E-15	5.94E-13	52	48.90
104	1.35E-13	1.07E-11	8.60E-14	5.82E-12	51	48.80
1 004	9.21E-13	6.73E-11	8.28E-13	5.41E-11	52	48.90
10 004	1.29E-11	8.86E-10	8.60E-12	5.25E-10	52	49.00
100 004	1.24E-10	7.60E-09	8.24E-11	5.07E-09	52	49.10

Tabuľka *CG 12:*

CR metóda, $n = 100$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
Riedkosť = 5%						
4	3.55E-15	3.20E-14	1.95E-15	1.72E-14	68	62.91
14	7.88E-15	7.46E-14	4.04E-15	3.24E-14	69	63.94
104	8.19E-14	1.28E-12	3.60E-14	3.05E-13	69	63.93
1 004	8.96E-13	1.07E-11	3.56E-13	3.09E-12	69	63.87
10 004	6.69E-12	9.79E-11	3.56E-12	2.83E-11	69	64.08
100 004	6.47E-11	8.67E-10	3.44E-11	2.86E-10	69	64.05
Riedkosť = 10%						
4	2.66E-15	4.26E-14	1.67E-15	2.82E-14	56	51.16
14	5.68E-15	1.42E-13	3.32E-15	5.83E-14	55	51.82
104	8.59E-14	1.99E-12	2.91E-14	5.23E-13	58	51.74
1 004	5.44E-13	1.57E-11	3.06E-13	5.56E-12	56	51.58
10 004	8.96E-12	2.28E-10	3.01E-12	5.36E-11	56	51.66
100 004	6.54E-11	1.43E-09	2.86E-11	5.21E-10	57	51.68
Riedkosť = 15%						
4	2.66E-15	7.11E-14	1.50E-15	3.83E-14	51	46.45
14	6.00E-15	2.27E-13	3.26E-15	8.88E-14	51	47.00
104	5.52E-14	2.05E-12	2.91E-14	7.69E-13	51	47.03
1 004	8.53E-13	2.72E-11	2.93E-13	7.97E-12	51	47.02
10 004	5.23E-12	1.70E-10	2.83E-12	7.52E-11	51	47.10
100 004	5.65E-11	1.66E-09	2.67E-11	7.29E-10	51	47.12
Riedkosť = 20%						
4	2.22E-15	8.53E-14	1.48E-15	4.93E-14	47	42.91
14	5.11E-15	2.27E-13	3.10E-15	1.13E-13	46	43.32
104	6.17E-14	2.40E-12	2.73E-14	9.37E-13	46	43.39
1 004	5.36E-13	2.12E-11	2.73E-13	9.70E-12	47	43.39
10 004	4.81E-12	2.01E-10	2.68E-12	9.40E-11	46	43.44
100 004	4.76E-11	1.93E-09	2.66E-11	9.65E-10	46	43.38
Riedkosť = 25%						
4	2.66E-15	1.42E-13	1.53E-15	6.62E-14	50	46.67
14	6.66E-15	3.55E-13	3.06E-15	1.41E-13	51	47.33
104	7.36E-14	4.52E-12	2.97E-14	1.44E-12	51	47.26
1 004	6.44E-13	3.90E-11	2.92E-13	1.37E-11	51	47.27
10 004	5.05E-12	2.60E-10	2.81E-12	1.29E-10	51	47.30
100 004	8.77E-11	5.56E-09	2.99E-11	1.41E-09	51	47.22

Tabuľka *CR 1:*

$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	Pres_min $\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	Pres_min $\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	Pres_avg $\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	Pres_avg $\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	Iterácie max	Iterácie avg
CR *metóda,* $n = 200$						
Riedkosť = 5%						
4	2.89E-15	8.53E-14	1.83E-15	3.28E-14	59	53.88
14	1.26E-14	3.27E-13	3.81E-15	7.54E-14	59	54.62
104	7.73E-14	2.02E-12	3.61E-14	7.11E-13	59	54.65
1 004	8.52E-13	2.00E-11	3.42E-13	6.75E-12	59	54.69
10 004	5.93E-12	1.43E-10	3.35E-12	6.51E-11	59	54.71
100 004	7.83E-11	1.90E-09	3.58E-11	7.04E-10	59	54.51
Riedkosť = 10%						
4	2.44E-15	8.53E-14	1.59E-15	5.24E-14	48	44.67
14	6.44E-15	2.84E-13	3.51E-15	1.34E-13	49	45.11
104	6.17E-14	2.67E-12	3.29E-14	1.24E-12	48	45.12
1 004	5.38E-13	2.34E-11	3.23E-13	1.20E-11	48	45.19
10 004	6.59E-12	2.56E-10	3.20E-12	1.16E-10	48	45.16
100 004	5.74E-11	2.83E-09	3.23E-11	1.22E-09	48	45.12
Riedkosť = 15%						
4	2.00E-15	1.42E-13	1.48E-15	7.64E-14	45	41.42
14	6.44E-15	4.55E-13	3.20E-15	1.75E-13	46	41.94
104	6.08E-14	3.70E-12	3.25E-14	1.78E-12	45	42.02
1 004	6.92E-13	4.74E-11	3.12E-13	1.73E-11	45	41.96
10 004	8.19E-12	5.33E-10	3.15E-12	1.71E-10	45	41.99
100 004	5.93E-11	3.70E-09	3.19E-11	1.72E-09	45	41.98
Riedkosť = 20%						
4	2.22E-15	1.42E-13	1.50E-15	9.45E-14	43	39.59
14	7.55E-15	6.82E-13	3.50E-15	2.58E-13	43	39.93
104	6.00E-14	4.46E-12	3.13E-14	2.25E-12	44	40.01
1 004	5.99E-13	4.27E-11	3.16E-13	2.30E-11	43	39.94
10 004	7.40E-12	6.06E-10	3.12E-12	2.27E-10	43	40.03
100 004	7.65E-11	6.13E-09	3.35E-11	2.46E-09	43	39.99
Riedkosť = 25%						
4	2.22E-15	2.27E-13	1.44E-15	1.17E-13	42	38.52
14	9.10E-15	8.24E-13	3.56E-15	3.26E-13	43	39.02
104	8.17E-14	9.49E-12	3.24E-14	2.97E-12	42	38.99
1 004	5.79E-13	6.94E-11	3.30E-13	3.10E-11	42	38.93
10 004	6.09E-12	6.21E-10	3.36E-12	3.10E-10	43	38.97
100 004	6.04E-11	7.14E-09	3.28E-11	3.01E-09	42	39.00

Tabuľka *CR 2:*

CR metóda, $n = 300$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\lVert \boldsymbol{x}_1 - \boldsymbol{x}^* \rVert_\infty$	$\lVert \boldsymbol{x}_i - \boldsymbol{x}^* \rVert_\infty$	$\lVert \boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^* \rVert_\infty$	$\lVert \boldsymbol{x}_i - \boldsymbol{x}^* \rVert_\infty$	$\lVert \boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^* \rVert_\infty$	*max*	*avg*
Riedkosť = 6.7%						
4	2.44E-15	8.53E-14	1.70E-15	5.60E-14	49	45.24
14	7.55E-15	3.13E-13	3.74E-15	1.41E-13	49	45.75
104	6.93E-14	3.01E-12	3.57E-14	1.37E-12	49	45.80
1 004	7.83E-13	3.37E-11	3.44E-13	1.31E-11	49	45.80
10 004	7.33E-12	3.01E-10	3.47E-12	1.34E-10	49	45.67
100 004	8.57E-11	3.72E-09	3.42E-11	1.31E-09	50	45.74
Riedkosť = 10%						
4	2.22E-15	1.42E-13	1.61E-15	8.00E-14	46	41.84
14	6.99E-15	4.41E-13	3.58E-15	1.98E-13	46	42.29
104	1.10E-13	7.38E-12	3.46E-14	1.94E-12	45	42.33
1 004	6.21E-13	4.01E-11	3.37E-13	1.88E-11	46	42.38
10 004	7.77E-12	5.30E-10	3.14E-12	1.74E-10	46	42.41
100 004	8.16E-11	5.18E-09	3.56E-11	1.98E-09	46	42.28
Riedkosť = 16.7%						
4	2.22E-15	2.27E-13	1.51E-15	1.27E-13	42	38.93
14	8.44E-15	7.96E-13	3.55E-15	3.16E-13	42	39.41
104	8.86E-14	9.46E-12	3.53E-14	3.24E-12	43	39.38
1 004	8.12E-13	8.50E-11	3.53E-13	3.24E-11	43	39.35
10 004	7.02E-12	6.84E-10	3.40E-12	3.16E-10	43	39.38
100 004	6.87E-11	6.87E-09	3.32E-11	2.98E-09	43	39.40
Riedkosť = 20%						
4	2.44E-15	2.31E-13	1.45E-15	1.43E-13	41	38.32
14	8.44E-15	1.08E-12	3.69E-15	4.07E-13	42	38.66
104	7.66E-14	9.89E-12	3.63E-14	3.97E-12	42	38.63
1 004	6.53E-13	7.70E-11	3.76E-13	4.10E-11	41	38.58
10 004	7.26E-12	8.53E-10	3.36E-12	3.64E-10	42	38.71
100 004	8.52E-11	9.07E-09	3.50E-11	3.85E-09	42	38.68
Riedkosť = 25%						
4	2.22E-15	3.41E-13	1.60E-15	1.97E-13	43	39.99
14	6.99E-15	1.05E-12	3.76E-15	5.13E-13	44	40.37
104	8.39E-14	1.23E-11	3.66E-14	4.94E-12	43	40.35
1 004	6.40E-13	9.02E-11	3.50E-13	4.79E-11	43	40.36
10 004	7.70E-12	1.14E-09	3.67E-12	4.93E-10	43	40.41
100 004	9.80E-11	1.46E-08	3.63E-11	4.92E-09	44	40.41

Tabuľka *CR 3:*

CR metóda, $n = 500$							
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
Riedkosť = 5%						
4	2.66E-15	1.14E-13	1.82E-15	7.48E-14	53	48.61
14	7.99E-15	4.55E-13	3.97E-15	1.93E-13	53	49.23
104	7.74E-14	4.41E-12	3.72E-14	1.77E-12	54	49.37
1 004	7.44E-13	4.31E-11	3.89E-13	1.81E-11	54	49.28
10 004	8.58E-12	5.20E-10	3.64E-12	1.73E-10	54	49.32
100 004	8.78E-11	4.13E-09	3.88E-11	1.81E-09	53	49.25
Riedkosť = 10%						
4	2.44E-15	2.56E-13	1.62E-15	1.33E-13	43	39.65
14	7.33E-15	6.82E-13	3.86E-15	3.45E-13	43	40.08
104	1.20E-13	1.28E-11	3.64E-14	3.29E-12	44	40.20
1 004	6.72E-13	7.02E-11	3.69E-13	3.36E-11	44	40.13
10 004	9.17E-12	9.79E-10	3.89E-12	3.56E-10	44	40.10
100 004	6.37E-11	6.09E-09	3.63E-11	3.23E-09	44	40.14
Riedkosť = 15%						
4	2.22E-15	3.41E-13	1.67E-15	2.04E-13	44	41.00
14	5.93E-15	9.66E-13	4.00E-15	5.25E-13	46	41.58
104	8.36E-14	1.32E-11	3.88E-14	5.19E-12	46	41.56
1 004	8.13E-13	1.19E-10	3.76E-13	4.96E-11	45	41.59
10 004	8.55E-12	1.42E-09	3.77E-12	4.98E-10	46	41.59
100 004	7.33E-11	1.18E-08	3.81E-11	5.20E-09	45	41.54
Riedkosť = 20%						
4	3.11E-15	6.25E-13	1.55E-15	2.67E-13	40	37.49
14	7.55E-15	1.36E-12	4.23E-15	7.56E-13	41	37.92
104	9.50E-14	1.88E-11	4.10E-14	7.32E-12	41	38.01
1 004	8.93E-13	1.83E-10	4.19E-13	7.42E-11	41	37.95
10 004	1.01E-11	2.15E-09	4.19E-12	7.59E-10	41	37.84
100 004	8.58E-11	1.52E-08	4.01E-11	7.07E-09	41	38.02
Riedkosť = 25%						
4	2.66E-15	6.82E-13	1.87E-15	4.16E-13	50	46.25
14	1.03E-14	2.66E-12	5.15E-15	1.19E-12	50	46.81
104	1.12E-13	2.47E-11	4.71E-14	1.08E-11	49	46.83
1 004	7.85E-13	2.00E-10	4.95E-13	1.12E-10	50	46.82
10 004	7.21E-12	1.79E-09	4.76E-12	1.07E-09	50	46.85
100 004	8.93E-11	2.17E-08	4.74E-11	1.10E-08	50	46.79

Tabuľka *CR 4:*

$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	Pres_min $\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	Pres_min $\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	Pres_avg $\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	Pres_avg $\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	Iterácie max	Iterácie avg
			Riedkosť = 5%			
4	2.44E-15	2.56E-13	1.73E-15	1.42E-13	43	40.05
14	8.58E-15	9.38E-13	4.43E-15	4.09E-13	44	40.56
104	8.93E-14	8.87E-12	4.16E-14	3.80E-12	44	40.67
1 004	8.83E-13	8.32E-11	4.13E-13	3.72E-11	43	40.58
10 004	9.82E-12	8.29E-10	4.12E-12	3.66E-10	43	40.65
100 004	6.83E-11	7.10E-09	4.04E-11	3.66E-09	44	40.65
			Riedkosť = 10%			
4	2.66E-15	4.55E-13	1.69E-15	2.74E-13	40	37.83
14	8.41E-15	1.68E-12	4.61E-15	8.02E-13	41	38.29
104	7.57E-14	1.32E-11	4.33E-14	7.59E-12	41	38.34
1 004	9.58E-13	1.75E-10	4.22E-13	7.33E-11	41	38.44
10 004	7.52E-12	1.38E-09	4.31E-12	7.57E-10	41	38.33
100 004	7.59E-11	1.40E-08	4.21E-11	7.18E-09	41	38.41
			Riedkosť = 15%			
4	2.66E-15	7.96E-13	1.70E-15	4.22E-13	40	37.33
14	1.07E-14	3.07E-12	5.05E-15	1.29E-12	40	37.62
104	8.31E-14	2.35E-11	4.71E-14	1.22E-11	41	37.73
1 004	9.39E-13	2.61E-10	4.59E-13	1.20E-10	41	37.65
10 004	8.20E-12	2.37E-09	4.44E-12	1.17E-09	40	37.66
100 004	8.85E-11	2.28E-08	4.41E-11	1.16E-08	41	37.70
			Riedkosť = 20%			
4	2.66E-15	1.02E-12	1.78E-15	5.93E-13	40	37.47
14	9.33E-15	3.47E-12	5.43E-15	1.95E-12	40	37.72
104	1.22E-13	4.33E-11	5.12E-14	1.81E-11	40	37.74
1 004	9.42E-13	3.76E-10	5.11E-13	1.86E-10	41	37.78
10 004	1.43E-11	5.62E-09	5.16E-12	1.85E-09	41	37.72
100 004	8.05E-11	3.35E-08	4.91E-11	1.76E-08	40	37.79
			Riedkosť = 25%			
4	3.33E-15	1.71E-12	1.92E-15	8.46E-13	38	35.56
14	1.29E-14	6.37E-12	6.50E-15	2.95E-12	38	35.87
104	1.02E-13	5.35E-11	6.23E-14	2.86E-11	38	35.81
1 004	1.07E-12	5.72E-10	6.14E-13	2.84E-10	38	35.83
10 004	1.06E-11	5.36E-09	5.91E-12	2.75E-09	38	35.82
100 004	9.02E-11	4.45E-08	5.78E-11	2.65E-08	38	35.89

CR *metóda,* $n = 1\,000$

Tabuľka *CR 5:*

CR metóda, $n = 2\,000$						
	Pres_min		Pres_avg		Iterácie	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	max	avg
			Riedkosť = 5%			
4	2.22E-15	3.98E-13	1.81E-15	2.92E-13	42	38.40
14	8.44E-15	1.62E-12	4.81E-15	8.31E-13	42	38.82
104	7.53E-14	1.41E-11	4.46E-14	7.63E-12	42	38.94
1 004	8.28E-13	1.34E-10	4.50E-13	7.79E-11	42	38.89
10 004	6.44E-12	1.09E-09	4.41E-12	7.63E-10	42	38.86
100 004	9.18E-11	1.57E-08	4.63E-11	7.93E-09	42	38.86
			Riedkosť = 10%			
4	2.66E-15	1.14E-12	1.80E-15	5.78E-13	39	36.95
14	7.77E-15	2.96E-12	5.07E-15	1.70E-12	39	37.43
104	8.91E-14	2.81E-11	4.83E-14	1.61E-11	40	37.41
1 004	9.62E-13	3.48E-10	5.15E-13	1.70E-10	40	37.41
10 004	8.38E-12	2.88E-09	4.95E-12	1.64E-09	40	37.47
100 004	7.29E-11	2.52E-08	4.71E-11	1.57E-08	40	37.46
			Riedkosť = 15%			
4	2.66E-15	1.59E-12	1.89E-15	9.22E-13	39	36.85
14	9.77E-15	5.23E-12	5.98E-15	3.10E-12	40	37.20
104	7.86E-14	4.14E-11	5.43E-14	2.78E-11	40	37.24
1 004	8.19E-13	4.54E-10	5.51E-13	2.76E-10	39	37.25
10 004	9.13E-12	4.99E-09	5.40E-12	2.74E-09	39	37.25
100 004	9.63E-11	5.40E-08	5.45E-11	2.79E-08	39	37.22
			Riedkosť = 20%			
4	5.33E-15	4.32E-12	2.12E-15	1.44E-12	40	37.38
14	1.78E-14	1.36E-11	7.59E-15	5.37E-12	40	37.57
104	1.18E-13	8.90E-11	6.49E-14	4.59E-11	40	37.73
1 004	1.61E-12	1.19E-09	6.92E-13	4.75E-10	40	37.67
10 004	1.08E-11	8.13E-09	6.80E-12	4.80E-09	40	37.70
100 004	1.27E-10	9.75E-08	6.59E-11	4.71E-08	40	37.69
			Riedkosť = 25%			
4	3.55E-15	3.41E-12	2.40E-15	2.10E-12	38	35.53
14	1.45E-14	1.48E-11	8.93E-15	8.14E-12	38	35.76
104	1.08E-13	1.01E-10	7.95E-14	7.33E-11	38	35.79
1 004	1.49E-12	1.50E-09	8.01E-13	7.41E-10	38	35.77
10 004	1.54E-11	1.43E-08	8.17E-12	7.58E-09	38	35.85
100 004	1.35E-10	1.35E-07	8.13E-11	7.36E-08	38	35.80

Tabuľka *CR 6:*

CR *metóda*, $n = 3\,000$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
Riedkosť = 0.15%						
4	4.00E-15	7.11E-14	3.03E-15	3.38E-14	85	77.82
14	1.95E-14	3.09E-13	8.58E-15	1.04E-13	84	78.94
104	2.36E-13	3.37E-12	8.51E-14	1.03E-12	87	78.93
1 004	2.57E-12	3.66E-11	8.67E-13	1.04E-11	83	78.81
10 004	1.55E-11	2.31E-10	8.30E-12	9.78E-11	84	78.92
100 004	2.28E-10	3.23E-09	8.84E-11	1.07E-09	86	78.90
Riedkosť = 5%						
4	2.44E-15	7.96E-13	1.84E-15	5.80E-13	40	37.42
14	7.77E-15	2.72E-12	5.21E-15	1.75E-12	40	37.88
104	1.02E-13	3.68E-11	4.86E-14	1.62E-11	40	37.90
1 004	1.15E-12	3.69E-10	5.08E-13	1.67E-10	40	37.93
10 004	7.67E-12	2.59E-09	4.93E-12	1.64E-09	40	37.95
100 004	1.28E-10	4.66E-08	4.93E-11	1.65E-08	40	37.90
Riedkosť = 10%						
4	2.66E-15	1.36E-12	1.85E-15	9.16E-13	39	36.76
14	9.77E-15	5.23E-12	5.73E-15	2.87E-12	39	37.16
104	1.18E-13	6.83E-11	5.72E-14	2.85E-11	40	37.13
1 004	9.37E-13	4.75E-10	5.74E-13	2.78E-10	39	37.17
10 004	8.47E-12	4.56E-09	5.36E-12	2.60E-09	39	37.16
100 004	9.13E-11	4.57E-08	5.33E-11	2.61E-08	39	37.20
Riedkosť = 15%						
4	4.44E-15	4.09E-12	2.13E-15	1.70E-12	38	35.75
14	1.04E-14	9.44E-12	7.25E-15	5.92E-12	38	35.93
104	1.27E-13	1.18E-10	7.24E-14	5.98E-11	38	35.96
1 004	1.21E-12	1.04E-09	7.13E-13	6.10E-10	38	35.91
10 004	1.21E-11	1.13E-08	6.93E-12	5.93E-09	38	35.96
100 004	1.27E-10	1.16E-07	7.11E-11	5.83E-08	38	36.01

Tabuľka *CR 7:*

CR metóda, $n = 5\,000$						
	Pres_min		Pres_avg		Iterácie	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	max	avg
Riedkosť = 0.1%						
4	6.44E-15	9.95E-14	3.29E-15	4.08E-14	84	79.84
14	2.02E-14	2.52E-13	1.02E-14	1.26E-13	85	80.77
104	2.23E-13	3.24E-12	9.98E-14	1.24E-12	87	80.83
1 004	2.21E-12	3.26E-11	9.45E-13	1.20E-11	86	80.98
10 004	2.39E-11	3.21E-10	9.60E-12	1.20E-10	86	80.87
100 004	2.62E-10	3.75E-09	1.04E-10	1.31E-09	87	80.74
Riedkosť = 3%						
4	2.66E-15	7.96E-13	1.90E-15	4.70E-13	40	37.93
14	1.09E-14	2.84E-12	5.39E-15	1.36E-12	41	38.36
104	1.02E-13	2.74E-11	5.17E-14	1.32E-11	41	38.32
1 004	9.15E-13	2.28E-10	5.02E-13	1.27E-10	41	38.36
10 004	9.13E-12	2.37E-09	5.02E-12	1.26E-09	41	38.33
100 004	1.18E-10	3.26E-08	5.21E-11	1.33E-08	41	38.33
Riedkosť = 5%						
4	3.11E-15	1.14E-12	1.92E-15	9.27E-13	39	37.15
14	9.77E-15	5.34E-12	5.77E-15	2.78E-12	40	37.57
104	8.30E-14	4.24E-11	5.48E-14	2.67E-11	40	37.58
1 004	9.73E-13	5.04E-10	5.58E-13	2.66E-10	40	37.53
10 004	8.59E-12	4.43E-09	5.57E-12	2.73E-09	40	37.50
100 004	8.08E-11	3.99E-08	5.38E-11	2.60E-08	40	37.55
Riedkosť = 10%						
4	2.66E-15	2.27E-12	2.11E-15	1.61E-12	38	36.50
14	9.77E-15	6.48E-12	7.74E-15	5.41E-12	38	36.80
104	8.57E-14	5.68E-11	6.28E-14	4.52E-11	39	37.20
1 004	6.65E-13	5.73E-10	6.11E-13	4.98E-10	38	36.90
10 004	9.50E-12	8.43E-09	6.86E-12	5.58E-09	39	37.00
100 004	8.73E-11	5.79E-08	6.85E-11	4.95E-08	38	37.00

Tabuľka *CR 8:*

			***CR** metóda,* $n = 10\,000$			
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
			Riedkosť = 0.05%			
4	7.11E-15	1.07E-13	3.48E-15	4.52E-14	86	82.25
14	2.73E-14	3.91E-13	1.17E-14	1.53E-13	89	83.18
104	2.77E-13	3.65E-12	1.20E-13	1.56E-12	88	83.10
1 004	3.45E-12	4.77E-11	1.14E-12	1.47E-11	88	83.18
10 004	2.29E-11	3.45E-10	1.18E-11	1.50E-10	89	83.17
100 004	2.58E-10	3.64E-09	1.19E-10	1.59E-09	89	83.20
			Riedkosť = 0.5%			
4	2.66E-15	2.84E-13	2.13E-15	1.89E-13	44	41.60
14	1.55E-14	1.62E-12	6.40E-15	6.19E-13	45	42.04
104	1.23E-13	1.23E-11	5.99E-14	5.71E-12	45	42.12
1 004	9.69E-13	1.02E-10	5.92E-13	5.55E-11	45	42.03
10 004	1.02E-11	1.05E-09	5.89E-12	5.68E-10	45	42.05
100 004	1.06E-10	1.08E-08	5.94E-11	5.69E-09	45	42.09
			Riedkosť = 1%			
4	3.11E-15	6.25E-13	2.11E-15	3.43E-13	44	40.73
14	1.55E-14	3.35E-12	5.72E-15	1.01E-12	45	41.30
104	9.62E-14	1.78E-11	5.59E-14	9.77E-12	44	41.28
1 004	9.76E-13	1.87E-10	5.60E-13	9.89E-11	44	41.25
10 004	8.16E-12	1.55E-09	5.46E-12	9.64E-10	44	41.27
100 004	8.91E-11	1.90E-08	5.55E-11	9.79E-09	44	41.24
			Riedkosť = 2%			
4	2.22E-15	7.96E-13	1.93E-15	6.48E-13	39	37.70
14	8.44E-15	2.73E-12	6.02E-15	1.96E-12	39	38.00
104	5.50E-14	1.92E-11	4.92E-14	1.66E-11	40	38.00
1 004	6.60E-13	2.25E-10	5.21E-13	1.73E-10	40	38.10
10 004	6.15E-12	2.10E-09	5.24E-12	1.81E-09	39	38.00
100 004	6.28E-11	2.11E-08	5.44E-11	1.76E-08	40	38.10

Tabuľka *CR 9:*

CR metóda, $n = 20\,000$						
	Pres_min		Pres_avg		Iterácie	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	max	avg
Riedkosť = 0.025%						
4	9.55E-15	1.28E-13	4.10E-15	5.69E-14	89	84.69
14	5.31E-14	7.96E-13	1.57E-14	2.19E-13	89	85.25
104	4.76E-13	7.23E-12	1.46E-13	2.02E-12	90	85.41
1 004	4.70E-12	7.59E-11	1.40E-12	1.92E-11	91	85.49
10 004	2.94E-11	4.42E-10	1.38E-11	1.89E-10	91	85.66
100 004	2.60E-10	4.62E-09	1.48E-10	2.01E-09	90	85.48
Riedkosť = 0.15%						
4	3.77E-15	2.98E-13	2.47E-15	1.43E-13	50	47.18
14	1.80E-14	1.25E-12	7.77E-15	5.02E-13	51	47.61
104	1.51E-13	1.04E-11	7.29E-14	4.59E-12	51	47.70
1 004	1.22E-12	9.94E-11	7.33E-13	4.66E-11	51	47.62
10 004	1.43E-11	9.65E-10	7.29E-12	4.62E-10	51	47.65
100 004	1.11E-10	8.54E-09	7.47E-11	4.78E-09	51	47.63
Riedkosť = 0.25%						
4	3.11E-15	2.84E-13	2.29E-15	2.01E-13	45	42.38
14	1.31E-14	1.39E-12	6.95E-15	6.82E-13	46	42.69
104	1.28E-13	1.13E-11	6.63E-14	6.45E-12	45	42.70
1 004	1.74E-12	1.85E-10	6.66E-13	6.40E-11	45	42.72
10 004	9.55E-12	1.01E-09	6.42E-12	6.21E-10	46	42.78
100 004	1.20E-10	1.06E-08	6.45E-11	6.21E-09	45	42.74
Riedkosť = 0.375%						
4	2.22E-15	2.84E-13	2.22E-15	2.70E-13	46	42.60
14	8.66E-15	1.42E-12	6.44E-15	9.02E-13	46	43.00
104	8.06E-14	1.25E-11	6.48E-14	9.18E-12	46	43.00
1 004	1.48E-12	2.38E-10	6.74E-13	9.39E-11	46	43.10
10 004	9.10E-12	1.27E-09	6.78E-12	9.37E-10	46	43.00
100 004	7.13E-11	9.89E-09	6.21E-11	8.16E-09	46	43.20

Tabuľka *CR 10:*

$\|x_1 - x^*\|_\infty$	Pres_min $\|x_i - x^*\|_\infty$	Pres_min $\|Ax_i - x^*\|_\infty$	Pres_avg $\|x_i - x^*\|_\infty$	Pres_avg $\|Ax_i - x^*\|_\infty$	Iterácie max	Iterácie avg
CR metóda, $n = 30\,000$						
Riedkosť = 0.015%						
4	7.99E-15	1.14E-13	4.12E-15	5.49E-14	92	85.81
14	4.49E-14	7.74E-13	1.75E-14	2.49E-13	92	86.45
104	3.49E-13	5.25E-12	1.60E-13	2.26E-12	92	86.42
1 004	3.90E-12	5.75E-11	1.64E-12	2.30E-11	94	86.43
10 004	4.10E-11	6.16E-10	1.60E-11	2.23E-10	92	86.37
100 004	4.05E-10	7.34E-09	1.58E-10	2.18E-09	91	86.52
Riedkosť = 0.05%						
4	7.99E-15	5.12E-13	2.83E-15	1.20E-13	58	54.14
14	1.67E-14	7.32E-13	8.81E-15	4.17E-13	59	54.63
104	1.69E-13	9.81E-12	8.85E-14	4.10E-12	59	54.56
1 004	1.60E-12	8.46E-11	8.71E-13	4.13E-11	58	54.66
10 004	1.60E-11	9.62E-10	8.80E-12	4.12E-10	58	54.61
100 004	1.86E-10	1.06E-08	8.98E-11	4.22E-09	59	54.59
Riedkosť = 0.1%						
4	2.89E-15	1.71E-13	2.53E-15	1.42E-13	50	47.70
14	1.24E-14	8.17E-13	8.10E-15	5.05E-13	50	48.30
104	1.14E-13	7.26E-12	8.36E-14	5.41E-12	50	47.90
1 004	8.93E-13	6.13E-11	6.66E-13	4.24E-11	51	48.60
10 004	1.57E-11	1.06E-09	8.25E-12	5.28E-10	50	48.30
100 004	1.07E-10	6.44E-09	8.13E-11	4.92E-09	51	48.30
Riedkosť = 0.15%						
4	2.44E-15	2.27E-13	2.24E-15	1.99E-13	45	42.60
14	8.58E-15	9.09E-13	7.08E-15	6.98E-13	46	43.00
104	1.03E-13	1.01E-11	7.10E-14	6.67E-12	46	43.10
1 004	8.88E-13	9.02E-11	7.21E-13	7.09E-11	46	43.00
10 004	8.12E-12	8.93E-10	6.22E-12	6.00E-10	46	43.20
100 004	6.81E-11	6.99E-09	6.29E-11	6.01E-09	46	43.30

Tabuľka *CR 11:*

CR metóda, $n = 50\,000$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
			Riedkosť = 0.01%			
4	1.69E-14	2.98E-13	4.70E-15	6.93E-14	92	87.35
14	5.06E-14	8.53E-13	1.98E-14	2.94E-13	96	87.84
104	6.08E-13	1.11E-11	1.92E-13	2.78E-12	94	87.95
1 004	4.79E-12	8.48E-11	1.90E-12	2.75E-11	92	87.97
10 004	4.52E-11	8.78E-10	1.95E-11	2.82E-10	92	87.76
100 004	4.00E-10	6.15E-09	1.84E-10	2.66E-09	95	87.98
			Riedkosť = 0.02%			
4	1.09E-14	1.71E-13	3.88E-15	9.35E-14	84	76.50
14	5.84E-14	9.45E-13	1.56E-14	3.99E-13	85	76.96
104	6.49E-13	1.11E-11	1.44E-13	3.53E-12	85	77.17
1 004	4.03E-12	9.24E-11	1.30E-12	3.40E-11	85	77.30
10 004	6.03E-11	2.01E-09	1.38E-11	3.68E-10	85	77.13
100 004	6.74E-10	8.54E-09	1.47E-10	3.53E-09	86	77.08
			Riedkosť = 0.04%			
4	4.44E-15	2.84E-13	2.85E-15	1.24E-13	60	55.29
14	1.73E-14	9.38E-13	1.01E-14	4.86E-13	60	55.75
104	1.80E-13	9.07E-12	9.59E-14	4.49E-12	60	55.74
1 004	1.81E-12	8.47E-11	9.44E-13	4.44E-11	60	55.79
10 004	1.47E-11	8.20E-10	9.38E-12	4.46E-10	61	55.79
100 004	1.65E-10	9.04E-09	9.74E-11	4.62E-09	61	55.69
			Riedkosť = 0.06%			
4	3.89E-15	2.70E-13	2.76E-15	1.62E-13	51	48.50
14	9.66E-15	6.11E-13	8.00E-15	5.11E-13	52	49.10
104	1.20E-13	7.91E-12	8.25E-14	5.45E-12	52	49.00
1 004	9.80E-13	6.69E-11	8.06E-13	5.36E-11	52	49.20
10 004	1.10E-11	7.19E-10	8.22E-12	5.33E-10	52	49.10
100 004	1.17E-10	8.15E-09	8.75E-11	5.79E-09	52	49.20

Tabuľka *CR 12:*

CGNR *metóda*, $n = 100$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
			Riedkosť = 5%			
4	3.33E-10	5.72E-10	1.63E-11	3.79E-11	100	100.0
14	1.30E-09	2.16E-09	8.62E-11	1.97E-10	100	100.0
104	1.38E-08	2.26E-08	8.06E-10	1.85E-09	100	100.0
1 004	1.41E-07	2.30E-07	8.02E-09	1.87E-08	100	100.0
10 004	1.43E-06	2.34E-06	8.74E-08	1.95E-07	100	100.0
100 004	1.36E-05	2.22E-05	8.30E-07	1.87E-06	100	100.0
			Riedkosť = 10%			
4	2.80E-14	3.06E-13	2.74E-15	4.24E-14	98	89.16
14	3.24E-14	3.06E-13	4.94E-15	7.91E-14	100	90.68
104	7.23E-13	4.82E-12	5.26E-14	7.15E-13	100	90.67
1 004	3.30E-12	2.86E-11	4.12E-13	6.23E-12	100	90.77
10 004	1.87E-11	2.66E-10	3.88E-12	6.45E-11	100	90.72
100 004	2.33E-10	1.79E-09	3.74E-11	5.93E-10	100	90.80
			Riedkosť = 15%			
4	2.89E-15	1.14E-13	2.08E-15	5.19E-14	84	75.15
14	1.12E-14	3.77E-13	3.95E-15	1.06E-13	85	76.22
104	9.53E-14	3.11E-12	3.47E-14	9.46E-13	84	76.48
1 004	8.60E-13	2.86E-11	3.54E-13	9.72E-12	84	76.40
10 004	7.89E-12	2.75E-10	3.49E-12	8.82E-11	85	76.37
100 004	7.62E-11	2.70E-09	3.46E-11	9.05E-10	85	76.40
			Riedkosť = 20%			
4	2.66E-15	9.95E-14	1.84E-15	6.21E-14	72	65.03
14	1.11E-14	5.26E-13	4.22E-15	1.52E-13	72	65.92
104	6.16E-14	2.39E-12	3.40E-14	1.17E-12	72	66.01
1 004	5.07E-13	2.23E-11	3.25E-13	1.13E-11	72	66.18
10 004	7.20E-12	3.10E-10	3.33E-12	1.18E-10	72	66.12
100 004	7.38E-11	3.47E-09	3.31E-11	1.17E-09	72	66.10
			Riedkosť = 25%			
4	6.99E-15	1.56E-13	2.11E-15	8.87E-14	84	74.54
14	1.11E-14	3.98E-13	4.17E-15	1.85E-13	84	75.84
104	2.78E-13	6.81E-12	4.18E-14	1.85E-12	85	75.69
1 004	1.05E-12	5.53E-11	3.83E-13	1.76E-11	84	75.72
10 004	1.22E-11	5.91E-10	3.69E-12	1.69E-10	84	75.74
100 004	1.07E-10	5.13E-09	3.89E-11	1.78E-09	84	75.76

Tabuľka *CGNR 1:*

CGNR *metóda*, $n = 200$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
Riedkosť = 5%						
4	1.02E-13	5.83E-13	6.25E-15	6.23E-14	114	103.7
14	6.48E-14	4.26E-13	7.29E-15	9.70E-14	118	105.9
104	1.15E-12	7.30E-12	9.43E-14	1.01E-12	117	105.9
1 004	1.15E-11	7.40E-11	9.21E-13	9.81E-12	117	105.7
10 004	1.74E-10	1.21E-09	1.33E-11	1.22E-10	117	105.7
100 004	1.10E-09	7.03E-09	9.40E-11	1.01E-09	117	105.7
Riedkosť = 10%						
4	3.11E-15	9.95E-14	2.13E-15	6.84E-14	79	71.38
14	8.33E-15	3.53E-13	4.38E-15	1.61E-13	80	72.55
104	8.57E-14	3.87E-12	4.14E-14	1.51E-12	80	72.55
1 004	1.10E-12	5.23E-11	4.13E-13	1.55E-11	79	72.57
10 004	9.48E-12	4.57E-10	4.16E-12	1.54E-10	82	72.44
100 004	1.01E-10	4.86E-09	4.33E-11	1.67E-09	79	72.38
Riedkosť = 15%						
4	2.89E-15	1.71E-13	2.00E-15	1.00E-13	68	62.12
14	9.55E-15	5.97E-13	4.46E-15	2.48E-13	69	62.94
104	1.05E-13	7.59E-12	4.52E-14	2.52E-12	69	62.97
1 004	8.33E-13	5.90E-11	4.36E-13	2.35E-11	68	62.93
10 004	9.58E-12	6.23E-10	4.28E-12	2.36E-10	69	63.01
100 004	8.76E-11	5.33E-09	4.32E-11	2.38E-09	70	63.01
Riedkosť = 20%						
4	3.11E-15	2.56E-13	1.96E-15	1.27E-13	64	57.04
14	9.33E-15	8.24E-13	4.64E-15	3.46E-13	64	57.79
104	9.80E-14	8.60E-12	4.52E-14	3.42E-12	64	57.84
1 004	1.10E-12	9.91E-11	4.45E-13	3.31E-11	64	57.80
10 004	1.13E-11	1.00E-09	4.74E-12	3.58E-10	64	57.82
100 004	1.33E-10	1.22E-08	4.60E-11	3.48E-09	65	57.82
Riedkosť = 25%						
4	3.11E-15	3.69E-13	1.91E-15	1.55E-13	62	54.53
14	1.01E-14	1.08E-12	5.10E-15	4.77E-13	63	55.24
104	1.57E-13	1.81E-11	5.18E-14	5.02E-12	61	55.10
1 004	1.23E-12	1.29E-10	4.94E-13	4.59E-11	62	55.22
10 004	1.31E-11	1.44E-09	4.78E-12	4.58E-10	63	55.17
100 004	1.33E-10	1.42E-08	4.95E-11	4.77E-09	62	55.14

Tabuľka *CGNR 2:*

***CGNR** metóda,* $n = 300$						
	Pres_ min		*Pres_ avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
			Riedkosť = 6.7%			
4	3.11E-15	1.71E-13	2.33E-15	7.54E-14	82	74.01
14	1.33E-14	6.68E-13	4.94E-15	1.91E-13	86	75.04
104	1.01E-13	4.73E-12	4.85E-14	1.95E-12	85	74.89
1 004	1.16E-12	5.62E-11	4.73E-13	1.86E-11	86	75.10
10 004	1.48E-11	6.58E-10	4.58E-12	1.79E-10	84	75.13
100 004	9.35E-11	4.10E-09	4.54E-11	1.75E-09	84	75.04
			Riedkosť = 10%			
4	2.89E-15	1.71E-13	2.14E-15	1.04E-13	71	63.65
14	1.22E-14	1.02E-12	4.82E-15	2.76E-13	71	64.46
104	1.86E-13	1.23E-11	5.00E-14	2.91E-12	71	64.37
1 004	1.36E-12	9.04E-11	5.20E-13	2.97E-11	72	64.32
10 004	1.37E-11	8.97E-10	4.77E-12	2.73E-10	71	64.38
100 004	1.40E-10	9.42E-09	4.74E-11	2.74E-09	72	64.44
			Riedkosť = 16.7%			
4	3.11E-15	3.41E-13	1.98E-15	1.60E-13	61	55.79
14	1.12E-14	1.22E-12	5.51E-15	5.17E-13	62	56.28
104	1.31E-13	1.50E-11	5.75E-14	5.51E-12	63	56.27
1 004	1.33E-12	1.48E-10	5.30E-13	5.02E-11	61	56.20
10 004	1.14E-11	1.30E-09	5.33E-12	5.08E-10	62	56.18
100 004	1.73E-10	1.87E-08	5.58E-11	5.25E-09	62	56.09
			Riedkosť = 20%			
4	3.11E-15	3.41E-13	2.05E-15	2.03E-13	60	54.02
14	1.80E-14	2.44E-12	6.00E-15	6.82E-13	59	54.20
104	1.81E-13	2.17E-11	5.88E-14	6.74E-12	59	54.21
1 004	1.86E-12	2.25E-10	5.87E-13	6.67E-11	59	54.21
10 004	1.63E-11	2.14E-09	5.66E-12	6.43E-10	59	54.37
100 004	1.59E-10	1.86E-08	5.63E-11	6.33E-09	60	54.33
			Riedkosť = 25%			
4	3.33E-15	4.55E-13	2.12E-15	2.69E-13	64	58.56
14	1.51E-14	2.33E-12	5.93E-15	7.99E-13	66	59.09
104	1.16E-13	1.73E-11	5.42E-14	7.35E-12	66	59.23
1 004	9.47E-13	1.38E-10	5.54E-13	7.53E-11	64	59.11
10 004	1.01E-11	1.50E-09	5.50E-12	7.51E-10	66	59.29
100 004	1.14E-10	1.73E-08	5.35E-11	7.29E-09	66	59.21

Tabuľka *CGNR 3:*

CGNR metóda, $n = 500$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
			Riedkosť = 5%			
4	7.29E-14	1.23E-12	4.33E-15	1.35E-13	98	83.56
14	5.90E-13	7.95E-12	1.63E-14	4.30E-13	97	84.74
104	5.36E-12	7.23E-11	1.52E-13	4.11E-12	97	84.93
1 004	5.51E-11	7.50E-10	1.52E-12	4.07E-11	97	84.83
10 004	5.48E-10	7.43E-09	1.49E-11	4.02E-10	98	84.84
100 004	5.50E-09	7.49E-08	1.48E-10	3.99E-09	97	84.81
			Riedkosť = 10%			
4	3.55E-15	3.98E-13	2.14E-15	1.80E-13	63	57.48
14	1.73E-14	1.68E-12	6.12E-15	5.73E-13	64	58.18
104	1.61E-13	1.80E-11	5.64E-14	5.34E-12	64	58.30
1 004	1.76E-12	1.83E-10	6.16E-13	5.85E-11	63	58.20
10 004	1.56E-11	1.55E-09	5.94E-12	5.58E-10	63	58.22
100 004	1.20E-10	1.26E-08	6.03E-11	5.74E-09	64	58.13
			Riedkosť = 15%			
4	3.55E-15	5.68E-13	2.26E-15	2.73E-13	68	61.56
14	1.43E-14	2.24E-12	6.51E-15	8.95E-13	68	62.19
104	1.58E-13	2.78E-11	6.40E-14	8.87E-12	69	62.30
1 004	1.36E-12	2.06E-10	6.04E-13	8.30E-11	69	62.32
10 004	1.15E-11	1.86E-09	6.27E-12	8.65E-10	69	62.35
100 004	3.01E-10	4.72E-08	6.79E-11	9.46E-09	69	62.25
			Riedkosť = 20%			
4	3.55E-15	7.39E-13	2.19E-15	3.79E-13	56	51.80
14	1.98E-14	4.04E-12	7.55E-15	1.42E-12	57	52.19
104	1.41E-13	2.93E-11	6.77E-14	1.27E-11	57	52.32
1 004	2.06E-12	4.33E-10	7.28E-13	1.36E-10	56	52.31
10 004	1.68E-11	3.49E-09	7.07E-12	1.31E-09	57	52.32
100 004	1.71E-10	3.52E-08	7.06E-11	1.33E-08	57	52.30
			Riedkosť = 25%			
4	4.88E-15	1.25E-12	2.64E-15	5.73E-13	84	78.69
14	2.62E-14	7.05E-12	8.13E-15	1.97E-12	85	79.77
104	2.04E-13	5.55E-11	7.81E-14	1.88E-11	85	79.76
1 004	3.16E-12	7.92E-10	7.61E-13	1.84E-10	86	79.94
10 004	1.73E-11	4.49E-09	7.47E-12	1.79E-09	87	79.79
100 004	2.87E-10	7.46E-08	7.61E-11	1.80E-08	86	79.82

Tabuľka *CGNR 4:*

CGNR metóda, $n = 1\,000$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
			Riedkosť = 5%			
4	*4.44E-15*	*4.55E-13*	*2.35E-15*	*1.97E-13*	*65*	*58.88*
14	*1.68E-14*	*1.77E-12*	*7.17E-15*	*6.86E-13*	*66*	*59.50*
104	*2.00E-13*	*2.08E-11*	*7.27E-14*	*6.97E-12*	*67*	*59.52*
1 004	*1.62E-12*	*1.65E-10*	*7.03E-13*	*6.69E-11*	*66*	*59.59*
10 004	*1.96E-11*	*2.13E-09*	*6.92E-12*	*6.55E-10*	*66*	*59.62*
100 004	*1.24E-10*	*1.24E-08*	*6.61E-11*	*6.29E-09*	*67*	*59.49*
			Riedkosť = 10%			
4	*4.22E-15*	*8.53E-13*	*2.21E-15*	*3.68E-13*	*57*	*53.09*
14	*1.89E-14*	*3.64E-12*	*8.19E-15*	*1.50E-12*	*58*	*53.48*
104	*1.68E-13*	*3.15E-11*	*7.81E-14*	*1.41E-11*	*57*	*53.45*
1 004	*1.71E-12*	*3.50E-10*	*7.20E-13*	*1.31E-10*	*58*	*53.59*
10 004	*1.33E-11*	*2.37E-09*	*6.80E-12*	*1.22E-09*	*59*	*53.64*
100 004	*1.64E-10*	*3.16E-08*	*7.59E-11*	*1.37E-08*	*58*	*53.48*
			Riedkosť = 15%			
4	*5.77E-15*	*1.71E-12*	*2.47E-15*	*6.37E-13*	*56*	*51.43*
14	*2.03E-14*	*5.97E-12*	*9.10E-15*	*2.46E-12*	*57*	*51.70*
104	*1.49E-13*	*4.02E-11*	*8.08E-14*	*2.16E-11*	*57*	*51.83*
1 004	*2.28E-12*	*6.67E-10*	*8.85E-13*	*2.43E-10*	*56*	*51.68*
10 004	*1.78E-11*	*5.20E-09*	*8.24E-12*	*2.22E-09*	*57*	*51.70*
100 004	*1.36E-10*	*3.89E-08*	*8.25E-11*	*2.20E-08*	*57*	*51.71*
			Riedkosť = 20%			
4	*5.77E-15*	*2.16E-12*	*2.59E-15*	*9.14E-13*	*56*	*51.54*
14	*2.28E-14*	*9.15E-12*	*1.07E-14*	*3.99E-12*	*56*	*51.69*
104	*2.26E-13*	*9.35E-11*	*9.86E-14*	*3.67E-11*	*56*	*51.80*
1 004	*2.40E-12*	*9.48E-10*	*1.00E-12*	*3.77E-10*	*56*	*51.81*
10 004	*1.89E-11*	*7.73E-09*	*9.78E-12*	*3.61E-09*	*56*	*51.82*
100 004	*2.55E-10*	*1.03E-07*	*1.01E-10*	*3.73E-08*	*56*	*51.74*
			Riedkosť = 25%			
4	*6.00E-15*	*2.73E-12*	*2.85E-15*	*1.31E-12*	*51*	*47.71*
14	*3.77E-14*	*1.91E-11*	*1.24E-14*	*5.97E-12*	*51*	*48.00*
104	*3.15E-13*	*1.63E-10*	*1.16E-13*	*5.56E-11*	*51*	*48.00*
1 004	*2.74E-12*	*1.36E-09*	*1.18E-12*	*5.69E-10*	*52*	*47.96*
10 004	*2.58E-11*	*1.26E-08*	*1.14E-11*	*5.47E-09*	*51*	*48.09*
100 004	*2.38E-10*	*1.21E-07*	*1.12E-10*	*5.36E-08*	*51*	*48.02*

Tabuľka *CGNR 5:*

CGNR metóda, $n = 2\,000$						
	Pres_min		Pres_avg		Iterácie	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	max	avg
			Riedkosť = 5%			
4	5.33E-15	1.08E-12	2.43E-15	4.12E-13	60	54.31
14	1.51E-14	3.01E-12	8.38E-15	1.53E-12	60	54.81
104	2.08E-13	4.30E-11	8.63E-14	1.55E-11	60	54.74
1 004	2.88E-12	5.90E-10	7.86E-13	1.42E-10	61	54.86
10 004	2.50E-11	5.53E-09	8.45E-12	1.52E-09	61	54.81
100 004	1.89E-10	3.56E-08	8.20E-11	1.47E-08	61	54.85
			Riedkosť = 10%			
4	4.00E-15	1.48E-12	2.49E-15	8.17E-13	54	50.98
14	2.13E-14	7.79E-12	9.90E-15	3.46E-12	55	51.29
104	1.55E-13	5.63E-11	8.85E-14	3.07E-11	55	51.46
1 004	1.74E-12	6.80E-10	9.28E-13	3.25E-10	55	51.37
10 004	1.94E-11	7.06E-09	9.06E-12	3.16E-09	55	51.47
100 004	1.66E-10	6.56E-08	9.48E-11	3.30E-08	55	51.33
			Riedkosť = 15%			
4	5.33E-15	2.73E-12	2.96E-15	1.46E-12	54	50.61
14	1.87E-14	1.09E-11	1.08E-14	5.61E-12	54	51.07
104	1.93E-13	1.13E-10	1.04E-13	5.57E-11	54	50.93
1 004	1.72E-12	9.65E-10	1.05E-12	5.59E-10	54	51.08
10 004	2.30E-11	1.28E-08	1.08E-11	5.71E-09	54	50.97
100 004	1.90E-10	1.06E-07	1.05E-10	5.60E-08	54	50.97
			Riedkosť = 20%			
4	7.11E-15	5.68E-12	3.42E-15	2.44E-12	55	52.03
14	2.75E-14	2.10E-11	1.37E-14	1.01E-11	55	52.20
104	3.12E-13	2.47E-10	1.34E-13	9.93E-11	56	52.28
1 004	2.88E-12	2.26E-09	1.35E-12	9.89E-10	56	52.17
10 004	2.74E-11	1.94E-08	1.29E-11	9.40E-09	55	52.24
100 004	2.24E-10	1.64E-07	1.27E-10	9.29E-08	55	52.29
			Riedkosť = 25%			
4	7.77E-15	7.73E-12	3.86E-15	3.58E-12	51	47.70
14	4.09E-14	4.07E-11	1.78E-14	1.69E-11	51	47.82
104	3.19E-13	3.18E-10	1.69E-13	1.60E-10	52	47.82
1 004	2.88E-12	2.91E-09	1.55E-12	1.46E-09	51	47.91
10 004	2.59E-11	2.49E-08	1.56E-11	1.47E-08	50	47.85
100 004	2.91E-10	3.01E-07	1.59E-10	1.52E-07	51	47.95

Tabuľka *CGNR 6:*

CGNR *metóda*, $n = 3\,000$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
			Riedkosť = 0.15%			
4	2.52E-12	3.49E-12	1.57E-13	2.43E-13	261	237.1
14	2.68E-12	3.95E-12	1.89E-13	3.27E-13	272	243.0
104	4.90E-11	6.74E-11	3.83E-12	5.99E-12	271	243.4
1 004	2.36E-10	3.42E-10	1.72E-11	3.07E-11	267	243.7
10 004	2.51E-09	3.65E-09	1.73E-10	3.13E-10	267	243.6
100 004	3.85E-08	5.49E-08	2.79E-09	4.45E-09	269	244.0
			Riedkosť = 5%			
4	4.88E-15	1.82E-12	2.62E-15	8.77E-13	55	51.92
14	1.85E-14	6.88E-12	9.77E-15	3.41E-12	56	52.31
104	1.84E-13	6.93E-11	9.15E-14	3.16E-11	57	52.38
1 004	1.60E-12	5.97E-10	9.17E-13	3.14E-10	57	52.39
10 004	1.67E-11	6.12E-09	9.12E-12	3.15E-09	57	52.38
100 004	1.65E-10	5.79E-08	9.51E-11	3.30E-08	57	52.35
			Riedkosť = 10%			
4	4.88E-15	2.73E-12	3.04E-15	1.47E-12	54	50.52
14	1.67E-14	9.09E-12	1.03E-14	5.22E-12	55	50.98
104	1.79E-13	1.00E-10	1.01E-13	5.18E-11	55	50.93
1 004	2.14E-12	9.20E-10	9.90E-13	5.05E-10	55	50.97
10 004	2.09E-11	1.17E-08	1.03E-11	5.27E-09	55	50.95
100 004	2.35E-10	1.35E-07	1.01E-10	5.21E-08	54	50.97
			Riedkosť = 15%			
4	4.88E-15	4.32E-12	3.38E-15	2.81E-12	51	47.98
14	2.74E-14	2.53E-11	1.38E-14	1.20E-11	52	48.21
104	2.75E-13	2.48E-10	1.26E-13	1.11E-10	51	48.26
1 004	2.46E-12	2.31E-09	1.32E-12	1.16E-09	51	48.27
10 004	2.16E-11	1.82E-08	1.31E-11	1.15E-08	51	48.24
100 004	2.97E-10	2.71E-07	1.29E-10	1.14E-07	51	48.28

Tabuľka *CGNR 7:*

CGNR *metóda*, $n = 5\,000$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
Riedkosť = 0.1%						
4	2.23E-12	2.97E-12	2.09E-13	3.16E-13	275	251.5
14	5.06E-12	7.37E-12	5.48E-13	8.22E-13	285	257.2
104	6.53E-11	9.10E-11	6.40E-12	9.37E-12	285	258.3
1 004	6.02E-10	8.38E-10	5.64E-11	8.32E-11	279	257.5
10 004	6.92E-09	9.63E-09	6.50E-10	9.42E-10	282	257.4
100 004	1.06E-07	1.50E-07	1.00E-08	1.46E-08	281	257.3
Riedkosť = 3%						
4	4.44E-15	1.25E-12	2.72E-15	6.88E-13	58	53.08
14	4.05E-14	1.22E-11	1.00E-14	2.68E-12	59	53.47
104	2.21E-13	6.25E-11	9.45E-14	2.50E-11	58	53.50
1 004	2.27E-12	6.45E-10	9.27E-13	2.43E-10	58	53.59
10 004	1.99E-11	5.73E-09	9.59E-12	2.51E-09	58	53.54
100 004	2.61E-10	7.84E-08	9.83E-11	2.60E-08	59	53.44
Riedkosť = 5%						
4	5.11E-15	2.73E-12	2.84E-15	1.35E-12	54	51.45
14	1.95E-14	1.07E-11	1.09E-14	5.38E-12	55	51.67
104	2.34E-13	1.28E-10	1.03E-13	5.24E-11	55	51.77
1 004	2.24E-12	1.20E-09	1.03E-12	5.22E-10	55	51.71
10 004	1.97E-11	1.01E-08	1.03E-11	5.18E-09	55	51.74
100 004	1.84E-10	8.76E-08	1.02E-10	5.21E-08	55	51.74
Riedkosť = 10%						
4	5.33E-15	4.77E-12	3.29E-15	2.53E-12	52	49.70
14	2.31E-14	1.93E-11	1.30E-14	1.09E-11	53	50.01
104	3.02E-13	2.81E-10	1.20E-13	1.01E-10	53	50.21
1 004	2.13E-12	1.82E-09	1.18E-12	9.91E-10	53	50.18
10 004	2.89E-11	2.57E-08	1.18E-11	9.61E-09	53	50.22
100 004	1.59E-10	1.44E-07	1.17E-10	9.59E-08	53	50.19

Tabuľka *CGNR 8:*

CGNR *metóda*, $n = 10\,000$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
Riedkosť = 0.05%						
4	8.31E-12	1.07E-11	5.56E-13	7.74E-13	296	268.6
14	2.12E-11	2.82E-11	1.42E-12	1.96E-12	299	273.3
104	3.49E-10	4.64E-10	1.87E-11	2.58E-11	302	274.7
1 004	2.77E-09	3.69E-09	1.47E-10	2.03E-10	306	274.6
10 004	1.02E-08	1.38E-08	6.45E-10	9.23E-10	304	274.8
100 004	1.84E-07	2.46E-07	9.84E-09	1.37E-08	309	274.6
Riedkosť = 0.5%						
4	6.22E-15	6.82E-13	3.20E-15	3.02E-13	68	63.08
14	3.58E-14	4.04E-12	1.22E-14	1.24E-12	70	63.45
104	2.36E-13	2.74E-11	1.14E-13	1.18E-11	70	63.45
1 004	2.44E-12	2.84E-10	1.20E-12	1.23E-10	70	63.34
10 004	5.94E-11	7.12E-09	1.27E-11	1.30E-09	70	63.33
100 004	2.90E-10	3.43E-08	1.16E-10	1.19E-08	70	63.55
Riedkosť = 1%						
4	6.66E-15	1.31E-12	3.10E-15	5.34E-13	65	60.27
14	1.97E-14	3.92E-12	1.12E-14	2.08E-12	65	60.79
104	3.52E-13	7.46E-11	1.12E-13	2.07E-11	65	60.74
1 004	2.70E-12	5.71E-10	1.11E-12	2.05E-10	65	60.79
10 004	2.29E-11	4.41E-09	1.10E-11	2.03E-09	66	60.76
100 004	2.87E-10	5.48E-08	1.07E-10	1.96E-08	67	60.81
Riedkosť = 2%						
4	3.55E-15	1.25E-12	2.82E-16	9.66E-14	55	52.40
14	1.55E-14	5.57E-12	1.16E-14	3.96E-12	55	52.60
104	1.66E-13	5.96E-11	1.18E-13	4.02E-11	56	52.80
1 004	1.25E-12	4.41E-10	9.44E-13	3.28E-10	56	53.00
10 004	1.68E-11	5.74E-09	1.15E-11	3.95E-09	56	52.70
100 004	1.46E-10	5.39E-08	9.73E-11	3.28E-08	56	52.80

Tabuľka *CGNR 9:*

$\|x_1 - x^*\|_\infty$	$\|x_i - x^*\|_\infty$	$\|Ax_i - x^*\|_\infty$	$\|x_i - x^*\|_\infty$	$\|Ax_i - x^*\|_\infty$	*max*	*avg*
	***CGNR** metóda,* $n = 20\,000$					
	Pres_min		*Pres_avg*		*Iterácie*	
			Riedkosť = 0.025%			
4	1.52E-12	2.02E-12	1.50E-13	2.14E-13	307	287.1
14	1.20E-11	1.60E-11	1.18E-12	1.61E-12	312	291.6
104	9.85E-11	1.31E-10	8.93E-12	1.23E-11	316	290.9
1 004	9.29E-10	1.23E-09	8.82E-11	1.23E-10	312	291.8
10 004	7.47E-09	9.89E-09	6.91E-10	9.70E-10	315	291.7
100 004	1.30E-07	1.73E-07	1.16E-08	1.57E-08	322	292.2
			Riedkosť = 0.15%			
4	1.01E-14	7.23E-13	4.13E-15	2.71E-13	88	80.45
14	4.31E-14	3.23E-12	1.67E-14	1.17E-12	89	80.90
104	5.39E-13	3.86E-11	1.66E-13	1.10E-11	89	81.02
1 004	4.83E-12	3.36E-10	1.67E-12	1.12E-10	88	80.88
10 004	3.62E-11	2.52E-09	1.61E-11	1.10E-09	89	80.87
100 004	4.98E-10	3.79E-08	1.66E-10	1.14E-08	89	81.06
			Riedkosť = 0.25%			
4	6.77E-15	6.54E-13	3.49E-15	3.47E-13	70	64.93
14	2.93E-14	3.15E-12	1.42E-14	1.47E-12	72	65.17
104	3.15E-13	3.12E-11	1.39E-13	1.45E-11	70	65.20
1 004	3.59E-12	4.34E-10	1.39E-12	1.44E-10	70	65.29
10 004	3.39E-11	3.92E-09	1.40E-11	1.45E-09	71	65.28
100 004	3.22E-10	3.32E-08	1.36E-10	1.39E-08	71	65.30
			Riedkosť = 0.375%			
4	6.88E-15	1.14E-12	3.82E-16	5.57E-14	71	65.70
14	2.84E-14	4.66E-12	1.60E-14	2.40E-12	72	66.80
104	2.10E-13	2.95E-11	1.55E-13	2.28E-11	72	66.50
1 004	1.97E-12	2.84E-10	1.18E-12	1.69E-10	72	66.60
10 004	2.11E-11	3.13E-09	1.18E-11	1.72E-09	73	67.00
100 004	2.24E-10	3.51E-08	1.24E-10	1.77E-08	72	66.90

Tabuľka *CGNR 10:*

$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	Pres_min $\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	Pres_min $\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	Pres_avg $\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	Pres_avg $\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	Iterácie max	Iterácie avg
CGNR metóda, $n = 30\,000$						
Riedkosť = 0.015%						
4	1.41E-12	1.87E-12	1.81E-13	2.50E-13	319	296.3
14	1.29E-12	1.78E-12	2.95E-13	5.01E-13	320	301.3
104	4.97E-11	6.44E-11	5.05E-12	7.54E-12	318	298.9
1 004	6.25E-10	8.10E-10	6.47E-11	9.08E-11	319	299.9
10 004	4.83E-09	6.25E-09	5.10E-10	7.32E-10	320	299.9
100 004	3.21E-08	4.12E-08	3.44E-09	5.39E-09	322	300.0
Riedkosť = 0.05%						
4	1.00E-13	1.08E-12	8.47E-15	2.47E-13	122	106.6
14	4.35E-13	4.87E-12	4.10E-14	1.19E-12	119	107.1
104	8.86E-12	9.53E-11	6.46E-13	1.33E-11	121	107.0
1 004	3.07E-11	3.43E-10	3.37E-12	1.20E-10	119	106.9
10 004	3.05E-10	3.40E-09	3.27E-11	1.12E-09	119	107.2
100 004	3.08E-09	3.44E-08	3.10E-10	1.00E-08	121	107.1
Riedkosť = 0.1%						
4	1.60E-14	1.31E-12	4.47E-15	3.05E-13	89	81.75
14	4.57E-14	3.52E-12	1.86E-14	1.31E-12	89	82.32
104	3.94E-13	2.95E-11	1.87E-13	1.30E-11	91	82.13
1 004	3.95E-12	2.67E-10	1.72E-12	1.18E-10	89	82.34
10 004	4.82E-11	3.51E-09	1.82E-11	1.28E-09	88	82.14
100 004	5.23E-10	4.13E-08	1.89E-10	1.32E-08	90	82.13
Riedkosť = 0.15%						
4	4.44E-15	5.12E-13	3.52E-16	3.75E-14	69	65.40
14	2.26E-14	2.59E-12	1.53E-14	1.60E-12	70	65.90
104	1.88E-13	2.05E-11	1.41E-13	1.45E-11	70	65.60
1 004	2.04E-12	2.08E-10	1.50E-12	1.52E-10	70	65.50
10 004	2.04E-11	2.25E-09	1.48E-11	1.52E-09	70	66.00
100 004	2.18E-10	2.35E-08	1.42E-10	1.48E-08	70	65.40

Tabuľka *CGNR 11:*

CGNR *metóda,* $n = 50\,000$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
			Riedkosť = 0.01%			
4	1.17E-12	1.50E-12	1.56E-13	2.31E-13	333	307.6
14	6.46E-12	7.83E-12	6.04E-13	8.67E-13	339	311.5
104	7.84E-11	9.58E-11	6.60E-12	9.26E-12	338	312.0
1 004	6.51E-10	7.98E-10	5.82E-11	8.70E-11	335	311.2
10 004	5.76E-09	7.08E-09	5.22E-10	7.73E-10	332	311.1
100 004	8.47E-08	1.03E-07	7.40E-09	1.03E-08	342	310.9
			Riedkosť = 0.02%			
4	1.09E-11	3.73E-11	1.59E-13	5.21E-13	242	215.9
14	4.96E-11	1.63E-10	6.87E-12	2.21E-11	238	216.5
104	5.64E-10	1.86E-09	7.51E-11	2.42E-10	238	216.9
1 004	3.12E-09	1.03E-08	4.00E-10	1.33E-09	246	216.4
10 004	1.21E-07	3.97E-07	1.61E-08	5.16E-08	244	215.8
100 004	1.44E-06	4.76E-06	1.92E-07	6.17E-07	238	216.0
			Riedkosť = 0.04%			
4	3.25E-13	3.55E-12	3.14E-14	4.76E-13	122	110.8
14	7.67E-13	7.95E-12	8.19E-14	1.72E-12	122	111.4
104	1.18E-11	1.22E-10	1.16E-12	1.95E-11	126	111.5
1 004	1.04E-10	1.20E-09	1.15E-11	1.98E-10	121	111.6
10 004	5.95E-10	6.16E-09	6.35E-11	1.46E-09	126	111.5
100 004	8.52E-09	8.83E-08	8.46E-10	1.69E-08	123	111.6
			Riedkosť = 0.06%			
4	8.44E-15	6.54E-13	5.48E-16	3.87E-14	92	84.50
14	3.33E-14	2.68E-12	1.92E-14	1.40E-12	93	85.90
104	3.16E-13	2.55E-11	2.24E-13	1.62E-11	93	85.10
1 004	4.81E-12	3.56E-10	2.64E-12	1.93E-10	91	85.10
10 004	4.09E-11	3.14E-09	2.17E-11	1.59E-09	92	85.60
100 004	3.81E-10	3.07E-08	2.03E-10	1.46E-08	92	85.60

Tabuľka *CGNR 12:*

GMRes *metóda*, $n = 100$						
	Pres_ min		*Pres_ avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
Riedkosť = 5%						
4	4.77E-15	5.33E-14	2.60E-15	2.34E-14	68	62.62
14	1.73E-14	1.76E-13	9.68E-15	8.78E-14	69	62.68
104	2.20E-13	2.86E-12	9.84E-14	8.76E-13	67	62.51
1 004	2.59E-12	2.55E-11	9.36E-13	8.72E-12	66	62.60
10 004	1.64E-11	2.45E-10	9.14E-12	8.37E-11	69	62.59
100 004	1.62E-10	1.95E-09	9.67E-11	9.10E-10	67	62.42
Riedkosť = 10%						
4	3.77E-15	7.82E-14	2.40E-15	4.39E-14	54	50.50
14	1.67E-14	3.69E-13	8.84E-15	1.65E-13	56	50.75
104	1.67E-13	3.59E-12	8.63E-14	1.62E-12	56	50.59
1 004	1.29E-12	2.71E-11	8.08E-13	1.51E-11	55	50.76
10 004	1.73E-11	4.93E-10	8.28E-12	1.54E-10	56	50.65
100 004	1.95E-10	3.89E-09	8.47E-11	1.61E-09	54	50.57
Riedkosť = 15%						
4	5.33E-15	1.99E-13	2.32E-15	6.56E-14	50	46.00
14	1.80E-14	6.11E-13	8.93E-15	2.33E-13	50	46.19
104	1.44E-13	4.38E-12	7.89E-14	2.12E-12	50	46.30
1 004	1.83E-12	4.92E-11	8.07E-13	2.23E-11	50	46.25
10 004	1.48E-11	5.12E-10	7.98E-12	2.11E-10	50	46.22
100 004	1.40E-10	4.30E-09	8.36E-11	2.27E-09	50	46.12
Riedkosť = 20%						
4	7.99E-15	4.12E-13	2.47E-15	8.90E-14	46	42.44
14	1.98E-14	9.81E-13	9.64E-15	3.43E-13	45	42.55
104	2.09E-13	9.58E-12	8.88E-14	3.10E-12	46	42.57
1 004	1.88E-12	7.75E-11	8.52E-13	2.97E-11	46	42.63
10 004	1.76E-11	6.99E-10	8.44E-12	2.93E-10	46	42.61
100 004	2.14E-10	1.06E-08	8.58E-11	3.06E-09	46	42.62
Riedkosť = 25%						
4	4.88E-15	2.13E-13	2.54E-15	1.19E-13	50	46.17
14	2.73E-14	1.24E-12	1.03E-14	4.79E-13	51	46.41
104	1.99E-13	1.03E-11	8.89E-14	4.24E-12	50	46.50
1 004	2.05E-12	1.09E-10	9.23E-13	4.30E-11	50	46.41
10 004	1.66E-11	9.45E-10	9.32E-12	4.37E-10	50	46.36
100 004	1.78E-10	9.06E-09	9.03E-11	4.23E-09	50	46.41

Tabuľka *GMRes 1:*

GMRes *metóda*, $n = 200$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\Vert \boldsymbol{x}_1 - \boldsymbol{x}^* \Vert_\infty$	$\Vert \boldsymbol{x}_i - \boldsymbol{x}^* \Vert_\infty$	$\Vert \boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^* \Vert_\infty$	$\Vert \boldsymbol{x}_i - \boldsymbol{x}^* \Vert_\infty$	$\Vert \boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^* \Vert_\infty$	*max*	*avg*
			Riedkosť = 5%			
4	*5.11E-15*	*1.28E-13*	*2.66E-15*	*5.07E-14*	*58*	*53.41*
14	*2.22E-14*	*5.19E-13*	*1.00E-14*	*1.98E-13*	*58*	*53.71*
104	*1.86E-13*	*5.25E-12*	*9.45E-14*	*1.85E-12*	*58*	*53.65*
1 004	*2.14E-12*	*5.69E-11*	*9.75E-13*	*1.93E-11*	*58*	*53.59*
10 004	*2.40E-11*	*6.36E-10*	*9.61E-12*	*1.95E-10*	*59*	*53.61*
100 004	*1.53E-10*	*3.60E-09*	*9.38E-11*	*1.85E-09*	*58*	*53.49*
			Riedkosť = 10%			
4	*6.66E-15*	*2.84E-13*	*2.78E-15*	*9.97E-14*	*48*	*44.07*
14	*1.81E-14*	*8.74E-13*	*1.02E-14*	*3.73E-13*	*48*	*44.33*
104	*1.68E-13*	*7.61E-12*	*9.69E-14*	*3.58E-12*	*47*	*44.18*
1 004	*1.83E-12*	*7.74E-11*	*9.54E-13*	*3.53E-11*	*47*	*44.29*
10 004	*1.63E-11*	*7.03E-10*	*9.78E-12*	*3.54E-10*	*47*	*44.26*
100 004	*1.77E-10*	*6.77E-09*	*9.92E-11*	*3.65E-09*	*47*	*44.27*
			Riedkosť = 15%			
4	*4.88E-15*	*2.56E-13*	*2.84E-15*	*1.49E-13*	*45*	*40.94*
14	*2.22E-14*	*1.15E-12*	*1.10E-14*	*5.78E-13*	*44*	*41.17*
104	*1.83E-13*	*1.12E-11*	*1.03E-13*	*5.35E-12*	*44*	*41.20*
1 004	*3.23E-12*	*1.84E-10*	*1.03E-12*	*5.35E-11*	*44*	*41.21*
10 004	*2.14E-11*	*1.45E-09*	*1.01E-11*	*5.31E-10*	*44*	*41.26*
100 004	*1.59E-10*	*1.08E-08*	*1.04E-10*	*5.37E-09*	*45*	*41.18*
			Riedkosť = 20%			
4	*5.77E-15*	*3.98E-13*	*3.12E-15*	*2.15E-13*	*43*	*39.05*
14	*2.02E-14*	*1.62E-12*	*1.24E-14*	*8.76E-13*	*42*	*39.18*
104	*2.20E-13*	*1.61E-11*	*1.11E-13*	*7.66E-12*	*42*	*39.26*
1 004	*1.78E-12*	*1.56E-10*	*1.13E-12*	*7.94E-11*	*42*	*39.13*
10 004	*1.98E-11*	*1.67E-09*	*1.11E-11*	*7.65E-10*	*43*	*39.28*
100 004	*2.32E-10*	*1.92E-08*	*1.18E-10*	*8.46E-09*	*42*	*39.13*
			Riedkosť = 25%			
4	*4.88E-15*	*5.68E-13*	*3.18E-15*	*2.69E-13*	*41*	*38.00*
14	*2.26E-14*	*2.15E-12*	*1.30E-14*	*1.16E-12*	*42*	*38.12*
104	*2.06E-13*	*1.74E-11*	*1.17E-13*	*1.06E-11*	*42*	*38.22*
1 004	*1.88E-12*	*1.77E-10*	*1.19E-12*	*1.05E-10*	*42*	*38.26*
10 004	*2.16E-11*	*2.21E-09*	*1.26E-11*	*1.13E-09*	*42*	*38.17*
100 004	*1.97E-10*	*1.62E-08*	*1.19E-10*	*1.05E-08*	*42*	*38.23*

Tabuľka *GMRes 2:*

GMRes *metóda,* $n = 300$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
Riedkosť = 6.7%						
4	6.22E-15	2.98E-13	3.02E-15	1.08E-13	48	44.78
14	1.82E-14	8.38E-13	1.10E-14	4.07E-13	48	44.84
104	2.05E-13	8.85E-12	1.05E-13	3.93E-12	48	44.96
1 004	1.91E-12	8.58E-11	1.05E-12	4.01E-11	48	44.89
10 004	1.67E-11	6.72E-10	1.03E-11	3.73E-10	48	44.92
100 004	2.01E-10	8.73E-09	1.08E-10	4.01E-09	48	44.83
Riedkosť = 10%						
4	6.66E-15	3.98E-13	3.14E-15	1.65E-13	45	41.33
14	1.81E-14	9.99E-13	1.16E-14	6.02E-13	45	41.46
104	2.06E-13	1.11E-11	1.09E-13	5.88E-12	45	41.44
1 004	1.97E-12	1.09E-10	1.10E-12	5.97E-11	45	41.46
10 004	1.74E-11	1.09E-09	1.11E-11	5.96E-10	45	41.45
100 004	1.78E-10	1.27E-08	1.07E-10	5.74E-09	45	41.54
Riedkosť = 16.7%						
4	7.11E-15	6.25E-13	3.71E-15	3.12E-13	42	38.37
14	2.13E-14	1.91E-12	1.28E-14	1.13E-12	42	38.53
104	2.26E-13	2.12E-11	1.25E-13	1.10E-11	42	38.60
1 004	1.95E-12	1.80E-10	1.22E-12	1.04E-10	42	38.60
10 004	1.82E-11	1.91E-09	1.21E-11	1.05E-09	42	38.60
100 004	2.09E-10	1.99E-08	1.22E-10	1.08E-08	41	38.52
Riedkosť = 20%						
4	5.33E-15	5.68E-13	3.56E-15	3.57E-13	41	37.76
14	2.35E-14	2.39E-12	1.41E-14	1.48E-12	41	37.85
104	2.16E-13	2.40E-11	1.37E-13	1.43E-11	41	37.89
1 004	2.02E-12	2.60E-10	1.33E-12	1.40E-10	41	37.85
10 004	2.18E-11	2.73E-09	1.32E-11	1.40E-09	41	37.94
100 004	2.01E-10	2.25E-08	1.31E-10	1.38E-08	41	37.84
Riedkosť = 25%						
4	7.55E-15	1.19E-12	3.98E-15	5.14E-13	42	39.29
14	2.29E-14	3.52E-12	1.46E-14	1.86E-12	42	39.42
104	2.17E-13	2.73E-11	1.45E-13	1.84E-11	42	39.39
1 004	2.44E-12	2.88E-10	1.45E-12	1.83E-10	42	39.45
10 004	2.06E-11	3.08E-09	1.41E-11	1.79E-09	42	39.41
100 004	2.49E-10	2.74E-08	1.41E-10	1.78E-08	43	39.43

Tabuľka *GMRes 3:*

GMRes *metóda,* $n = 500$						
	Pres_ min		*Pres_ avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
			Riedkosť = 5%			
4	5.33E-15	2.98E-13	3.08E-15	1.37E-13	52	48.04
14	2.31E-14	1.17E-12	1.24E-14	5.61E-13	52	48.31
104	2.06E-13	1.09E-11	1.12E-13	5.11E-12	53	48.44
1 004	2.04E-12	1.07E-10	1.14E-12	5.23E-11	52	48.36
10 004	1.99E-11	9.63E-10	1.10E-11	4.98E-10	53	48.45
100 004	1.84E-10	1.01E-08	1.10E-10	4.96E-09	53	48.48
			Riedkosť = 10%			
4	6.66E-15	7.39E-13	3.80E-15	3.25E-13	42	38.96
14	2.43E-14	2.32E-12	1.39E-14	1.19E-12	42	39.19
104	2.51E-13	2.68E-11	1.35E-13	1.14E-11	43	39.30
1 004	2.12E-12	2.32E-10	1.35E-12	1.12E-10	43	39.33
10 004	2.09E-11	1.82E-09	1.36E-11	1.16E-09	43	39.26
100 004	2.42E-10	1.91E-08	1.32E-10	1.12E-08	43	39.31
			Riedkosť = 15%			
4	6.66E-15	8.53E-13	3.88E-15	4.92E-13	43	40.36
14	2.76E-14	3.81E-12	1.67E-14	2.17E-12	44	40.51
104	2.56E-13	3.16E-11	1.49E-13	1.93E-11	44	40.63
1 004	2.57E-12	3.40E-10	1.49E-12	1.92E-10	44	40.64
10 004	2.35E-11	3.16E-09	1.48E-11	1.91E-09	44	40.64
100 004	2.29E-10	3.28E-08	1.51E-10	1.99E-08	44	40.61
			Riedkosť = 20%			
4	8.88E-15	1.59E-12	4.93E-15	8.43E-13	39	36.82
14	2.69E-14	4.32E-12	1.79E-14	3.01E-12	40	37.01
104	2.57E-13	4.44E-11	1.73E-13	2.93E-11	40	37.05
1 004	2.64E-12	4.95E-10	1.77E-12	2.97E-10	40	37.08
10 004	2.91E-11	5.40E-09	1.82E-11	3.07E-09	40	37.06
100 004	2.76E-10	4.15E-08	1.76E-10	2.97E-08	39	37.01
			Riedkosť = 25%			
4	9.10E-15	2.05E-12	5.36E-15	1.20E-12	48	45.25
14	3.24E-14	8.00E-12	2.03E-14	4.67E-12	49	45.52
104	3.19E-13	7.66E-11	1.84E-13	4.14E-11	48	45.71
1 004	3.18E-12	7.84E-10	1.81E-12	4.09E-10	49	45.59
10 004	3.13E-11	8.28E-09	1.88E-11	4.28E-09	48	45.61
100 004	2.95E-10	7.29E-08	1.88E-10	4.24E-08	48	45.62

Tabuľka *GMRes 4:*

GMRes *metóda*, $n = 1\,000$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
			Riedkosť = 5%			
4	8.44E-15	8.24E-13	4.16E-15	3.48E-13	42	39.43
14	2.40E-14	2.06E-12	1.55E-14	1.36E-12	43	39.70
104	2.10E-13	1.93E-11	1.47E-13	1.28E-11	43	39.74
1 004	2.52E-12	2.36E-10	1.49E-12	1.34E-10	43	39.73
10 004	2.17E-11	2.06E-09	1.41E-11	1.24E-09	43	39.76
100 004	2.45E-10	2.27E-08	1.49E-10	1.31E-08	43	39.77
			Riedkosť = 10%			
4	7.99E-15	1.48E-12	5.17E-15	8.33E-13	40	37.18
14	2.83E-14	4.90E-12	1.95E-14	3.28E-12	40	37.41
104	3.22E-13	5.79E-11	1.74E-13	2.92E-11	40	37.48
1 004	3.69E-12	6.29E-10	1.78E-12	3.01E-10	40	37.53
10 004	2.68E-11	5.24E-09	1.78E-11	2.99E-09	40	37.49
100 004	3.54E-10	5.90E-08	1.75E-10	2.91E-08	40	37.50
			Riedkosť = 15%			
4	1.02E-14	2.61E-12	6.34E-15	1.59E-12	40	36.48
14	3.29E-14	8.98E-12	2.38E-14	5.83E-12	40	36.73
104	2.96E-13	6.99E-11	2.18E-13	5.37E-11	40	36.77
1 004	3.16E-12	8.36E-10	2.25E-12	5.53E-10	39	36.72
10 004	2.96E-11	7.65E-09	2.26E-11	5.59E-09	40	36.71
100 004	2.96E-10	7.51E-08	2.19E-10	5.39E-08	39	36.74
			Riedkosť = 20%			
4	1.27E-14	3.87E-12	6.95E-15	2.29E-12	39	36.63
14	4.31E-14	1.43E-11	2.77E-14	9.47E-12	39	36.76
104	3.37E-13	1.23E-10	2.33E-13	7.96E-11	39	36.84
1 004	3.47E-12	1.29E-09	2.41E-12	8.28E-10	39	36.77
10 004	3.94E-11	1.50E-08	2.36E-11	8.01E-09	39	36.86
100 004	3.63E-10	1.40E-07	2.31E-10	7.83E-08	39	36.82
			Riedkosť = 25%			
4	1.51E-14	6.82E-12	8.24E-15	3.65E-12	37	34.78
14	4.69E-14	2.11E-11	3.11E-14	1.35E-11	37	34.91
104	4.37E-13	1.90E-10	2.92E-13	1.30E-10	37	34.94
1 004	4.78E-12	1.99E-09	2.85E-12	1.26E-09	37	34.90
10 004	4.87E-11	1.93E-08	2.98E-11	1.32E-08	37	34.87
100 004	4.70E-10	2.08E-07	2.99E-10	1.32E-07	37	34.90

Tabuľka *GMRes 5:*

***GMRes** metóda,* $n = 2\,000$						
	Pres_ min		*Pres_ avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
Riedkosť = 5%						
4	9.99E-15	1.65E-12	6.08E-15	1.01E-12	41	37.58
14	2.88E-14	4.95E-12	2.16E-14	3.47E-12	41	37.90
104	2.97E-13	5.05E-11	1.99E-13	3.25E-11	41	37.92
1 004	2.97E-12	4.86E-10	2.01E-12	3.25E-10	41	37.92
10 004	3.19E-11	5.27E-09	2.03E-11	3.29E-09	41	37.92
100 004	2.82E-10	4.48E-08	2.01E-10	3.24E-08	41	37.90
Riedkosť = 10%						
4	1.33E-14	4.66E-12	7.83E-15	2.59E-12	38	36.07
14	4.10E-14	1.28E-11	2.89E-14	9.29E-12	39	36.39
104	3.97E-13	1.33E-10	2.80E-13	9.17E-11	39	36.36
1 004	4.02E-12	1.29E-09	2.71E-12	8.85E-10	39	36.36
10 004	4.23E-11	1.33E-08	2.72E-11	8.87E-09	39	36.36
100 004	4.13E-10	1.24E-07	2.76E-10	8.97E-08	39	36.36
Riedkosť = 15%						
4	1.38E-14	6.59E-12	9.17E-15	4.40E-12	38	35.90
14	5.95E-14	3.05E-11	3.58E-14	1.81E-11	38	36.10
104	3.82E-13	2.08E-10	2.75E-13	1.33E-10	38	36.26
1 004	3.92E-12	2.14E-09	2.77E-12	1.35E-09	38	36.27
10 004	4.12E-11	2.02E-08	2.77E-11	1.33E-08	39	36.26
100 004	3.66E-10	1.72E-07	2.75E-10	1.32E-07	38	36.22
Riedkosť = 20%						
4	1.87E-14	1.23E-11	1.16E-14	7.87E-12	38	36.34
14	5.51E-14	3.66E-11	4.04E-14	2.68E-11	38	36.49
104	5.79E-13	3.77E-10	3.93E-13	2.65E-10	38	36.50
1 004	5.38E-12	3.66E-09	3.91E-12	2.63E-09	39	36.57
10 004	5.38E-11	3.53E-08	3.96E-11	2.66E-08	38	36.50
100 004	5.91E-10	3.88E-07	3.95E-10	2.64E-07	38	36.56
Riedkosť = 25%						
4	1.69E-14	1.50E-11	1.24E-14	1.04E-11	36	34.52
14	6.82E-14	6.14E-11	4.47E-14	3.92E-11	37	34.77
104	4.96E-13	4.91E-10	3.89E-13	3.23E-10	37	34.83
1 004	4.92E-12	4.15E-09	3.86E-12	3.19E-09	37	34.87
10 004	5.27E-11	4.21E-08	3.85E-11	3.14E-08	37	34.84
100 004	6.26E-10	4.69E-07	3.89E-10	3.21E-07	37	34.83

Tabuľka *GMRes 6:*

GMRes metóda, $n = 3\,000$						
	Pres_min		Pres_avg		Iterácie	
$\lVert \boldsymbol{x}_1 - \boldsymbol{x}^* \rVert_\infty$	$\lVert \boldsymbol{x}_i - \boldsymbol{x}^* \rVert_\infty$	$\lVert \boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^* \rVert_\infty$	$\lVert \boldsymbol{x}_i - \boldsymbol{x}^* \rVert_\infty$	$\lVert \boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^* \rVert_\infty$	max	avg
Riedkosť = 0.15%						
4	1.75E-14	2.77E-13	4.93E-15	6.08E-14	82	76.91
14	3.97E-14	5.58E-13	1.98E-14	2.47E-13	82	77.34
104	4.39E-13	6.86E-12	1.99E-13	2.68E-12	82	77.09
1 004	5.01E-12	6.71E-11	2.06E-12	2.65E-11	84	77.18
10 004	5.03E-11	7.67E-10	2.01E-11	2.65E-10	82	77.22
100 004	6.18E-10	1.12E-08	1.98E-10	2.52E-09	83	77.39
Riedkosť = 5%						
4	1.04E-14	3.41E-12	7.42E-15	2.34E-12	39	36.56
14	4.88E-14	1.56E-11	3.08E-14	1.00E-11	39	36.77
104	3.60E-13	1.32E-10	2.53E-13	8.31E-11	39	36.91
1 004	4.17E-12	1.33E-09	2.46E-12	8.15E-10	39	36.91
10 004	3.78E-11	1.29E-08	2.41E-11	7.90E-09	39	36.95
100 004	3.98E-10	1.32E-07	2.41E-10	7.96E-08	39	36.90
Riedkosť = 10%						
4	1.55E-14	7.73E-12	1.05E-14	5.05E-12	38	35.75
14	4.93E-14	2.55E-11	3.42E-14	1.63E-11	38	36.07
104	5.07E-13	2.19E-10	3.31E-13	1.56E-10	38	36.06
1 004	4.35E-12	2.13E-09	3.27E-12	1.54E-09	38	36.07
10 004	4.22E-11	1.97E-08	3.28E-11	1.54E-08	38	36.07
100 004	4.92E-10	2.30E-07	3.34E-10	1.56E-07	38	36.07
Riedkosť = 15%						
4	2.18E-14	1.82E-11	1.28E-14	1.06E-11	36	34.61
14	6.17E-14	4.89E-11	4.66E-14	3.75E-11	37	34.85
104	5.61E-13	4.56E-10	4.38E-13	3.62E-10	37	34.89
1 004	6.03E-12	4.88E-09	4.49E-12	3.69E-09	37	34.88
10 004	6.14E-11	4.98E-08	4.43E-11	3.67E-08	37	34.86
100 004	6.19E-10	5.27E-07	4.45E-10	3.68E-07	37	34.83

Tabuľka *GMRes 7:*

GMRes *metóda*, $n = 5\,000$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
Riedkosť = 0.1%						
4	1.64E-14	2.84E-13	5.21E-15	7.02E-14	85	78.98
14	5.03E-14	7.25E-13	2.33E-14	3.07E-13	84	79.15
104	3.46E-13	6.75E-12	2.13E-13	2.80E-12	84	79.30
1 004	4.23E-12	7.17E-11	2.17E-12	2.94E-11	86	79.19
10 004	6.71E-11	1.01E-09	2.22E-11	2.94E-10	85	79.13
100 004	7.79E-10	1.18E-08	2.30E-10	3.05E-09	85	79.32
Riedkosť = 3%						
4	1.02E-14	2.50E-12	6.84E-15	1.67E-12	39	37.13
14	4.10E-14	1.08E-11	2.84E-14	6.78E-12	40	37.34
104	3.66E-13	8.73E-11	2.59E-13	6.31E-11	40	37.30
1 004	4.01E-12	9.46E-10	2.60E-12	6.32E-10	40	37.37
10 004	3.39E-11	8.09E-09	2.59E-11	6.32E-09	40	37.34
100 004	3.32E-10	8.04E-08	2.54E-10	6.23E-08	40	37.37
Riedkosť = 5%						
4	1.80E-14	8.87E-12	9.24E-15	4.44E-12	39	36.23
14	5.65E-14	2.55E-11	3.96E-14	1.86E-11	39	36.34
104	5.38E-13	2.62E-10	3.48E-13	1.69E-10	39	36.40
1 004	5.33E-12	2.65E-09	3.45E-12	1.69E-09	39	36.39
10 004	5.03E-11	2.53E-08	3.35E-11	1.63E-08	39	36.43
100 004	5.09E-10	2.51E-07	3.31E-10	1.62E-07	39	36.40
Riedkosť = 10%						
4	1.73E-14	1.39E-11	1.23E-14	9.60E-12	37	35.24
14	7.68E-14	6.05E-11	4.88E-14	3.95E-11	37	35.49
104	6.90E-13	5.75E-10	3.62E-13	2.79E-10	38	35.75
1 004	4.72E-12	3.95E-09	3.63E-12	2.82E-09	38	35.75
10 004	4.77E-11	3.62E-08	3.62E-11	2.78E-08	38	35.79
100 004	4.76E-10	3.77E-07	3.63E-10	2.77E-07	38	35.77

Tabuľka *GMRes 8:*

GMRes *metóda*, $n = 10\,000$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
Riedkosť = 0.05%						
4	1.38E-14	2.34E-13	6.24E-15	8.83E-14	86	81.34
14	6.68E-14	1.01E-12	2.71E-14	3.84E-13	88	81.66
104	5.57E-13	1.15E-11	2.71E-13	3.84E-12	88	81.51
1 004	8.33E-12	1.60E-10	2.68E-12	3.66E-11	87	81.61
10 004	5.72E-11	1.02E-09	2.71E-11	3.81E-10	87	81.60
100 004	6.74E-10	1.26E-08	2.65E-10	3.67E-09	87	81.48
Riedkosť = 0.5%						
4	7.77E-15	7.39E-13	5.35E-15	4.77E-13	44	40.95
14	2.78E-14	3.15E-12	2.02E-14	1.74E-12	44	41.25
104	2.58E-13	2.34E-11	1.89E-13	1.59E-11	44	41.26
1 004	3.33E-12	3.63E-10	1.88E-12	1.60E-10	44	41.28
10 004	2.63E-11	2.60E-09	1.84E-11	1.57E-09	44	41.26
100 004	3.05E-10	3.26E-08	1.84E-10	1.59E-08	44	41.31
Riedkosť = 1%						
4	9.33E-15	1.65E-12	6.54E-15	1.09E-12	43	39.95
14	3.33E-14	5.40E-12	2.54E-14	4.07E-12	44	40.23
104	3.24E-13	5.48E-11	2.36E-13	3.88E-11	44	40.27
1 004	3.42E-12	5.50E-10	2.38E-12	3.91E-10	44	40.29
10 004	3.35E-11	5.53E-09	2.37E-11	3.92E-09	44	40.24
100 004	3.12E-10	5.47E-08	2.37E-10	3.91E-08	44	40.26
Riedkosť = 2%						
4	1.33E-14	4.66E-12	8.30E-15	2.61E-12	39	36.68
14	4.82E-14	1.44E-11	3.51E-14	1.11E-11	39	36.82
104	3.68E-13	1.24E-10	2.87E-13	9.37E-11	39	36.88
1 004	3.86E-12	1.23E-09	2.93E-12	9.57E-10	39	36.92
10 004	5.15E-11	1.68E-08	2.93E-11	9.58E-09	39	36.92
100 004	4.50E-10	1.39E-07	2.93E-10	9.56E-08	39	36.89

Tabuľka *GMRes 9:*

GMRes metóda, $n = 20\,000$						
	Pres_min		Pres_avg		Iterácie	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	max	avg
Riedkosť = 0.025%						
4	1.29E-14	2.06E-13	7.15E-15	1.03E-13	90	83.58
14	9.41E-14	1.43E-12	3.62E-14	5.36E-13	89	83.70
104	8.18E-13	1.31E-11	3.18E-13	4.69E-12	90	83.96
1 004	7.31E-12	1.31E-10	3.14E-12	4.62E-11	89	83.91
10 004	9.63E-11	1.66E-09	3.12E-11	4.61E-10	90	83.99
100 004	6.77E-10	1.14E-08	3.20E-10	4.72E-09	90	83.89
Riedkosť = 0.15%						
4	7.55E-15	4.55E-13	5.14E-15	3.00E-13	49	46.53
14	2.96E-14	1.87E-12	1.93E-14	1.12E-12	50	46.89
104	2.51E-13	1.71E-11	1.78E-13	1.01E-11	50	46.93
1 004	2.58E-12	1.54E-10	1.76E-12	9.87E-11	50	47.02
10 004	2.82E-11	2.13E-09	1.81E-11	1.05E-09	50	46.95
100 004	2.60E-10	1.74E-08	1.79E-10	1.03E-08	50	46.96
Riedkosť = 0.25%						
4	7.99E-15	9.09E-13	5.60E-15	4.88E-13	45	41.68
14	3.01E-14	2.93E-12	2.19E-14	1.95E-12	44	41.83
104	2.97E-13	3.01E-11	1.98E-13	1.82E-11	44	41.90
1 004	4.02E-12	3.62E-10	2.01E-12	1.85E-10	44	41.88
10 004	2.92E-11	2.78E-09	1.95E-11	1.80E-09	45	41.90
100 004	3.29E-10	3.07E-08	2.03E-10	1.85E-08	45	41.90
Riedkosť = 0.375%						
4	9.33E-15	1.14E-12	6.57E-15	8.16E-13	45	41.80
14	2.71E-14	3.89E-12	2.40E-14	2.99E-12	45	42.10
104	2.48E-13	3.07E-11	2.19E-13	2.72E-11	45	42.00
1 004	3.03E-12	3.82E-10	2.38E-12	3.01E-10	45	42.00
10 004	2.29E-11	3.13E-09	2.16E-11	2.71E-09	45	42.00
100 004	2.59E-10	3.27E-08	2.29E-10	2.79E-08	45	42.10

Tabuľka *GMRes 10:*

GMRes metóda, $n = 30\,000$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
			Riedkosť = 0.015%			
4	1.55E-14	2.49E-13	7.52E-15	1.10E-13	90	84.57
14	1.05E-13	1.80E-12	3.66E-14	5.56E-13	91	84.73
104	8.29E-13	1.40E-11	3.41E-13	5.21E-12	91	84.85
1 004	9.13E-12	1.39E-10	3.63E-12	5.43E-11	90	84.73
10 004	8.12E-11	1.31E-09	3.47E-11	5.18E-10	90	84.70
100 004	8.42E-10	1.40E-08	3.58E-10	5.31E-09	91	84.75
			Riedkosť = 0.05%			
4	8.44E-15	4.55E-13	5.17E-15	2.15E-13	57	53.49
14	3.29E-14	1.73E-12	2.04E-14	8.96E-13	58	53.85
104	3.88E-13	1.96E-11	1.98E-13	8.66E-12	58	53.69
1 004	3.21E-12	1.39E-10	1.95E-12	8.56E-11	58	53.79
10 004	3.83E-11	2.22E-09	1.97E-11	8.80E-10	58	53.79
100 004	3.24E-10	1.54E-08	1.95E-10	8.48E-09	58	53.80
			Riedkosť = 0.1%			
4	9.55E-15	5.97E-13	5.34E-15	3.14E-13	50	46.91
14	2.98E-14	2.18E-12	2.08E-14	1.25E-12	50	47.22
104	2.86E-13	1.80E-11	1.98E-13	1.13E-11	51	47.29
1 004	2.67E-12	1.83E-10	1.89E-12	1.10E-10	50	47.30
10 004	2.38E-11	1.60E-09	1.90E-11	1.09E-09	50	47.34
100 004	2.97E-10	2.45E-08	1.92E-10	1.10E-08	50	47.25
			Riedkosť = 0.15%			
4	7.99E-15	7.39E-13	6.15E-15	5.43E-13	44	42.20
14	3.11E-14	2.79E-12	2.31E-14	2.09E-12	45	42.20
104	2.40E-13	2.55E-11	1.95E-13	1.77E-11	45	42.30
1 004	2.22E-12	2.01E-10	1.92E-12	1.71E-10	45	42.50
10 004	2.39E-11	2.52E-09	1.98E-11	1.84E-09	45	42.30
100 004	2.42E-10	2.87E-08	1.94E-10	1.75E-08	45	42.30

Tabuľka *GMRes 11:*

GMRes *metóda*, $n = 50\,000$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
			Riedkosť = 0.01%			
4	2.07E-14	3.27E-13	8.74E-15	1.30E-13	92	86.09
14	8.35E-14	1.71E-12	4.27E-14	6.74E-13	92	86.27
104	9.27E-13	1.68E-11	4.18E-13	6.52E-12	92	86.30
1 004	1.43E-11	2.40E-10	4.27E-12	6.62E-11	93	86.19
10 004	8.41E-11	1.49E-09	4.14E-11	6.44E-10	93	86.33
100 004	8.67E-10	1.46E-08	4.07E-10	6.18E-09	92	86.32
			Riedkosť = 0.02%			
4	1.82E-14	3.98E-13	6.62E-15	1.69E-13	83	75.53
14	9.75E-14	2.20E-12	2.83E-14	7.00E-13	83	76.04
104	9.78E-13	1.79E-11	2.69E-13	6.70E-12	83	76.03
1 004	1.03E-11	1.28E-10	2.67E-12	6.68E-11	84	76.16
10 004	6.26E-11	1.63E-09	2.63E-11	7.18E-10	85	76.05
100 004	9.93E-10	2.24E-08	2.88E-10	7.32E-09	84	75.96
			Riedkosť = 0.04%			
4	7.99E-15	4.41E-13	5.70E-15	2.47E-13	59	54.52
14	3.69E-14	1.79E-12	2.13E-14	9.49E-13	59	54.91
104	3.02E-13	1.54E-11	1.99E-13	8.78E-12	59	55.01
1 004	3.05E-12	1.65E-10	2.01E-12	8.99E-11	59	54.95
10 004	3.13E-11	1.75E-09	1.98E-11	9.01E-10	59	54.95
100 004	2.65E-10	1.43E-08	1.97E-10	8.86E-09	59	55.00
			Riedkosť = 0.06%			
4	6.88E-15	3.98E-13	5.62E-15	3.06E-13	50	47.80
14	2.53E-14	1.59E-12	2.19E-14	1.30E-12	51	48.00
104	2.30E-13	1.55E-11	1.92E-13	1.14E-11	51	48.30
1 004	2.36E-12	1.43E-10	2.00E-12	1.16E-10	51	48.20
10 004	2.41E-11	1.50E-09	1.96E-11	1.22E-09	51	48.30
100 004	2.67E-10	1.66E-08	2.05E-10	1.27E-08	51	48.20

Tabuľka *GMRes 12:*

LSQR *metóda,* $n = 100$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
			Riedkosť = 5%			
4	5.33E-10	1.45E-09	4.40E-11	1.18E-10	100	100.0
14	2.42E-09	5.29E-09	2.32E-10	6.22E-10	100	100.0
104	2.51E-08	5.45E-08	2.15E-09	5.77E-09	100	100.0
1 004	2.58E-07	6.29E-07	2.22E-08	5.95E-08	100	100.0
10 004	2.19E-06	4.91E-06	2.21E-07	5.90E-07	100	100.0
100 004	2.49E-05	5.46E-05	2.18E-06	5.82E-06	100	100.0
			Riedkosť = 10%			
4	7.13E-14	7.55E-13	1.20E-14	1.67E-13	99	88.78
14	7.97E-14	1.13E-12	4.38E-14	5.98E-13	100	89.93
104	4.70E-12	4.11E-11	4.75E-13	6.05E-12	100	89.67
1 004	2.29E-11	1.93E-10	4.62E-12	6.33E-11	100	89.70
10 004	3.75E-10	3.44E-09	4.97E-11	6.60E-10	100	89.56
100 004	1.10E-09	1.30E-08	4.17E-10	5.65E-09	100	89.68
			Riedkosť = 15%			
4	1.72E-14	4.67E-13	1.01E-14	2.43E-13	83	74.99
14	1.14E-13	3.25E-12	4.31E-14	1.00E-12	91	75.29
104	9.00E-13	2.37E-11	4.22E-13	1.00E-11	97	75.58
1 004	9.26E-12	2.48E-10	4.02E-12	9.26E-11	92	75.60
10 004	1.02E-10	2.67E-09	3.91E-11	8.83E-10	91	75.73
100 004	9.64E-10	2.66E-08	4.03E-10	9.48E-09	88	75.45
			Riedkosť = 20%			
4	1.90E-14	6.73E-13	1.01E-14	3.23E-13	76	64.90
14	9.36E-14	3.69E-12	4.23E-14	1.35E-12	75	65.26
104	7.80E-13	2.87E-11	4.00E-13	1.29E-11	79	65.20
1 004	8.16E-12	2.84E-10	4.07E-12	1.32E-10	79	65.17
10 004	1.11E-10	4.09E-09	4.23E-11	1.39E-09	78	65.03
100 004	8.64E-10	3.34E-08	4.09E-10	1.35E-08	100	66.03
			Riedkosť = 25%			
4	1.82E-14	8.60E-13	1.08E-14	4.21E-13	83	74.27
14	9.56E-14	3.95E-12	4.52E-14	1.79E-12	86	74.73
104	8.36E-13	4.08E-11	4.19E-13	1.70E-11	91	74.83
1 004	9.01E-12	3.95E-10	4.24E-12	1.68E-10	91	74.68
10 004	8.87E-11	4.16E-09	4.20E-11	1.71E-09	87	74.79
100 004	9.56E-10	4.17E-08	4.16E-10	1.62E-08	85	74.74

Tabuľka *LSQR 1:*

LSQR metóda, $n = 200$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\|x_1 - x^*\|_\infty$	$\|x_i - x^*\|_\infty$	$\|Ax_i - x^*\|_\infty$	$\|x_i - x^*\|_\infty$	$\|Ax_i - x^*\|_\infty$	*max*	*avg*
			Riedkosť = 5%			
4	4.05E-13	3.09E-12	3.26E-14	3.77E-13	113	102.7
14	1.25E-12	1.03E-11	1.21E-13	1.48E-12	116	103.3
104	2.10E-11	1.64E-10	1.51E-12	1.65E-11	114	103.3
1 004	1.81E-10	1.47E-09	1.49E-11	1.61E-10	114	103.4
10 004	1.16E-09	9.30E-09	1.19E-10	1.50E-09	113	103.1
100 004	1.10E-08	8.92E-08	1.20E-09	1.42E-08	115	103.3
			Riedkosť = 10%			
4	3.49E-14	1.36E-12	1.75E-14	5.71E-13	90	70.88
14	1.46E-13	5.71E-12	7.43E-14	2.42E-12	81	70.85
104	1.87E-12	5.92E-11	7.30E-13	2.41E-11	78	70.84
1 004	1.53E-11	5.45E-10	7.05E-12	2.38E-10	77	70.83
10 004	1.30E-10	4.69E-09	7.63E-11	2.57E-09	77	70.71
100 004	1.71E-09	6.56E-08	7.70E-10	2.62E-08	78	70.75
			Riedkosť = 15%			
4	4.87E-14	2.83E-12	1.75E-14	8.56E-13	68	61.38
14	1.96E-13	1.13E-11	8.09E-14	4.09E-12	72	61.51
104	1.61E-12	8.63E-11	7.43E-13	3.88E-11	67	61.48
1 004	1.92E-11	9.46E-10	7.84E-12	3.98E-10	68	61.54
10 004	2.20E-10	1.22E-08	7.84E-11	4.00E-09	72	61.57
100 004	2.08E-09	1.15E-07	7.65E-10	3.86E-08	80	61.83
			Riedkosť = 20%			
4	4.69E-14	3.32E-12	1.89E-14	1.27E-12	62	56.29
14	1.86E-13	1.36E-11	8.16E-14	5.48E-12	63	56.37
104	2.00E-12	1.45E-10	8.27E-13	5.58E-11	62	56.50
1 004	2.21E-11	1.64E-09	8.10E-12	5.51E-10	73	56.59
10 004	1.52E-10	1.07E-08	7.66E-11	5.10E-09	68	56.62
100 004	1.68E-09	1.32E-07	7.72E-10	5.25E-08	73	56.68
			Riedkosť = 25%			
4	3.51E-14	3.05E-12	1.68E-14	1.37E-12	60	53.95
14	2.10E-13	2.01E-11	7.81E-14	6.58E-12	61	54.12
104	2.12E-12	2.05E-10	8.14E-13	6.78E-11	62	54.00
1 004	1.64E-11	1.54E-09	7.86E-12	6.63E-10	61	54.01
10 004	2.13E-10	2.04E-08	8.35E-11	7.02E-09	61	53.90
100 004	2.24E-09	2.04E-07	8.34E-10	7.00E-08	61	53.93

Tabuľka *LSQR 2:*

$\|x_1 - x^*\|_\infty$	Pres_min $\|x_i - x^*\|_\infty$	Pres_min $\|Ax_i - x^*\|_\infty$	Pres_avg $\|x_i - x^*\|_\infty$	Pres_avg $\|Ax_i - x^*\|_\infty$	Iterácie max	Iterácie avg
LSQR metóda, $n = 300$						
Riedkosť = 6.7%						
4	5.49E-14	2.15E-12	2.54E-14	8.52E-13	82	72.82
14	2.61E-13	1.06E-11	1.09E-13	3.77E-12	81	72.98
104	2.91E-12	1.16E-10	1.11E-12	3.80E-11	82	72.96
1 004	2.31E-11	9.76E-10	1.13E-11	3.85E-10	89	73.05
10 004	2.65E-10	1.00E-08	1.14E-10	3.88E-09	83	72.93
100 004	2.36E-09	8.82E-08	1.18E-09	4.01E-08	89	73.10
Riedkosť = 10%						
4	5.73E-14	3.10E-12	2.53E-14	1.26E-12	70	62.56
14	2.99E-13	1.55E-11	1.03E-13	5.27E-12	71	62.81
104	2.32E-12	1.34E-10	1.01E-12	5.17E-11	70	62.77
1 004	3.16E-11	1.84E-09	1.04E-11	5.21E-10	71	62.85
10 004	2.00E-10	1.15E-08	1.07E-10	5.49E-09	71	62.73
100 004	2.42E-09	1.44E-07	1.03E-09	5.24E-08	70	62.87
Riedkosť = 16.7%						
4	5.38E-14	4.77E-12	2.62E-14	2.14E-12	60	54.82
14	2.46E-13	2.18E-11	1.13E-13	9.43E-12	61	54.80
104	2.85E-12	2.34E-10	9.97E-13	8.25E-11	60	54.91
1 004	2.52E-11	2.25E-09	1.22E-11	1.04E-09	61	54.68
10 004	2.00E-10	1.72E-08	1.08E-10	9.11E-09	60	54.86
100 004	2.66E-09	2.51E-07	1.06E-09	8.81E-08	60	54.90
Riedkosť = 20%						
4	7.91E-14	8.20E-12	2.64E-14	2.56E-12	58	52.99
14	3.05E-13	3.29E-11	1.12E-13	1.13E-11	58	53.11
104	2.82E-12	2.94E-10	1.08E-12	1.08E-10	58	53.10
1 004	2.43E-11	2.36E-09	1.06E-11	1.05E-09	59	53.15
10 004	2.84E-10	3.06E-08	1.12E-10	1.13E-08	59	53.05
100 004	3.20E-09	3.27E-07	1.13E-09	1.11E-07	59	53.04
Riedkosť = 25%						
4	6.34E-14	8.03E-12	2.61E-14	3.18E-12	62	57.45
14	3.19E-13	4.18E-11	1.21E-13	1.48E-11	63	57.41
104	3.15E-12	4.37E-10	1.10E-12	1.36E-10	63	57.55
1 004	3.30E-11	4.56E-09	1.18E-11	1.46E-09	63	57.43
10 004	2.63E-10	3.34E-08	1.20E-10	1.52E-08	63	57.41
100 004	3.13E-09	4.21E-07	1.28E-09	1.60E-07	63	57.40

Tabuľka *LSQR 3:*

LSQR metóda, $n = 500$						
	Pres_min		Pres_avg		Iterácie	
$\|x_1 - x^*\|_\infty$	$\|x_i - x^*\|_\infty$	$\|Ax_i - x^*\|_\infty$	$\|x_i - x^*\|_\infty$	$\|Ax_i - x^*\|_\infty$	max	avg
Riedkosť = 5%						
4	7.76E-13	1.99E-11	5.34E-14	1.92E-12	92	81.73
14	1.41E-12	2.66E-11	2.06E-13	8.18E-12	94	82.01
104	2.06E-11	4.30E-10	2.29E-12	8.66E-11	96	82.00
1 004	1.34E-10	2.48E-09	1.92E-11	7.38E-10	95	82.13
10 004	2.04E-09	4.27E-08	2.08E-10	7.61E-09	96	82.24
100 004	1.38E-08	2.79E-07	1.99E-09	7.74E-08	94	82.00
Riedkosť = 10%						
4	8.38E-14	7.54E-12	3.80E-14	3.14E-12	61	56.27
14	5.40E-13	4.78E-11	1.86E-13	1.59E-11	61	56.34
104	4.51E-12	4.11E-10	1.83E-12	1.57E-10	62	56.31
1 004	3.20E-11	2.89E-09	1.66E-11	1.39E-09	62	56.52
10 004	4.43E-10	4.15E-08	1.55E-10	1.28E-08	62	56.57
100 004	5.23E-09	4.83E-07	1.90E-09	1.60E-07	61	56.32
Riedkosť = 15%						
4	1.21E-13	1.63E-11	4.07E-14	5.04E-12	66	60.05
14	5.39E-13	7.39E-11	1.95E-13	2.42E-11	67	60.16
104	4.26E-12	5.56E-10	1.74E-12	2.14E-10	67	60.25
1 004	3.81E-11	4.93E-09	1.74E-11	2.15E-09	67	60.24
10 004	4.53E-10	5.96E-08	1.79E-10	2.20E-08	67	60.24
100 004	4.03E-09	5.07E-07	1.70E-09	2.08E-07	67	60.31
Riedkosť = 20%						
4	1.01E-13	1.69E-11	3.70E-14	6.00E-12	55	50.70
14	4.97E-13	8.16E-11	1.78E-13	2.94E-11	55	50.89
104	5.76E-12	9.75E-10	1.96E-12	3.28E-10	55	50.77
1 004	4.22E-11	7.47E-09	1.80E-11	2.96E-09	55	50.91
10 004	4.99E-10	8.48E-08	1.83E-10	3.03E-08	56	50.84
100 004	4.66E-09	7.55E-07	1.93E-09	3.20E-07	56	50.84
Riedkosť = 25%						
4	8.63E-14	1.88E-11	4.14E-14	8.37E-12	83	76.63
14	4.14E-13	9.38E-11	1.97E-13	4.05E-11	83	77.14
104	5.17E-12	1.09E-09	1.95E-12	4.05E-10	83	77.08
1 004	4.99E-11	1.07E-08	1.83E-11	3.75E-09	83	77.23
10 004	4.82E-10	1.05E-07	2.06E-10	4.33E-08	83	77.01
100 004	4.54E-09	1.02E-06	1.82E-09	3.78E-07	83	77.28

Tabuľka *LSQR 4:*

LSQR *metóda*, $n = 1\,000$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
			Riedkosť = 5%			
4	1.77E-13	1.48E-11	7.07E-14	5.78E-12	63	57.08
14	9.12E-13	7.80E-11	3.58E-13	3.02E-11	63	57.07
104	8.33E-12	7.31E-10	3.38E-12	2.81E-10	63	57.18
1 004	8.09E-11	7.55E-09	3.44E-11	2.97E-09	64	57.18
10 004	7.65E-10	7.05E-08	3.25E-10	2.76E-08	63	57.19
100 004	8.44E-09	7.56E-07	3.41E-09	2.85E-07	63	57.24
			Riedkosť = 10%			
4	1.70E-13	2.55E-11	6.64E-14	1.06E-11	55	51.44
14	8.09E-13	1.38E-10	3.39E-13	5.51E-11	56	51.51
104	9.37E-12	1.50E-09	3.38E-12	5.55E-10	56	51.57
1 004	9.76E-11	1.62E-08	3.44E-11	5.65E-09	56	51.49
10 004	1.12E-09	1.93E-07	3.58E-10	5.80E-08	55	51.48
100 004	7.72E-09	1.28E-06	3.28E-09	5.37E-07	56	51.62
			Riedkosť = 15%			
4	1.65E-13	4.07E-11	6.70E-14	1.61E-11	55	49.81
14	8.87E-13	2.12E-10	3.40E-13	8.22E-11	56	50.03
104	1.09E-11	2.77E-09	3.81E-12	9.20E-10	55	49.80
1 004	1.06E-10	2.65E-08	3.37E-11	8.16E-09	55	49.85
10 004	7.91E-10	2.00E-07	3.46E-10	8.32E-08	55	49.89
100 004	1.10E-08	2.80E-06	3.63E-09	8.88E-07	55	49.88
			Riedkosť = 20%			
4	2.17E-13	7.05E-11	7.47E-14	2.40E-11	54	49.98
14	9.84E-13	3.34E-10	3.53E-13	1.15E-10	54	50.13
104	9.25E-12	3.16E-09	3.81E-12	1.23E-09	54	49.99
1 004	9.29E-11	2.98E-08	3.53E-11	1.15E-08	54	50.05
10 004	8.72E-10	2.99E-07	3.94E-10	1.29E-07	54	49.94
100 004	1.05E-08	3.49E-06	3.74E-09	1.22E-06	53	50.03
			Riedkosť = 25%			
4	2.19E-13	8.64E-11	9.02E-14	3.68E-11	49	46.16
14	8.50E-13	3.60E-10	3.69E-13	1.51E-10	49	46.48
104	1.09E-11	4.33E-09	3.90E-12	1.58E-09	49	46.44
1 004	1.06E-10	4.53E-08	3.87E-11	1.57E-08	49	46.40
10 004	1.15E-09	4.68E-07	3.57E-10	1.46E-07	49	46.53
100 004	1.11E-08	4.64E-06	3.91E-09	1.59E-06	49	46.48

Tabuľka *LSQR 5:*

LSQR *metóda*, $n = 2\,000$						
	Pres_ min		*Pres_ avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
			Riedkosť = 5%			
4	*4.03E-13*	*6.49E-11*	*1.48E-13*	*2.39E-11*	*57*	*51.79*
14	*1.55E-12*	*2.53E-10*	*5.94E-13*	*9.67E-11*	*58*	*52.23*
104	*1.97E-11*	*3.27E-09*	*6.60E-12*	*1.07E-09*	*58*	*52.14*
1 004	*1.82E-10*	*2.99E-08*	*6.67E-11*	*1.08E-08*	*58*	*52.18*
10 004	*1.62E-09*	*2.81E-07*	*6.41E-10*	*1.05E-07*	*59*	*52.18*
100 004	*2.16E-08*	*3.57E-06*	*6.59E-09*	*1.07E-06*	*59*	*52.23*
			Riedkosť = 10%			
4	*4.21E-13*	*1.37E-10*	*1.43E-13*	*4.45E-11*	*53*	*48.73*
14	*2.10E-12*	*6.60E-10*	*7.10E-13*	*2.26E-10*	*53*	*49.01*
104	*1.96E-11*	*6.36E-09*	*7.74E-12*	*2.46E-09*	*53*	*48.89*
1 004	*2.26E-10*	*7.07E-08*	*6.67E-11*	*2.12E-08*	*53*	*49.02*
10 004	*1.76E-09*	*5.76E-07*	*5.99E-10*	*1.91E-07*	*53*	*49.07*
100 004	*2.15E-08*	*6.98E-06*	*7.26E-09*	*2.31E-06*	*53*	*48.91*
			Riedkosť = 15%			
4	*4.60E-13*	*2.24E-10*	*1.50E-13*	*7.06E-11*	*52*	*48.34*
14	*2.08E-12*	*9.96E-10*	*7.13E-13*	*3.38E-10*	*52*	*48.69*
104	*1.69E-11*	*8.17E-09*	*6.98E-12*	*3.36E-09*	*52*	*48.73*
1 004	*2.26E-10*	*1.10E-07*	*7.00E-11*	*3.31E-08*	*52*	*48.67*
10 004	*1.84E-09*	*8.87E-07*	*6.75E-10*	*3.18E-07*	*52*	*48.73*
100 004	*2.42E-08*	*1.18E-05*	*6.66E-09*	*3.17E-06*	*52*	*48.78*
			Riedkosť = 20%			
4	*5.23E-13*	*3.42E-10*	*1.54E-13*	*9.77E-11*	*53*	*49.86*
14	*1.74E-12*	*1.15E-09*	*7.32E-13*	*4.66E-10*	*53*	*49.99*
104	*2.32E-11*	*1.52E-08*	*7.48E-12*	*4.80E-09*	*54*	*49.95*
1 004	*1.92E-10*	*1.24E-07*	*7.42E-11*	*4.80E-08*	*53*	*49.94*
10 004	*2.38E-09*	*1.55E-06*	*7.59E-10*	*4.87E-07*	*53*	*49.93*
100 004	*1.91E-08*	*1.22E-05*	*7.62E-09*	*4.90E-06*	*53*	*50.00*
			Riedkosť = 25%			
4	*4.70E-13*	*3.95E-10*	*1.51E-13*	*1.23E-10*	*49*	*45.93*
14	*2.49E-12*	*2.07E-09*	*7.58E-13*	*6.18E-10*	*49*	*46.06*
104	*1.87E-11*	*1.58E-08*	*7.61E-12*	*6.22E-09*	*49*	*46.08*
1 004	*1.78E-10*	*1.53E-07*	*7.29E-11*	*5.96E-08*	*49*	*46.04*
10 004	*1.95E-09*	*1.57E-06*	*7.78E-10*	*6.31E-07*	*49*	*46.07*
100 004	*1.85E-08*	*1.47E-05*	*8.10E-09*	*6.59E-06*	*49*	*45.99*

Tabuľka *LSQR 6:*

LSQR metóda, $n = 3\,000$						
	Pres_min		Pres_avg		Iterácie	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	max	avg
Riedkosť = 0.15%						
4	4.11E-11	8.55E-11	3.75E-12	8.82E-12	248	224.4
14	9.88E-11	2.03E-10	9.46E-12	2.36E-11	251	227.0
104	9.68E-10	2.07E-09	9.38E-11	2.32E-10	252	227.1
1 004	1.14E-08	2.42E-08	1.07E-09	2.58E-09	249	227.2
10 004	9.57E-08	2.04E-07	9.26E-09	2.32E-08	249	227.0
100 004	8.12E-07	1.75E-06	8.07E-08	2.09E-07	254	227.0
Riedkosť = 5%						
4	4.56E-13	1.47E-10	1.91E-13	5.95E-11	53	49.41
14	3.51E-12	1.14E-09	1.12E-12	3.55E-10	55	49.58
104	3.56E-11	1.16E-08	9.94E-12	3.15E-09	54	49.58
1 004	3.18E-10	1.06E-07	9.84E-11	3.11E-08	53	49.61
10 004	2.55E-09	8.37E-07	9.54E-10	3.03E-07	54	49.62
100 004	3.08E-08	9.62E-06	1.04E-08	3.27E-06	53	49.51
Riedkosť = 10%						
4	5.69E-13	2.77E-10	2.22E-13	1.04E-10	51	47.92
14	3.46E-12	1.66E-09	1.13E-12	5.34E-10	51	48.18
104	2.85E-11	1.40E-08	1.01E-11	4.77E-09	52	48.36
1 004	3.66E-10	1.75E-07	1.06E-10	5.01E-08	51	48.24
10 004	2.60E-09	1.28E-06	8.59E-10	4.06E-07	53	48.44
100 004	3.09E-08	1.52E-05	1.09E-08	5.15E-06	52	48.24
Riedkosť = 15%						
4	5.91E-13	4.66E-10	2.29E-13	1.80E-10	49	45.68
14	3.13E-12	2.57E-09	1.14E-12	9.01E-10	49	45.92
104	2.32E-11	1.85E-08	9.50E-12	7.54E-09	49	46.00
1 004	2.87E-10	2.25E-07	1.05E-10	8.31E-08	49	45.96
10 004	3.08E-09	2.43E-06	1.07E-09	8.54E-07	49	45.90
100 004	3.35E-08	2.72E-05	1.16E-08	9.20E-06	49	45.89

Tabuľka *LSQR 7:*

LSQR metóda, $n = 5\,000$						
	Pres_min		Pres_avg		Iterácie	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	max	avg
Riedkosť = 0.1%						
4	1.10E-10	1.88E-10	1.52E-11	3.20E-11	258	233.2
14	2.93E-10	5.96E-10	3.32E-11	7.21E-11	257	236.9
104	2.37E-09	4.81E-09	2.82E-10	6.21E-10	259	237.3
1 004	1.13E-08	2.34E-08	1.41E-09	3.57E-09	259	237.1
10 004	1.72E-07	3.50E-07	2.11E-08	4.91E-08	259	237.3
100 004	1.72E-06	3.49E-06	2.11E-07	4.79E-07	262	238.1
Riedkosť = 3%						
4	8.53E-13	2.09E-10	3.13E-13	7.39E-11	55	50.12
14	6.00E-12	1.46E-09	1.75E-12	4.17E-10	55	50.17
104	5.17E-11	1.26E-08	1.71E-11	4.08E-09	55	50.20
1 004	5.19E-10	1.25E-07	1.56E-10	3.72E-08	55	50.28
10 004	4.52E-09	1.09E-06	1.52E-09	3.61E-07	55	50.32
100 004	4.33E-08	1.06E-05	1.60E-08	3.84E-06	55	50.24
Riedkosť = 5%						
4	1.07E-12	5.03E-10	3.64E-13	1.71E-10	51	48.39
14	5.66E-12	2.59E-09	1.79E-12	8.40E-10	52	48.56
104	5.64E-11	2.70E-08	1.62E-11	7.58E-09	51	48.65
1 004	5.59E-10	2.66E-07	1.51E-10	7.06E-08	52	48.76
10 004	4.47E-09	2.16E-06	1.68E-09	7.90E-07	53	48.62
100 004	4.21E-08	2.01E-05	1.54E-08	7.22E-06	52	48.66
Riedkosť = 10%						
4	1.14E-12	8.93E-10	3.53E-13	2.75E-10	50	46.73
14	5.31E-12	4.16E-09	1.74E-12	1.36E-09	50	47.16
104	4.02E-11	3.22E-08	1.65E-11	1.29E-08	50	47.24
1 004	4.62E-10	3.59E-07	1.55E-10	1.21E-07	50	47.28
10 004	4.26E-09	3.40E-06	1.57E-09	1.22E-06	50	47.28
100 004	5.91E-08	4.71E-05	1.72E-08	1.34E-05	50	47.16

Tabuľka *LSQR 8:*

LSQR *metóda*, $n = 10\,000$							
	Pres_min		*Pres_avg*		*Iterácie*		
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
			Riedkosť = 0.05%			
4	2.24E-10	4.20E-10	2.22E-11	4.71E-11	274	245.0
14	9.51E-10	1.73E-09	8.66E-11	1.75E-10	279	248.6
104	5.74E-09	1.03E-08	5.08E-10	1.11E-09	269	248.3
1 004	7.56E-08	1.37E-07	6.53E-09	1.37E-08	275	249.2
10 004	1.47E-06	2.75E-06	1.42E-07	2.81E-07	279	248.7
100 004	9.38E-06	1.72E-05	8.39E-07	1.71E-06	277	248.8
			Riedkosť = 0.5%			
4	1.84E-12	1.53E-10	7.49E-13	5.96E-11	65	58.53
14	7.90E-12	7.40E-10	3.33E-12	2.79E-10	65	58.79
104	1.04E-10	9.26E-09	3.23E-11	2.70E-09	65	58.78
1 004	1.06E-09	9.31E-08	3.57E-10	3.04E-08	64	58.71
10 004	1.02E-08	8.34E-07	2.94E-09	2.45E-07	64	58.92
100 004	8.76E-08	7.88E-06	2.88E-08	2.42E-06	66	59.01
			Riedkosť = 1%			
4	1.80E-12	2.93E-10	6.39E-13	1.01E-10	62	56.15
14	8.21E-12	1.41E-09	3.29E-12	5.28E-10	63	56.29
104	7.15E-11	1.21E-08	2.88E-11	4.62E-09	63	56.42
1 004	1.12E-09	1.84E-07	2.94E-10	4.73E-08	61	56.40
10 004	9.79E-09	1.61E-06	3.11E-09	5.06E-07	62	56.37
100 004	1.15E-07	1.90E-05	3.27E-08	5.30E-06	63	56.36
			Riedkosť = 2%			
4	1.45E-12	4.65E-10	4.10E-13	1.25E-10	52	49.30
14	5.32E-12	1.75E-09	2.81E-12	8.91E-10	51	49.20
104	6.69E-11	2.07E-08	2.89E-11	9.08E-09	52	49.10
1 004	8.58E-10	2.79E-07	4.12E-10	1.29E-07	51	48.80
10 004	4.14E-09	1.38E-06	1.87E-09	5.95E-07	52	49.60
100 004	5.05E-08	1.65E-05	3.21E-08	1.00E-05	52	48.90

Tabuľka *LSQR 9:*

***LSQR** metóda,* $n = 20\,000$						
	Pres_ min		*Pres_ avg*		*Iterácie*	
$\|x_1 - x^*\|_\infty$	$\|x_i - x^*\|_\infty$	$\|Ax_i - x^*\|_\infty$	$\|x_i - x^*\|_\infty$	$\|Ax_i - x^*\|_\infty$	*max*	*avg*
			Riedkosť = 0.025%			
4	2.06E-10	3.79E-10	3.10E-11	7.83E-11	285	253.6
14	6.89E-10	1.25E-09	1.06E-10	2.39E-10	279	259.0
104	6.59E-09	1.19E-08	1.15E-09	2.39E-09	279	260.5
1 004	4.75E-08	8.56E-08	8.06E-09	1.80E-08	284	260.9
10 004	6.33E-07	1.14E-06	1.02E-07	2.18E-07	289	260.4
100 004	6.81E-06	1.23E-05	1.15E-06	2.38E-06	282	260.8
			Riedkosť = 0.15%			
4	5.64E-12	2.14E-10	2.22E-12	9.74E-11	81	73.04
14	2.18E-11	1.00E-09	7.79E-12	3.61E-10	81	73.86
104	1.52E-10	8.52E-09	7.16E-11	3.65E-09	80	73.81
1 004	1.61E-09	9.05E-08	6.35E-10	3.26E-08	81	73.98
10 004	1.56E-08	8.21E-07	6.42E-09	3.33E-07	80	73.92
100 004	1.66E-07	8.71E-06	6.36E-08	3.25E-06	81	74.05
			Riedkosť = 0.25%			
4	5.61E-12	4.26E-10	1.74E-12	1.36E-10	65	59.38
14	1.71E-11	1.46E-09	6.85E-12	5.51E-10	65	59.79
104	2.30E-10	2.15E-08	6.59E-11	5.56E-09	65	59.76
1 004	1.58E-09	1.35E-07	6.33E-10	5.26E-08	64	59.82
10 004	1.87E-08	1.64E-06	6.47E-09	5.48E-07	65	59.77
100 004	1.58E-07	1.41E-05	6.41E-08	5.39E-06	65	59.75
			Riedkosť = 0.375%			
4	1.74E-12	2.18E-10	1.14E-12	1.40E-10	65	60.40
14	1.09E-11	1.24E-09	6.92E-12	8.28E-10	66	60.50
104	2.02E-10	2.63E-08	9.81E-11	1.26E-08	66	60.10
1 004	9.68E-10	1.22E-07	6.59E-10	8.11E-08	66	60.40
10 004	1.27E-08	1.58E-06	7.28E-09	8.76E-07	67	60.20
100 004	2.11E-07	2.63E-05	7.54E-08	9.60E-06	66	60.40

Tabuľka *LSQR 10:*

$\|x_1 - x^*\|_\infty$	Pres_min $\|x_i - x^*\|_\infty$	Pres_min $\|Ax_i - x^*\|_\infty$	Pres_avg $\|x_i - x^*\|_\infty$	Pres_avg $\|Ax_i - x^*\|_\infty$	Iterácie max	Iterácie avg
LSQR metóda, $n = 30\,000$						
Riedkosť = 0.015%						
4	4.00E-10	7.36E-10	7.70E-11	1.71E-10	275	257.3
14	6.74E-10	1.25E-09	1.65E-10	3.80E-10	279	263.1
104	2.82E-09	5.33E-09	7.34E-10	2.08E-09	287	264.8
1 004	5.91E-08	1.09E-07	1.40E-08	3.08E-08	285	264.9
10 004	3.25E-07	5.91E-07	8.07E-08	2.19E-07	291	264.6
100 004	3.26E-06	7.03E-06	8.08E-07	2.18E-06	282	264.8
Riedkosť = 0.05%						
4	2.07E-11	4.62E-10	4.99E-12	1.30E-10	105	95.10
14	7.75E-11	1.50E-09	1.82E-11	5.08E-10	106	95.96
104	4.80E-10	7.57E-09	1.19E-10	3.48E-09	109	96.81
1 004	4.80E-09	9.30E-08	1.23E-09	3.81E-08	108	96.62
10 004	4.77E-08	1.18E-06	1.26E-08	3.96E-07	110	96.60
100 004	4.77E-07	8.78E-06	1.18E-07	3.65E-06	110	96.80
Riedkosť = 0.1%						
4	1.25E-11	5.28E-10	3.69E-12	1.65E-10	80	73.51
14	3.45E-11	1.55E-09	1.24E-11	5.73E-10	81	74.33
104	3.25E-10	1.72E-08	1.08E-10	5.49E-09	80	74.42
1 004	3.30E-09	1.76E-07	1.01E-09	5.07E-08	81	74.60
10 004	2.30E-08	1.34E-06	1.03E-08	5.40E-07	80	74.36
100 004	2.51E-07	1.40E-05	9.73E-08	5.04E-06	81	74.55
Riedkosť = 0.15%						
4	4.91E-12	4.20E-10	2.80E-12	2.30E-10	66	59.50
14	1.47E-11	1.22E-09	7.42E-12	6.24E-10	64	60.30
104	1.77E-10	1.47E-08	8.69E-11	7.25E-09	64	60.00
1 004	1.72E-09	1.54E-07	8.88E-10	7.84E-08	64	60.00
10 004	1.34E-08	1.09E-06	7.11E-09	5.95E-07	66	60.40
100 004	1.35E-07	1.04E-05	7.45E-08	5.91E-06	64	60.20

Tabuľka *LSQR 11:*

		LSQR *metóda,* $n = 50\,000$				
	Pres_min		*Pres_avg*		*Iterácie*	
$\lVert \boldsymbol{x}_1 - \boldsymbol{x}^* \rVert_\infty$	$\lVert \boldsymbol{x}_i - \boldsymbol{x}^* \rVert_\infty$	$\lVert \boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^* \rVert_\infty$	$\lVert \boldsymbol{x}_i - \boldsymbol{x}^* \rVert_\infty$	$\lVert \boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^* \rVert_\infty$	*max*	*avg*
			Riedkosť = 0.01%			
4	4.38E-10	7.57E-10	8.13E-11	2.29E-10	284	263.1
14	1.09E-09	2.07E-09	2.67E-10	6.49E-10	293	269.4
104	1.01E-08	1.79E-08	2.33E-09	5.43E-09	303	270.9
1 004	9.18E-08	1.63E-07	2.11E-08	5.03E-08	292	271.5
10 004	7.24E-07	1.32E-06	1.69E-07	4.38E-07	294	271.2
100 004	7.98E-06	1.45E-05	1.89E-06	4.90E-06	290	270.5
			Riedkosť = 0.02%			
4	1.73E-09	6.06E-09	2.32E-10	9.73E-10	208	187.3
14	5.85E-09	2.54E-08	7.26E-10	3.32E-09	210	191.8
104	1.50E-07	6.06E-07	1.83E-08	7.69E-08	222	193.1
1 004	4.43E-07	1.98E-06	5.63E-08	2.73E-07	225	192.7
10 004	5.44E-06	2.42E-05	6.78E-07	3.12E-06	222	193.9
100 004	3.60E-05	1.63E-04	4.63E-06	2.22E-05	214	193.3
			Riedkosť = 0.04%			
4	1.78E-11	4.33E-10	1.00E-11	2.84E-10	106	97.40
14	8.27E-11	2.07E-09	4.13E-11	1.07E-09	107	98.20
104	2.95E-10	1.19E-08	2.20E-10	7.08E-09	108	99.20
1 004	3.72E-09	1.11E-07	2.41E-09	7.44E-08	108	99.10
10 004	4.20E-08	1.03E-06	1.93E-08	5.34E-07	108	99.90
100 004	3.95E-07	1.24E-05	2.05E-07	6.66E-06	110	99.70
			Riedkosť = 0.06%			
4	1.64E-11	6.66E-10	8.66E-12	3.72E-10	80	75.10
14	6.40E-11	2.52E-09	4.00E-11	1.62E-09	82	75.60
104	4.36E-10	2.41E-08	2.02E-10	1.04E-08	84	76.70
1 004	2.95E-09	1.20E-07	1.88E-09	9.03E-08	83	76.20
10 004	3.34E-08	1.96E-06	1.79E-08	8.99E-07	81	76.40
100 004	4.37E-07	2.42E-05	1.99E-07	1.00E-05	83	76.30

Tabuľka *LSQR 12:*

***SymmLQ** metóda,* $n = 100$						
	Pres_ min		*Pres_ avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
			Riedkosť = 5%			
4	2.13E-14	3.04E-13	6.70E-15	8.56E-14	100	71.11
14	7.02E-14	9.17E-13	2.81E-14	3.27E-13	100	71.47
104	6.80E-13	9.29E-12	2.66E-13	3.20E-12	100	71.79
1 004	6.49E-12	1.00E-10	2.63E-12	3.18E-11	96	72.59
10 004	6.73E-11	1.02E-09	2.55E-11	3.11E-10	100	71.75
100 004	7.35E-10	1.07E-08	2.63E-10	3.22E-09	98	71.48
			Riedkosť = 10%			
4	1.80E-14	4.41E-13	6.10E-15	1.49E-13	88	58.47
14	9.23E-14	2.60E-12	2.89E-14	7.13E-13	81	57.98
104	1.18E-12	3.26E-11	2.72E-13	6.64E-12	80	58.04
1 004	7.15E-12	1.93E-10	2.62E-12	6.44E-11	85	58.12
10 004	8.60E-11	2.20E-09	2.71E-11	6.53E-10	83	58.53
100 004	1.08E-09	3.20E-08	2.45E-10	6.00E-09	83	58.11
			Riedkosť = 15%			
4	1.37E-14	5.44E-13	5.38E-15	1.85E-13	75	50.58
14	9.27E-14	3.89E-12	2.90E-14	1.03E-12	75	51.73
104	1.01E-12	4.32E-11	2.56E-13	9.09E-12	77	51.69
1 004	8.67E-12	3.42E-10	2.29E-12	8.03E-11	80	50.43
10 004	1.01E-10	3.64E-09	2.63E-11	9.42E-10	75	50.98
100 004	8.85E-10	3.55E-08	2.58E-10	9.33E-09	82	50.91
			Riedkosť = 20%			
4	2.04E-14	1.08E-12	6.29E-15	2.73E-13	66	45.57
14	1.13E-13	6.05E-12	2.71E-14	1.23E-12	82	45.67
104	7.19E-13	3.81E-11	2.35E-13	1.06E-11	72	45.51
1 004	8.73E-12	4.52E-10	2.63E-12	1.20E-10	79	45.90
10 004	7.71E-11	3.82E-09	2.46E-11	1.13E-09	72	45.93
100 004	1.17E-09	5.96E-08	2.21E-10	9.99E-09	70	45.98
			Riedkosť = 25%			
4	2.07E-14	1.25E-12	6.23E-15	3.49E-13	64	49.94
14	1.05E-13	6.74E-12	3.10E-14	1.82E-12	76	50.66
104	8.12E-13	5.28E-11	2.55E-13	1.48E-11	73	49.85
1 004	1.18E-11	7.65E-10	2.77E-12	1.62E-10	70	50.41
10 004	7.58E-11	4.99E-09	2.70E-11	1.56E-09	78	50.34
100 004	1.05E-09	6.96E-08	2.73E-10	1.59E-08	76	50.49

Tabuľka *SymmLQ 1:*

SymmLQ *metóda*, $n = 200$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
			Riedkosť = 5%			
4	6.08E-14	1.84E-12	1.25E-14	3.46E-13	85	58.31
14	2.09E-13	6.31E-12	5.61E-14	1.56E-12	88	59.77
104	1.47E-12	4.36E-11	5.13E-13	1.43E-11	91	59.06
1 004	2.02E-11	6.11E-10	5.84E-12	1.65E-10	88	59.23
10 004	1.55E-10	4.66E-09	4.97E-11	1.37E-09	85	58.95
100 004	1.52E-09	4.74E-08	5.82E-10	1.65E-08	88	59.02
			Riedkosť = 10%			
4	3.02E-14	1.65E-12	1.00E-14	4.72E-13	63	46.64
14	1.63E-13	8.54E-12	4.79E-14	2.29E-12	62	46.42
104	1.80E-12	9.63E-11	4.55E-13	2.21E-11	64	46.49
1 004	1.51E-11	8.42E-10	4.06E-12	1.99E-10	62	46.39
10 004	1.21E-10	6.26E-09	4.18E-11	2.02E-09	85	46.89
100 004	1.28E-09	6.87E-08	4.07E-10	1.98E-08	62	46.14
			Riedkosť = 15%			
4	2.15E-14	1.48E-12	7.27E-15	4.57E-13	59	43.06
14	1.36E-13	9.75E-12	4.20E-14	2.83E-12	59	43.00
104	1.45E-12	1.12E-10	3.41E-13	2.31E-11	59	43.52
1 004	1.33E-11	1.03E-09	3.39E-12	2.30E-10	58	43.08
10 004	1.08E-10	7.69E-09	3.33E-11	2.22E-09	59	43.16
100 004	1.11E-09	8.91E-08	3.48E-10	2.37E-08	57	43.02
			Riedkosť = 20%			
4	2.71E-14	2.53E-12	7.35E-15	6.05E-13	57	41.14
14	1.03E-13	9.81E-12	3.41E-14	2.93E-12	54	40.87
104	9.79E-13	8.95E-11	3.04E-13	2.64E-11	48	40.88
1 004	1.00E-11	9.31E-10	3.36E-12	2.89E-10	50	40.91
10 004	1.13E-10	1.05E-08	3.01E-11	2.59E-09	55	40.96
100 004	9.47E-10	8.70E-08	2.98E-10	2.59E-08	56	41.32
			Riedkosť = 25%			
4	2.01E-14	2.42E-12	7.92E-15	8.04E-13	51	39.86
14	1.08E-13	1.33E-11	3.17E-14	3.35E-12	53	40.09
104	9.08E-13	9.28E-11	3.36E-13	3.60E-11	54	40.05
1 004	1.10E-11	1.24E-09	3.23E-12	3.47E-10	53	40.06
10 004	1.15E-10	1.32E-08	3.21E-11	3.44E-09	52	40.03
100 004	1.17E-09	1.28E-07	3.36E-10	3.60E-08	53	39.84

Tabuľka *SymmLQ 2:*

SymmLQ *metóda,* $n = 300$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
			Riedkosť = 6.7%			
4	*4.69E-14*	*2.60E-12*	*1.32E-14*	*6.41E-13*	*63*	*46.85*
14	*1.91E-13*	*1.06E-11*	*5.76E-14*	*2.80E-12*	*62*	*47.16*
104	*1.93E-12*	*9.50E-11*	*6.07E-13*	*3.06E-11*	*67*	*47.41*
1 004	*3.29E-11*	*1.80E-09*	*5.69E-12*	*2.87E-10*	*64*	*47.03*
10 004	*2.06E-10*	*1.12E-08*	*5.54E-11*	*2.78E-09*	*65*	*47.44*
100 004	*2.49E-09*	*1.38E-07*	*5.69E-10*	*2.82E-08*	*65*	*47.16*
			Riedkosť = 10%			
4	*4.91E-14*	*3.64E-12*	*1.10E-14*	*7.24E-13*	*57*	*43.11*
14	*1.54E-13*	*1.14E-11*	*4.54E-14*	*3.12E-12*	*58*	*43.12*
104	*2.14E-12*	*1.60E-10*	*4.86E-13*	*3.35E-11*	*58*	*43.25*
1 004	*1.82E-11*	*1.33E-09*	*4.69E-12*	*3.25E-10*	*59*	*43.25*
10 004	*1.27E-10*	*8.30E-09*	*4.50E-11*	*3.06E-09*	*62*	*43.40*
100 004	*1.74E-09*	*1.38E-07*	*4.76E-10*	*3.31E-08*	*59*	*43.29*
			Riedkosť = 16.7%			
4	*4.31E-14*	*4.97E-12*	*9.70E-15*	*9.80E-13*	*49*	*40.04*
14	*1.53E-13*	*1.74E-11*	*3.84E-14*	*4.08E-12*	*43*	*40.07*
104	*1.32E-12*	*1.52E-10*	*3.96E-13*	*4.29E-11*	*54*	*40.20*
1 004	*1.75E-11*	*2.05E-09*	*4.10E-12*	*4.42E-10*	*45*	*40.09*
10 004	*1.52E-10*	*1.82E-08*	*3.83E-11*	*4.07E-09*	*55*	*40.38*
100 004	*1.50E-09*	*1.83E-07*	*4.17E-10*	*4.51E-08*	*53*	*40.12*
			Riedkosť = 20%			
4	*2.87E-14*	*4.18E-12*	*9.75E-15*	*1.18E-12*	*42*	*39.20*
14	*1.63E-13*	*2.17E-11*	*4.21E-14*	*5.31E-12*	*43*	*39.26*
104	*1.63E-12*	*2.17E-10*	*4.33E-13*	*5.53E-11*	*42*	*39.28*
1 004	*1.05E-11*	*1.51E-09*	*3.59E-12*	*4.47E-10*	*50*	*39.48*
10 004	*1.54E-10*	*2.03E-08*	*3.76E-11*	*4.68E-09*	*43*	*39.37*
100 004	*1.56E-09*	*2.10E-07*	*3.71E-10*	*4.62E-08*	*43*	*39.40*
			Riedkosť = 25%			
4	*2.38E-14*	*3.89E-12*	*8.43E-15*	*1.20E-12*	*44*	*41.04*
14	*1.03E-13*	*1.65E-11*	*3.69E-14*	*5.51E-12*	*44*	*41.11*
104	*8.98E-13*	*1.50E-10*	*3.48E-13*	*5.32E-11*	*56*	*41.46*
1 004	*8.27E-12*	*1.40E-09*	*3.39E-12*	*5.16E-10*	*55*	*41.29*
10 004	*1.37E-10*	*2.21E-08*	*3.57E-11*	*5.53E-09*	*44*	*41.18*
100 004	*1.25E-09*	*2.08E-07*	*3.41E-10*	*5.17E-08*	*44*	*41.25*

Tabuľka *SymmLQ 3:*

SymmLQ metóda, $n = 500$						
	Pres_ min		*Pres_ avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
Riedkosť = 5%						
4	6.88E-14	4.44E-12	1.88E-14	1.11E-12	54	49.65
14	3.70E-13	2.33E-11	9.17E-14	5.60E-12	63	49.82
104	3.44E-12	2.19E-10	8.22E-13	5.02E-11	66	50.00
1 004	3.48E-11	2.16E-09	8.97E-12	5.50E-10	66	50.04
10 004	3.03E-10	1.93E-08	9.30E-11	5.69E-09	62	49.89
100 004	2.78E-09	1.83E-07	8.58E-10	5.28E-08	57	49.88
Riedkosť = 10%						
4	3.82E-14	4.04E-12	1.27E-14	1.30E-12	44	40.50
14	2.15E-13	2.39E-11	5.88E-14	6.29E-12	57	40.77
104	1.68E-12	1.96E-10	5.07E-13	5.41E-11	51	40.78
1 004	2.03E-11	2.33E-09	5.57E-12	5.98E-10	45	40.65
10 004	1.55E-10	1.77E-08	5.82E-11	6.30E-09	44	40.63
100 004	2.28E-09	2.54E-07	5.88E-10	6.32E-08	44	40.63
Riedkosť = 15%						
4	3.55E-14	5.83E-12	1.26E-14	1.85E-12	59	42.06
14	2.24E-13	3.68E-11	6.01E-14	9.36E-12	58	42.23
104	2.20E-12	3.62E-10	4.78E-13	7.42E-11	60	42.49
1 004	1.55E-11	2.57E-09	4.54E-12	6.95E-10	46	42.25
10 004	1.92E-10	3.33E-08	4.59E-11	7.11E-09	58	42.35
100 004	1.55E-09	2.71E-07	5.22E-10	8.20E-08	45	42.14
Riedkosť = 20%						
4	6.35E-14	1.36E-11	1.07E-14	2.05E-12	41	38.36
14	1.47E-13	3.24E-11	4.39E-14	8.66E-12	42	38.62
104	1.22E-12	2.52E-10	4.10E-13	8.17E-11	42	38.63
1 004	1.71E-11	3.64E-09	4.73E-12	9.49E-10	42	38.62
10 004	1.41E-10	2.98E-08	4.45E-11	8.83E-09	43	38.55
100 004	1.48E-09	3.06E-07	4.16E-10	8.24E-08	42	38.62
Riedkosť = 25%						
4	4.02E-14	1.08E-11	1.54E-14	3.91E-12	51	47.52
14	3.08E-13	8.70E-11	7.20E-14	1.92E-11	50	47.76
104	2.45E-12	6.81E-10	6.62E-13	1.76E-10	51	47.86
1 004	2.47E-11	6.99E-09	7.96E-12	2.13E-09	59	47.84
10 004	2.78E-10	7.78E-08	7.13E-11	1.89E-08	51	47.72
100 004	2.90E-09	8.20E-07	7.08E-10	1.89E-07	51	47.82

Tabuľka *SymmLQ 4:*

SymmLQ *metóda*, $n = 1\,000$						
	Pres_ min		*Pres_ avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
			Riedkosť = 5%			
4	8.75E-14	9.83E-12	2.11E-14	2.20E-12	44	40.70
14	2.52E-13	2.63E-11	8.06E-14	8.67E-12	45	41.05
104	3.21E-12	3.86E-10	8.94E-13	9.79E-11	45	40.89
1 004	3.36E-11	3.84E-09	7.92E-12	8.63E-10	44	40.98
10 004	2.14E-10	2.35E-08	8.31E-11	9.06E-09	44	40.95
100 004	3.51E-09	3.99E-07	8.11E-10	8.76E-08	44	41.00
			Riedkosť = 10%			
4	5.51E-14	1.16E-11	1.45E-14	2.69E-12	42	38.52
14	2.25E-13	4.64E-11	6.73E-14	1.30E-11	42	38.78
104	1.80E-12	3.55E-10	5.64E-13	1.10E-10	45	38.93
1 004	1.61E-11	3.34E-09	5.46E-12	1.06E-09	43	38.90
10 004	2.31E-10	4.55E-08	6.30E-11	1.24E-08	41	38.78
100 004	2.34E-09	4.91E-07	5.94E-10	1.16E-07	42	38.87
			Riedkosť = 15%			
4	4.36E-14	1.33E-11	1.40E-14	3.78E-12	41	37.96
14	1.68E-13	5.07E-11	5.47E-14	1.52E-11	41	38.22
104	1.18E-12	3.64E-10	4.59E-13	1.29E-10	41	38.32
1 004	2.17E-11	6.53E-09	5.79E-12	1.66E-09	41	38.18
10 004	2.03E-10	6.15E-08	5.32E-11	1.51E-08	41	38.32
100 004	1.52E-09	4.65E-07	5.09E-10	1.42E-07	41	38.32
			Riedkosť = 20%			
4	3.64E-14	1.38E-11	1.59E-14	5.72E-12	41	38.08
14	3.04E-13	1.26E-10	6.81E-14	2.60E-11	46	38.41
104	2.08E-12	8.72E-10	5.86E-13	2.24E-10	41	38.35
1 004	2.11E-11	8.83E-09	6.40E-12	2.48E-09	49	38.37
10 004	1.18E-10	4.89E-08	4.99E-11	1.91E-08	45	38.44
100 004	1.77E-09	7.34E-07	5.89E-10	2.27E-07	42	38.36
			Riedkosť = 25%			
4	6.84E-14	3.66E-11	1.74E-14	8.17E-12	41	36.21
14	2.69E-13	1.46E-10	6.78E-14	3.27E-11	40	36.52
104	1.54E-12	7.86E-10	5.99E-13	2.90E-10	50	36.70
1 004	2.13E-11	1.06E-08	5.75E-12	2.82E-09	39	36.64
10 004	2.03E-10	1.10E-07	6.39E-11	3.15E-08	47	36.61
100 004	1.44E-09	7.20E-07	5.89E-10	2.84E-07	51	36.71

Tabuľka *SymmLQ 5:*

$\|x_1 - x^*\|_\infty$	Pres_min $\|x_i - x^*\|_\infty$	Pres_min $\|Ax_i - x^*\|_\infty$	Pres_avg $\|x_i - x^*\|_\infty$	Pres_avg $\|Ax_i - x^*\|_\infty$	Iterácie max	Iterácie avg
			Riedkosť = 5%			
4	6.39E-14	1.24E-11	2.14E-14	3.95E-12	42	38.79
14	3.09E-13	6.23E-11	8.45E-14	1.63E-11	43	39.20
104	3.80E-12	8.15E-10	8.60E-13	1.70E-10	43	39.18
1 004	2.73E-11	5.60E-09	7.93E-12	1.55E-09	45	39.29
10 004	3.26E-10	6.63E-08	9.49E-11	1.86E-08	42	39.18
100 004	2.15E-09	4.50E-07	8.12E-10	1.59E-07	42	39.21
			Riedkosť = 10%			
4	8.73E-14	3.41E-11	2.00E-14	6.89E-12	40	37.48
14	1.98E-13	8.04E-11	7.77E-14	2.83E-11	41	37.82
104	3.08E-12	1.31E-09	7.55E-13	2.80E-10	41	37.76
1 004	2.22E-11	8.78E-09	6.85E-12	2.50E-09	40	37.87
10 004	2.43E-10	9.87E-08	7.03E-11	2.58E-08	41	37.89
100 004	2.37E-09	9.70E-07	7.18E-10	2.66E-07	40	37.84
			Riedkosť = 15%			
4	3.77E-14	2.24E-11	1.94E-14	1.00E-11	40	37.30
14	2.24E-13	1.31E-10	7.27E-14	3.91E-11	40	37.75
104	3.18E-12	1.87E-09	6.74E-13	3.68E-10	40	37.77
1 004	1.51E-11	8.07E-09	6.42E-12	3.49E-09	40	37.83
10 004	1.40E-10	7.94E-08	6.27E-11	3.36E-08	41	37.92
100 004	1.89E-09	1.13E-06	6.43E-10	3.51E-07	41	37.88
			Riedkosť = 20%			
4	5.60E-14	4.34E-11	2.34E-14	1.70E-11	40	37.89
14	2.31E-13	1.80E-10	8.08E-14	5.85E-11	51	38.35
104	1.92E-12	1.52E-09	7.10E-13	5.25E-10	41	38.36
1 004	2.99E-11	2.42E-08	7.78E-12	5.74E-09	41	38.25
10 004	2.09E-10	1.67E-07	7.48E-11	5.51E-08	41	38.31
100 004	1.88E-09	1.51E-06	7.52E-10	5.58E-07	41	38.33
			Riedkosť = 25%			
4	6.04E-14	6.00E-11	2.58E-14	2.34E-11	42	36.07
14	2.65E-13	2.61E-10	9.56E-14	9.08E-11	39	36.35
104	2.60E-12	2.61E-09	8.26E-13	7.87E-10	47	36.60
1 004	2.06E-11	2.07E-08	7.90E-12	7.47E-09	43	36.47
10 004	1.60E-10	1.65E-07	7.67E-11	7.27E-08	46	36.58
100 004	2.75E-09	2.87E-06	8.18E-10	7.78E-07	47	36.53

SymmLQ *metóda,* $n = 2\,000$

Tabuľka *SymmLQ 6:*

***SymmLQ** metóda,* $n = 3\,000$						
	Pres_ min		*Pres_ avg*		*Iterácie*	
$\lVert \boldsymbol{x}_1 - \boldsymbol{x}^* \rVert_\infty$	$\lVert \boldsymbol{x}_i - \boldsymbol{x}^* \rVert_\infty$	$\lVert \boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^* \rVert_\infty$	$\lVert \boldsymbol{x}_i - \boldsymbol{x}^* \rVert_\infty$	$\lVert \boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^* \rVert_\infty$	*max*	*avg*
Riedkosť = 0.15%						
4	1.44E-12	3.04E-11	2.75E-13	5.40E-12	98	79.98
14	3.71E-12	7.22E-11	1.02E-12	2.00E-11	107	81.19
104	5.82E-11	1.11E-09	9.27E-12	1.84E-10	115	81.94
1 004	3.60E-10	6.87E-09	8.78E-11	1.74E-09	100	80.86
10 004	3.00E-09	5.87E-08	8.24E-10	1.62E-08	102	81.03
100 004	4.38E-08	8.97E-07	1.00E-08	1.98E-07	102	81.01
Riedkosť = 5%						
4	8.33E-14	3.17E-11	2.42E-14	8.17E-12	40	37.84
14	3.32E-13	1.29E-10	8.76E-14	3.16E-11	41	38.24
104	2.44E-12	9.21E-10	7.85E-13	2.81E-10	41	38.33
1 004	2.30E-11	8.78E-09	8.16E-12	2.97E-09	41	38.28
10 004	2.43E-10	9.15E-08	8.03E-11	2.93E-08	41	38.25
100 004	1.79E-09	6.94E-07	7.22E-10	2.62E-07	41	38.39
Riedkosť = 10%						
4	5.22E-14	2.89E-11	2.35E-14	1.19E-11	39	37.22
14	2.60E-13	1.49E-10	7.93E-14	4.18E-11	40	37.58
104	1.96E-12	1.12E-09	8.13E-13	4.37E-10	40	37.49
1 004	2.77E-11	1.58E-08	7.69E-12	4.09E-09	40	37.59
10 004	2.36E-10	1.32E-07	7.56E-11	4.04E-08	40	37.65
100 004	2.13E-09	1.25E-06	8.04E-10	4.37E-07	40	37.63
Riedkosť = 15%						
4	7.33E-14	6.78E-11	2.39E-14	2.00E-11	38	36.09
14	2.15E-13	2.08E-10	8.64E-14	7.51E-11	38	36.39
104	1.99E-12	1.90E-09	8.07E-13	7.27E-10	39	36.50
1 004	2.35E-11	2.18E-08	8.10E-12	7.17E-09	39	36.49
10 004	1.99E-10	1.87E-07	8.04E-11	7.23E-08	40	36.48
100 004	2.05E-09	2.00E-06	7.59E-10	6.56E-07	39	36.50

Tabuľka *SymmLQ 7:*

***SymmLQ** metóda,* $n = 5\,000$						
	Pres_ min		*Pres_ avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
Riedkosť = 0.1%						
4	4.61E-12	9.10E-11	8.16E-13	1.66E-11	113	81.21
14	6.54E-12	1.38E-10	1.91E-12	3.92E-11	105	81.65
104	9.13E-11	1.87E-09	1.82E-11	3.77E-10	105	83.06
1 004	1.03E-09	2.13E-08	1.80E-10	3.71E-09	111	83.31
10 004	7.70E-09	1.65E-07	1.69E-09	3.49E-08	104	82.88
100 004	9.23E-08	1.98E-06	1.41E-08	2.92E-07	109	83.40
Riedkosť = 3%						
4	1.12E-13	3.58E-11	3.60E-14	9.71E-12	41	38.18
14	5.16E-13	1.63E-10	1.31E-13	3.65E-11	42	38.50
104	3.23E-12	9.98E-10	1.02E-12	2.84E-10	41	38.63
1 004	3.11E-11	9.49E-09	1.10E-11	3.13E-09	42	38.64
10 004	3.39E-10	9.85E-08	1.11E-10	3.15E-08	41	38.69
100 004	4.03E-09	1.22E-06	1.09E-09	3.07E-07	41	38.64
Riedkosť = 5%						
4	6.68E-14	3.81E-11	2.88E-14	1.43E-11	40	37.54
14	3.09E-13	1.70E-10	9.97E-14	5.14E-11	40	37.92
104	2.68E-12	1.53E-09	9.52E-13	4.98E-10	40	37.95
1 004	2.45E-11	1.37E-08	8.69E-12	4.61E-09	40	38.01
10 004	2.43E-10	1.40E-07	8.91E-11	4.70E-08	40	37.93
100 004	2.81E-09	1.54E-06	8.76E-10	4.56E-07	41	38.03
Riedkosť = 10%						
4	1.01E-13	9.28E-11	3.10E-14	2.57E-11	39	36.64
14	2.46E-13	2.32E-10	1.04E-13	8.84E-11	39	37.08
104	2.02E-12	1.83E-09	8.64E-13	7.27E-10	42	37.36
1 004	3.54E-11	3.20E-08	8.42E-12	7.23E-09	40	37.33
10 004	2.82E-10	2.58E-07	9.07E-11	7.84E-08	42	37.32
100 004	3.42E-09	3.06E-06	8.96E-10	7.59E-07	39	37.27

Tabuľka *SymmLQ 8:*

***SymmLQ** metóda,* $n = 10\,000$						
	Pres_ min		*Pres_ avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
			Riedkosť = 0.05%			
4	1.07E-11	2.24E-10	1.45E-12	3.05E-11	108	82.52
14	1.76E-11	3.80E-10	3.79E-12	8.02E-11	97	82.82
104	1.90E-10	4.11E-09	3.70E-11	7.84E-10	118	84.28
1 004	1.89E-09	4.10E-08	3.46E-10	7.30E-09	119	84.17
10 004	1.60E-08	3.33E-07	3.69E-09	7.88E-08	119	84.24
100 004	1.39E-07	2.98E-06	3.63E-08	7.73E-07	106	83.71
			Riedkosť = 0.5%			
4	5.41E-13	6.28E-11	1.50E-13	1.69E-11	44	41.15
14	2.53E-12	2.96E-10	6.05E-13	7.00E-11	45	41.38
104	1.73E-11	2.14E-09	4.92E-12	5.75E-10	45	41.55
1 004	2.47E-10	3.12E-08	4.94E-11	5.72E-09	45	41.51
10 004	1.16E-09	1.31E-07	4.23E-10	4.88E-08	46	41.70
100 004	1.45E-08	1.65E-06	4.31E-09	4.98E-07	45	41.69
			Riedkosť = 1%			
4	2.67E-13	5.52E-11	7.91E-14	1.56E-11	43	40.66
14	1.16E-12	2.55E-10	2.82E-13	5.63E-11	44	41.02
104	1.11E-11	2.59E-09	2.42E-12	4.90E-10	44	41.17
1 004	7.55E-11	1.56E-08	2.07E-11	4.20E-09	44	41.24
10 004	8.77E-10	1.92E-07	2.32E-10	4.72E-08	44	41.15
100 004	7.33E-09	1.55E-06	2.20E-09	4.45E-07	44	41.24
			Riedkosť = 2%			
4	1.29E-13	5.01E-11	5.24E-14	1.80E-11	40	37.59
14	5.84E-13	2.23E-10	1.81E-13	6.52E-11	41	38.06
104	5.54E-12	2.18E-09	1.48E-12	5.40E-10	41	38.25
1 004	4.70E-11	1.85E-08	1.45E-11	5.27E-09	41	38.21
10 004	5.14E-10	1.96E-07	1.48E-10	5.47E-08	41	38.20
100 004	4.32E-09	1.67E-06	1.38E-09	5.03E-07	41	38.16

Tabuľka *SymmLQ 9:*

SymmLQ metóda, $n = 20\,000$						
	Pres_ min		*Pres_ avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
			Riedkosť = 0.025%			
4	3.63E-11	7.85E-10	5.41E-12	1.18E-10	101	82.16
14	4.80E-11	1.09E-09	1.15E-11	2.53E-10	110	84.46
104	3.62E-10	7.97E-09	8.24E-11	1.81E-09	107	84.99
1 004	3.42E-09	7.60E-08	8.28E-10	1.82E-08	118	84.94
10 004	5.87E-08	1.30E-06	1.00E-08	2.22E-07	99	84.09
100 004	3.40E-07	7.67E-06	7.43E-08	1.64E-06	103	85.21
			Riedkosť = 0.15%			
4	3.61E-12	3.03E-10	7.26E-13	5.65E-11	57	45.59
14	8.39E-12	7.40E-10	2.37E-12	1.87E-10	59	46.28
104	7.14E-11	5.75E-09	1.55E-11	1.23E-09	60	46.70
1 004	4.41E-10	3.73E-08	1.51E-10	1.19E-08	60	46.72
10 004	4.86E-09	3.99E-07	1.38E-09	1.09E-07	58	46.67
100 004	5.61E-08	4.72E-06	1.27E-08	1.01E-06	61	46.73
			Riedkosť = 0.25%			
4	1.79E-12	2.05E-10	3.21E-13	3.68E-11	45	41.39
14	3.92E-12	4.82E-10	1.05E-12	1.21E-10	45	41.82
104	3.53E-11	4.12E-09	9.22E-12	1.07E-09	45	41.90
1 004	3.95E-10	4.85E-08	7.58E-11	8.85E-09	46	42.12
10 004	2.75E-09	3.28E-07	6.82E-10	7.91E-08	45	42.07
100 004	3.13E-08	3.86E-06	7.53E-09	8.73E-07	46	42.10
			Riedkosť = 0.375%			
4	3.12E-13	4.94E-11	1.62E-13	2.55E-11	45	41.80
14	1.28E-12	2.16E-10	7.66E-13	1.26E-10	45	42.40
104	4.87E-12	7.84E-10	3.23E-12	5.22E-10	45	43.00
1 004	1.28E-10	2.11E-08	5.63E-11	9.23E-09	45	42.80
10 004	1.50E-09	2.60E-07	4.55E-10	7.51E-08	46	42.90
100 004	9.84E-09	1.61E-06	5.55E-09	9.11E-07	45	42.60

Tabuľka *SymmLQ 10:*

***SymmLQ** metóda,* $n = 30\,000$						
	Pres_min		*Pres_avg*		*Iterácie*	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	*max*	*avg*
Riedkosť = 0.015%						
4	*3.75E-11*	*8.43E-10*	*7.12E-12*	*1.57E-10*	*104*	*82.53*
14	*1.36E-10*	*3.20E-09*	*1.93E-11*	*4.32E-10*	*124*	*84.55*
104	*7.70E-10*	*1.74E-08*	*1.41E-10*	*3.15E-09*	*109*	*84.47*
1 004	*9.76E-09*	*2.17E-07*	*1.49E-09*	*3.34E-08*	*123*	*84.59*
10 004	*9.85E-08*	*2.22E-06*	*1.50E-08*	*3.36E-07*	*113*	*84.68*
100 004	*7.48E-07*	*1.77E-05*	*1.52E-07*	*3.40E-06*	*112*	*84.46*
Riedkosť = 0.05%						
4	*1.20E-11*	*7.12E-10*	*2.80E-13*	*1.66E-11*	*53*	*50.40*
14	*3.69E-11*	*2.27E-09*	*8.16E-12*	*4.91E-10*	*64*	*53.00*
104	*8.81E-11*	*5.42E-09*	*3.03E-11*	*1.83E-09*	*65*	*53.50*
1 004	*7.54E-10*	*4.48E-08*	*3.62E-10*	*2.18E-08*	*55*	*52.00*
10 004	*7.70E-09*	*4.66E-07*	*3.18E-09*	*1.88E-07*	*55*	*52.20*
100 004	*6.65E-08*	*3.73E-06*	*3.00E-08*	*1.73E-06*	*65*	*53.60*
Riedkosť = 0.1%						
4	*6.78E-12*	*5.72E-10*	*1.06E-12*	*8.46E-11*	*50*	*45.85*
14	*1.41E-11*	*1.16E-09*	*2.99E-12*	*2.37E-10*	*50*	*46.31*
104	*7.16E-11*	*5.77E-09*	*2.09E-11*	*1.66E-09*	*61*	*46.66*
1 004	*6.98E-10*	*6.07E-08*	*2.00E-10*	*1.60E-08*	*50*	*46.66*
10 004	*1.33E-08*	*1.15E-06*	*2.01E-09*	*1.61E-07*	*61*	*46.80*
100 004	*7.08E-08*	*6.15E-06*	*1.90E-08*	*1.53E-06*	*50*	*46.77*
Riedkosť = 0.15%						
4	*8.57E-13*	*1.02E-10*	*4.01E-13*	*4.63E-11*	*44*	*41.70*
14	*3.35E-12*	*4.00E-10*	*1.42E-12*	*1.67E-10*	*44*	*41.80*
104	*5.92E-11*	*7.13E-09*	*1.72E-11*	*1.98E-09*	*44*	*42.00*
1 004	*2.47E-10*	*2.95E-08*	*8.99E-11*	*1.07E-08*	*44*	*42.10*
10 004	*1.73E-09*	*1.95E-07*	*7.00E-10*	*8.03E-08*	*45*	*42.70*
100 004	*3.20E-08*	*3.82E-06*	*1.12E-08*	*1.32E-06*	*44*	*42.20*

Tabuľka *SymmLQ 11:*

	SymmLQ metóda, $n = 50\,000$					
	Pres_min		Pres_avg		Iterácie	
$\|\boldsymbol{x}_1 - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	$\|\boldsymbol{A}\boldsymbol{x}_i - \boldsymbol{x}^*\|_\infty$	max	avg
			Riedkosť = 0.01%			
4	7.88E-11	1.85E-09	1.69E-11	3.88E-10	101	82.44
14	2.26E-10	5.25E-09	3.38E-11	7.82E-10	113	85.94
104	1.18E-09	2.77E-08	2.55E-10	5.86E-09	144	86.08
1 004	1.39E-08	3.24E-07	2.80E-09	6.44E-08	113	85.38
10 004	1.20E-07	2.79E-06	2.38E-08	5.47E-07	112	85.43
100 004	1.64E-06	3.86E-05	3.02E-07	6.95E-06	105	84.87
			Riedkosť = 0.02%			
4	2.36E-11	9.14E-10	8.36E-13	3.08E-11	88	72.00
14	5.45E-11	2.11E-09	1.92E-11	7.13E-10	84	72.80
104	5.45E-10	1.98E-08	1.55E-10	5.61E-09	87	74.20
1 004	3.61E-09	1.33E-07	1.79E-09	6.50E-08	80	71.90
10 004	3.91E-08	1.52E-06	1.84E-08	6.90E-07	86	72.70
100 004	4.46E-07	1.58E-05	1.75E-07	6.34E-06	103	75.00
			Riedkosť = 0.04%			
4	2.09E-11	1.21E-09	5.44E-12	3.20E-10	67	51.98
14	6.97E-11	4.53E-09	1.40E-11	8.43E-10	65	53.08
104	2.78E-10	1.66E-08	6.38E-11	3.84E-09	69	54.12
1 004	2.06E-09	1.27E-07	5.95E-10	3.55E-08	66	54.02
10 004	2.27E-08	1.39E-06	6.60E-09	3.96E-07	66	53.81
100 004	2.27E-07	1.39E-05	6.35E-08	3.84E-06	68	53.76
			Riedkosť = 0.06%			
4	7.18E-12	6.24E-10	2.61E-12	2.14E-10	48	46.10
14	3.83E-11	3.30E-09	1.33E-11	1.08E-09	49	46.40
104	1.02E-10	8.68E-09	4.40E-11	3.60E-09	50	47.00
1 004	2.13E-09	1.84E-07	4.53E-10	3.82E-08	50	47.00
10 004	8.87E-09	7.59E-07	4.16E-09	3.44E-07	49	47.00
100 004	8.86E-08	7.59E-06	3.48E-08	2.93E-06	51	47.20

Tabuľka *SymmLQ 12:*

Príloha 2

Zdrojkový kód programu

```
%--------------------------------------------
% A = Gen(n, d)
%--------------------------------------------
% Funkcia vrati kladne definitnu, symetricku,
% diagonalne dominantnu maticu
%
% pocet nenulovych prvkov k = n*d,
% pricom rozmery matice su n x n

function A = Gen(n, d)

n2 = n/2;
n4 = n2/2;

d2 = d - 1;

s1 = n/d2;
s2 = 1 - s1;

for h = 1 : d2
    j((h - 1)*n4 + 1 : h*n4) = round(s2*rand(n4, 1) + h*s1);
end

pom = 1 : n4;

i = pom;

for h = 2 : d2
    i = [i, pom];
end

clear pom

Value = [-1, 1, -2, 2];

P = sparse(i, j, Value(random('Discrete Uniform', 4, 1, n4*d2)), n4, n);

C = P(:, n2 + 1 : n);
```

```
clear P(:, n2 + 1 : n)

if C(1, 1) ~= 0
    C(1, 2) = C(1, 1);
    C(1, 1) = 0;
end

k = find(diag(C) ~= 0);

for h = 1 : length(k)
    if C(k(h), k(h) - 1) == 0
        C(k(h), k(h) - 1) = C(k(h), k(h));
    else
        C(k(h), k(h) + 1) = C(k(h), k(h));
    end
    C(k(h), k(h)) = 0;
end

if P(1, 1) ~= 0
    P(1, 2) = P(1, 1);
    P(1, 1) = 0;
end

k = find(diag(P) ~= 0);

for h = 1 : length(k)
    if P(k(h), k(h) - 1) == 0
        P(k(h), k(h) - 1) = P(k(h), k(h));
    else
        P(k(h), k(h) + 1) = P(k(h), k(h));
    end
    P(k(h), k(h)) = 0;
end

C = [C; P(:, n4 + 1 : n2)', P(:, 1 : n4)'];
clear P

L = tril(C, -1);
L = fliplr(flipud(L));
U = triu(C, 1);
U = fliplr(flipud(U));

L = L + L';
U = U + U';

D = sum(abs(U)) + sum(abs(C)) + 1;
D = sparse(1 : n2, 1 : n2, D, n2, n2);
U = U + D;
```

```
D = sum(abs(L)) + sum(abs(C')) + 1;
D = sparse(1 : n2, 1 : n2, D, n2, n2);
L = L + D;

clear D

A = [L, C; C', U];
```

```
%----------------------------------------------
% [x, Pres, Iter] = CG(A, x_ries, x1)
%----------------------------------------------
% Funkcia vrati riesenie sustavy
% Ax = b, kde
% A - je kladne definitna, symetricka,
% diagonalne dominantna matica n x n
%
% CG - The conjugate gradient algorithm


function [Pres, Iter] = CG(A, x_ries, x1)

n = length(x1);

b = A*x_ries;
x = x1;
r = A*x - b;
Pres(1, 1) = norm(x - x_ries, inf);
Pres(1, 2) = norm(r, inf);
p = r;
q = A*p;
s1 = r'*r;
u = p'*q;

i = 0;

Log = 1;

while i < n & u ~= 0 & s1 ~= 0 & Log > .1
    i = i + 1;

    u = s1/u;
    x = x - p*u;
    r = r - q*u;
    s2 = r'*r;
    v = s2/s1;
    p = r + p*v;
    q = A*p;
    u = p'*q;
    s1 = s2;

    Pres(i + 1, 1) = norm(x_ries - x, inf);
    Pres(i + 1, 2) = norm(A*x - b, inf);

    if i > 13
        Log = abs(log10(max(Pres(i- 3:i+1, 2))/min(Pres(i-13:i+1, 2))));
    end
end
```

```
Iter = i;
```

```
%--------------------------------------------
% [x, Pres, Iter] = CR(A, x_ries, x1)
%--------------------------------------------
% Funkcia vrati riesenie sustavy
% Ax = b, kde
% A - je kladne definitna, symetricka,
% diagonalne dominantna matica n x n
%
% CR - The conjugate residual algorithm


function [Pres, Iter] = CR(A, x_ries, x1)

n = length(x1);

b = A*x_ries;
x = x1;
r = A*x - b;
Pres(1, 1) = norm(x - x_ries, inf);
Pres(1, 2) = norm(r, inf);
p = r;
q = A*p;
d = q'*q;
s = q;

i = 0;
v = s'*r;

Log = 1;

while i < n & d ~= 0 & v ~= 0 & Log > .1
    i = i + 1;

    u = v/d;

    x = x - u*p;
    r = r - u*q;

    s = A*r;
    v = (s'*r)/v;
    p = r + v*p;
    q = s + v*q;

    d = q'*q;
    v = s'*r;

    Pres(i + 1, 1) = norm(x_ries - x, inf);
    Pres(i + 1, 2) = norm(A*x - b, inf);
```

```
    if i > 13
        Log = abs(log10(max(Pres(i-13 : i+1, 2))/min(Pres(i-13 : i+1, 2))));
    end
end

Iter = i;
```

```
%---------------------------------------------
% [x, Pres, Iter] = CGNR(A, x_ries, x1)
%---------------------------------------------
% Funkcia vrati riesenie sustavy
% Ax = b, kde
% A - je kladne definitna, symetricka,
% diagonalne dominantna matica n x n
%
% CGNR - The CG normal residuals algorithm


function [Pres, Iter] = CGNR(A, x_ries, x1)

n = length(x1);

b = A*x_ries;
x = x1;
r = A*x - b;
Pres(1, 1) = norm(x - x_ries, inf);
Pres(1, 2) = norm(r, inf);
p = A*r;
q = A*p;
h = q'*q;
s1 = p'*p;

i = 0;

Log = 1;

while i < n & h ~= 0 & s1 ~= 0 & Log > .1
    i = i + 1;

    u = s1/h;
    x = x - p*u;
    r = r - q*u;
    v = A*r;
    s2 = v'*v;
    p = v + p*(s2/s1);
    q = A*p;
    h = q'*q;
    s1 = s2;

    Pres(i + 1, 1) = norm(x - x_ries, inf);
    Pres(i + 1, 2) = norm(A*x - b, inf);

    if i > 13
        Log = abs(log10(max(Pres(i-13 : i+1, 2))/min(Pres(i-13: i+1, 2))));
    end
end
```

```
Iter = i;
```

```
%---------------------------------------------
% [x, Pres, Iter] = GMRes(A, x_ries, x1)
%---------------------------------------------
% Funkcia vrati riesenie sustavy
% Ax = b, kde
% A - je kladne definitna, symetricka,
% diagonalne dominantna matica n x n
%
% GMRes - The generalised minimal residuals


function [Pres, Iter] = GMRes(A, x_ries, x1)

n = length(x1);

b = A*x_ries;
x = x1;
r = A*x - b;
Pres(1, 1) = norm(x - x_ries, inf);
Pres(1, 2) = norm(r, inf);
gama = norm(r);
i = 1;
p1 = r/gama;
q = A*p1;
h1 = p1'*q ;
p2 = q - h1*p1;
h2 = norm(p2);
if h2 == 0 | i == n
    p2 = zeros(n, 1);
    u1 = h1;
else
    p2 = p2/h2;
    u1 = sqrt(h1^2 + h2^2);
end
z1 = p1/u1;
cos1 = h1/u1;
sin1 = -h2/u1;
x = x - z1*gama*cos1;
gama = gama*sin1;
%------------------------------------------------------------
Pres(2, 1) = norm(x - x_ries, inf);
Pres(2, 2) = norm(A*x - b, inf);

i = 2;
q = A*p2;
h1 = p2'*q;
p1 = q - p1*h2 -p2*h1;
h_ = sin1*h2 + cos1*h1;
h3 = norm(p1);
```

```
if h3 == 0 | i == n
    p1 = zeros(n, 1);
    u2 = h_;
else
    p1 = p1/h3;
    u2 = sqrt(h_^2 + h3^2);
end
u1 = cos1*h2 - sin1*h1;
cos2 = h_/u2;
sin2 = -h3/u2;
z2 = (p2 - z1*u1)/u2;
x = x - z2*gama*cos2;
gama = gama*sin2;
%-----------------------------------------------------------
Pres(3, 1) = norm(x - x_ries, inf);
Pres(3, 2) = norm(A*x - b, inf);

p3 = p1;

Log = 1;

while i < n & gama ~= 0 & Log > .1
    i = i + 1;

    h1 = h3;

    p1 = p2;
    p2 = p3;

    q = A*p2;
    h2 = p2'*q;
    p3 = q - p1*h1 - p2*h2;
    h_ = sin2*cos1*h1 + cos2*h2;
    h3 = norm(p3);
    if h3 == 0 | i == n
       p3 = zeros(n, 1);
       u3 = h_;
    else
       p3 = p3/h3;
       u3 = sqrt(h_^2 + h3^2);
    end
    u2 = cos2*cos1*h1 - sin2*h2;
    u1 = -sin1*h1;

    cos1 = cos2;
    sin1 = sin2;

    cos2 = h_/u3;
    sin2 = -h3/u3;
```

```
    z3 = (p2 - z1*u1 - z2*u2)/u3;
    x = x - z3*gama*cos2;
    gama = gama*sin2;

    z1 = z2;
    z2 = z3;

    Pres(i + 1, 1) = norm(x - x_ries, inf);
    Pres(i + 1, 2) = norm(A*x - b, inf);

    if i > 13
        Log = abs(log10(max(Pres(i-13 : i+1, 2))/min(Pres(i-13 : i+1, 2))));
    end

end

Iter = i;
```

```
%-------------------------------------------
% [x, Pres, Iter] = LSQR(A, x_ries, x1)
%-------------------------------------------
% Funkcia vrati riesenie sustavy
% Ax = b, kde
% A - je kladne definitna, symetricka,
% diagonalne dominantna matica n x n
%
% LSQR - The least-square QR


function [Pres, Iter] = LSQR(A, x_ries, x1)

n = length(x1);

b = A*x_ries;
x = x1;
r = A*x - b;
Pres(1, 1) = norm(x - x_ries, inf);
Pres(1, 2) = norm(r, inf);
i = 1;
norm_r1 = norm(r);
if norm_r1 == 0
    return
end
s = r/norm_r1;
v = A*s;
alfa = norm(v);
if alfa == 0
    return
end
v = v/alfa;
s = A*v - alfa*s;
beta = norm(s);
if beta == 0
    u2 = alfa;
    cos = 1;
    sin = 0;
else
    u2 = sqrt(alfa^2 + beta^2);
    cos = alfa/u2;
    sin = -beta/u2;
    s = s/beta;
end
w = v/u2;
q = A*w;
p = q'*r;
x = x - w*p;
r = r - q*p;
```

```
%-----------------------------------------------------
Pres(2, 1) = norm(x_ries - x, inf);
Pres(2, 2) = norm(A*x - b, inf);

v = A*s - beta*v;
alfa = norm(v);

Log = 1;

while i < n & alfa ~= 0 & Log > .1
    i = i + 1;

    if alfa == 0
        return
    end
    v = v/alfa;
    alfa_ = cos*alfa;
    u1 = -sin*alfa;
    s = A*v - alfa*s;
    beta = norm(s);
    if beta == 0
        u2 = alfa_;
        cos = 1;
        sin = 0;
    else
        u2 = sqrt(alfa_^2 + beta^2);
        cos = alfa_/u2;
        sin = -beta/u2;
        s = s/beta;
    end
    w = (v - w*u1)/u2;
    q = A*w;
    p = q'*r;
    x = x - w*p;
    r = r - p*q;
    v = A*s - beta*v;
    alfa = norm(v);

    Pres(i + 1, 1) = norm(x_ries - x, 2);
    Pres(i + 1, 2) = norm(A*x - b, 2);

    if i > 13
        Log = abs(log10(max(Pres(i-13 : i+1, 2))/min(Pres(i-13: i+1, 2))));
    end
end

Iter = i;
```

```
%---------------------------------------------
% [x, Pres, Iter] = SymmLQ(A, x_ries, x1)
%---------------------------------------------
% Funkcia vrati riesenie sustavy
% Ax = b, kde
% A - je kladne definitna, symetricka,
% diagonalne dominantna matica n x n
%
% SymmLQ - The Symmetric LQ


function [Pres, Iter] = SymmLQ(A, x_ries, x1)

n = length(x1);

b = A*x_ries;
i = 0;
x = x1;
r = A*x - b;
Pres(1, 1) = norm(x - x_ries, inf);
Pres(1, 2) = norm(r, inf);
norm_r1 = norm(r);
if norm_r1 == 0
    return
end
i = 1;
p1 = r/norm_r1;
q = A*p1;
h1 = p1'*q ;
p2 = q - h1*p1;
h2 = norm(p2);
if h2 == 0 | i == n
    p2 = zeros(n, 1);
    u1 = h1;
else
    p2 = p2/h2;
    u1 = sqrt(h1^2 + h2^2);
end
z1 = p1/u1;
y1 = q/u1;
cos1 = h1/u1;
sin1 = -h2/u1;
s1 = r'*z1;
x = x - y1*s1;
r = r - A*y1*s1;
%-----------------------------------------------------------
Pres(2, 1) = norm(x - x_ries, inf);
Pres(2, 2) = norm(A*x - b, inf);
```

```
i = 2;
q = A*p2;
h1 = p2'*q;
p1 = q - p1*h2 - p2*h1;
h_ = sin1*h2 + cos1*h1;
h3 = norm(p1);
if h3 == 0 | i == n
    p1 = zeros(n, 1);
    u2 = h_;
else
    p1 = p1/h3;
    u2 = sqrt(h_^2 + h3^2);
end
u1 = cos1*h2 - sin1*h1;
cos2 = h_/u2;
sin2 = -h3/u2;
z2 = (p2 - z1*u1)/u2;
y2 = (q - y1*u1)/u2;
s2 = r'*z2;
x = x - y2*s2;
r = r - A*y2*s2;
%-----------------------------------------------------------
Pres(3, 1) = norm(x - x_ries, inf);
Pres(3, 2) = norm(A*x - b, inf);

p3 = p1;

Log = 1;

while i < n & Log > .1
    i = i + 1;

    h1 = h3;

    p1 = p2;
    p2 = p3;

    q = A*p2;
    h2 = p2'*q;
    p3 = q - p1*h1 - p2*h2;
    h_ = sin2*cos1*h1 + cos2*h2;
    h3 = norm(p3);
    if h3 == 0 | i == n
       p3 = zeros(n, 1);
       u3 = h_;
    else
       p3 = p3/h3;
       u3 = sqrt(h_^2 + h3^2);
    end
```

```
    u2 = cos2*cos1*h1 - sin2*h2;
    u1 = -sin1*h1;

    cos1 = cos2;
    sin1 = sin2;

    cos2 = h_/u3;
    sin2 = -h3/u3;
    z3 = (p2 - z1*u1 - z2*u2)/u3;
    y3 = (q - y1*u1 - y2*u2)/u3;
    s3 = r'*z3;
    x = x - y3*s3;
    r = r - A*y3*s3;

    z1 = z2;
    z2 = z3;

    y1 = y2;
    y2 = y3;

    Pres(i + 1, 1) = norm(x - x_ries, inf);
    Pres(i + 1, 2) = norm(A*x - b, inf);

    if i > 13
        Log = abs(log10(max(Pres(i-13 : i+1, 2))/min(Pres(i-13: i+1, 2))));
    end
end

Iter = i;
```

```
%-----------------------------------------------------------------------------
% [Pres_Max,Pres_Avg,Pres_Iter_Max,Iter_Max,Iter_Avg,Iter_Min] = Test100(n, d)
%-----------------------------------------------------------------------------

function [Pres_Min, Pres_Avg, Iter_Max, Iter_Avg] = Test100(n,d)

Values = (-2 : 2)';

Iter_Max1 = -1;
Iter_Avg1 = 0;
Iter_Min1 = n + 1;

Iter_Max2 = -1;
Iter_Avg2 = 0;
Iter_Min2 = n + 1;

Iter_Max3 = -1;
Iter_Avg3 = 0;
Iter_Min3 = n + 1;

Iter_Max4 = -1;
Iter_Avg4 = 0;
Iter_Min4 = n + 1;

Iter_Max5 = -1;
Iter_Avg5 = 0;
Iter_Min5 = n + 1;

Iter_Max6 = -1;
Iter_Avg6 = 0;
Iter_Min6 = n + 1;

rand('state', 11684)

x1 = Values(random('Discrete Uniform', 5, n, 1));
x_ries = Values(random('Discrete Uniform', 5, n, 1));

for k = 1 : 100
    disp(['k = ' int2str(k)])
    A = Gen(n, d);
    %-----------------------------------
    % x1
    %-----------------------------------
    [Pres, Iter] = CG(A, x_ries, x1);

    Iter_Avg1  = Iter_Avg1 + Iter;

    if Iter_Min1  > Iter + 1
        Iter_Min1  = Iter + 1;
```

```
end

if k == 1
    Pres_Min1 = zeros(Iter_Min1, 2);
    Pres_Avg1 = zeros(Iter_Min1, 2);
end

Pres_Min1(1:Iter_Min1, 1) =
                 max([Pres_Min1(1:Iter_Min1, 1), Pres(1:Iter_Min1, 1)]')';
Pres_Min1(1:Iter_Min1, 2) =
                 max([Pres_Min1(1:Iter_Min1, 2), Pres(1:Iter_Min1, 2)]')';
Pres_Avg1 = Pres_Avg1(1:Iter_Min1, :) + Pres(1:Iter_Min1, :);

if Iter_Max1  < Iter + 1
    Iter_Max1  = Iter + 1;
end
%------------------------------------
% x1 + 10
%------------------------------------
[Pres, Iter] = CG(A, x_ries, x1 + 10);

Iter_Avg2  = Iter_Avg2 + Iter;

if Iter_Min2  > Iter + 1
    Iter_Min2  = Iter + 1;
end

if k == 1
    Pres_Min2 = zeros(Iter_Min2, 2);
    Pres_Avg2 = zeros(Iter_Min2, 2);
end

Pres_Min2(1:Iter_Min2, 1) =
                 max([Pres_Min2(1:Iter_Min2, 1), Pres(1:Iter_Min2, 1)]')';
Pres_Min2(1:Iter_Min2, 2) =
                 max([Pres_Min2(1:Iter_Min2, 2), Pres(1:Iter_Min2, 2)]')';
Pres_Avg2 = Pres_Avg2(1:Iter_Min2, :) + Pres(1:Iter_Min2, :);

if Iter_Max2  < Iter + 1
    Iter_Max2  = Iter + 1;
end
%------------------------------------
% x1 + 100
%------------------------------------
[Pres, Iter] = CG(A, x_ries, x1 + 100);

Iter_Avg3  = Iter_Avg3 + Iter;

if Iter_Min3  > Iter + 1
```

```
    Iter_Min3  = Iter + 1;
end

if k == 1
    Pres_Min3 = zeros(Iter_Min3, 2);
    Pres_Avg3 = zeros(Iter_Min3, 2);
end

Pres_Min3(1:Iter_Min3, 1) =
                 max([Pres_Min3(1:Iter_Min3, 1), Pres(1:Iter_Min3, 1)]')';
Pres_Min3(1:Iter_Min3, 2) =
                 max([Pres_Min3(1:Iter_Min3, 2), Pres(1:Iter_Min3, 2)]')';
Pres_Avg3 = Pres_Avg3(1:Iter_Min3, :) + Pres(1:Iter_Min3, :);

if Iter_Max3  < Iter + 1
    Iter_Max3  = Iter + 1;
end
%------------------------------------
% x1 + 1000
%------------------------------------
[Pres, Iter] = CG(A, x_ries, x1 + 1000);

Iter_Avg4  = Iter_Avg4 + Iter;

if Iter_Min4  > Iter + 1
    Iter_Min4  = Iter + 1;
end

if k == 1
    Pres_Min4 = zeros(Iter_Min4, 2);
    Pres_Avg4 = zeros(Iter_Min4, 2);
end

Pres_Min4(1:Iter_Min4, 1) =
                 max([Pres_Min4(1:Iter_Min4, 1), Pres(1:Iter_Min4, 1)]')';
Pres_Min4(1:Iter_Min4, 2) =
                 max([Pres_Min4(1:Iter_Min4, 2), Pres(1:Iter_Min4, 2)]')';
Pres_Avg4 = Pres_Avg4(1:Iter_Min4, :) + Pres(1:Iter_Min4, :);

if Iter_Max4  < Iter + 1
    Iter_Max4  = Iter + 1;
end
%------------------------------------
% x1 + 10 000
%------------------------------------
[Pres, Iter] = CG(A, x_ries, x1 + 10000);

Iter_Avg5  = Iter_Avg5 + Iter;
```

```
    if Iter_Min5  > Iter + 1
        Iter_Min5  = Iter + 1;
    end

    if k == 1
        Pres_Min5 = zeros(Iter_Min5, 2);
        Pres_Avg5 = zeros(Iter_Min5, 2);
    end

    Pres_Min5(1:Iter_Min5, 1) =
                  max([Pres_Min5(1:Iter_Min5, 1), Pres(1:Iter_Min5, 1)]')';
    Pres_Min5(1:Iter_Min5, 2) =
                  max([Pres_Min5(1:Iter_Min5, 2), Pres(1:Iter_Min5, 2)]')';
    Pres_Avg5 = Pres_Avg5(1:Iter_Min5, :) + Pres(1:Iter_Min5, :);

    if Iter_Max5  < Iter + 1
        Iter_Max5  = Iter + 1;
    end
    %------------------------------------
    % x1 + 100 000
    %------------------------------------
    [Pres, Iter] = CG(A, x_ries, x1 + 100000);

    Iter_Avg6  = Iter_Avg6 + Iter;

    if Iter_Min6  > Iter + 1
        Iter_Min6  = Iter + 1;
    end

    if k == 1
        Pres_Min6 = zeros(Iter_Min6, 2);
        Pres_Avg6 = zeros(Iter_Min6, 2);
    end

    Pres_Min6(1:Iter_Min6, 1) =
                  max([Pres_Min6(1:Iter_Min6, 1), Pres(1:Iter_Min6, 1)]')';
    Pres_Min6(1:Iter_Min6, 2) =
                  max([Pres_Min6(1:Iter_Min6, 2), Pres(1:Iter_Min6, 2)]')';
    Pres_Avg6 = Pres_Avg6(1:Iter_Min6, :) + Pres(1:Iter_Min6, :);

    if Iter_Max6  < Iter + 1
        Iter_Max6  = Iter + 1;
    end
end

Iter_Avg1 = .01*Iter_Avg1;
Pres_Avg1 = .01*Pres_Avg1;

Iter_Avg2 = .01*Iter_Avg2;
```

```
Pres_Avg2 = .01*Pres_Avg2;

Iter_Avg3 = .01*Iter_Avg3;
Pres_Avg3 = .01*Pres_Avg3;

Iter_Avg4 = .01*Iter_Avg4;
Pres_Avg4 = .01*Pres_Avg4;

Iter_Avg5 = .01*Iter_Avg5;
Pres_Avg5 = .01*Pres_Avg5;

Iter_Avg6 = .01*Iter_Avg6;
Pres_Avg6 = .01*Pres_Avg6;

Pom = max([Iter_Min1,Iter_Min2,Iter_Min3,Iter_Min4,Iter_Min5,Iter_Min6]);
Pres_Min = -ones(Pom, 12);
Pres_Avg = -ones(Pom, 12);

Pres_Min(1 : Iter_Min1, 1) = Pres_Min1(1 : Iter_Min1, 1);
Pres_Avg(1 : Iter_Min1, 1) = Pres_Avg1(1 : Iter_Min1, 1);

Pres_Min(1 : Iter_Min2, 2) = Pres_Min2(1 : Iter_Min2, 1);
Pres_Avg(1 : Iter_Min2, 2) = Pres_Avg2(1 : Iter_Min2, 1);

Pres_Min(1 : Iter_Min3, 3) = Pres_Min3(1 : Iter_Min3, 1);
Pres_Avg(1 : Iter_Min3, 3) = Pres_Avg3(1 : Iter_Min3, 1);

Pres_Min(1 : Iter_Min4, 4) = Pres_Min4(1 : Iter_Min4, 1);
Pres_Avg(1 : Iter_Min4, 4) = Pres_Avg4(1 : Iter_Min4, 1);

Pres_Min(1 : Iter_Min5, 5) = Pres_Min5(1 : Iter_Min5, 1);
Pres_Avg(1 : Iter_Min5, 5) = Pres_Avg5(1 : Iter_Min5, 1);

Pres_Min(1 : Iter_Min6, 6) = Pres_Min6(1 : Iter_Min6, 1);
Pres_Avg(1 : Iter_Min6, 6) = Pres_Avg6(1 : Iter_Min6, 1);

Pres_Min(1 : Iter_Min1, 7) = Pres_Min1(1 : Iter_Min1, 2);
Pres_Avg(1 : Iter_Min1, 7) = Pres_Avg1(1 : Iter_Min1, 2);
clear Pres_Min1 Pres_Avg1

Pres_Min(1 : Iter_Min2, 8) = Pres_Min2(1 : Iter_Min2, 2);
Pres_Avg(1 : Iter_Min2, 8) = Pres_Avg2(1 : Iter_Min2, 2);
clear Pres_Min2 Pres_Avg2

Pres_Min(1 : Iter_Min3, 9) = Pres_Min3(1 : Iter_Min3, 2);
Pres_Avg(1 : Iter_Min3, 9) = Pres_Avg3(1 : Iter_Min3, 2);
clear Pres_Min3 Pres_Avg3

Pres_Min(1 : Iter_Min4, 10) = Pres_Min4(1 : Iter_Min4, 2);
```

```
Pres_Avg(1 : Iter_Min4, 10) = Pres_Avg4(1 : Iter_Min4, 2);
clear Pres_Min4 Pres_Avg4

Pres_Min(1 : Iter_Min5, 11) = Pres_Min5(1 : Iter_Min5, 2);
Pres_Avg(1 : Iter_Min5, 11) = Pres_Avg5(1 : Iter_Min5, 2);
clear Pres_Min5 Pres_Avg5

Pres_Min(1 : Iter_Min6, 12) = Pres_Min6(1 : Iter_Min6, 2);
Pres_Avg(1 : Iter_Min6, 12) = Pres_Avg6(1 : Iter_Min6, 2);
clear Pres_Min6 Pres_Avg6


Iter_Max = [Iter_Max1; Iter_Max2; Iter_Max3; Iter_Max4;Iter_Max5; Iter_Max6];
Iter_Avg = [Iter_Avg1; Iter_Avg2; Iter_Avg3; Iter_Avg4; Iter_Avg5; Iter_Avg6];
```

```
%---------------------------
% Zapis_Test100(Filename, n)
%---------------------------

function Zapis_Test100(Filename, n)
Filename = [Filename '.txt'];
Error_Message = ['Problemy so suborom ', Filename];

for i = 1 : length(n)
    n_ = n(i);

    switch n_
          case 100
            d = [5,  10,  15,  20,  25     ];
          case 200
            d = [5,  10,  20,  30,  40,  50];
          case 300
            d = [5,  20,  30,  50,  60,  75];
          case 500
            d = [5,  25,  50,  75, 100, 125];
          case 1000
            d = [5,  50, 100, 150, 200, 250];
          case 2000
            d = [5, 100, 200, 300, 400, 500];
          case 3000
            d = [5, 200, 300, 500, 600, 750];
          case 5000
            d = [5, 150, 300, 500, 600     ];
          case 10000
            d = [5,  50, 100, 200, 300     ];
          case 20000
            d = [5,  30,  50,  75, 100, 150];
          case 30000
            d = [5,  20,  30,  50,  75, 100];
          case 50000
            d = [5,  10,  20,  30,  50,  60];
          case 100000
            d = [5,  10,  15,  20,  25,  30];
          otherwise
            disp('Unknown ''n''!')
            d = [];
        end

    for j = 1 : length(d)
        disp(['n = ' int2str(n_)])
        disp(['d = ' int2str(d(j))])

        [Pres_Min,Pres_Avg,Iter_Max,Iter_Avg] = Test100(n(i), d(j));
```

```
        [fid, Error_Message] = fopen(Filename, 'a');

        fprintf(fid, ['n=', int2str(n(i)), '\n']);
        fprintf(fid, ['d=', int2str(d(j)), '\n']);
        fprintf(fid, ['k=n*d=', int2str(n(i)*d(j)), '\n']);
        fprintf(fid, ['riedkost=', num2str(n(i)*d(j)*100/n(i)^2, 6), '%%\n\n']);

        fprintf(fid, 'Pres_Min_Error\n');
        fprintf(fid, 'x1  x1+10  x1+100  x1+1000 x1+10000 x1+100000\n');
        fprintf(fid, '%0.12g %0.12g %0.12g %0.12g %0.12g %0.12g\n',
                                                Pres_Min(:,1:6)');

        fprintf(fid, 'Pres_Min_Residuals\n');
        fprintf(fid, 'x1  x1+10  x1+100  x1+1000  x1+10000 x1+100000\n');
        fprintf(fid, '%0.12g %0.12g %0.12g %0.12g %0.12g %0.12g\n',
                                               Pres_Min(:,7:12)');

        fprintf(fid, 'Pres_Avg_Error\n');
        fprintf(fid, 'x1  x1+10  x1+100  x1+1000  x1+10000 x1+100000\n');
        fprintf(fid, '%0.12g %0.12g %0.12g %0.12g %0.12g %0.12g\n',
                                                Pres_Avg(:,1:6)');

        fprintf(fid, 'Pres_Avg_Residuals\n');
        fprintf(fid, 'x1  x1+10  x1+100  x1+1000 x1+10000 x1+100000\n');
        fprintf(fid, '%0.12g %0.12g %0.12g %0.12g %0.12g %0.12g\n',
                                               Pres_Avg(:,7:12)');

        fprintf(fid, 'Iter_Max\n');
        fprintf(fid, 'x1  x1+10  x1+100  x1+1000 x1+10000 x1+100000\n');
        fprintf(fid, '%0.12g %0.12g %0.12g %0.12g %0.12g %0.12g\n',
                                                       Iter_Max');

        fprintf(fid, 'Iter_Avg\n');
        fprintf(fid, 'x1  x1+10  x1+100  x1+1000 x1+10000 x1+100000\n');
        fprintf(fid, '%0.12g %0.12g %0.12g %0.12g %0.12g %0.12g\n\n',
                                                         Iter_Avg');

        fclose(fid);

    end
end

disp('HOTOVO')
```