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Mean-Variance Hedging with Uncertain Trade Execution

(Master Thesis)

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Kvadratické Zaistenie s Neistým Zobchodovaným Množstvom

(Diplomová práca)

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I declare this thesis was written on my own, with the only help provided by my supervisor and the referred-to literature and sources.

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Abstract

The purpose of mean-variance hedging is to find a dynamic hedging strategy minimizing the unconditional squared hedging error (where error equals the difference between the value of a self-financing portfolio and the value of a derivative asset at maturity). In the thesis we focused on discrete-time models for which the standard mean-variance theory (in the most general setting with locally square-integrable price processes) is described in Černý and Kallsen (2007). We extended this theory to situations where there is non-zero probability that a trade may not be executed at any given time. The economic rationale for such a model is derived from limit market orders. Compared to classical mean-variance hedging theory the new problem contains an additional endogenous state variable. We identified this state variable and derived full dynamic programming solution for the optimal strategy.

Keywords: mean-variance hedging, stock liquidity, dynamic programming.

Abstrakt

Cieľom kvadratického zaistenia je nájsť dynamickú zaisťovaciu stratégiu, ktorá minimalizuje nepodmienenú kvadratickú zaisťovaciu chybu (kde chyba je definovaná ako rozdiel medzi hodnotou samofinancovaného portfólia a hodnotou finančného derivátu v čase expirácie). V tejto práci sme sa zamerali na diskrétne časové modely, pre ktoré je teória kvadratického zaisťovania opísaná v článku Černého a Kallsena (2007). Rozšírili sme túto teóriu o situácie, kde je nenulová pravdepodobnosť, že časť obchodu nebude nikdy zrealizovaná. Ekonomicky sa tento model dá interpretovať ako zadanie limitného pokynu. V porovnaní s klasickou kvadratickou zaisťovacou teóriou sa nový problém rozširuje o novú endogénnu stavovú premennú. Indentifikovali sme túto premennú a odvodili sme optimálnu stratégiu.

Kľúčové slová: kvadratické zaistenie, likvidita akcie, dynamické programovanie.

Contents

Chapter 1

Introduction

Derivative securities essential part of the modern financial markets in the last years. Practical needs of investors led to formation of modern financial instruments such as equity and currency options, swaps, forwards etc., which originally served as a tool to transfer risk. The main contribution to the rapid development of the derivative market was made by breaking result of Black and Scholes [BS73]. Well-known model introduced by Black and Scholes makes some explicit assumptions, which allow creating a portfolio which perfectly hedges an option (or any other derivative). Then an arbitrage-free argument is used: "If options are correctly priced in the market, it should not be possible to make sure profits by creating portfolios of long and short positions in options and their underlying stocks." [BS73]. However, assumptions of the model often do not hold in the real market. A price of an asset usually does not follow a pure difussion process, but may jump in response to surprise events which lead to inability to hedge a contingent claim perfectly - the market is incomplete.

The problem of pricing and hedging contingent claims in incomplete markets has led to the development of various valuation methodologies. A classical one is the mean-variance hedging approach. It minimizes the expectation of the square hedging error (square difference between the value of the self-financing portfolio and the contingent claim at the maturity) among all self-financing strategies. This approach was introduced (in the martingale case) by Föllmer and Sonderman [FS86]. Subsequent extensions to the general semimartingale case were done in the mean-time. For more detailed overview of the literature we refer the reader to Černý and Kallsen [eK07].

Our thesis extends the work of Černý and Kallsen [eK07], in which the standard mean-variance theory is described in the general setting with locally square-integrable price processes. We work with a more specific discrete-time model described in Černý and Kallsen [eK09] and broaden the standard mean-variance theory to situations where there is non-zero probability that a trade may not be executed at any given time. We indentify a new endogenous state variable, derive full dynamic programming solution for the optimal strategy and perform a recursive computation of unconditional expected squared hedging error in the modified model which we compare to the results of the standard model.

The thesis is organized as follows. Chapter 2 briefly describes classical discrete-time mean-variance hedging theory. In Chapter 3 we describe local and global risk minimization for the extended model; these processes are closely related to each other. We also introduce the full dynamic programming solution for the extended model. Numerical implementation of presented theory in chapter 4 compares the results of extended model with the standard model. Chapter 5 presents results and conclusions of this thesis. Proofs and most of the derivations are defered to appendix which is in the sixth chapter.

Chapter 2

Mean-Variance Hedging in discrete-time

In this chapter we present the notation and assumptions used in the following sections and briefly introduce the general discrete time mean-variance theory. Presentation is based on the paper of Černý and Kallsen [eK09]. Discrete-time model is a special case of the general setting with locally square-integrable price processes, described in Černý and Kallsen [eK07]. The main emphasis of this chapter is put on explanation of difference between locally and globally optimal hedging strategy. We also underline why it is interesting and relevant to consider a local risk minimization approach.

The mean-variance hedging approach minimizes the *unconditional* (time 0) expected squared hedging error at maturity of contingent claim

$$\inf_{\vartheta} E((\upsilon + \vartheta \bullet S_T - H)^2),$$

over all admissible trading strategies ϑ (number of stocks held in the portfolio) and all admissible initial endowments υ. The random variable H denotes the payoff of the contingent claim to be hedged, S represents a discounted price process of the underlying asset and $\vartheta \bullet S_T$ represents gains from trading in the time interval $[0, T]$. The variables υ, S, H are expressed in terms of appropriate numeraire, most comonly the risk-free bank account.

2.1 Notation and Assumptions

Consider a time horizon T and the set of trading dates $\tau := \{0, 1, ..., T\}$. We fix a probability space $(\Omega, P, \mathcal{F})$, a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in\tau}$, $\mathcal{F}_T = \mathcal{F}$ and an $\mathcal{F}_T$-measurable contingent claim $H \in L^2(P)$. We introduce the following notation for conditional expectations,

$$\begin{aligned} E_t(X) &: = E(X|\mathcal{F}_t), \\ Var_t(X) &: = E_t(X^2) - (E_t(X))^2. \end{aligned}$$

The discounted stock price process $\{S_t\}_{t\in\tau}$ is adapted to $\mathbb{F}$ and we assume that S is locally square-integrable, i.e. for $\Delta S_{t+1} = S_{t+1} - S_t$ we have

$$E_t((\Delta S_{t+1})^2) < \infty \ for \ t < T.$$

Definition 1. *We say that process S admits no arbitrage , if for all $t \in \tau\backslash\{0\}$ and all $\mathcal{F}_{t-1}$-measurable portfolios ϑ_t we have that $\vartheta_t \Delta S_t \geq 0$ almost surely implies $\vartheta_t \Delta S_t = 0$ almost surely.*

We assume, that S is arbitrage-free in the sense of above definition.

Proposition 8.4. in [eK09] claims that in a discrete-time model the definition of admissibility of the strategy changes from the more involving general form into the simpler one.

Definition 2. *We say that (υ, ϑ) is an admissible endowment-strategy pair if and only if υ is $\mathcal{F}_0$-measurable, $\vartheta = \{\vartheta_t\}_{t\in\tau\backslash\{0\}}$ is predictable, meaning that ϑ_t is $\mathcal{F}_{t-1}$-measurable, and*

$$\upsilon + \vartheta \bullet S_T = \upsilon + \sum_{t=1}^{T} \vartheta_t \Delta S_t \in L^2(P).$$

The set of adimissible trading strategies with initial endowment υ is denoted $\overline{\Theta}(\upsilon)$. We write $\overline{\Theta}$ as a shortland for $\overline{\Theta}(0)$.

As ϑ_t is number of units of stocks in portfolio at time t, it represents what is commonly known as the option delta.

2.2 Comparison of Locally and Globally Optimal Strategy

The problem defined above can be solved by globally optimal strategy, which minimizes the resulting square difference between values of hedging portfolio and payoff of contingent claim at maturity. This strategy is path dependent. It means that if we are in a particular trading date, we need to know the actual price of stock, time remaining to maturity **and** the actual value of the hedging portfolio to compute an optimal hedging strategy (optimal number of stock held in the portfolio up to the next trading time). Actual value of the hedging portfolio is determined by previous trading, so we need to know the history of our trading to make a right decision to the future. This property can be sometimes unwelcome.

An example of path-independent strategy is a locally optimal strategy presented by Föllmer and Schweizer [FS88], who suggested a way how to evaluate an option in the incomplete market using a sequential regression. They also showed that their pricing formula reduces to the Black-Scholes formula in the complete market and the continuous-time hedging. In their theory the global optimization problem is separated into the one-period conditional optimization problems solved by least-squares regression. Instead of minimizing the total squared hedging error at maturity, one minimizes just a one-step ahead squared hedging error.

The whole trick of local hedging lies in the fact, that we pretend we can choose an arbitrary value of hedging portfolio in each trading date. In other words, we find the optimal hedging strategy as if we do not require hedging portfolio to be self-financing. At time $T-1$ we define

$$\begin{aligned}\{V_{T-1}, \xi_T\} &= \operatorname*{arg\,min}_{\upsilon_{T-1}, \vartheta_T} E_{T-1}((\upsilon_{T-1} + \vartheta_T \Delta S_T - V_T)^2), \\ V_T &:= H.\end{aligned}$$

After obtaining optimal value of the hedging portfolio at time $T-1$, we move one trading interval backwards and define

$$\{V_{T-2}, \xi_{T-1}\} = \arg \min_{\upsilon_{T-2}, \vartheta_{T-1}} E_{T-2}((\upsilon_{T-2} + \vartheta_{T-1} \Delta S_{T-1} - V_{T-1})^2),$$

i.e. the trading date $T-2$ we minimize expected squared difference between the value of the hedging portfolio and the *optimal value* of the hedging portfolio at time $T-1$.

Generally, we solve a set of one-period optimization problems

$$\begin{aligned} \{V_{t-1}, \xi_t\} &= \arg\min_{\upsilon_{t-1},\vartheta_t} \{E_t((\upsilon_{t-1} + \vartheta_t \Delta S_t - V_t)^2) : \upsilon_{t-1}, \vartheta_t \text{ are } \mathcal{F}_{t-1}\text{- measurable}\}, \\ V_T &:= H. \end{aligned}$$

Note that to compute locally optimal strategy we need to know just the actual price of stock and time remaining to maturity because the process V_t can be derived as expected value under a special martingale measure from terminal payoffs.

"Since in practice no one uses path-dependent hedging coefficients, it is important to ask how much a BlackScholes-like hedge would take away from the optimal performance", [Čer06]. If we measure the performance of locally and globally optimal strategy by the unconditional squared hedging error, globally optimal strategy will always outperform locally optimal strategy by construction. The interesting point is that the locally optimal strategy performs just slightly worse than globally optimal strategy. This is shown for example in [Čer06].

The only problem with locally optimal strategy is, that generally it does not have to be admissible, whereas the globally optimal strategy is always admissible as it is shown later. But according to [Hol09], under the assumptions of IID stock returns, the found locally optimal hedging strategy is admissible.

2.3 Formulas for locally and globally optimal strategy

For simplicity, the following brief explanation is made under the assumption that stock returns are IID. At the end of the chapter we will introduce a theorem which deals with well definition of all variables and admissibility of globally optimal hedging strategy in the general case.

Locally optimal strategy is obtained from one-period minimization problem

$$\{V_{t-1}, \xi_t\} = \arg\min_{\upsilon_{t-1},\vartheta_t} E_t((\upsilon_{t-1} + \vartheta_t \Delta S_t - V_t)^2).$$

Using least squares we have

$$\xi_t = \frac{\mathrm{Cov}_t(V_{t-1}, \Delta S_t)}{\mathrm{Var}_{t-1}(\Delta S_t)},$$
$$V_{t-1} = E_{t-1}\left(\frac{1-\tilde{\lambda}_t \Delta S_t}{1-\Delta\tilde{K}_t} V_t\right),$$

where

$$\tilde{\lambda}_t = \frac{E(\Delta S_t)}{E(\Delta S_t^2)}, \Delta\tilde{K}_t = \frac{(E_{t-1}(\Delta S_t))^2}{E_{t-1}(\Delta S_t^2)},$$

and (possibly signed) measure Q defined as

$$\frac{dQ}{dP} = \prod_{t=1}^{T} \frac{1-\tilde{\lambda}_t \Delta S_t}{1-\Delta\tilde{K}_t}$$

is a martingale measure (shown in [eK09]). Let us denote the value of the self-financing hedging portfolio at the time t as $G_t^{\upsilon,\vartheta} := \upsilon + \vartheta \bullet S_t$, $G_0^{\upsilon,\vartheta} = \upsilon$. The self-financing condition is then

$$G_{t+1}^{\upsilon,\varphi(\upsilon)} = G_t^{\upsilon,\varphi(\upsilon)} + \vartheta_{t+1}\Delta S_{t+1}.$$

In the global risk minimization approach, we are trying to minimize the time 0 expected squared hedging error at maturity:

$$\min_{\vartheta} E_0((G_T^{\upsilon,\vartheta} - V_T)^2).$$

Let us denote the globally optimal strategy as $\varphi(\upsilon) = \{\varphi_t(\upsilon)\}_{t\in\mathcal{T}}$, where

$$\varphi(\upsilon) = \arg\min_{\vartheta} E_0((G_T^{\upsilon,\vartheta} - V_T)^2 + \psi_T) .$$

Using law of iterated expectations, the definition of a self-financing strategy and the optimality of φ, we obtain

$$\begin{aligned}\min_{\upsilon,\vartheta_1..\vartheta_T} E_0((G_T^{\upsilon,\vartheta} - V_T)^2) &= \min_{\upsilon,\vartheta_1..\vartheta_{T-1}} E_0(\min_{\vartheta_T} E_{T-1}((G_T^{\upsilon,\vartheta} - V_T)^2)) \\ &= \min_{\upsilon,\vartheta_1..\vartheta_{T-2}} E_0(\min_{\vartheta_{T-1}} E_{T-2}(\min_{\vartheta_T} E_{T-1}((G_T^{\upsilon,\vartheta} - V_T)^2)))...\end{aligned}$$

It turns out, that under the assumption of IID stock returns the global optimization can be separated to the set of one-period problems

$$\varphi_t(\upsilon) = \arg\min_{\vartheta} E_t((G_{t-1}^{\upsilon,\varphi(\upsilon)} + \vartheta_t\Delta S_t - V_t)^2),$$

so

$$\varphi_t(\upsilon) = \frac{E_{t-1}((G^{\upsilon,\varphi(\upsilon)}_{t-1} - V_t)\Delta S_t)}{E_{t-1}(\Delta S_t^2)} = \xi_t + \tilde{\lambda}_t(V_{t-1} - G^{\upsilon,\varphi(\upsilon)}_{t-1}).$$

Non-IID case is solved by introducing a new probability measure P^*, for more details see [eK09].

2.4 Unconditional squared hedging error of locally and globally optimal strategy

Let us denote the squared hedging error process obtained by implementation of globally optimal strategy as $\varepsilon^2_{GS} = \{\varepsilon^2_{tGS}\}_{t\in\tau}$ and local hedging squared error process as $\varepsilon^2_{LS} = \{\varepsilon^2_{tLS}\}_{t\in\tau}$, where

$$\begin{aligned} \varepsilon^2_{TGS} &= (G_T^{\upsilon,,\varphi(\upsilon)} - V_T)^2, \quad \varepsilon^2_{tGS} = \min_{\vartheta_t} E_t(\varepsilon^2_{t+1GS}), \\ \varepsilon^2_{TLS} &= (G_T^{\upsilon,\xi} - V_T)^2, \quad \varepsilon^2_{tLS} = \min_{G_t,\vartheta_t} E_t(\varepsilon^2_{t+1LS}). \end{aligned}$$

To obtain the unconditional hedging error of locally optimal strategy (ε^2_{0LS}) starting with capital υ, denote the value of portfolio in time t obtained by trading according to locally optimal strategy as $G_t^{\upsilon,\xi} := \upsilon + \xi \bullet S_t$. Then using the self-financing property $G^{\upsilon,\xi}_{t+1} = G^{\upsilon,\xi}_t + \xi_{t+1}\Delta S_{t+1}$ and law of iterated expectations, we have

$$\varepsilon^2_{T-1LS} = E_{T-1}((G_T^{\upsilon,\xi} - V_T)^2) = E_{T-1}((G^{\upsilon,\xi}_{T-1} - V_{T-1} + V_{T-1} + \xi_T\Delta S_T - V_T)^2).$$

The locally optimal hedging error from the least square regression $e_T = V_{T-1} + \xi_T\Delta S_T - V_T$ is orthogonal to the explanatory variables 1 and ΔS_T. That is why $E_{T-1}(e_T) = 0$ and

$$\varepsilon^2_{T-1LS} = (G^{\upsilon,\xi}_{T-1} - V_{T-1})^2 + \psi_T,$$

where

$$\psi_{t+1} = E_t(e^2_{t+1}) = \mathrm{Var}_t(V_{t+1}) - \frac{(\mathrm{Cov}_t(\Delta S_{t+1}, V_{t+1}))^2}{\mathrm{Var}_t(\Delta S_{t+1})}.$$

After recursive application and using the law of iterated expectations, we obtain total hedging error in time t of strategy ξ which is denoted by ε^2_{tLS}:

$$\begin{aligned}
\varepsilon^2_{tLS} &= \min_{\vartheta_t} E_t(\varepsilon^2_{t+1LS}) = E_t((G^{v,\xi}_{t+1} - V_{t+1})^2 + \sum_{i=t+2}^{T} E_{t+1}(\psi_i)) \\
&= (G^{v,\xi}_t - V_t)^2 + \psi_{t+1} + E_t(\sum_{i=t+2}^{T} E_{t+1}(\psi_i)) \\
&= (G^{v,\xi}_t - V_t)^2 + \sum_{i=t+1}^{T} E_t(L_i\psi_i),
\end{aligned}$$

$$\varepsilon^2_{0LS} = E_0((G^{v,\xi}_T - V_T)^2) = (v - V_0)^2 + \sum_{t=1}^{T} E_0(\psi_t).$$

The form for ε^2_{tGS} can be derived by analogy:

$$\begin{aligned}
\varepsilon^2_{T-1GS} &= E_t((G^{v,\varphi(v)}_T - V_T)^2) = E_t((G^{v,\varphi(v)}_{T-1} + \varphi_T(v)\Delta S_T - V_T)^2) \\
&= (1 - \Delta\tilde{K}_T)(G^{v,\varphi(v)}_{T-1} - V_{T-1})^2 + \psi_T.
\end{aligned}$$

Denote

$$L_t = E_t(L_{t+1}(1 - \Delta\tilde{K}_{t+1})), \qquad L_T = 1.$$

After recursive application we get total hedging error of strategy $\varphi(v)$ in time t which is expressed by ε^2_t:

$$\begin{aligned}
\varepsilon^2_{tGS} &= \min_{\vartheta_t} E_t(\varepsilon^2_{t+1GS}) = E_t(L_{t+1}(G^{v,\varphi(v)}_{t+1} - V_{t+1})^2 + \sum_{i=t+2}^{T} E_{t+1}(L_i\psi_i)) \\
&= L_t(G^{v,\varphi(v)}_t - V_t)^2 + E_t(L_{t+1}\psi_{t+1}) + E_t(\sum_{i=t+2}^{T} E_{t+1}(L_i\psi_i)) \\
&= L_t(G^{v,\varphi(v)}_t - V_t)^2 + \sum_{i=t+1}^{T} E_t(L_i\psi_i).
\end{aligned}$$

Then total (time 0, unconditional) global hedging error is ε^2_{0GS} :

$$\varepsilon^2_{0GS} = E_0((G^{v,\varphi(v)}_T - V_T)^2) = L_0(v - V_0)^2 + \sum_{t=1}^{T} E_0(L_t\psi_t).$$

2.5 Admissibility of globally optimal strategy and well-definedness of processes

The following theorem treats the general case where S is a multidimensional process. According to [eK09]

Theorem 3. *Under the assumptions of section 1.1 the process L given by*

$$\begin{aligned} L_T &= 1, \\ L_{t-1} &= E_{t-1}(L_t(1 - E_{t-1}(L_t\Delta S_t)^\top E_{t-1}((L_t\Delta S_t\Delta S_t^\top)^{-1}\Delta S_t))), \end{aligned}$$

is (0,1]-valued and the opportunity-neutral measure P^*,

$$\frac{dP^*}{dP} := \prod_{t=1}^{T} \frac{L_t}{E_{t-1}(L_t)},$$

is well defined. The processes $\tilde{\lambda}^*, V^*$ and ξ^* given by

$$\begin{aligned} \tilde{\lambda}_t^* &= E_{t-1}(L_t\Delta S_t)^\top E_{t-1}(L_t\Delta S_t\Delta S_t^\top)^{-1} \\ &= E_{t-1}^{P^*}(\Delta S_t)^\top E_{t-1}^{P^*}(\Delta S_t\Delta S_t^\top)^{-1}, \\ V_{t-1}^* &= E_{t-1}^{P^*}\left(\frac{1-\tilde{\lambda}_t^*\Delta S_t}{1-\Delta\tilde{K}_t^*}V_t^*\right), \qquad V_T^* = H, \\ \Delta\tilde{K}_t^* &= E_{t-1}^{P^*}(\Delta S_t)^\top E_{t-1}^{P^*}(\Delta S_t\Delta S_t^\top)^{-1}E_{t-1}^{P^*}(\Delta S_t), \\ \xi^* &= E_{t-1}^{P^*}((V_t^* - V_{t-1}^*)\Delta S_t)^\top E_{t-1}^{P^*}(\Delta S_t\Delta S_t^\top)^{-1}. \end{aligned}$$

are well-defined. For a fixed admissible initial endowment $\upsilon \in \overline{U}$ the strategy $\varphi(\upsilon)$ given by

$$\varphi_t(\upsilon) = \xi_t^* + \tilde{\lambda}_t^*(V_{t-1}^* - G_{t-1}^{\upsilon,\varphi(\upsilon)}),$$

is admissible and minimizes the expected squared hedging error among all adimissible strategies with initial endowment υ, while $(V_0^*, \varphi(V_0^*))$ is the optimal endowment - strategy pair if the hedging error is minimized over the initial endowment as well.

For proof see [eK09].

Chapter 3

Global and Local Hedging in Discrete-time Extended Model

The purpose of this chapter is to extend a discrete-time version of the mean-variance theory described in work of Černý and Kallsen [eK09] to the case where there is a non-zero probability that a trade may not be executed at any given time. Let us call the original model "standard model" and modified model "extended model". We will introduce a new process $\{\lambda_t\}_{t\in\tau}$ which represents a proportion of trade which is executed at time t. Due to this process a new endogenous state variable will appear.

As it turned out in the standard model, performance of locally optimal strategy (in sense of chapter 2) measured by unconditional expected squared hedging error is just slightly worse than performance of globally optimal strategy. Moreover, the locally optimal strategy has the advantage of path independence, therefore we deal with local risk minimization approach too.

The formula for globally optimal strategy in extended model is derived here, locally optimal strategy is defined and it is proven that these strategies are well defined. Our intuition says that admissibility of globally optimal strategy is a natural consequence of proven admissibility in standard model, however rigorous proof has to be done. Locally optimal strategy is not admissible in general.

We will recursively derive an unconditional expected squared hedging error of globally and locally optimal hedging strategies. Comparison of these two errors is performed by numerical implementation in chapter 4.

Finally the link between extended and standard model considering λ_t as constant is shown. Proofs and most derivations are deferred to appendix (chapter 6).

3.1 Introduction to the extended model

In the extension of the original model we will consider only one risky asset to hedge the contingent claim. Notation remains the same as in the previous chapter. Properties of a stock price process $\{S_t\}_{t\in\tau}$ remain the same, but the definition of admissible endowment-strategy pair is slightly changed.

In a discrete-time mean-variance extended theory we wish to minimize the *unconditional* (time 0) expected squared hedging error of contingent claim at maturity

$$\min_{\{\vartheta_t\}t\in\tau\setminus\{0\}} E((G_T - H)^2).$$

To derive the expression for the terminal value of hedging portfolio G_T, note that in the standard model the following self-financing condition holds:

$$G_{t+1}^{v,\vartheta} = G_t^{v,\vartheta} + \vartheta_{t+1}\Delta S_{t+1} \quad \forall t \in \tau,$$

but in the extended model the above condition is not valid any more, since we consider that there is a non-zero probability that a trade (or part of it) may not be executed at any given time. Firstly one should realise, that at time $t > 0$ there is already some amount of stocks in the hedging portfolio. Let us denote it by $\tilde{\vartheta}_t$ ($\tilde{\vartheta}_0 = 0$). To get into the situation in which the hedging portfolio contains ϑ_{t+1} units of stock, one needs to buy (or sell) just $\vartheta_{t+1} - \tilde{\vartheta}_t$ stocks.

Consider process $\lambda = \{\lambda_t\}_{t\in\tau}$ adapted to $\mathbb{F}$, which can take values over the interval $[0, 1]$. It is obvious that this process is locally square integrable, i.e.

$$E_t(\lambda_{t+1}^2) < \infty \text{ for } t < T.$$

Random variable λ_t will represent a proportion of trade which is executed straight after placing a market order at the time $t - 1$. Let us name this process **liquidity process**.

When one places a market order to trade $\vartheta_{t+1} - \tilde{\vartheta}_t$ units of a stock at the time t, just $(\vartheta_{t+1} - \tilde{\vartheta}_t)\lambda_{t+1}$ units are actually traded. This implies that at the time $t+1$ the hedging portfolio will contain $\tilde{\vartheta}_{t+1} = \tilde{\vartheta}_t + (\vartheta_{t+1} - \tilde{\vartheta}_t)\lambda_{t+1}$ stocks.

To summarize, in comparison with the standard model we introduced liquidity process $\{\lambda_t\}_{t\in\tau}$ and as a consequence new endogenous state variable $\tilde{\vartheta}_t$ appears. It denotes the number of stocks in hedging portfolio at the time $t \in \tau$, where $\tilde{\vartheta}_0 = 0$.

Self-financing conditions for $G_t^{v,\vartheta}$ and $\tilde{\vartheta}_t^{\vartheta}$ have the form:

$$\begin{aligned} \tilde{\vartheta}_{t+1}^{\vartheta} &= \tilde{\vartheta}_t^{\vartheta}(1-\lambda_{t+1}) + \vartheta_{t+1}\lambda_{t+1} \quad \text{for all} \quad t \in \tau, & (3.1)\\ G_{t+1}^{v,\vartheta} &= G_t^{v,\vartheta} + \tilde{\vartheta}_{t+1}^{\vartheta}\Delta S_{t+1} \\ &= G_t^{v,\vartheta} + \tilde{\vartheta}_t^{\vartheta}(1-\lambda_{t+1})\Delta S_{t+1} + \vartheta_{t+1}\lambda_{t+1}\Delta S_{t+1} \qquad \forall t \in \tau & (3.2) \end{aligned}$$

A general formula for G_T can be derived from 3.1 and 3.2:

$$\begin{aligned} G_T^{v,\vartheta} &= v + \sum_{t=1}^{T} \tilde{\vartheta}_t^{\vartheta}\Delta S_t \\ &= v + \sum_{t=1}^{T} \tilde{\vartheta}_{t-1}^{\vartheta}(1-\lambda_t)\Delta S_t + \sum_{t=1}^{T} \vartheta_t\lambda_t\Delta S_t \\ &= v + \sum_{t=1}^{T-1} \tilde{\vartheta}_t^{\vartheta}(1-\lambda_{t+1})\Delta S_{t+1} + \sum_{t=1}^{T} \vartheta_t\lambda_t\Delta S_t \\ &= v + \sum_{i=1}^{T-1}\sum_{j=1}^{i} \vartheta_j\lambda_j\Delta S_{j+T-i}\prod_{k=j+1}^{j+T-i}(1-\lambda_k) + \sum_{t=1}^{T} \vartheta_t\lambda_t\Delta S_t. \end{aligned}$$

Because of the new form of $G_T^{v,\vartheta}$, we introduce a new definition of admissibility:

Definition 4. *We say that (v, ϑ) is an admissible endowment-strategy pair if and only if v is $\mathcal{F}_0$-measurable, $\vartheta = \{\vartheta_t\}_{t\in\tau\setminus\{0\}}$ is predictable, meaning that ϑ_t is $\mathcal{F}_{t-1}$-measurable, and*

$$v + \sum_{i=1}^{T-1}\sum_{j=1}^{i} \vartheta_j\lambda_j\Delta S_{j+T-i}\prod_{k=j+1}^{j+T-i}(1-\lambda_k) + \sum_{t=1}^{T} \vartheta_t\lambda_t\Delta S_t \in L^2(P).$$

The set of admissible trading strategies with initial endowment v is denoted $\overline{\Theta}(v)$. We write $\overline{\Theta}$ as a shorthand for $\overline{\Theta}(0)$.

From now we take $\mathcal{F}_0$ trivial. Let us introduce the following substitutions for the rest of this work:

$\tilde{S}_t = \lambda_t \Delta S_t,$
$\tilde{L}_t = (1 - \lambda_t)\Delta S_t,$
$\tilde{A}_t = \mathrm{Cov}^2(\tilde{L}_t, \tilde{S}_t) - \mathrm{Var}(\tilde{S}_t)\mathrm{Var}(\tilde{L}_t),$
$\tilde{B}_t = E(\tilde{L}_t)\mathrm{Cov}(\tilde{L}_t, \tilde{S}_t) - E(\tilde{S}_t)\mathrm{Var}(\tilde{L}_t),$
$\tilde{C}_t = E(\tilde{S}_t)\mathrm{Cov}(\tilde{L}_t, \tilde{S}_t) - E(\tilde{L}_t)\mathrm{Var}(\tilde{S}_t).$

3.2 Global hedging

Let us now find the solution for the global risk minimization

$$\min_{\{\vartheta_t\}t\in\tau\setminus\{0\}} E((G_T^{\upsilon,\vartheta} - V_T)^2), \quad V_T = H,$$

with fixed initial wealth υ. Denote a globally optimal strategy $\varphi(\upsilon)$,

$$\varphi(\upsilon) = \underset{\{\vartheta_t\}t\in\tau\setminus\{0\}}{\arg\min}\, E((G_T^{\upsilon,\varphi(\upsilon)} - V_T)^2).$$

To find the optimal strategy, we rewrite the general optimization problem into the set of the one-period optimization problems using the law of iterated expectations:

$$\min_{\vartheta_1..\vartheta_T} E_0((G_T^{\upsilon,\varphi(\upsilon)} - V_T)^2) = \min_{\vartheta_1..\vartheta_{T-1}} E_0(\min_{\vartheta_T} E_{T-1}((G_T^{\upsilon,\varphi(\upsilon)} - V_T)^2))...$$

The optimal hedging strategy is thus obtained from a series of one-period problems, where we denote conditional expected squared hedging error ε^2_{tGE} by J_t:

$$\begin{aligned} J_t &= \min_{\vartheta_t} E_t(J_{t+1}), \\ J_T &= (G_T^{\upsilon,\varphi(\upsilon)} - V_T)^2, \quad V_T := H. \end{aligned}$$

How J_{t+1} is affected by the choice of ϑ_{t+1} is given by the self-financing conditions

$$\begin{aligned} \tilde{\vartheta}_{t+1}^{\varphi(\upsilon)} &= \tilde{\vartheta}_t^{\varphi(\upsilon)}(1 - \lambda_{t+1}) + \vartheta_{t+1}\lambda_{t+1}, \\ G_{t+1}^{\upsilon,\varphi(\upsilon)} &= G_t^{\upsilon,\varphi(\upsilon)} + \tilde{\vartheta}_t^{\varphi(\upsilon)}(1 - \lambda_{t+1})\Delta S_{t+1} + \vartheta_{t+1}\lambda_{t+1}\Delta S_{t+1}. \end{aligned}$$

It would be nice to find a general recursive form for J_t. In order to do this, let us evaluate the expectation $E_{T-1}((G_T^{\upsilon,\vartheta} - V_T)^2)$. First of all, we introduce the following substitutions:

$\vartheta = \vartheta_T,$
$G = G_{T-1}^{\upsilon,\varphi(\upsilon)},$
$\tilde{\vartheta} = \tilde{\vartheta}_{T-1}^{\varphi(\upsilon)}.$

$$E_{T-1}(J_T) = E_{T-1}((G + \tilde{\vartheta}\tilde{L}_t + \vartheta\tilde{S}_t - V_T)^2)$$
$$= E_{T-1}(((G + \tilde{\vartheta}\tilde{L}_t - V_T)^2 + 2\vartheta\tilde{S}_t(G + \tilde{\vartheta}\tilde{L}_t - V_T) + \vartheta^2\tilde{S}_t^2))$$
$$= E_{T-1}((G + \tilde{\vartheta}\tilde{L}_t - V_T)^2) + 2\vartheta E_{T-1}(\tilde{S}_t(G + \tilde{\vartheta}\tilde{L}_t - V_T)) + \vartheta^2 E_{T-1}(\tilde{S}_t^2).$$

FOC for $\varphi_T(\upsilon)$ is:

$$2E_{T-1}(\tilde{S}_t(G + \tilde{\vartheta}\tilde{L}_t - V_T)) + 2\varphi_T(\upsilon)E_{T-1}(\tilde{S}_t^2) = 0,$$

so that

$$\varphi_T(\upsilon) = -\frac{E_{T-1}(\tilde{S}_t(G + \tilde{\vartheta}\tilde{L}_t - V_T))}{E_{T-1}(\tilde{S}_t^2)}.$$

Substituting optimal $\varphi_T(\upsilon)$ into 3.2 yields

$$J_{T-1} = \min_{\vartheta_T} E_{T-1}(J_T) = E_{T-1}((G + \tilde{\vartheta}\tilde{L}_t - V_T)^2) - \frac{(E_{T-1}(\tilde{S}_t(G + \tilde{\vartheta}\tilde{L}_t - V_T)))^2}{E_{T-1}(\tilde{S}_t^2)}.$$

Now we should somehow rearrange the form of J_{T-1} so that the form of J_{T-2} will look similar to J_{T-1}. It is not difficult to say that the form above will turn into the expression which will contain elements as $G_{T-1}^{\upsilon,\varphi(\upsilon)}$, $\left(G_{T-1}^{\upsilon,\varphi(\upsilon)}\right)^2$, $\tilde{\vartheta}_{T-1}^{\varphi(\upsilon)}$, $\left(\tilde{\vartheta}_{T-1}^{\varphi(\upsilon)}\right)^2$, $\tilde{\vartheta}_{T-1}^{\varphi(\upsilon)}G_{T-1}^{\upsilon,\varphi(\upsilon)}$ multiplied by some exogenous variables. The fact that self-financing conditions for $\left(G_{t+1}^{\upsilon,\varphi(\upsilon)}\right)^2$ and $\tilde{\vartheta}_{t+1}^{\varphi(\upsilon)}$ are linear functions in all (endogenous) input variables, leads to the conclusion that form of J_{T-2} will contain just the elements $G_{T-2}^{\upsilon,\varphi(\upsilon)}$, $\left(G_{T-2}^{\upsilon,\varphi(\upsilon)}\right)^2$, $\tilde{\vartheta}_{T-2}^{\varphi(\upsilon)}$, $\left(\tilde{\vartheta}_{T-2}^{\varphi(\upsilon)}\right)^2$, $\tilde{\vartheta}_{T-2}^{\varphi(\upsilon)}G_{T-2}^{\upsilon,\varphi(\upsilon)}$ which enable us to believe that such a general form exists. We can rewrite J_{T-1} into the form

$$\begin{aligned} J_{T-1} &= \min_{\vartheta_T} E_{T-1}(J_T) = \begin{bmatrix} G_{T-1}^{\upsilon,\varphi(\upsilon)} & \tilde{\vartheta}_{T-1}^{\varphi(\upsilon)} & 1 \end{bmatrix} \begin{bmatrix} \alpha_{T-1} & \beta_{T-1} & \gamma_{T-1} \\ \beta_{T-1} & \delta_{T-1} & \epsilon_{T-1} \\ \gamma_{T-1} & \epsilon_{T-1} & \zeta_{T-1} \end{bmatrix} \begin{bmatrix} G_{T-1}^{\upsilon,\varphi(\upsilon)} \\ \tilde{\vartheta}_{T-1}^{\varphi(\upsilon)} \\ 1 \end{bmatrix} \\ &= x_{T-1}^\top Q_{T-1} x_{T-1}. \end{aligned}$$

We can also rewrite $J_T = (G_T - V_T)^2$ as $J_T = x_T^\top Q_T x_T$, where

$$Q_T = \begin{bmatrix} 1 & 0 & -V_T \\ 0 & 0 & 0 \\ -V_T & 0 & V_T^2 \end{bmatrix}.$$

Note that Q_T is a symmetric positive-semidefinite matrix.

In abstract terms we are dealing with a specific Linear-quadratic control problem with one control and two endogenous state variables.

$$\min_{\{\vartheta_t\}} J_0 = E_0(x_T^\top Q_T x_T) \quad s.t. \quad x_{t+1} = A_{t+1}x_t + b_{t+1}\vartheta_{t+1},$$

where

$$\begin{aligned} A_{t+1} &= \begin{bmatrix} 1 & (1-\lambda_{t+1})\Delta S_{t+1} & 0 \\ 0 & (1-\lambda_{t+1}) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad b_{t+1} = \begin{bmatrix} \lambda_{t+1}\Delta S_{t+1} \\ \lambda_{t+1} \\ 0 \end{bmatrix}, \\ x_t^{v,\vartheta} &:= \begin{bmatrix} G_t^{v,\vartheta} \\ \tilde{\vartheta}_t^{\vartheta} \\ 1 \end{bmatrix}, \quad \vartheta_{t+1} \in \mathbb{R}. \end{aligned}$$

At the time t matrices $Q_{t+1}, A_{t+1}, b_{t+1}$ contains random variables, whose realisations are unknown.

In the linear-quadratic control literature it is customary to express the value function in the form $\tilde{V}_t = x_t^\top Q_t x_t \quad Q_t \geq 0$ for all t. We would prefer J_t to have the same form. It turns out, that it really has. We will state and prove it in the following theorem.

Theorem 5. *Under the assumptions of chapter 2 the sequence of matrices Q_t given by*

$$Q_T = \begin{bmatrix} 1 & 0 & -V_T \\ 0 & 0 & 0 \\ -V_T & 0 & V_T^2 \end{bmatrix},$$

$$Q_t = E_t(A_{t+1}^\top Q_{t+1} A_{t+1}) - \frac{(E_t(b_{t+1}^\top Q_{t+1} A_t))^\top E_t(b_{t+1}^\top Q_{t+1} A_{t+1})}{E_t(b_{t+1}^\top Q_{t+1} b_{t+1})}$$

for $E_t(b_{t+1}^\top Q_{t+1} b_{t+1}) > 0$
and

$$Q_t = E_t(A_{t+1}^\top Q_{t+1} A_{t+1})$$

for $E_t(b_{t+1}^\top Q_{t+1} b_{t+1}) = 0$ *are well defined, symmetric and positive-semidefinite for* $\forall t \in \tau$. *For a fixed admissible initial endowment* υ *the strategy* $\varphi(\upsilon)$ *given by*

$$\varphi_t(\upsilon) = -\frac{E_t(b_{t+1}^\top Q_{t+1} A_{t+1})}{E_t(b_{t+1}^\top Q_{t+1} b_{t+1})} x_t^{\upsilon,\varphi(\upsilon)} \quad \text{if} \quad E_t(b_{t+1}^\top Q_{t+1} b_{t+1}) > 0,$$

and

$$\varphi_t(\upsilon) = 0 \quad \text{if} \quad E_t(b_{t+1}^\top Q_{t+1} b_{t+1}) = 0,$$

is well defined and minimizes the unconditional squared hedging error among all strategies with initial endowment υ. *The unconditional squared hedging error of this strategy has the form*

$$J_0 = x_0 Q_0 x_0.$$

where $x_0 = (\upsilon \ 0 \ 1)^\top$

For proof see appendix (6.1).

3.3 Local hedging

3.3.1 One-period risk minimization problem

In contrast with globally optimal strategy, the locally optimal strategy ξ could be defined in various ways in the extended model. We will define it as follows and we find out that the process V_t turns out to be a martingale process under risk-neutral measure Q (as it is in standard model) which enable us to believe that our definition is reasonable. problem.

$$\begin{aligned}\{V_0, \tilde{\vartheta}_0^*, \xi_1\} &= \arg\min_{\upsilon,\tilde{\vartheta}_0,\vartheta_1} E((\upsilon + \tilde{\vartheta}_0^{\vartheta}(1-\lambda_1)\Delta S_1 + \vartheta_1 \lambda_1 \Delta S_1 - V_1)^2) \\ &= \arg\min_{\upsilon,\tilde{\vartheta}_0,\vartheta_1} E((\upsilon + \tilde{\vartheta}_0^{\vartheta}\tilde{L}_1 + \vartheta_1 \tilde{S}_1 - V_1)^2).\end{aligned}$$

Standard least squares regression leads to

$$\begin{aligned}
\xi_1 &= \frac{\mathrm{Cov}(V,\tilde{L}_1)\mathrm{Cov}(\tilde{L}_1,\tilde{S}_1) - \mathrm{Cov}(V_t,\tilde{S}_1)\mathrm{Var}(\tilde{L}_1)}{\tilde{A}_1},\\
\tilde{\vartheta}_0^* &= \frac{\mathrm{Cov}(V_t,\tilde{S}_1)\mathrm{Cov}(\tilde{L}_1,\tilde{S}_1) - \mathrm{Cov}(V_t,\tilde{L}_1)\mathrm{Var}(\tilde{S}_1)}{\tilde{A}_1},\\
V_0 &= E(V) - \tilde{\vartheta}_t^* E(\tilde{L}_1) - \xi_{t+1}E(\tilde{S}_1).
\end{aligned}$$

However, we provide an alternative and more useful expressions for $\xi_1, \tilde{\vartheta}_0^*$ and V_0 using a Frisch-Waugh-Lovell theorem [DM92], for more details see appendix (6.2.1).

$$\begin{aligned}
V_0 &= E(qV_1)\\
&= E\left(\left(1 - \frac{(\tilde{S}_1 - E(\tilde{S}_1))\tilde{B}_1 + ((\tilde{L}_1 - E(\tilde{L}_1))\tilde{C}_1}{\tilde{A}_1}\right)V_1\right),\\
\tilde{\vartheta}_0^* &= E\left(\frac{E(\tilde{S}_1)\tilde{L}_1 - E(\tilde{L}_1\tilde{S}_1)\tilde{S}_1}{E(\tilde{L}_1)E(\tilde{S}_1) - E^2(\tilde{S}_1\tilde{L}_1)}(V_1 - V_0)\right),\\
\xi_1 &= \frac{E(\tilde{S}_1(V_1 - V_0 - \tilde{\vartheta}_0^*\tilde{L}_1))}{E(\tilde{S}_1^2)}.
\end{aligned}$$

Well definidness of all variables is shown in appendix (6.2.2).

Measure Q (possibly signed) defined by

$$\frac{dQ}{dP} := q,$$

is a martingale measure. To see this mathematically we observe

$$\begin{aligned}
E\left(\frac{dQ}{dP}\right) &= E(q) = 1,\\
E^Q(\Delta S_1) &= E\left(q\Delta\tilde{S}_1\right) = 0\ .
\end{aligned}$$

The first result says that Q has total mass 1, the second one states that the stock is priced correctly by Q.

Remark 6. *If we set $\lambda_1 = 1$, measure transformation q will turn into the standard model change of measure*

$$q_{(\lambda_1=1)} = \frac{1 - \tilde{\lambda}_1\Delta S_t}{1 - \Delta\tilde{K}_1 E(\Delta S_1)}.$$

It is not conincidence, that measure Q is a martingale measure. We prove this in the following lemma.

Lemma 7. *Let us have a minimization problem*

$$\min_{\beta_i} E_t((\beta_1 + \sum \beta_i x_i - y)^2),$$

where y is a random variable and $\vec{x} = (x_1 ... x_k) = (1\ x_2 ... x_k)$ is a vector of random variables such that $\vec{1}$ does not belong to the vector space spanned by the columns of x_i, $i \in \{2....k\}$. Let us denote a vector of optimal values $\hat{\beta} = (\hat{\beta}_1 .... \hat{\beta}_k)'$. Then the optimal value $\hat{\beta}_1$ has the form $\hat{\beta}_1 = E_t(F_1 y)$, where F_1 is a random variable and it satisfies $E_t(F_1) = 1$, $E(F_1 x_i) = 0$ for all $i \in \{2....k\}$.

For proof see appendix (6.2.1).

3.3.2 Local risk minimization by sequential regression

Similarly as in the standard model, the local hedging approach leads to set of one-period optimization problems

$$\begin{aligned}
\{V_{t-1}, \tilde{\vartheta}^*_{t-1}, \xi_t\} &= \arg \min_{\upsilon_{t-1}, \tilde{\vartheta}_{t-1}, \vartheta_t} E_{t-1}((\upsilon_{t-1} + \tilde{\vartheta}_{t-1}(1-\lambda_t)\Delta S_t + \vartheta_t \lambda_t \Delta S_t - V_t)^2) \\
&= \arg \min_{\upsilon_{t-1}, \tilde{\vartheta}_{t-1}, \vartheta_t} E_t((\upsilon_{t-1} + \tilde{\vartheta}_{t-1}\tilde{L}_t + \vartheta_t \tilde{S}_t - V_t)^2), \quad V_T := H.
\end{aligned}$$

By standard least squared regression we obtain locally optimal strategy ξ_t, variables V_{t-1} and $\tilde{\vartheta}^*_{t-1}$:

$$\begin{aligned}
V_{t-1} &= E_{t-1}\left(\left(1 - \frac{(\tilde{S}_t - E_{t-1}(\tilde{S}_t))\tilde{B}_t + ((\tilde{L}_t - E_{t-1}(\tilde{L}_t))\tilde{C}_t}{\tilde{A}_t}\right) V_t\right) =: E^Q_{t-1}(V_t), \\
\xi_t &= \frac{\mathrm{Cov}(V, \tilde{L}_t)\mathrm{Cov}(\tilde{L}_t, \tilde{S}_t) - \mathrm{Cov}(V_t, \tilde{S}_t)\mathrm{Var}(\tilde{L}_t)}{\tilde{A}_t}, \\
\tilde{\vartheta}^*_{t-1} &= \frac{\mathrm{Cov}(V_t, \tilde{S}_t)\mathrm{Cov}(\tilde{L}_t, \tilde{S}_t) - \mathrm{Cov}(V_t], \tilde{L}_t)\mathrm{Var}(\tilde{S}_t)}{\tilde{A}_t}.
\end{aligned}$$

or, expressing alternatively

$$\begin{aligned}
\tilde{\vartheta}^*_{t-1} &= E_{t-1}\left(\frac{E_{t-1}(\tilde{S}_t)\tilde{L}_t - E_{t-1}(\tilde{L}_t\tilde{S}_t)\tilde{S}_t}{E_{t-1}(\tilde{L}_t)E(\tilde{S}_t) - (E_{t-1}(\tilde{S}_t\tilde{L}_t))^2}(V_t - V_{t-1})\right), \\
\xi_t &= \frac{E_{t-1}(\tilde{S}_t(V_t - V_{t-1} - \tilde{\vartheta}^*_{t-1}\tilde{L}_t))}{E_{t-1}(\tilde{S}_t^2)}.
\end{aligned}$$

$$\frac{dQ}{dP} := \prod_{t=1}^{T} \left(1 - \frac{(\tilde{S}_t - E_{t-1}(\tilde{S}_t))\tilde{B}_t + ((\tilde{L}_t - E_{t-1}(\tilde{L}_t))\tilde{C}_t}{\tilde{A}_t} \right).$$

In appendix (6.2.2) we cope with the problem arised when $\tilde{A}_t = 0$ or $\mathrm{Var}(\tilde{L}_t) = 0$ and provide detailed derivation of these formulas.

3.3.3 Unconditional hedging error of the local hedging strategy

Now we want to calculate *uncoditional* hedging error of the local hedging strategy. To derive it, we use the self-financing conditions for the value of hedging portfolio at time t obtained by trading according to locally optimal strategy:

$$G_t^{\upsilon,\xi} = G_{t-1}^{\upsilon,\xi} + \tilde{\vartheta}_{t-1}^{\upsilon,\xi}\tilde{L}_t + \xi_t \lambda_t \tilde{S}_t,$$

and number of stocks held in such a portfolio at the time t:

$$\tilde{\vartheta}_t^{\upsilon,\xi} = \tilde{\vartheta}_{t-1}(1 - \lambda_t) + \xi_t \lambda_t.$$

Rewriting them into the matrix form we obtain:

$$x_{t+1}^{\upsilon,\xi} = A_{t+1} x_t^{\upsilon,\xi} + b_{t+1}\xi_{t+1}.$$

Theorem 8. *Unconditional squared error of local hedging has the form*

$$\varepsilon_{0LE}^2 = E((G_T - V_T)^2) = x_0^\top P_0 x_0 + 2c_0^\top x_0 + \sum_{t=1}^{T} E(\phi_t),$$

where

$$P_T = Q_T, c_T = \vec{0}, x_0 = (\upsilon\ 0\ 1)^\top$$

$$\begin{aligned} P_t &= E_t(A_{t+1}^\top P_{t+1} A_{t+1}), \\ c_t &= \xi_{t+1} E_t(b_{t+1}^\top P_{t+1} A_{t+1}) + E_t(c_{t+1}^\top A_{t+1}), \end{aligned}$$

$$\phi_t = \xi_{t+1}^2 E_t(b_{t+1}^\top P_{t+1} b_{t+1}) + 2\xi_t E_t(c_{t+1}^\top b_{t+1}).$$

For proof see appendix (6.2.3 chapter 6).

3.4 Extended model in terms of standard model

Let us show, how can the extended global model be rewritten into the standard global model, if we set $\lambda = 1$. For simplicity assume stock returns IID.

$$Q_T = \begin{bmatrix} 1 & 0 & -V_T \\ 0 & 0 & 0 \\ -V_T & 0 & V_T^2 \end{bmatrix},$$

$$\alpha_T = 1, \qquad \gamma_T = -V_T, \qquad \zeta_T = V_T^2.$$

If $\lambda = 1$, matrices Q_t, $t < T$ have the following structure:

$$Q_t = \begin{bmatrix} \alpha_t & 0 & \gamma_t \\ 0 & 0 & 0 \\ \gamma_t & 0 & \zeta_t \end{bmatrix},$$

$$\begin{aligned}
\alpha_t &= E_t(\alpha_{t+1}) - \frac{(E_t(\alpha_{t+1}\Delta S_{t+1}))^2}{E_t(\alpha_{t+1}\Delta \tilde{S}_{tt+1}^2)}, \\
\gamma_t &= E_t(\gamma_{t+1}) - \frac{E_t(\alpha_{t+1}\Delta S_{t+1})E_t(\gamma_{t+1}\Delta S_{t+1})}{E_t(\alpha_{t+1}\Delta \tilde{S}_{tt+1}^2)}, \\
\zeta_t &= E_t(\zeta_{t+1}) - \frac{\left(E_t(\gamma_{t+1}\Delta S_{t+1})\right)^2}{E_t(\alpha_{t+1}\Delta \tilde{S}_{tt+1}^2)}.
\end{aligned}$$

Then the elements of matrix Q_t are assigned to the variables from standard model as follows:

$$\begin{aligned}
\alpha_t &= L_t, \\
\gamma_t &= -L_t V_t^* = -\alpha_t V_t^*\ , \\
\zeta_t &= \sum_{i=t+1}^{T} E_t(L_i \psi_i^*) + L_t V_t^{*2} = \sum_{i=t+1}^{T} E_t(L_i \psi_i) + \frac{\gamma_t^2}{\alpha_t}.
\end{aligned}$$

For justification see appendix (6.3).

Remark 9. *If we set λ predictable (meaning that λ_t is $\mathcal{F}_{t-1}$ measurable), $\varphi_{t+1}^*(\upsilon)$ will be given as a function of $\varphi_{t+1}(\upsilon)_{\lambda=1}$ so that together with λ_t and $\tilde{\vartheta}_{t+1}$ it will form the standard $\varphi_{t+1}(\upsilon)_{\lambda=1}$:*

$$\begin{aligned}
\tilde{\vartheta}_{t+1}^{\varphi(\upsilon)_{\lambda=1}} &= \tilde{\vartheta}_{t}^{\varphi(\upsilon)_{\lambda=1}}(1-\lambda_{t+1}) + \varphi_{t+1}^{*}(\upsilon)\lambda_{t+1}, \\
\varphi_{t+1}(\upsilon)_{\lambda=1} &= \tilde{\vartheta}_t(1-\lambda_{t+1}) + \varphi_{t+1}^{*}(\upsilon)\ \lambda_{t+1}, \\
\varphi_{t+1}^{*}(\upsilon) &= \frac{\varphi_{t+1}(\upsilon)_{\lambda=1}}{\lambda_{t+1}} + \tilde{\vartheta}_t\left(1-\frac{1}{\lambda_{t+1}}\right).
\end{aligned}$$

That is why the form of Q_t and error ϵ_{tGE}^2 will remain the same, as if $\lambda_t = 1$.

Chapter 4

Numerical Implementation of Unconditional Expected Squared Hedging Error

In the previous chapter we derived a general formula for the unconditional squared hedging error in globally and locally optimal strategy for extended model. The purpose of this chapter is to investigate empirically the performance of the hedging strategies when the rehedge interval approaches zero and point out the qualitative differences between the standard and the extended model unconditional squared hedging errors. To enable numerical evaluation for different rehedging intervals and trading environment, we will introduce a reasonable model of λ and adopt a multinomial model of the stock price from [Čer03]. Part of MATLAB code integrated in [Čer03] which provides a simulation of the multinomial model will also be used.

The recursive scheme for computing squared error process ε_t^2 derived in chapter 2 is suitable for the simple computer implementation.

To summarize, the unconditional squared hedging error of the **globally** optimal strategy in **extended model** has the form

$$\varepsilon_{0EG}^2 = E((G_T^{v,\varphi(v)} - H)^2) = x_0^{v,\varphi(v)^\top} Q_0 x_0^{v,\varphi(v)},$$

for the **locally** optimal strategy its form turns into

$$\varepsilon_{0EL}^2 = E((G_T^{\upsilon,\xi} - H)^2) = x_0^{\upsilon,\xi^\top} P_0 x_0^{\upsilon,\xi} + 2c_0^\top x_0^{\upsilon,\xi} + \sum_{t=1}^{T} E(\phi_t).$$

If the liquidity process is a constant ($\lambda = 1$), the above formulas come into the **standard model** form:

$$\varepsilon_{0SG}^2 = E((G_T^{\upsilon,\varphi(\upsilon)} - H)^2) = L_0(\upsilon - V_0^*)^2 + \sum_{t=1}^{T} E(L_t \psi_t^*),$$

for **globally** optimal strategy and

$$\varepsilon_{0SL}^2 = E((G_T^{\upsilon,\xi} - H)^2) = L_0(\upsilon - V_0)^2 + \sum_{t=1}^{T} E(\psi_t),$$

for **locally** optimal strategy. For further details see previous chapter and [eK09].

4.1 Model Specification

Numerical implementation has to be performed on the particular model of stock price and liquidity process. The model should be realistic but simple enough due to computational burden in order to investigate the limit performance of the hedging strategies.

For simplicity we assume that stock price log returns process and liquidity are independent, both IID. To keep the model realistic to some extent we selected the following criteria to be satisfied:

1. Stock price process follows a geometric Brownian motion as $\Delta t \to 0$.
2. In every trading time t, liquidity process λ takes values over the interval $[0, 1]$.
3. The liquidity of the stock is not changed by the frequency of hedging.

For the stock price we adopt a multinomial model from [Čer03]. Modelled stock price process follows a geometric Brownian motion as $\Delta t \to 0$:

$$\ln \frac{S_t}{S_{t-\Delta t}} \sim N(\mu \Delta t, \sigma^2 \Delta t).$$

If we define a log return as a random variable $R \sim N(\mu\Delta t, \sigma^2\Delta t)$, for the stock price at time $t \in \tau$ we obtain

$$S_t = S_{t-\Delta t}e^R.$$

We will model a random variable R using discrete distribution which can take n possible values with certain probabilities. Denote these values $R_1 \dots R_n$ and probabilities assigned to them $p_1 \dots p_n$. In our model we allow seven values of weekly log returns in one node ($n = 7$). The values will be spaced out regularly so that the tree recombines and moreover, the tree contains a zero log return: $R_i - R_{i-1} = h$, $0 \in R_1, \dots R_n$.

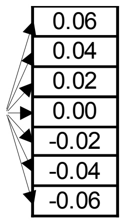


Figure 4.1 -Model of weekly log return

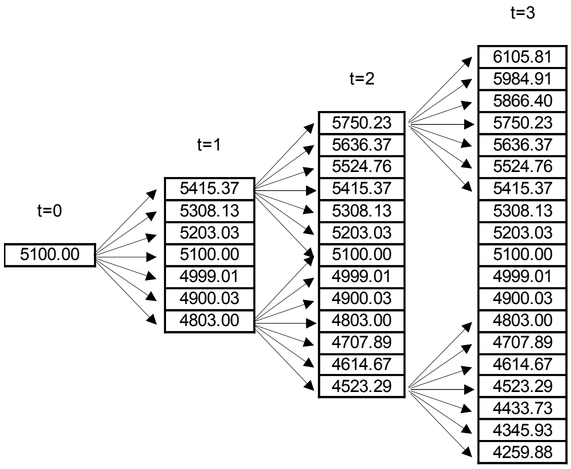
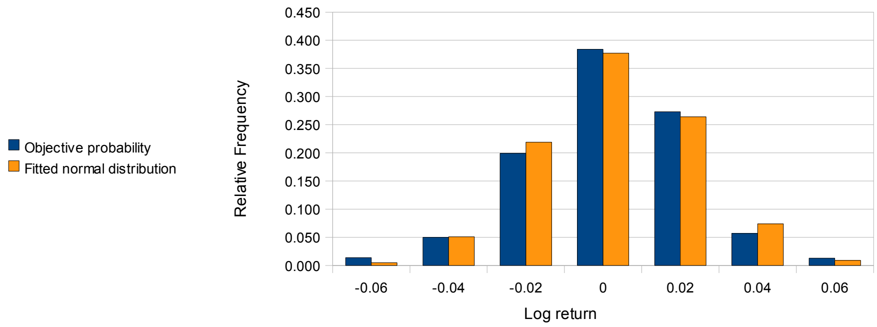


Figure 4.2 -Lattice of stock prices

The empirical distribution of stock returns is obtained from weekly returns of FTSE 100 Index in the period 1984-2001.



	-0.06	-0.04	-0.02	0	0.02	0.04	0.06
Objective probability	0.014	0.050	0.199	0.384	0.273	0.057	0.013
Fitted normal distribution	0.005	0.051	0.219	0.377	0.264	0.074	0.009

Figure 4.3 - Historgram of weekly returns[1]

For simplicity, it is assumed that the weekly returns are distributed independently and the risk-free rate has been constant between 1984 and 2001. To produce an appropriate histogram the log returns are divided into seven categories as can be seen in the figure above. Length of one category is $h = 2\%$. Mathematically speaking

$$S_t \in \{e^{\ln S_0 + kh}; k \in \mathbb{Z}, -3t \leq k \leq 3t\}$$

As we change the length of rehedge interval Δt, weekly log returns are calibrated to give Brownian motion in the limit (as the rehedge interval approaches zero). It is done as following:

$$dt = \frac{\Delta t}{week},$$

$$R_i(dt) = \mu dt + (R_i - \mu)\sqrt{dt},$$

where $\mu = \sum_{i=1}^{7} R_i p_i$.

To model the liquidity process with required qualities, the following method is used: we firstly construct an auxiliary homogenous Poisson process which is used as a decision rule. At each trading (hedging) date the rule decides whether the stock is traded to some extent or it is illiquid. If the stock is traded, λ is modeled by the continuous uniform distribution over the interval $[0, 1]$. The following definitions are taken from [Ros80].

[1]Figures 4.1, 4.2 and 4.3 taken from [Čer03]

Definition 10. *A counting process is a stochastic proces $\{N(t), t \geq 0\}$ that posesses the following properties:*

1. $N(t) \geq 0$.
2. $N(t) \in \mathbb{Z}$.
3. *if* $s < t$ *then* $N(s) \leq N(t)$.

If $s < t$ then $N(t) - N(s)$ is the number of events occured during the interval $(s, t]$.

Definition 11. *The counting process $\{N(t), t \geq 0\}$ is said to be Poisson process having rate κ,$\kappa \geq 0$, if that possesses the following properties:*

1. $N(0) = 0$.
2. *the process has independent increments.*
3. *the number of events in any interval of length t is Poisson distributed with mean κt. That is, for all $s, t \geq 0$*

$$P(N(t+s) - N(s) = k) = \frac{e^{-\kappa t}(\kappa t)^k}{k!} \qquad k = 0, 1, 2...$$

The Poisson process defined above is also known as homogenous Poisson process as the parameter κ is time independent variable. It follows from the condition 3, that a Poisson process has stationary increments and also that

$$E(N(t)) = \kappa t.$$

The rate parameter κ is therefore an expected number of events that occur per unit time. For the homogenous Poisson process, the inter-arrival times are exponentially distributed with parameter κ.

Denote the length of rehedge interval as Δt such that T is divisible by Δt. Then we define a new set of trading dates $\tau := \{0, \Delta t, 2\Delta t..., T\}$ and the liquidity process λ as

$$\lambda_t = \langle \begin{array}{c} X_t \text{ for } N(t) - N(t - \Delta t) \geq 1 \\ 0 \text{ for } N(t) - N(t - \Delta t) = 0 \end{array} \quad \text{for all } t \in \tau \backslash \{0\},$$

where $X_t \sim U(0,1)$ for all $t \in \tau \backslash \{0\}$ and $N(t)$ is the homogenous Poisson process with a rate parameter κ.

Because we assume the liquidity process independent on the stock price process, for the computational purposes only formulas for $E_{t-\Delta t}(\lambda_t)$ and $E_{t-\Delta t}(\lambda_t^2)$ are needed. To obtain them let us express

$$\begin{aligned} P(N(t) - N(t-\Delta t) &= 0) = e^{-\kappa \Delta t} \qquad \text{for all } t \in \tau \backslash \{0\}, \\ P(N(t) - N(t-\Delta t) &\geq 1) = 1 - e^{-\kappa \Delta t} \quad \text{for all } t \in \tau \backslash \{0\}. \end{aligned}$$

Therefore

$$\begin{aligned} E_{t-\Delta t}(\lambda_t) &= E_{t-\Delta t}(X_t)(1 - e^{-\kappa \Delta t}) + 0.e^{-\kappa \Delta t} = 0.5(1 - e^{-\kappa \Delta t}), \\ E_{t-\Delta t}(\lambda_t^2) &= E_{t-1}(X_t^2)(1 - e^{-\kappa \Delta t}) + 0.e^{-\kappa \Delta t} = \frac{(1 - e^{-\kappa t \Delta t})}{3}. \end{aligned}$$

Example 12. *Let us set a time unit equal to 1 minute, time horizon $T = 240$, $\kappa = \frac{1}{60}$, rehedge interval $\Delta t = 120$. During every rehedge interval two events should occur on average, which means that $\lambda_t = X_t$ and non-zero volume of stock will be traded every 120 minutes on average. When setting $\Delta t = 60$, one event should occur during the rehedge interval, therefore non-zero volume of stock will be traded every 60 minutes in average. More importantly, when changing rehedge interval to $\Delta t = 30$, the event will occur approximately during every second rehedge interval, therefore one out of two market orders will not be executed at a medium. As a consequence, the non-zero volume of stock will be traded approximately every 60 minutes as before.*

Having all the necessary variables defined let us present the results of numerical implementation.

4.2 Numerical Implementation Results

The computation was performed on European call option under following conditions:

initial stock price	$S_0 = 5100$
strike price of the option	$K = 5355$
time to maturity	$T = 6$ weeks
1 week	5 trading days
1 trading date	8 trading hours
time unit	1 minute

Note that in the following tables and graphs a more natural variable 'inter-arrival time' $I = \frac{1}{\kappa}$ was employed. We choose the length of rehedge interval Δt so that T is divisible by Δt. basic results are shown in the following tables:

Unconditional Expected Squared Error of Global Hedging							
	Uncertained Trade Execution - ϵ^2_{0EG}						**Standard**
RI \IAT*	8hrs	2hrs	1hour	30min	15min	1min	**Model - ϵ^2_{0SG}**
8hrs	751.4153	412.345	400.9999	400.7919	400.7918	400.7918	243.5157
2hrs	667.3452	203.747	132.5488	109.1405	106.0155	105.9583	62.9527
1hour	660.5024	185.9942	104.584	67.71	55.634	53.9944	31.8871
30min	658.188	179.8346	95.3034	53.3037	34.3898	27.3759	16.0992
15min	657.5672	177.5631	92.0669	48.5308	27.0428	13.8415	8.1194
5min	657.5668	176.4105	90.4808	46.3856	23.9852	4.7266	2.7364
*RI - Rehedge Interval; IAT - Inter-Arrival Time=$1/\kappa$							

Table 1 - Global hedging

Unconditional Expected Squared Error of Local Hedging							
	Uncertained Trade Execution - ϵ^2_{0EL}						**Standard**
RI \IAT*	8hrs	2hrs	1hour	30min	15min	1min	**Model - ϵ^2_{0SL}**
8hrs	898.6733	502.7339	489.6076	489.3671	489.367	489.367	246.0946
2hrs	779.6777	241.1896	159.6307	132.8823	129.3139	129.2485	63.6246
1hour	767.8808	216.8516	123.6334	81.5332	67.7616	65.8923	32.2257
30min	763.2184	207.7403	110.9425	62.9887	41.4215	33.4279	16.2693
15min	761.482	204.0894	106.1775	56.4622	31.9557	16.9086	8.2046
5min	760.787	202.0603	103.6436	53.282	27.729	5.772	2.7649
*RI - Rehedge Interval; IAT - Inter-Arrival Time=$1/\kappa$							

Table 2 - Local hedging

Table 1 and 2 indicate several interesting facts:

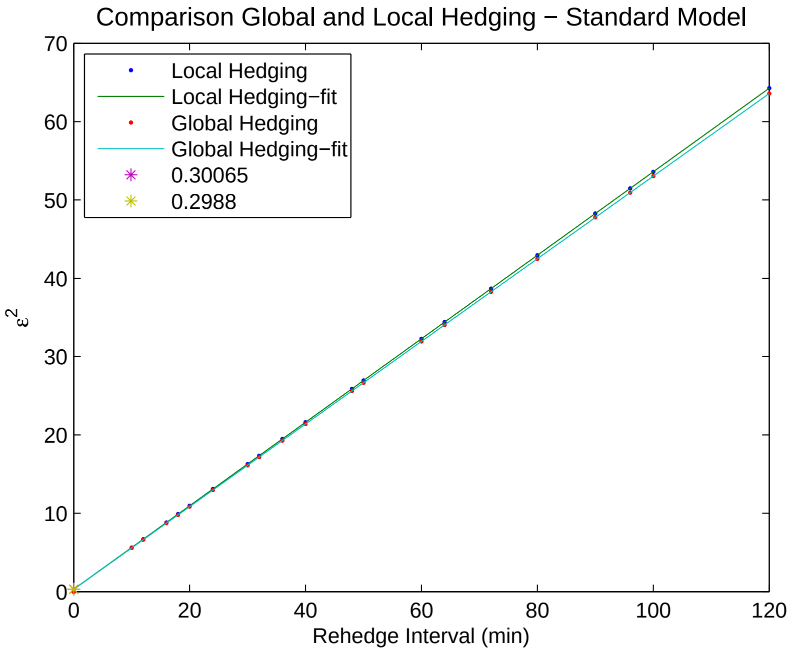
1. In the standard model the performance of local strategy is just a slightly worse than performance of global strategy (this is no news, as the same result was obtained in [Čer03]).

2. In the extended model the unconditional squared hedging error of local strategy is significantly larger than in global strategy.

3. It seems that in the standard model the unconditional squared hedging error goes to zero as the rehedge interval approaches zero.

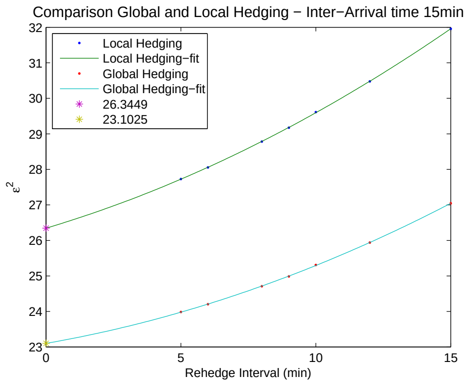
4. In the extended model the unconditional squared hedging error goes to positive number as the rehedge interval approaches zero.

To justify these suggestions, we performed more detailed computations. Results can be seen in the following graphs.

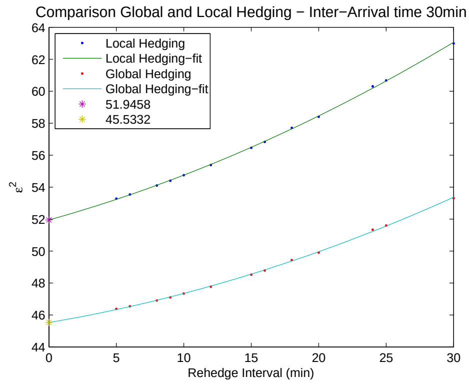


Graph 4.1 - Standard model

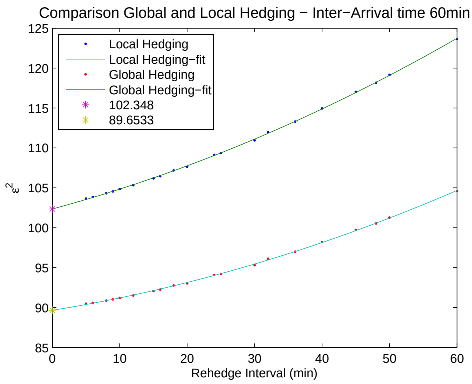
Graph 4.1 demonstrates the empirical limit of the unconditional squared hedging error of standard model, as well as a very similar performance of local and global strategy. Because of the computational burden the shortest evaluated rehedging interval was set to 5 minutes. The limit for rehedging interval approaching zero was obtained by fitting the OLS line across the data.



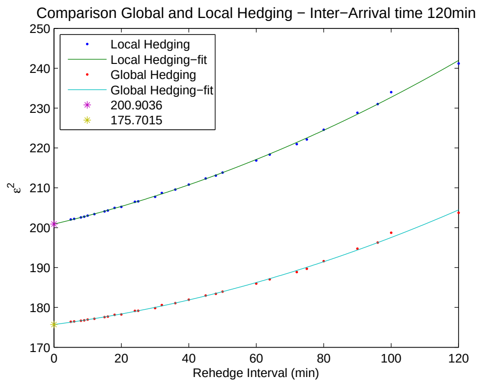
Graph 4.2 - $IAT = 15$min



Graph 4.3 - $IAT = 30$min



Graph 4.4 - $IAT = 60$min

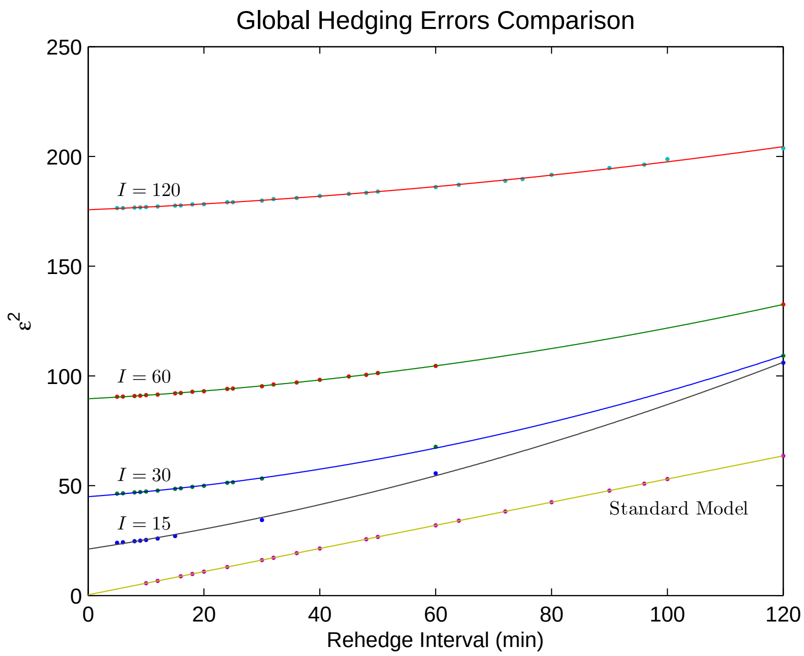


Graph 4.5 - $IAT = 120$min

Graphs 4.2-4.5 show the empirical limits of unconditional squared hedging error of local and global strategy for different inter-arrival times. The limit for rehedging interval approaching zero was obtained by fitting the OLS quadratic curve across the data.

We can explain a positive empirical limit of unconditional square error by the fact that a frequent hedging does not eliminate a low liquidity of the stock. Therefore the trade is not executed in many hedging dates as the rehedge interval approaches zero.

Note that differences between the unconditional squared errors of global and local strategy diminish as the inter-arrival time shortens and the ratio of the difference and the total squared error of global strategy remains approximately constant. This evidence leads us to conclusion that a resulting difference is caused by the fact, that relatively small local difference between the global and local strategy in actual node is enlarged during 'no stock traded' nodes. To compare, in the standard model, the local difference is adjusted in every trading time.



Graph 4.6 - Global hedging comparison

In the graph 4.6 we can see how the global unconditional squared hedging error behaves for different inter-arrival times and standard model.

Remark 13. *Interpretations of the numeric results presented in this chapter are conditional on the model specification to some extent. We have to realize, that liquidity process employed in our model is not realistic in some respects. Theory assumes that the market order is either executed straight after placing or not executed at all. The chosen model respects this attribute. The problem arises when we take closer look at the decision rule, since it is independent on when the event during the rehedging interval occurs. It is asymmetric in the sense that for the inter-arrival time longer than rehedging intervals, reducing the length of rehedging interval leads almost surely to later execution of the hedging orders. This unwilling attribute of our model could be eliminated by employing more sophisticated decision rule.*

Chapter 5

Results and Conclusions

Mean-Variance hedging (MVH) theory was proposed by Černý and Kallsen (2007). This thesis extends discrete-time models of MVH theory (Černý, Kallsen 2009) with introducing stock liquidity process λ_t, taking into account the fact that there is non-zero probability that trade may not be executed at any given time.

The main contribution of this work is derivation of the explicit form of optimal hedging strategies for local and global hedging and proof that these strategies are well defined. Our intuition says that admissibility of these strategies for local hedging is a natural consequence of proven admissibility in standard model, however rigorous proof has to be done. Derivation of recursive formulas for unconditional mean squared hedging error for both local and global optimal hedging strategies are also provided in this thesis.

In computational part we studied the local and global hedging strategies with different inter-arrival times and rehedge-intervals. Numerical implementations led to several findings. In standard model local and global hedging strategies provided similar performance with zero mean squared hedging error as the rehedge-interval approaches zero, which is in accordance with theory. Unconditional mean squared errors of local hedging are significantly larger than those from global hedging in extended model for different inter-arrival times and rehedge-intervals. Non-zero empirical limit follows the intuition that even frequent hedging will not eliminate a low liquidity of the stock.

The possible extensions of this work might be to expand this theory to multivariate case so that more stocks can be used for hedging. Another potential extention is to incorporate more realistic assumptions about stock liquidity process, particularly the fact that liquidity evolved gradually over time and that it might be correlated with stock returns.

Chapter 6

Appendix

6.1 Global hedging

Proof of Theorem 5. Let us prove this theorem using mathematical induction:

1° : For $t = T : J_T = (G_T - V_T)^2 = x_T^\top Q_T x_T$,

$Q_T \geq 0$ (is positive semidefinite), Q_T is symmetric.

2° :

$$\begin{aligned} J_t &= \min_{\vartheta_{t+1}} E_t(J_{t+1}) = \min_{\vartheta_{t+1}} E_t((A_{t+1}x_t + b_t\vartheta_{t+1})^\top Q_{t+1}(A_{t+1}x_t + b_t\vartheta_{t+1})) \\ &= E_t(x_t{}^\top A_{t+1}^\top Q_{t+1}A_{t+1}x_t) + \min_{\vartheta_{t+1}}(2\vartheta_{t+1}E_t(b_t^\top Q_{t+1}A_{t+1}x_t) + \vartheta_{t+1}^2 E_t(b_t^\top Q_{t+1}b_t)). \end{aligned}$$

FOC for ϑ:

$$2E_t(b_t^\top Q_{t+1}A_{t+1}x_t) + 2\varphi_{t+1}(\upsilon)E_t(b_{t+1}^\intercal Q_{t+1}b_t) = 0,$$

so that

$$\varphi_{t+1}(\upsilon) = -\frac{E_t(b_{t+1}^\intercal Q_{t+1}A_t x_t)}{E_t(b_{t+1}^\intercal Q_{t+1}b_t)} = -\frac{E_t(b_{t+1}^\intercal Q_{t+1}A_t)}{E_t(b_{t+1}^\intercal Q_{t+1}b_t)}x_t.$$

If $E_t(b_{t+1}^\intercal Q_{t+1}b_{t+1}) > 0$ then $\varphi_{t+1}(\upsilon)$ obtained from FOC is minimum and it is well defined. From induction hypotesis we have $Q_{t+1} \geq 0$, so $E_t(b_{t+1}^\intercal Q_{t+1}b_t)$ is at least non-negative and Q_{t+1} has an unique square root $Q_{t+1}^{1/2}$ such that $(Q_{t+1}^{1/2})^2 = Q_{t+1}$.

Then the following equation holds:

$$\begin{aligned}
E_t(b_{t+1}^\intercal Q_{t+1} b_t) &= 0 \\
&\Leftrightarrow\ E((Q_{t+1}^{1/2} b_{t+1})^\intercal (Q_{t+1}^{1/2} b_{t+1})) = 0 \\
&\Leftrightarrow\ (Q_{t+1}^{1/2} b_{t=1}) = \vec{0} \\
&\Rightarrow\ E(b_{t+1}^\intercal Q_{t+1} A_{t+1} x_t) = E((Q_{t+1}^{1/2} b_{t+1})^\top Q_{t+1}^{1/2} A_{t+1} x_t) = 0 \\
&\Rightarrow\ \varphi_{t+1}(v) \quad \text{is ambiguous.}
\end{aligned}$$

If $\varphi_{t+1}(v)$ is ambiguous, we set it 0. For $E_t(b_{t+1}^\intercal Q_{t+1} b_t) > 0$ we can express J_t as

$$\begin{aligned}
J_t &= {x_t}^\top \left(E_t(A_{t+1} Q_{t+1} A_{t+1}) - \frac{(E_t(b_{t+1}^\intercal Q_{t+1} A_{t+1}))^\top E_t(b_{t+1}^\intercal Q_{t+1} A_{t+1})}{E_t(b_{t+1}^\intercal Q_{t+1} b_{t+1})} \right) x_t, \\
Q_t &= E_t(A_{t+1} Q_{t+1} A_{t+1}) - \frac{(E_t(b_{t+1}^\intercal Q_{t+1} A_{t+1}))^\top E_t(b_{t+1}^\intercal Q_{t+1} A_{t+1})}{E_t(b_{t+1}^\intercal Q_{t+1} b_{t+1})}.
\end{aligned}$$

For $E_t(b_{t+1}^\intercal Q_{t+1} b_{t+1}) = 0$:

$$\begin{aligned}
J_t &= {x_t}^\top E_t(A_{t+1} Q_{t+1} A_{t+1}) x_t, \\
Q_t &= A_{t+1} Q_{t+1} A_{t+1}
\end{aligned}$$

Note that Q_t is symmetric in both cases. It remains to prove that if $Q_{t+1} \geq 0$ then $Q_t \geq 0$. We will do it by proving that $x^\top Q_t x \geq 0$ for any $x \in \mathbb{R}^3$. If $E_t(b_{t+1}^\intercal Q_{t+1} b_{t+1}) = 0$ then obviously $Q_t = E_t(A_{t+1} Q_{t+1} A_{t+1}) \geq 0$. To prove $Q_t \geq 0$ for $E_t(b_{t+1}^\intercal Q_{t+1} b_{t+1}) > 0$, we introduce the auxiliary univariate least squares. Let us have univariate least squares, dependent variable Y and explanatory variable $Z \neq \vec{0}$:

$$\min_{\alpha \in A_{t+1}^\intercal} E\left[(\alpha Z - Y)^2\right] = \min_{\alpha \in A_{t+1}^\intercal} \alpha^2 E(Z^2) - 2\alpha E(ZY) + E(Y^2).$$

FOC of this minimization problem:

$$2\hat{\alpha} E(Z^2) - 2E(ZY) = 0,$$

so that

$$\hat{\alpha} = \frac{E(ZY)}{E(Z^2)}.$$

Minimal expected squared error can be expressed as

$$\begin{aligned} E\left[(\hat{\alpha}Z - Y)^2\right] &= \hat{\alpha}^2 E(Z^2) - 2\hat{\alpha}E(ZY) + E(Y^2) \\ &= E(Y^2) - \frac{(E(ZY))^2}{E(Z^2)} \geq 0. \end{aligned}$$

We can notice, that $x^\top Q_t x$ has a similar form

$$x^\top Q_t x = E_t(x^\top A_{t+1}^\intercal Q_{t+1} A_{t+1} x) - \frac{(E_t(b_{t+1}^\intercal Q_{t+1} A_{t+1} x))^2}{E_t(b_{t+1}^\intercal Q_{t+1} b_{t+1})}.$$

Let us denote $x^\top A_{t+1}^\intercal Q_{t+1} A_{t+1} x$ as Y^2 and $b_{t+1}^\intercal Q_{t+1} b_{t+1}$ as Z^2. Because Q_{t+1} is a positive-semidefinite symetric matrix, it has a unique square root and we can write

$$\begin{aligned} Y^2 &= x^\top A_{t+1}^\intercal Q_{t+1} A_{t+1} x = (Q_{t+1}^{1/2} A_{t+1} x)^\intercal (Q_{t+1}^{1/2} A_{t+1} x) = y^\intercal y, \\ Z^2 &= b_{t+1}^\intercal Q_{t+1} b_{t+1} = (Q_{t+1}^{1/2} b)^\intercal (Q_{t+1}^{1/2} b) = z^\intercal z, \\ b_{t+1}^\intercal Q_{t+1} A_{t+1} x &= (Q_{t+1}^{1/2} b)^\intercal (Q_{t+1}^{1/2} A_{t+1} x) = z^\intercal y \leq \|z\| \|y\|, \\ &= ZY = \sqrt{(x^\top A_{t+1} Q_{t+1} A_{t+1} x)(b_{t+1}^\intercal Q_{t+1} b_t)}, \\ x^\intercal Q_t x &= E_t(x^\intercal A_{t+1}^\intercal Q_{t+1} A_{t+1} x) - \frac{(E_t(b_{t+1}^\intercal Q_{t+1} A_{t+1} x))^2}{E_t(b_{t+1}^\intercal Q_{t+1} b_t)} \\ &\geq E_t(x^\intercal A_{t+1}^\intercal Q_{t+1} A_{t+1} x) - \frac{\left(E_t\left[\sqrt{(x^\top A_{t+1} Q_{t+1} A_{t+1} x)(b_{t+1}^\intercal Q_{t+1} b_t)}\right]\right)^2}{E_t(b_{t+1}^\intercal Q_{t+1} b_t)} \\ &= E(Y^2) - \frac{(E(ZY))^2}{E(Z^2)} \geq 0. \end{aligned}$$

□

For the purposes of computer implementation it is useful to express the elements of matrix Q_t and $\varphi_{t+1}(v)$ terms of matrix Q_{t+1} not in the matrix form. Let us denote the elements of symetric matrix Q_t by $\alpha_t, \beta_t ... \varsigma_t$ and elements of matrix Q_{t+1} by

$\alpha, \beta ... \varsigma$, ΔS_{t+1} by ΔS and λ_{t+1} by λ. Then

$$\begin{aligned}
\alpha_t &= E_t(\alpha) - \frac{(E_t((\beta + \alpha\Delta S)\lambda))^2}{E_t((\delta + 2\beta\Delta S + \alpha\Delta S^2)\lambda^2)}, \\
\beta_t &= E_t(\beta + \alpha\Delta S) - \frac{E_t((\beta + \alpha\Delta S)\lambda)E_t((\delta + 2\beta\Delta S + \alpha\Delta S^2)\lambda)}{E_t((\delta + 2\beta\Delta S + \alpha\Delta S^2)\lambda^2)}, \\
\gamma_t &= E_t(\gamma) - \frac{E_t((\beta + \alpha\Delta S)\lambda)E_t((\epsilon + \gamma\Delta S)\lambda)}{E_t((\delta + 2\beta\Delta S + \alpha\Delta S^2)\lambda^2)}, \\
\delta_t &= E_t(\delta + 2\beta\Delta S + \alpha\Delta S^2) - \frac{(E_t((\delta + 2\beta\Delta S + \alpha\Delta S^2)\lambda))^2}{E_t((\delta + 2\beta\Delta S + \alpha\Delta S^2)\lambda^2)}, \\
\epsilon_t &= E_t((\epsilon + \gamma\Delta S)\lambda) - \frac{E_t((\epsilon + \gamma\Delta S)\lambda)E_t((\delta + 2\beta\Delta S + \alpha\Delta S^2)\lambda)}{E_t((\delta + 2\beta\Delta S + \alpha\Delta S^2)\lambda^2)}, \\
\zeta_t &= E_t(\zeta) - \frac{(E_t((\epsilon + \gamma\Delta S)\lambda))^2}{E_t((\delta + 2\beta\Delta S + \alpha\Delta S^2)\lambda^2)},
\end{aligned}$$

$$\varphi_{t+1}(\upsilon) = \tilde{\vartheta}_t - \frac{G_t E_t((\beta + \alpha\Delta S)\lambda) + E_t((\epsilon + \gamma\Delta S)\lambda) + \tilde{\vartheta}_t E_t((\delta + 2\beta\Delta S + \alpha\Delta S^2)\lambda)}{E_t((\delta + 2\beta\Delta S + \alpha\Delta S^2)\lambda^2)}.$$

6.2 Local hedging

6.2.1 One-period risk minimization problem

Frisch-Waugh-Lovell theorem says, that when we have the model expressed as follows

$$Y = X_1\beta_1 + X_2\beta_2 + \varepsilon,$$

and we are interesting in estimating β_2, we can use two alternative methods:

1. Regress Y on X obtaining the OLS estimator $\hat{\beta} = (\hat{\beta}_1^\top, \hat{\beta}_2^\top)^\top = (X^\top X)^{-1}X^\top Y$ using standard OLS.

2. Regress Y^*on X_2^* and obtain as estimate $\tilde{\beta}_2 = (X_2^{*\top} X_2^*)^{-1} X_2^{*\top} Y^*$

where $Y^* = M_1 Y$, $X_2^* = M_1 X_2$ and $M_1 = I - X_1(X_1^\top X_1)^{-1}X_1^\top$ is an orthogonal projection matrix that projects any vector in $\mathbb{R}^n$ onto the orthogonal complement of the linear space spanned by the columns of X_1.

Let us denote e_1 and e_2 the residual vectors of the regressions (1) and (2). The theorem says, that $\hat{\beta}_2 = \tilde{\beta}_2$ and $e_1 = e_2$.

In this section we just need to express $\hat{\beta}_2$, we do not mind residual vectors. For this purpose we can modify the model and set $Y^* = Y$. In this modified model the equation $\hat{\beta}_2 = \tilde{\beta}_2$ still holds, but the equation $e_1 = e_2$ does not hold any more. We can do this, because matrix is idempotent, i.e. $M_1^2 = M_1$.

We have the model:

$$\{V_{t-1}, \tilde{\vartheta}^*_{t-1}, \xi_t\} = \arg \min_{\upsilon_{t-1}, \tilde{\vartheta}_{t-1}, \vartheta_t} E_{t-1}((\upsilon_{t-1} + \tilde{\vartheta}_{t-1}\tilde{L}_t + \vartheta_t \tilde{S}_t - V_t)^2), \quad V_T := H.$$

To obtain V_{t-1} we will proceed as following:

1. Make an auxiliary regression of the variable $\tilde{S}_t$ onto the explanatory variable $\tilde{L}_t$ from which we will get S_t^L(the residual vector from this regression). Analogically make an auxiliary regression of the variable $\vec{1}$ onto the explanatory variable $\tilde{L}_t$ from which we will get $\vec{1}_t^L$. Then we obtain

$$\{V_{t-1}, \xi_t\} = \arg \min_{\upsilon_{t-1}, \tilde{\vartheta}_t} E_{t-1}((\upsilon_{t-1} 1_t^L + \vartheta_t S_t^L - V_t)^2).$$

2. Make an auxiliary regression of the variable $\vec{1}_t^L$ onto the explanatory variable $S_t^L$from which we will get $\vec{1}_t^{SL}$ (the residual vector from regression onto the residual vector from first regression). We get

$$V_{t-1} = \arg \min_{\upsilon_{t-1}} E_{t-1}((\upsilon_{t-1} 1^{SL} - V_t)^2).$$

3. Finally, we will obtain V_{t-1} using regression of the V_t onto the variable $\vec{1}^{SL}$.

Let us perform these steps now:

$$\begin{aligned}
\tilde{\lambda}_t &= \arg \min_{\tilde{\vartheta}_{t-1} \in \mathbb{R}} E_{t-1}((\tilde{S}_t - \tilde{\vartheta}_{t-1}\tilde{L}_t)^2) = \frac{E(\tilde{S}_t \tilde{L}_t)}{E(\tilde{L}_t^2)}, \\
\tilde{\kappa}_t &= \arg \min_{\tilde{\vartheta}_{t-1} \in \mathbb{R}} E_{t-1}((1 - \tilde{\vartheta}_{t-1}\tilde{L}_t)^2) = \frac{E(\tilde{L}_t)}{E(\tilde{L}_t^2)}, \\
S_t^L &= \tilde{S}_t - \tilde{\lambda}_t \tilde{L}_t, \\
\vec{1}_t^L &= 1 - \tilde{\kappa}_t \tilde{L}_t.
\end{aligned}$$

We obtain the optimization problem

$$\{V_{t-1}, \xi_t\} = \arg\min_{\upsilon_{t-1}, \tilde{\vartheta}_t} E_{t-1}((\upsilon_{t-1} 1_t^L + \vartheta_t S_t^L - V_t)^2).$$

We proceed:

$$\begin{aligned}
\tilde{\iota}_t &= \arg\min_{\tilde{\vartheta}_t \in \mathbb{R}} E_{t-1}((1_t^L - \vartheta_t S_t^L)^2) = \frac{E(S_t^L 1_t^L)}{E((S_t^L)^2)}, \\
\vec{1}_t^{SL} &= 1_t^L - \tilde{\iota}_t S_t^L,
\end{aligned}$$

$$\begin{aligned}
V_{t-1} &= \arg\min_{\upsilon_{t-1} \in \mathbb{R}} E_{t-1}((\upsilon_{t-1} 1_t^{SL} - V_t)^2) \\
&= \frac{E_{t-1}(1_t^{SL} V_t)}{E_{t-1}((1_t^{SL})^2)} = E_{t-1}\left(\frac{1_t^{SL}}{E_{t-1}((1_t^{SL})^2)} V_t\right) \\
&= E_{t-1}\left(\frac{\vec{1}_t^L - \tilde{\iota}_t S_t^L}{E_{t-1}((\vec{1}_t^L)^2) - \tilde{\iota}_t E_{t-1}(S_t^L 1_t^L)} V_t\right).
\end{aligned}$$

More detailed:

$$\begin{aligned}
\frac{1^{SL}}{E((1^{SL})^2)} &= 1 - \frac{(\tilde{S}_t - E_{t-1}(\tilde{S}_t))(E_{t-1}(\tilde{L}_t)\mathrm{Cov}(\tilde{L}_t, \tilde{S}_t) - E_{t-1}(\tilde{S}_t)\mathrm{Var}(\tilde{L}_t))}{\mathrm{Cov}^2(\tilde{L}_1, \tilde{S}_t) - \mathrm{Var}(\tilde{S}_t)\mathrm{Var}(\tilde{L}_t)} \\
&+ \frac{((\tilde{L}_t - E_{t-1}(\tilde{L}_t))(E_{t-1}(\tilde{S}_t)\mathrm{Cov}(\tilde{L}_t, \tilde{S}_t) - E_{t-1}(\tilde{L}_t)\mathrm{Var}(\tilde{S}_t))}{\mathrm{Cov}^2(\tilde{L}_t, S) - \mathrm{Var}(\tilde{S}_t)\mathrm{Var}(\tilde{L}_t)}.
\end{aligned}$$

We recover $\tilde{\vartheta}_{t-1}^*$ from regression

$$\{\tilde{\vartheta}_{t-1}^*, \xi_t\} = \arg\min_{\tilde{\vartheta}_{t-1}, \vartheta_t} E_{t-1}((\tilde{\vartheta}_{t-1}\tilde{L}_t + \vartheta_t \tilde{S}_t - (V_t - V_{t-1}))^2).$$

We obtain $\tilde{\vartheta}_t^*$ again by sequential regression. Firstly, we make an auxiliary regression of $\tilde{L}_t$ to $\tilde{S}_t$ and $(V_t - V_{t-1})$ to $\tilde{S}_t$. After that we make the regression of residuals $(V_t - V_{t-1})^S$ on residuals L_t^S and obtain a coefficient $\tilde{\vartheta}_t^*$:

$$\begin{aligned}
W_t &= \arg\min_{\vartheta_t \in \mathbb{R}} E_t((\tilde{L}_t - \vartheta_t \tilde{S}_t)^2) = \frac{E_{t-1}(\tilde{S}_t \tilde{L}_t)}{E_{t-1}(\tilde{S}_t^2)}, \\
Z_t &= \arg\min_{\vartheta_t \in \mathbb{R}} E_t(((V_t - V_{t-1}) - \vartheta_t \tilde{S}_t)^2) = \frac{E_{t-1}(\tilde{S}_t(V_t - V_{t-1}))}{E_{t-1}(\tilde{S}_t^2)}, \\
L_t^S &= \tilde{L}_t - W_t \tilde{S}_t, \\
(V_t - V_{t-1})^S &= (V_t - V_{t-1}) - Z_t \tilde{S}_t.
\end{aligned}$$

$$\begin{aligned}
\tilde{\vartheta}^*_{t-1} &= \arg\min_{\tilde{\vartheta}_{t-1}\in\mathbb{R}} (\tilde{\vartheta}_t L^S_t - (V_t - V_{t-1})^S) \\
&= \frac{E_{t-1}(L^S_t(V_t - V_{t-1})^S)}{E_{t-1}((L^S_t)^2)} = E_{t-1}\left(\frac{L^S_t}{E_{t-1}((L^S_t)^2)}(V_t - V_{t-1})^S\right) \\
&= E_{t-1}\left(\frac{E_{t-1}(\tilde{S}^2_t)\tilde{L}_t - E_{t-1}(\tilde{L}_t\tilde{S}_t)\tilde{S}_t}{E_{t-1}(\tilde{L}^2_t)E_{t-1}(\tilde{S}^2_t) - (E_{t-1}(\tilde{S}_t\tilde{L}_t))^2}(V_t - V_{t-1})\right).
\end{aligned}$$

Finally we get ξ_t from an univariate regression,

$$\begin{aligned}
\xi_t &= \arg\min_{\vartheta_t} E_{t-1}((\vartheta_t\tilde{S}_t - (V_t - V_{t-1} - \tilde{\vartheta}^*_{t-1}\tilde{L}_t))^2) \\
&= \frac{E(\tilde{S}_t(V - V_t - \tilde{\vartheta}^*_t\tilde{L}_t))}{E(\tilde{S}^2_t)}.
\end{aligned}$$

Proof of Lemma 7. The vector of the optimal values of the minimization problem has the form

$$\hat{\beta} = (E(x^\top x))^{-1}E(\vec{x}y) = E((E(x^\top x))^{-1}\vec{x}y).$$

We can rewrite this form into:

$$\begin{aligned}
\hat{\beta}_1 &= E(F_1 y), \\
&\vdots \\
\hat{\beta}_k &= E(F_k y),
\end{aligned}$$

where F_i is a random variable. From this formulation we can see, that $E(F_1 1)$ is an optimal value $\tilde{\beta}_1$ of the minimization problem

$$\min_{\beta_i} E_t((\beta_1 + \sum \beta_i x_i - 1)^2).$$

It is obvious, that the above expression is minimal, when $\tilde{\beta}_1 = 1$ a $\tilde{\beta}_2 \ldots \tilde{\beta}_k = 0$ (remember that we assume that $\vec{1}$ does not belong to the vector space spanned by columns of x_i). Thus we get

$$\begin{aligned}
E(F_1) &= E(F_1 1) = \tilde{\beta}_1 = 1, \\
E(F_2) &= E(F_2 1) = \tilde{\beta}_2 = 0, \\
&\vdots \\
E(F_k) &= E(F_k 1) = \tilde{\beta}_k = 0.
\end{aligned}$$

We can see that $E(F_1 x_i)$ is an optimal value $\bar{\beta}_1$ for the minimization problem

$$\min_{\beta_i} E_t((\beta_1 + \sum \beta_i x_i - x_i)^2).$$

Now let us assume, that the random variables x_i are linearly independent. Apparently, optimal values minimizing above expression are $\bar{\beta}_i = 1$, $\bar{\beta}_j = 0$ for $j \neq i$, therefore $\bar{\beta}_1 = 0$ and

$$E(F_1 x_i) = \bar{\beta}_1 = 0 \quad \forall i \in \{2....k\}.$$

If some random variables $x_{l_1}, x_{l_2}...x_{l_S}$ $l_i \neq 1$ are linearly dependent (remember that we assume that they are not linearly dependent on $\vec{1}$), then the space of solutions is such that $\sum_{j=1}^{s} \beta_{lj} x_{lj} = x_i$. Other optimal values are zero which implies $\beta_1 = 0$, so that

$$E(F_1 x_i) = \beta_1 = 0 \quad \forall i \in \{2....k\}.$$

□

6.2.2 Local risk minimization by sequential regression

To prove well-definedness of all variables from 3.3.2, we will firstly show how we derived them from least squares regression:

$$\{V_{t-1}, \tilde{\vartheta}^*_{t-1}, \xi_t\} = \underset{\upsilon_{t-1}, \tilde{\vartheta}_{t-1}, \vartheta_t}{\arg\min}\ E_{t-1}((\upsilon_{t-1} + \tilde{\vartheta}_{t-1}\tilde{L}_t + \vartheta_t \tilde{S}_t - V_t)^2).$$

FOC for $G, \tilde{\vartheta}, \vartheta$:

$$\begin{aligned}
2E_{t-1}(V_{t-1} + \tilde{\vartheta}^*_{t-1}\tilde{L}_t + \xi_t \tilde{S}_t - \tilde{V}_t) &= 0,\\
2E_{t-1}((V_{t-1} + \tilde{\vartheta}^*_{t-1}\tilde{L}_t + \xi_t \tilde{S}_t - V_t)\tilde{L}_t) &= 0,\\
2E_{t-1}((V_{t-1} + \tilde{\vartheta}^*_{t-1}\tilde{L}_t + \xi_t \tilde{S}_t - V_t)\tilde{S}_t) &= 0.
\end{aligned}$$

Now we have to solve the system of three equations with three unknown parameters $V_t, \tilde{\vartheta}^*_t, \xi_{t+1}$:

$$V_{t-1} + \tilde{\vartheta}^*_{t-1} E_{t-1}(\tilde{L}_t) + \xi_t E_{t-1}(\tilde{S}_t) - E_{t-1}(V_t) = 0, \tag{6.1}$$
$$V_{t-1}E_{t-1}(\tilde{L}_t) + \tilde{\vartheta}^*_{t-1} E_{t-1}(\tilde{L}_t^2) + \xi_t E_{t-1}(\tilde{S}_t\tilde{L}_t) - E_{t-1}(V_t\tilde{L}_t) = 0 \tag{6.2}$$
$$V_{t-1}E_{t-1}(\tilde{S}_t) + \tilde{\vartheta}^*_{t-1} E_{t-1}(\tilde{L}_t\tilde{S}_t) + \xi_t E_{t-1}(\tilde{S}_t^2) - E_{t-1}(V_t\tilde{S}_t) = 0. \tag{6.3}$$

If we express V_t from () and substitute into the (6.2) and (6.3), we obtain:

$$\begin{aligned}\tilde{\vartheta}^*_{t-1}\mathrm{Var}(\tilde{L}_t) + \xi_t \mathrm{Cov}(\tilde{L}_t, \tilde{S}_t) - \mathrm{Cov}(V_t, \tilde{L}_t) &= 0, &(6.4)\\ \tilde{\vartheta}^*_{t-1}\mathrm{Cov}(\tilde{L}_t, \tilde{S}_t) + \xi_t \mathrm{Var}(\tilde{S}_t) - \mathrm{Cov}(V_t, \tilde{S}_t) &= 0. &(6.5)\end{aligned}$$

If $\mathrm{Cov}^2(\tilde{L}_t, \tilde{S}_t) - \mathrm{Var}(\tilde{S}_t)\mathrm{Var}(\tilde{L}_t) > 0$ then we have

$$\begin{aligned}\xi_t &= \frac{\mathrm{Cov}(V, \tilde{L}_t)\mathrm{Cov}(\tilde{L}_t, \tilde{S}_t) - \mathrm{Cov}(V_t, \tilde{S}_t)\mathrm{Var}(\tilde{L}_t)}{\mathrm{Cov}^2(\tilde{L}_t, \tilde{S}_t) - \mathrm{Var}(\tilde{S}_t)\mathrm{Var}(\tilde{L}_t)},\\ \tilde{\vartheta}^*_{t-1} &= \frac{\mathrm{Cov}(V_t, \tilde{S}_t)\mathrm{Cov}(\tilde{L}_1, \tilde{S}_t) - \mathrm{Cov}(V_t, \tilde{L}_1)\mathrm{Var}(\tilde{S}_t)}{\mathrm{Cov}^2(\tilde{L}_1, \tilde{S}_t) - \mathrm{Var}(\tilde{S}_t)\mathrm{Var}(\tilde{L}_t)},\\ V_{t-1} &= E(V_t) - \tilde{\vartheta}^*_{t-1}E(\tilde{L}_1) - \xi_t E(\tilde{S}_t)\\ &= E_{t-1}\left(\left(1 - \frac{(\tilde{S}_t - E_{t-1}(\tilde{S}_t))\tilde{B}_t + ((\tilde{L}_t - E_{t-1}(\tilde{L}_t))\tilde{C}_t}{\mathrm{Cov}^2(\tilde{L}_1, \tilde{S}_t) - \mathrm{Var}(\tilde{S}_t)\mathrm{Var}(\tilde{L}_t)}\right) V_t\right),\\ \frac{dQ}{dP} &:= \prod_{t=1}^{\top}\left(1 - \frac{(\tilde{S}_t - E_{t-1}(\tilde{S}_t))\tilde{B}_t + ((\tilde{L}_t - E_{t-1}(\tilde{L}_t))\tilde{C}_t}{\tilde{A}_t}\right).\end{aligned}$$

If $\mathrm{Cov}^2(\tilde{L}_1, \tilde{S}_t) - \mathrm{Var}(\tilde{S}_t)\mathrm{Var}(\tilde{L}_t) = 0$, it means, that processes $\tilde{L}_t$ and $\tilde{S}_t$ are linearly depended, i.e. $\tilde{S}_t = c + a\tilde{L}_t$ where c and a are constants. Then

$$\begin{aligned}\mathrm{Var}(\tilde{S}_t) &= a^2\mathrm{Var}(\tilde{L}_t),\\ \mathrm{Cov}(\tilde{L}_t, \tilde{S}_t) &= a\mathrm{Var}(\tilde{L}_t),\\ \mathrm{Cov}(V_t, \tilde{S}_t) &= a\mathrm{Cov}(V_t, \tilde{L}_t).\end{aligned}$$

The expressions (6.4) and (6.5) turn into

$$\begin{aligned}\tilde{\vartheta}^*_{t-1}\mathrm{Var}(\tilde{L}_t) + a\xi_t\mathrm{Var}(\tilde{L}_t) - \mathrm{Cov}(V_t, \tilde{L}_t) &= 0, &(6.6)\\ a\tilde{\vartheta}^*_{t-1}\mathrm{Var}(\tilde{L}_t) + a^2\xi_t\mathrm{Var}(\tilde{L}_t) - a\mathrm{Cov}(V_t, \tilde{L}_t) &= 0. &(6.7)\end{aligned}$$

Equations (6.6) and (6.7) are dependent, so finally we obtain just one condition

$$\tilde{\vartheta}^*_{t-1}\mathrm{Var}(\tilde{L}_t) = \mathrm{Cov}(V_t, \tilde{L}_t) - a\xi_t\mathrm{Var}(\tilde{L}_t).$$

Again, for $\mathrm{Var}(\tilde{L}_t) > 0$

$$\begin{aligned}\tilde{\vartheta}^*_{t-1} &= \frac{\mathrm{Cov}(V_t, \tilde{L}_t) - a\xi_t\mathrm{Var}(\tilde{L}_t)}{\mathrm{Var}(\tilde{L}_t)}, \quad \xi_t \text{ is ambiguous,}\\ V_{t-1} &= E(V_t) - \tilde{\vartheta}^*_{t-1}E(\tilde{L}_1) - \xi_t E(\tilde{S}_t),\end{aligned}$$

and for $\mathrm{Var}(\tilde{L}_t) = 0 \Rightarrow \mathrm{Cov}(V_t, \tilde{L}_t) = 0$. Consequently, both $\tilde{\vartheta}^*_{t-1}$, ξ_t are ambiguous and $V_{t-1} = E(V_t) - \tilde{\vartheta}^*_{t-1}E(\tilde{L}_1) - \xi_t E(\tilde{S}_t)$.

6.2.3 Unconditional hedging error of the local hedging strategy

Proof of Theorem 8. We will start with

$$\varepsilon^2_{TLE} = (G^{\upsilon,\xi}_T - V_T)^2 = x^{\upsilon,\xi T}_T P_T x^{\upsilon,\xi}_T + 2c^\top_T x^{\upsilon,\xi}_T, \quad c_T = \vec{0}, \quad P_T = Q_T.$$

Using the self-financing condition

$$x^{\upsilon,\xi}_{t+1} = A_{t+1} x^{\upsilon,\xi}_t + b_{t+1}\xi_{t+1},$$

we express the hedging error in time $T-1$:

$$\begin{aligned}
\varepsilon^2_{T-1LE} &= E_{T-1}\left(x^{\upsilon,\xi T}_T P_T x^{\upsilon,\xi}_T + 2c^\top_T x^{\upsilon,\xi}_T\right) \\
&= E_{T-1}\left((A_{t+1}x^{\upsilon,\xi}_{T-1} + b_{t+1}\xi_T)^\top P_T(A_{t+1}x^{\upsilon,\xi}_{T-1} + b_{t+1}\xi_T) + 2c^\top_T(A_{t+1}x^{\upsilon,\xi}_{T-1} + b_{t+1}\xi_T)\right) \\
&= E_{T-1}(x^{\upsilon,\xi}_{T-1}{}^\top A^\intercal_{t+1} P_T A_{t+1} x^{\upsilon,\xi}_{T-1} + 2(\xi_T b^\intercal_{t+1} P_T A_{t+1} + c^\top_T A_{t+1}) x^{\upsilon,\xi}_{T-1} \\
&+ \xi^2_T b^\intercal_{t+1} P_T b_{t+1} + 2\xi_T c^\top_T b_{t+1}) \\
&= x^{\upsilon,\xi}_{T-1}{}^\top P_{T-1} x^{\upsilon,\xi}_{T-1} + 2c^\top_{T-1} x^{\upsilon,\xi}_{T-1} + E_{T-1}(\phi_{T-1}). \qquad (6.8)
\end{aligned}$$

After recursive application of (6.8) one obtains

$$E((G_T - V_T)^2) = x^\top_0 P_0 x_0 + c^\top_0 x_0 + \sum_{t=1}^{T} E(\phi_t).$$

□

6.3 Extended model in terms of standard model

Let assign elements of matrix Q_t to above variables:

$$\begin{aligned}
\alpha_t &= L_t, \\
\gamma_t &= -L_t V_t = -\alpha_t V_t \ , \\
\zeta_t &= \sum_{i=t+1}^{T} E_t(L_i\psi_i) + L_t V^2_t = \sum_{i=t+1}^{T} E_t(L_i\psi_i) + \frac{\gamma^2_t}{\alpha_t}.
\end{aligned}$$

. Now we will justify that it fits:

$$\alpha_t = \alpha_{t+1}\left(1 - \frac{(E(\Delta S_{t+1}))^2}{E(\Delta S^2_{t+1})}\right), \quad \alpha_T = 1.$$

which fits with

$$L_t = E_t(L_{t+1}(1 - \Delta\tilde{K}_{t+1})), \quad L_T = 1.$$

in standard model. In the case of IID stock returns $\alpha_t = b^{T-t}$ where $b = (1 - \Delta\tilde{K}_{t+1})$.

Using mathematical induction we show that $\gamma_t = -\alpha_t V_t$:

1° $\gamma_T = -1V_T$

2° $\gamma_{t+1} = -\alpha_{t+1}V_{t+1} \Rightarrow \gamma_t = -\alpha_t V_t$:

In the case of IID stock returns we obtain

$$\begin{aligned}
\gamma_t &= E_t(\gamma_{t+1}) - \frac{E_t(\alpha_{t+1}\Delta S_{t+1})E_t(\gamma_{t+1}\Delta S_{t+1})}{E_t(\alpha_{t+1}\Delta\tilde{S}^2_{t+1})} \\
&= -b^{T-t-1}E_t(V_{t+1}) + b^{T-t-1}\frac{E_t(\alpha_{t+1}\Delta S_{t+1})E_t(V_{t+1}\Delta S_{t+1})}{E_t(\alpha_{t+1}\Delta S^2_{t+1})} \\
&= E_t\left(b^{T-t-1}\left(1 - \frac{\alpha_{t+1}E_t(\Delta S_{t+1})\Delta S_{t+1}}{\alpha_{t+1}E_t(\Delta S^2_{t+1})}\right)V_{t+1}\right) \\
&= E_t\left(b^{T-t}\left(\frac{E_t(\Delta S^2_{t+1}) - E_t(\Delta S_{t+1})\Delta S_{t+1}}{E_t(\Delta S^2_{t+1}) - (E(\Delta S_{t+1}))^2}\right)V_{t+1}\right) = -\alpha_t V_t.
\end{aligned}$$

We know that in the standard model ψ_{t+1} in the case of IID stock returns has the form

$$\begin{aligned}
\psi_{t+1} &= E_t(e^2_{t+1}) = \mathrm{Var}_t(V_{t+1}) - \frac{(\mathrm{Cov}_t(\Delta S_{t+1}, V_{t+1}))^2}{\mathrm{Var}_t(\Delta S_{t+1})} \\
&= E_t(V^2_{t+1}) - bV_t^2 - \frac{(E_t(V_{t+1}\Delta S_{t+1}))^2}{E_t(\Delta S^2_{t+1})}.
\end{aligned}$$

Using mathematical induction we show that $\zeta_t = \sum_{i=t+1}^{T} E_t(L_i\psi_i) + L_t V_t^2$.

1° $\zeta_T = 0 + 1V_T^2$

2° $\zeta_{t+1} = \sum_{i=t+1}^{T} E_t(L_i\psi_i) + L_{t+1}V_{t+1}^2 \Rightarrow \zeta_t = \sum_{i=t+1}^{T} E_t(L_i\psi_i) + L_tV_t^2$:

$$
\begin{aligned}
\zeta_t &= E_t(\zeta_{t+1}) - \frac{\big(E_t(\gamma_{t+1}\Delta S_{t+1})\big)^2}{E_t(\alpha_{t+1}\Delta S_{t+1}^2)} \\
&= \sum_{i=t+2}^{T} b^{T-i}E_t(\psi_i) + b^{T-t-1}\left(E_t(V_{t+1}^2) - \frac{(E_t(V_{t+1}\Delta S_{t+1}))^2}{E_t(\Delta S_{t+1}^2)}\right) \\
&= \sum_{i=t+2}^{T} b^{T-i}E_t(\psi_i) + b^{T-t-1}\psi_{t+1} + b^{T-t-1}bV_t^2 \\
&= \sum_{i=t+1}^{T} b^{T-i}E_t(\psi_i) + L_tV_t^2
\end{aligned}
$$

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