Globálne optimálne gama zaisťovanie s transakčnými nákladmi

DIPLOMOVÁ PRÁCA

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(Master thesis)

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I hereby declare that I have written the thesis myself, using only referenced literature and my knowledge, under careful supervision of my thesis advisor.

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Zuzana Cel'uchová

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Abstract

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In 1989 Follmer and Schweizer [4] constructed a simple scheme for the computation of hedging strategies in an incomplete market, taking local risk minimization into consideration. Since then several significant steps have been taken in the development of global risk minimization theory. In 2007 Černý and Kallsen [2] provided derivation of the globally optimal hedging strategy by sequential regressions. In this thesis we concentrate on their results, applying their hedging strategy in a particular problem concerning portfolio rebalancing. We leave the idea of simply hedging the contingent claim. Instead, we add the element of changing the very structure of option portfolio. We derive the optimal strategy for a quadratic utility investor. Finally we present a numerical example based on historical data and evaluate the obtained results.

Keywords: hedging, financial derivatives, risk minimization, optimal investment strategy, gamma hedging

Abstrakt

CEĽUCHOVÁ, Zuzana: Globálne optimálne gama zaisťovanie s transakčnými nákladmi [Diplomová práca], Univerzita Komenského, Bratislava, Fakulta matematiky, fyziky a informatiky, Katedra aplikovanej matematiky a štatistiky; vedúci diplomovej práce: prof. Ing. Aleš Černý, PhD., Bratislava, 2009, 33 s.

V roku 1989 skonštruovali Follmer a Schweizer [4] jednoduchú schému na výpočet zaisťovacích stratégií na neúplnom trhu prostredníctvom lokálnej minimalizácie rizika. Odvtedy boli urobené výrazné pokroky vo vytváraní teórie globálnej minimalizácie rizika. V roku 2007 Černý a Kallsen [2] ukázali odvodenie globálne optimálnej zaisťovacej stratégie postupnosťou regresií. V tejto práci sa zameriame na ich výsledky a aplikujeme ich stratégiu na konkrétny problém týkajúci sa zmeny portfólia. Upustíme od myšlienky jednoduchého zaisťovania. Namiesto toho pridáme prvok zmeny štruktúry portfólia opcií. Odvodíme optimálnu stratégiu pre investora s kvadratickou funkciou užitočnosti. Napokon predstavíme konkrétny príklad založený na historických dátach a vyhodnotíme získané výsledky.

Kľúčové slová: zaisťovanie, finančné deriváty, minimalizácia rizika, optimálna investičná stratégia, gama zaisťovanie

Contents

Introduction

Trading in the financial markets naturally involves certain intrinsic risk, since the returns on the investment depend on stochastic processes which are impossible to predict exactly. As a result, many types of financial derivatives have been introduced as an effective tool for easing the risk and hedging the traders‘ portfolios. Finding the optimal hedging strategy is quite simple under the assumption of a complete market, where it is possible to find a perfect replicating portfolio. In reality, this assumption is hardly fulfilled, since the number of causes of uncertainty is greater than the number of assets held by the trader. The standard Black-Scholes model represents a good example helping us to understand the basic principles of option pricing and hedging strategies. However, the assumption of continuous trading and describing the asset price dynamics by a pure diffusion process keep the model far from reality. It is therefore essential that we look for a better approximation of the market dynamics allowing for price jumps and that we try to understand the computation of hedging strategies in an environment where perfect replication is not a matter of course.

A broad examination on this topic has been performed by Černý and Kallsen [2]. They considered the minimization of the unconditional squared hedging error $E((v + \vartheta \bullet S_T - H)^2)$, where v represents an admissible initial endowment, ϑ is an admissible trading strategy, S is a stock price and H is a contingent claim to be hedged. In other words, they tried to minimize the squared difference of the contingent claim and the total portfolio value at maturity. This approach is quite reasonable, since it is the total hedging error at maturity that really matters. They compared this so called dynamically optimal trading strategy to the local risk minimization introduced by Follmer and Schweizer in [4]. The purpose of this thesis is to apply their results to the solution of an intermediate option trading problem. In [2], the contingent claim to be hedged is fixed - we consider one derivative and seek a replicating portfolio. Instead of this approach, we now consider a whole portfolio consisting of several different derivatives (call them options). Moreover, we allow intermediate option trading, i.e. rebalancing the options portfolio throughout the trading. The objective of such a trading is maximizing the investor's utility. We show that the utility function maximization is equivalent to a special case of hedging error minimization, which gives us an opportunity to take comfort in the results of Černý and Kallsen [2].

The thesis is organized as follows. Chapter 1 presents the derivation of globally optimal trading strategy as introduced by Černý and Kallsen in [2]. We focus on their evaluation of the total hedging error. In Chapter 2 we introduce intermediate option trading in detail and provide a transformation of the original problem to a quadratic programming problem with constraints. In Chapter 3 we discuss the Lévy processes as a useful tool for modeling the stock price process. Specifically, we concentrate on normal inverse Gaussian (NIG) process, which we use in numerical implementation of the problem. We present NIG distribution and some of its properties. Chapter 4 comprises a numerical implementation of the theory provided in previous chapters. We examine a particular example based on historical data and perform a historical trading simulation.

Chapter 1

Mean–Variance Hedging

1.1 Notation and Assumptions

Consider a probability space $(\Omega, P, \mathcal{F})$, time horizon T and a set of trading dates $\mathcal{T} = \{0, 1, \ldots, T\}$. Let $\mathbb{F} = \{\mathcal{F}_t\}_{t\in\mathcal{T}}$ be a filtration of the given probability space and $H \in L^2(P)$ an $\mathcal{F}_t$ - measurable random variable representing the contingent claim to be hedged. We set $\mathcal{F}_0$ trivial.

Let $\{S_t\}_{t\in\mathcal{T}}$ be an $\mathbb{F}$-adapted stock price process satisfying the condition:

$$E_t((S_{t+1} - S_t)^2) = E_t((\Delta S_{t+1})^2) < \infty \text{ for } t < T.$$

Moreover, we assume that S satisfies the no arbitrage condition defined by

Definition 1.1.1. *We say that process S admits no arbitrage, if for all $t \in \mathcal{T}\backslash\{0\}$ and for all $\mathcal{F}_{t-1}$-measurable portfolios ϑ_t we have that $\vartheta_t \Delta S_t \geq 0$ implies $\vartheta_t \Delta S_t = 0$ almost surely.*

This requirement roughly means that the trader cannot achieve positive gains unless he invests a positive amount in the beginning. A trading strategy is described by a predictable process ϑ. This condition matches with the fact that the trading strategy at time t only depends on the previous development, not on what happens at t or later. If we denote the initial endowment v, then the portfolio value at maturity is $v + \vartheta \bullet S_T := v + \sum_{t=1}^{T} \vartheta_t \Delta S_t$. We assume that v is $\mathcal{F}_0$-measurable and

$$v + \vartheta \bullet S_T \in L^2(P).$$

1.2 Locally Optimal Hedging Strategy

In this section we examine the local risk minimization presented by Follmer and Schweizer. Their approach is based on minimizing the one-step expected squared hedging error recursively. This means solving the problem $\min\limits_{v_{t-1},\vartheta_t} E(v_{t-1}+\vartheta_t\Delta S_t - V_t)^2$ successively several times. An easy way to do so is performing sequential least squares regressions. Define

$$\begin{aligned} V_T &:= H, \\ \{V_{t-1},\xi_t\} &:= \arg\min_{v_{t-1},\vartheta_t} E_{t-1}((v_{t-1}+\vartheta_t\Delta S_t - V_t)^2).^{1} \end{aligned}$$

We see that the quantities V_{t-1},ξ_t are the least squares coefficients from the regression of V_t onto explanatory variables ΔS_t and constant. Following the Frisch–Waugh–Lovell theorem we can equivalently perform an auxiliary regression of the constant onto ΔS_t and then obtain V_{t-1} from the regression of V_t onto the residuals from the auxiliary regression. Hence we obtain

$$V_{t-1} = E_{t-1}\left(\frac{1-\tilde{\lambda}_t\Delta S_t}{1-\Delta\tilde{K}_t}V_t\right), \tag{1.1}$$

with $\tilde{\lambda}_t, \Delta\tilde{K}_t$ defined as follows:

$$\tilde{\lambda}_t := \arg\min_{\vartheta_1\in\mathbb{R}} E((\vartheta_1\Delta S_1 - 1)^2) = \frac{E_{t-1}(\Delta S_t)}{E_{t-1}((\Delta S_t)^2)}, \tag{1.2}$$

$$\Delta\tilde{K}_t := \min_{\vartheta_1\in\mathbb{R}} E((\vartheta_1\Delta S_1 - 1)^2) = \frac{(E_{t-1}(\Delta S_t))^2}{E_{t-1}((\Delta S_t)^2)} = 1 - E_{t-1}((1-\tilde{\lambda}_t\Delta S_t)^2). \tag{1.3}$$

Equation (1.1) suggests that V_{t-1} is a mean value of V_t under probability measure Q defined by

$$\frac{dQ}{dP} := \frac{1-\tilde{\lambda}_t\Delta S_t}{1-\Delta\tilde{K}_t}.$$

Such a conclusion would square with the risk-neutral pricing under risk neutral probability measure. To show that Q really is a risk neutral measure, we need to show that its total mass equals 1 and that the stock price process is a martingale under Q (i.e. $E(\Delta S_t) = 0$).

[1] minimize through the set of v_{t-1},ϑ_t being $\mathcal{F}_t$-measurable.

$$
\begin{aligned}
E\left(\frac{dQ}{dP}\right) &= E\left(\frac{1-\tilde{\lambda}_t\Delta S_t}{1-\Delta\tilde{K}_t}\right) = \frac{E(1-\tilde{\lambda}_t\Delta S_t)}{1-\tilde{\lambda}_t E(\Delta S_t)} = 1, \\
E^Q(\Delta S_t) &= E\left(\frac{1-\tilde{\lambda}_t\Delta S_t}{1-\Delta\tilde{K}_t}\Delta S_t\right) = \frac{E(\Delta S_t)-\tilde{\lambda}_t E((\Delta S_t)^2)}{1-\Delta\tilde{K}_t} = 0.
\end{aligned}
$$

□

On defining the martingale measure Q

$$
\frac{dQ}{dP} := \prod_{t=1}^{T} \frac{1-\tilde{\lambda}_t\Delta S_t}{1-\Delta\tilde{K}_t} \tag{1.4}
$$

we can write

$$
V_{t-1} = E^Q_{t-1}(V_t). \tag{1.5}
$$

The optimal trading strategy ξ_t is then easy to obtain from a univariate regression of $V_t - V_{t-1}$ onto the explanatory variable ΔS_t,

$$
\xi_t = \frac{Cov_{t-1}(V_t, \Delta S_t)}{Var_{t-1}(\Delta S_t)} = \frac{E_{t-1}((V_t - V_{t-1})\Delta S_t)}{E_{t-1}((\Delta S_t)^2)}. \tag{1.6}
$$

1.2.1 Evaluation of the hedging error

Following the locally optimal strategy derived in the previous section, one may ask what is the total hedging error of such trading. Assume we start with initial endowment v and hold ξ_t shares at time $t-1$. We assume that the portfolio is self-financing, i.e. no additional investment is provided during the trading. Mathematically, we write $G_t^{v,\xi} = G_{t-1}^{v,\xi} + \xi_t\Delta S_t$ for all $t \in \mathcal{T}$, where $G_t^{v,\xi}$ denotes the portfolio value at time t, starting with initial capital v and following the optimal strategy ξ. Set $V_T = H$. We wish to find the value of $E((G_T^{v,\xi} - V_T)^2)$. We obtain

$$
\begin{aligned}
E((G_T^{v,\xi} - V_T)^2) &= E(E_{T-1}((G_T^{v,\xi} - V_T)^2)) \\
&= E(E_{T-1}((G_T^{v,\xi} - V_{T-1} + V_{T-1} + \xi_t\Delta S_T - V_T)^2)).
\end{aligned} \tag{1.7}
$$

At this point we can take comfort in the fact that V_{T-1} and ξ_T are the coefficients from the least squares regression of V_T onto explanatory variables

ΔS_T and 1. The residuals in a least squares regression are orthogonal to the subspace generated by the explanatory variables. Hence the residuals are orthogonal to the subspace generated by ΔS_T and 1, which comprises their orthogonality to 1. [2] If we introduce the following notation

$$e_t = V_{t-1} + \xi_t \Delta S_t - V_t \qquad \text{for all } t \in \{1, 2, \ldots, T\}, \tag{1.8}$$

we can write $E_{T-1}(e_T) = 0$. (1.7) can then be rewritten as

$$E((G_T^{v;\xi} - V_T)^2) = E((G_{T-1}^{v;\xi} - V_{T_1})^2 + \psi_T), \tag{1.9}$$

with ψ_t defined as follows:

$$\psi_t = E_{t-1}(e_t^2) = Var_{t-1}(V_t) - \xi_t Cov_{t-1}(\Delta S_t, V_t). \tag{1.10}$$

Applying these formulas recursively for $t \in \{T-1, T-2, \ldots, 0\}$ we observe

$$E((G_T^{v;\xi} - V_T)^2) = (v - V_0)^2 + \sum_{t=1}^{T} E(\psi_t). \tag{1.11}$$

1.3 Globally Optimal Hedging Strategy

The previous sections show the way of computation of the unconditional squared expected hedging error using the locally optimal strategy ξ. However, the question remains whether this really is the minimum attainable expected squared hedging error value. Let us now examine the solution to the global risk minimization

$$\min_{\vartheta} E((G_T - V_T)^2), \qquad V_T = H. \tag{1.12}$$

Note that the locally optimal strategy minimizes the hedging error with respect to both v and ϑ_t. The self-financing property of the portfolio dictates that its value at time $t-1$ is precisely given by its initial value, the way of trading and the performance of stock price process. There is no way one could possibly influence the portfolio value $G_{t-1}^{v,\vartheta}$ at time $t-1$. The tool one holds in their hands is the number of shares held. In an incomplete market

[2] Note that the orthogonality here is meant in terms of probability.

one cannot count on the existence of a perfect replicating portfolio. The real portfolio value might undershoot or overshoot the ideal case value V_t significantly. The idea of inquiring about the optimal portfolio value at time $t-1$ is therefore seemingly useless. However, it provides an easy-to-compute information which can be later used in the derivation of globally optimal trading strategy.

We perform the minimization recursively again. Denote the optimal trading strategy by $\varphi(v)$. We begin with the choice of optimal strategy in the final period $t = T$. We minimize the expected squared hedging error

$$\min_{\vartheta_T} E_{T-1}((G_{T-1}^{v,\varphi(v)} + \vartheta_T \Delta S_T - V_T)^2). \tag{1.13}$$

Hence,

$$\varphi_T(v) = \arg\min_{\vartheta_T} E_{T-1}((G_{T-1}^{v,\varphi(v)} + \vartheta_T \Delta S_T - V_T)^2). \tag{1.14}$$

Note that this represents a least squares regression similar to the local risk minimization case. The difference lies in the dimension of the problem. Local risk minimization represents a bivariate regression, whereas in this case we only deal with a univariate regression of the amount $V_T - G_{T-1}^{v,\varphi(v)}$ onto the explanatory variable ΔS_T. This is a result of the previously mentioned fact that once we arrive at time $T-1$, the value of $G_{T-1}^{v,\varphi(v)}$ is fixed and our decision only lies on the optimal choice of $\varphi_T(v)$. By standard least squares we have

$$\begin{aligned}\varphi_T(v) &= \frac{E_{T-1}((V_T - G_{T-1}^{v,\varphi(v)})\Delta S_T)}{E_{T-1}((\Delta S_T)^2)} \\ &= \frac{E_{T-1}((V_T - V_{T-1} + V_{T-1} - G_{T-1}^{v,\varphi(v)})\Delta S_T)}{E_{T-1}((\Delta S_T)^2)} \\ &= \frac{E_{T-1}((V_T - V_{T-1}^{v,\varphi(v)})\Delta S_T)}{E_{T-1}((\Delta S_T)^2)} + \tilde{\lambda}_T(V_{T-1} - G_{T-1}^{v,\varphi(v)}) \\ &= \xi_T + \tilde{\lambda}_T(V_{T-1} - G_{T-1}^{v,\varphi(v)}).\end{aligned} \tag{1.15}$$

Let us now evaluate the hedging error of strategy $\varphi(v)$:

$$G_{T-1}^{v,\varphi(v)} + \varphi_T \Delta S_T - V_T = (G_{T-1}^{v,\varphi(v)} - V_{T-1})(1 - \tilde{\lambda}_T \Delta S_T) + e_T \tag{1.16}$$

with e_T defined in (1.8). We already know that e_T is orthogonal to 1 and ΔS_T, which implies

$$E_{T-1}((1 - \tilde{\lambda}_T \Delta S_T)e_T) = 0 \tag{1.17}$$

Equations (1.16) and (1.17) yield

$$E_{T-1}((G_{T-1}^{v,\varphi(v)} + \varphi_T \Delta S_T - V_T)^2) = (1 - \Delta\tilde{K}_T)(G_{T-1}^{v,\varphi(v)} - V_{T-1})^2 + \psi_T, \quad (1.18)$$

where ψ_T represents the one-step locally optimal squared hedging error defined in (1.10).

Now we can check the minimization at time $T-1$. Note that, if the stock returns are i.i.d. random variables, the quantity $L_{T-1} = 1 - \Delta\tilde{K}_T$ is deterministic. Moreover, the value of ψ_T does not depend on ϑ_{T-1}. Hence we have:

$$\begin{aligned}
\varphi_{T-1}(v) &= \arg\min_{\vartheta_{T-1}} E_{T-2}(L_{T-1}(G_{T-2}^{v,\varphi(v)} + \vartheta_{T-1}\Delta S_{T-1} - V_{T-1})^2 + \psi_T) \\
&= \arg\min_{\vartheta_{T-1}} L_{T-1} E_{T-2}((G_{T-2}^{v,\varphi(v)} + \vartheta_{T-1}\Delta S_{T-1} - V_{T-1})^2) \\
&= \arg\min_{\vartheta_{T-1}} E_{T-2}((G_{T-2}^{v,\varphi(v)} + \vartheta_{T-1}\Delta S_{T-1} - V_{T-1})^2).
\end{aligned}$$

We see that the minimization at time $T-1$ is basically the same as in the final period T. By analogy, we observe that the minimization has the same form for all $t \in \mathcal{T}$. Hence, the following statement follows:

$$\varphi_t(v) = \xi_t + \tilde{\lambda}_t(V_{t-1} - G_{t-1}^{v,\varphi(v)}) \qquad \forall t \in \mathcal{T} \quad (1.19)$$

If we define

$$L_t = \prod_{j=t+1}^{T} (1 - \Delta\tilde{K}_j), \qquad L_T = 1, \quad (1.20)$$

we obtain the desired formula for the total unconditional squared hedging error of a dynamically optimal strategy:

$$E((G_T^{v,\varphi(v)} - V_T)^2) = L_0(v - V_0)^2 + \sum_{t=1}^{T} E(L_t\psi_t) = L_0(v - V_0)^2 + \varepsilon_0^2(H), \quad (1.21)$$

with $\varepsilon_0^2(H)$ defined as

$$\varepsilon_0^2(H) := \sum_{t=1}^{T} E(L_t\psi_t). \quad (1.22)$$

Remark 1.3.1. *It follows from (1.19) that the dynamically optimal strategy reflects the locally optimal strategy ξ, but takes the portfolio value misalignment $V_{T-1} - G_{T-1}^{v,\varphi(v)}$ into consideration and makes a corresponding adjustment depending on the stock price performance and whether the portfolio value overshoots or undershoots the desired value V_{t-1}. Note that in a bull market the value of $\tilde{\lambda}_t$ is positive. If the portfolio value overshoots V_{t-1}, the dynamically optimal strategy performs a downward adjustment of ξ_t. Therefore $\varphi_t(v) < \xi_t$. On the other hand, in a bear market, the value of $\tilde{\lambda}_t$ is negative. Hence, the adjustment in the case of overshooting the desired portfolio value is upwards and $\varphi_t(v) > \xi_t$. This is a perfectly reasonable behaviour - if the market is expanding and the stock price is likely to increase and the desired portfolio value is lower then the actual one, we reduce the number of shares held.*

Chapter 2

Intermediate option trading

In this chapter we introduce the portfolio rebalancing problem. So far, we have supressed the dependence of φ and the mean-value process V_t on the contingent claim H. Now, by contrast, we shall show it off by introducing the notation

$$\varphi(v, H), \qquad V_t(H)$$

for the dynamically optimal strategy in hedging the contingent claim H and starting with initial capital v and for the locally optimal portfolio value defined by (1.1).

Consider an investor with a portfolio consisting of *several* different derivative securities in a short position (call them options). Suppose that he is rational and up to time $t := t_0$ he has been following the dynamically optimal hedging strategy $\varphi(v, H)$. At this point he considers changing the structure of his portfolio. The question is: if he changed the proportions of individual options in the portfolio, could he possibly increase his benefits from trading? It is intuitively clear that following the optimal hedging strategy does not mean maximum utility as a matter of course. It might happen that some portfolios bring higher utility than others, despite the fact that the investor follows the optimal strategy in all cases. At time $t = 0$, i.e. at the very beginning of trading with the portfolio which we consider, the investor has only little information about the future stock price performance. Throughout the trading, the stock price changes and so do the market prices of options. Hence it might easily happen that at time $t > 0$ the initial portfolio structure is not the best choice and it makes sense for the investor to inquire whether he should/should not rebalance his portfolio. As a criterion to compare the benefits from trading, we shall use the quadratic utility function. In doing so, we shall take transaction costs into consideration - we must not forget that trading options is not a free business. The ask/bid spreads represent an "obstacle" - it is essential that the investor examines thoroughly whether the benefits are really worth the costs of trading.

2.1 Formulation of the problem

Mathematically, we formulate the problem as follows.

Suppose the investor considers rebalancing his portfolio at time t. For the simplicity of notation, we shall only present a one–period decision making and adopt the view of a new starting point, i.e. we use the 0 lower index for quantities relating to time t and we think of T as the total time to maturity beginning at t. Note that the 0 indexation does not refer to the very beginning of trading, it only indicates that at the time of decision making, t is our starting point. Let G_0 denote the initial portfolio value at time t. Let ρ be the row vector of current amounts of options in the investor's portfolio before rebalancing. The investor considers selling η options, η being a row vector with positive elements representing selling and negative elements representing buying options. Furthermore, let H denote the column vector of payoffs of the individual options and C^a and C^b the column vectors of corresponding current ask and bid market prices of the options. We wish to maximize the utility function

$$\max_{\vartheta,\eta} U = -E_0(((\tilde{G}_T - (\rho + \eta)H) - V^*)^2), \tag{2.1}$$

where $\tilde{G}_T$ denotes the total portfolio value at maturity and V^* is the utility function bliss point.

The portfolio value immediately after option trading equals the investor's initial wealth reduced by the costs of trading, i.e.

$$\tilde{G}_0 = G_0 + \eta^+ C^b - \eta^- C^a. \tag{2.2}$$

Note that we wish to maximize (2.1) with respect to two variables. We wish to find the optimal change in portfolio structure η and optimal hedging strategy ϑ. We perform the maximization in two steps. First, we fix η and find the optimal strategy as a function of η, and then we find the optimal η. One can easily see that for a fixed value of η, (2.1) is a specific case of optimal hedging problem, starting with initial wealth $\tilde{G_0}$ and hedging the amount $\tilde{H} := (V^* + (\rho + \eta)H)$. We have tackled this issue in the previous chapter and we already know that the solution is given by the dynamically optimal trading strategy $\varphi(\tilde{G}_0, \tilde{H})$ and the corresponding value of the utility function can be expressed in terms of (1.21):

$$\hat{U} = -\left(L_0\left(\tilde{G_0} - V_0(\tilde{H})\right)^2 + \varepsilon_0^2(\tilde{H})\right), \tag{2.3}$$

where $\varepsilon_0^2 = \sum_{i=t}^{T} L_i\psi_i$ with ψ_t defined in (1.10).

Equivalently, we can minimize the function

$$f(\eta) = L_0\left(\tilde{G_0} - V_0(\tilde{H})\right)^2 + \varepsilon_0^2(\tilde{H}) \tag{2.4}$$

Our goal is to minimize this value with respect to η. To do so, we need to find the functional dependence of $f(\eta)$ on η. This seems rather tricky at first sight, since $\varphi(\tilde{G}_0, \tilde{H})$ itself depends on η and it is therefore not so easy to imagine what happens if η changes slightly. Let us examine the value of $f(\eta)$.

We know that $V_0(\tilde{H})$ is the expected value of $\tilde{H}$ with Q defined in (1.4) being the probability measure. Using the linearity of the expectation operator, we write

$$V_0(\tilde{H}) = V_0(V^* + (\rho + \eta)H) = V^* + \rho V_0(H) + \eta V_0(H). \tag{2.5}$$

Moreover, it is intuitively clear that

$$\varepsilon_0^2(\tilde{H}) = \varepsilon_0^2(V^* + (\rho + \eta)H) = \varepsilon_0^2((\rho + \eta)H), \tag{2.6}$$

since the constant amount V^* can be hedged perfectly.

Substituting (2.5) and (2.6) into (2.4) we obtain

$$\begin{aligned} f(\eta) = L_0\Big(&G_0 - V^* + (\eta^+)C^b - (\eta^-)C^a - \rho V_0(H) - \eta V_0(H)\Big)^2 \\ &+ \ \varepsilon_0^2((\rho + \eta)H). \end{aligned} \tag{2.7}$$

In the next section we shall prove that the quantity $\varepsilon_0^2((\rho+\eta)H)$ can be expressed in terms of individual options' hedging errors $\varepsilon_0^2(H_i)$ and this decomposition makes the whole optimization easier.

2.1.1 Hedging error of portfolio of options

Lemma 2.1.1. *Let $\hat{H}$ be the payoff of a portfolio consisting of n derivative securities with the same expiration time and underlying asset, $\hat{H} = \alpha H$. Then the value of hedging error $\varepsilon_0^2(\hat{H})$ equals*

$$\varepsilon_0^2(\hat{H}) = \alpha \Sigma \alpha^\top,$$

where Σ is an $n \times n$ matrix defined by:

$$\begin{aligned} \Sigma_{ii} &= \varepsilon_0^2(H_i) \\ \Sigma_{ij} &= \frac{\varepsilon_0^2(H_i+H_j) - \varepsilon_0^2(H_i-H_j)}{4}. \end{aligned}$$

Proof. We have

$$\varepsilon_0^2(H) = E_0\left[\left(V_0(H) + \varphi\left(V_0(H), H\right) \bullet S_T - H\right)^2\right]$$

Since $\hat{H} = \alpha H$ and from (1.5),(1.6) and (1.19) it follows that the functions $V_0(H)$ and $\varphi(x,H)$ are linear in H, we can rephrase the previous expression as:

$$\begin{aligned} \varepsilon_0^2(\hat{H}) &= E\left[\left(\sum_{i=1}^n \alpha_i V_0(H_i) + \sum_{i=1}^n \alpha_i \varphi(V_0(H_i), H_i) \bullet S_T - \sum_{i=1}^n \alpha_i H_i\right)^2\right] \\ &= E\left[\left(\sum_{i=1}^n \alpha_i (V_0(H_i) + \varphi(V_0(H_i), H_i) \bullet S_T - H_i)\right)^2\right] \\ &= \sum_{i=1}^n \alpha_i^2 \varepsilon^2(H_i) + E\Bigg[\sum_{i\neq j} \alpha_i\alpha_j \Big(V_0(H_i) + \varphi(V_0(H_i), H_i) \bullet S_T - H_i\Big) \\ &\quad \Big(V_0(H_j) + \varphi(V_0(H_j), H_j) \bullet S_T - H_j\Big)\Bigg] \end{aligned}$$

An easy computation shows that

$$\begin{aligned}
\varepsilon_0^2(H_i+H_j) &= \varepsilon^2(H_i) + 2E\Big((V_0(H_i)+\varphi(V_0(H_i),H_i)\bullet S_T - H_i) \\
&\quad (V_0(H_j)+\varphi(V_0(H_j),H_j)\bullet S_T - H_j)\Big) + \varepsilon_0^2(H_j), \\
\varepsilon_0^2(H_i-H_j) &= \varepsilon_0^2(H_i) - 2E\Big((V_0(H_i)+\varphi(V_0(H_i),H_i)\bullet S_T - H_i) \\
&\quad (V_0(H_j)+\varphi(V_0(H_j),H_j)\bullet S_T - H_j)\Big) + \varepsilon_0^2(H_j).
\end{aligned}$$

Hence,

$$\begin{aligned}
\varepsilon_0^2(H_i+H_j) - \varepsilon_0^2(H_i-H_j) &= 4E\Big((V_0(H_i)+\varphi(V_0(H_i),H_i)\bullet S_T - H_i) \\
&\quad (V_0(H_j)+\varphi(V_0(H_j),H_j)\bullet S_T - H_j)\Big).
\end{aligned}$$

It follows that

$$\begin{aligned}
\varepsilon_0^2(\hat{H}) &= \sum_{i=1}^{n} \alpha_i^2 \varepsilon_0^2(H_i) + \sum_{i\neq j} \alpha_i\alpha_j \frac{\varepsilon_0^2(H_i+H_j) - \varepsilon_0^2(H_i-H_j)}{2} \\
&= \alpha\Sigma\alpha^\top
\end{aligned}$$

□

Using Lemma 2.1.1 we can rewrite (2.7) as follows:

$$\begin{aligned}
f(\eta) = L_0\Big(G_0 - V^* + (\eta^+)C^b - (\eta^-)C^a - \rho V_0(H) - \eta V_0(H)\Big)^2 \\
+ (\rho+\eta)\Sigma(\rho+\eta)^\top.
\end{aligned} \tag{2.8}$$

In order to bring ellegant and transparent solution, we can normalize the problem to the case of unit initial wealth and unit local relative risk aversion[1] by introducing $\alpha = \rho(\gamma_U(G_0)/G_0)$ and $\zeta = \eta(\gamma_U(G_0)/G_0)$ as the amounts of options per unit of initial wealth and risk tolerance. Thus we obtain a "unit" optimal strategy, allowing for an easy adjustment to the real values of actual initial wealth and local risk aversion afterwards.

On dividing (2.8) by $(G_0/\gamma_U(G_0))^2$ we obtain

$$\begin{aligned}
\frac{f(\eta)}{(G_0/\gamma_U(G_0))^2} = L_0\Big(-1 + (\zeta^+)C^b - (\zeta^-)C^a - \alpha V_0(H) - \zeta V_0(H)\Big)^2 \\
+ (\alpha+\zeta)\Sigma(\alpha+\zeta)^\top.
\end{aligned} \tag{2.9}$$

[1]relative risk aversion measured by the Arrow-Pratt coefficient $\gamma_U(x) = -\frac{xU''(x)}{U'(x)}$

2.2 Transformation of the problem

The maximization of the objective function $\hat{U}$ is equivalent to minimization of the function $f(\eta)$. This task very much resembles a quadratic programming problem. The bar against a quick and efficient solution lies in the presence of vectors ζ^+ and ζ^-. To tackle this, we shall perform a transformation of the primary problem. We adopt the view of ζ^+ and ζ^- being two separate variables. The original variable ζ can then be expressed as their difference: $\zeta = \zeta^+ - \zeta^-$. Rewriting the whole minimization problem in terms of these two new variables gives us the following:

$$\begin{aligned}\min_{\zeta^+,\zeta^-} \frac{f(\eta)}{(G_0/\gamma_U(G_0))^2} &= L_0\Big(\zeta^+ C^b - \zeta^- C^a - (\zeta^+ - \zeta^-)V_0(H) - \alpha V_0(H) - 1\Big)^2 \\ &+ \ (\alpha + \zeta^+ - \zeta^-)\Sigma(\alpha + \zeta^+ - \zeta^-)^\top . \end{aligned} \tag{2.10}$$

We can easily transform the original problem to a standard quadratic programming problem with constraints by constructing a new variable

$$\bar{\zeta} = \begin{pmatrix} \zeta^+ & \zeta^- \end{pmatrix}.$$

The price we have to pay is the expansion of the problem dimension from n to $2n$, but as a recompense we get the comfort of quadratic programming. In order to rephrase the objective function, we introduce this notation:

$$\begin{aligned} \bar{\zeta} &= \begin{pmatrix} \zeta^+ & \zeta^- \end{pmatrix} \\ \bar{\alpha} &= \begin{pmatrix} \alpha & \alpha \end{pmatrix} \\ \bar{\Sigma} &= \begin{pmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{pmatrix} \quad \tilde{\Sigma} = \begin{pmatrix} \Sigma & -\Sigma \\ -\Sigma & \Sigma \end{pmatrix} \\ A &= \begin{pmatrix} C^b - V_0(H) \\ -C^a - V_0(H) \end{pmatrix} \end{aligned}$$

We get the following:

$$\min_{\bar{\zeta}} L_0(\bar{\zeta}A - \alpha V_0(H) - 1)^2 + \alpha\Sigma\alpha^\top + 2\bar{\alpha}\bar{\Sigma}\bar{\zeta}^\top + \bar{\zeta}\tilde{\Sigma}\bar{\zeta}^\top$$

$$\min_{\bar{\zeta}} L_0\Big(\bar{\zeta}(AA^\top)\bar{\zeta}^\top - 2(\alpha V_0(H)+1)\cdot\bar{\zeta}A + (\alpha V_0(H)+1)^2\Big) + \alpha\Sigma\alpha^\top + 2\bar{\alpha}\bar{\Sigma}\bar{\zeta}^\top + \bar{\zeta}\tilde{\Sigma}\bar{\zeta}^\top$$

Adding a constant has no impact on where the function takes the minimum value. Hence, this problem is equivalent to the following one:

$$\min_{\bar{\zeta}} \bar{\zeta}(L_0 \cdot AA^\top + \tilde{\Sigma})\bar{\zeta}^\top + 2(\bar{\alpha}\bar{\Sigma} - L_0(\alpha V_0(H)+1)A^\top)\bar{\zeta}^\top. \tag{2.11}$$

Hence we shall solve the quadratic programming problem

$$\min_{\bar{\zeta}} \frac{1}{2}\,\bar{\zeta}\mathbb{Q}\bar{\zeta}^\top + c\bar{\zeta}^\top, \qquad \bar{\zeta} \geq 0$$

with

$$\mathbb{Q} = L_0 \cdot AA^\top + \tilde{\Sigma}, \qquad c = \bar{\alpha}\bar{\Sigma} - L_0(\alpha V_0(H)+1)A^\top. \tag{2.12}$$

Chapter 3

Modeling the Stock Price Process

It is essential for the traders to approximate the market behavior as good as possible. Examining the stock price processes, we often observe that they have spikes or jumps, which mean large sudden price changes. Models based on Brownian motion do not capture this feature of the real market. Moreover, the distributions of log returns resemble normal distribution on the first sight, but they tend to have heavier tails and exhibit skewness. It is therefore tangible to look for other distributions and corresponding processes fitting the reality in a more accurate way. As an effective tool in this endeavor we introduce a more general class of stochastic processes, the Lévy processes. We focus specifically on normal-inverse gaussian process and present some of its features.

3.1 Lévy processes

Definition 3.1.1. *A càdlàg, adapted, real valued stochastic process* $L = \{L_t\}_{0 \le t \le T}$ *with* $L_0 = 0$ *a.s. is called a Lévy process, if the following conditions are satisfied:*

- *L has independent increments, i.e. $L_t - L_s$ is independent of $\mathcal{F}_s$ for any $0 \le s < t \le T$.*

- *L has stationary increments, i.e. for any $0 \le s, t \le T$ the distribution of $L_{t+s} - L_t$ does not depend on t.*

- *L is stochastically continuous, i.e. for every $0 \leq t \leq T$ and $\epsilon > 0$: $\lim_{s \to t} P(|L_t - L_s| > \epsilon) = 0$.*

Lévy processes are very closely related to the so called *infinitely divisible* probability distributions.

Definition 3.1.2. *We say that the law of a random variable X is infinitely divisible, if for all $n \in \mathbb{N}$ there exist n i.i.d. random variables $\{X_i^{(1/n)}\}_{i=1,2,\ldots,n}$ such that*

$$X = \sum_{i=1}^{n} X_i^{(1/n)}. \tag{3.1}$$

Equivalently, the law of a random variable X is infinitely divisible, if for all $n \in \mathbb{N}$ there exists a random variable $X^{(1/n)}$ such that

$$\varphi_X(z) = \left(\varphi_{X^{(1/n)}}(z)\right)^n, \tag{3.2}$$

where $\varphi_X(z)$ denotes the characteristic function of X.

The relationship between Lévy processes and infinitely divisible distributions is realized in two important mathematical results, The Lévy-Khintchine Formula and The Lévy-Itó Decomposition. The former states that for each Lévy process an associated infinitely divisible distribution can be found. The latter explains that, given a random variable X with an infinitely divisible distribution, we can construct a Lévy process $\{L_t\}_{t=1,2,\ldots,n}$ such that L_1 has the same distribution as X. For a deeper insight into this topic see Papapantoleon [7].

3.2 Normal Inverse Gaussian Distribution

The normal inverse Gaussian distribution has become one of great interest since its introduction by Barndorff-Nielsen in 1995. It was originally presented as a versatile model for modeling log returns of stock prices. We begin with its definition and then present some of its favourable features, which make it so attractive.

Definition 3.2.1. *Let $\alpha, \beta \in \mathbb{R}, \sigma > 0$. Let Y and V be independent random variables, $Y \sim \mathcal{N}(0,1)$ and $V \geq 0$ is a continuous probability distribution*

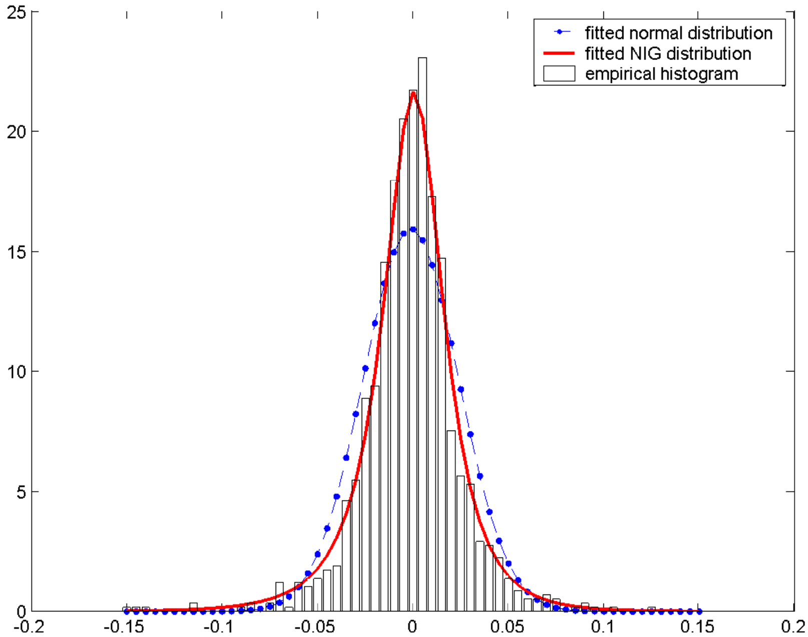

Figure 3.1: Histogram of daily Google returns from Aug 19, 2004 to Apr 15, 2009 compared to fitted normal and NIG distribution

with probability density function g. We call the probability distribution of a random variable X a normal mean-variance mixture*, if X has the form*

$$X = \alpha + \beta V + \sigma \sqrt{V} Y. \tag{3.3}$$

We call V a mixing probability distribution and g a mixing probability density.

A normal mean-variance mixture, where the mixing distribution is inverse gaussian, is called normal inverse Gaussian distribution. Equivalently, we can define it by the following definition:

Definition 3.2.2. *A random variable X is said to be normal inverse Gaussian, if it has a probability density function of the form*

$$f(x; \alpha, \beta, \delta, \mu) = \frac{\alpha \delta K_1 \left(\alpha \sqrt{\delta^2 + (x-\mu)^2} \right)}{\pi \sqrt{\delta^2 + (x-\mu)^2}} e^{\delta \gamma + \beta (x-\mu)}, \tag{3.4}$$

where $\gamma = \sqrt{\alpha^2 - \beta^2}$ *and* K_1 *is the modified Bessel function of second kind with index* 1. *Furthermore,* $0 \leq |\beta| < \alpha, \delta > 0$ *and* $|\mu| < \infty$.

The NIG distribution is specified by four parameters α, β, δ and μ, whose interpretations are following:

- α - describes the kurtosis of the density function - the higher the value of α, the steeper the density function. This also influences the tails of the distribution, as small values of α imply heavy tails and large values of α imply light tails.
- β - skewness parameter; it indicates the symmetry of the density function. Positive values imply a density skew to the right, negative values imply a density skew to the left. Distributions with $\beta = 0$ are symmetric around the mean.
- δ - scale parameter
- μ - location parameter.

3.2.1 Properties of the NIG distribution

Barndorff-Nielsen derived the moment-generating function of NIG distribution in 1997. It has the form

$$M_X(z) = E(e^{zX}) = e^{\mu z + \delta(\gamma - \sqrt{\alpha^2 - (\beta+z)^2})}. \tag{3.5}$$

By differentiating it we obtain the central moments of NIG distribution,

$$E(X^n) = \frac{\mathrm{d}^{(n)} M_X(z)}{\mathrm{d}z^n} \Bigg|_{z=0}$$

Then, using the known identity

$$E((X - E(X))^n) = \sum_{j=1}^{n} \binom{n}{j} E(X^j)(E(X))^{n-j}$$

we write the following expressions for the first four normalized moments:

$$E(X) = \mu + \frac{\beta\delta}{\gamma}, \qquad \sigma^2 = \frac{\alpha^2\delta}{\gamma^3},$$

$$\gamma_1 = \frac{3\beta}{\alpha\sqrt{\delta\gamma}}, \qquad \gamma_2 = \frac{3\alpha^2(\alpha^2 + 4\beta^2)}{\alpha^2\delta\gamma}.$$

We see that the expressions for the first four moments are quite simple and elegant.

Probably the most appealing property of the NIG distribution is that it is closed under convolution. It is the only distribution in the class of generalized hyperbolic distributions to have this property. If $X \sim \mathrm{NIG}(\alpha, \beta, \delta_1, \mu_1)$ and $Y \sim \mathrm{NIG}(\alpha, \beta, \delta_2, \mu_2)$ are two independent random variables, then

$$X + Y \sim \mathrm{NIG}(\alpha, \beta, \delta_1 + \delta_2, \mu_1 + \mu_2). \tag{3.6}$$

This is a powerful tool for transforming the distributions of observed data. Assuming that the stock price is driven by an exponential Lévy NIG process (the increments of log returns are NIG-distributed), we can very easily change the time scaling and adjust the probability density function. The next figure illustrates the histogram of weekly log returns of Google stocks together with an NIG probability density function produced by estimating the NIG distribution of the daily returns and then transforming it to the distribution of weekly returns.

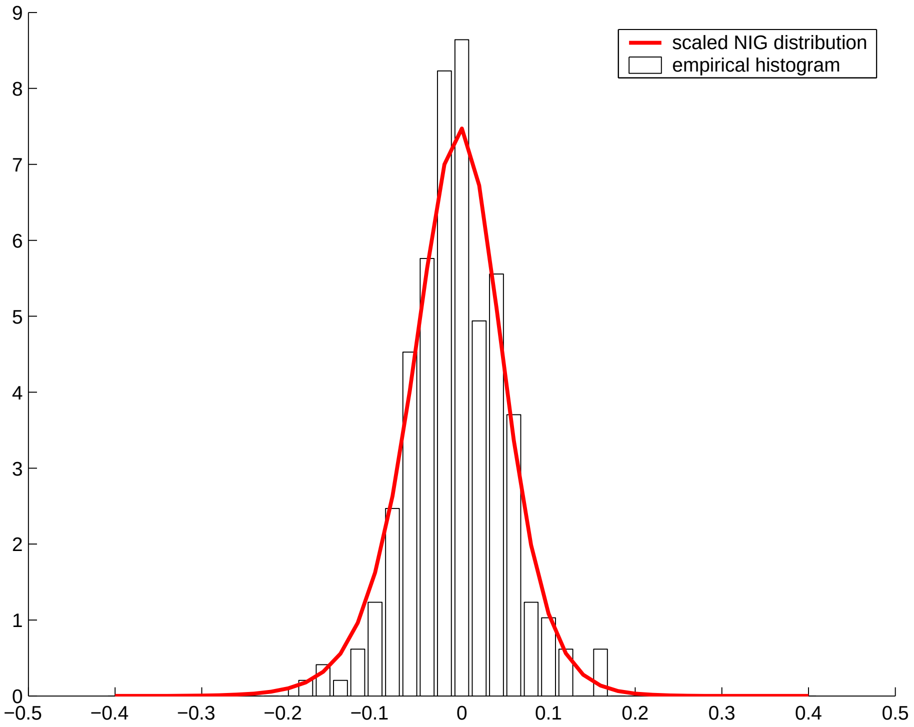

Figure 3.2: Histogram of weekly GOOG returns from Aug 19, 2004 to Apr 15, 2009 compared to NIG($\alpha, \beta, 5\delta, 5\mu$)

Chapter 4

Numerical implementation

In this chapter we describe a numerical implementation of the solution to the problem introduced in Chapter 2. We consider an investor who has a portfolio consisting of 2 different types of options with the same underlying stock and time to maturity, but with different strike prices (\$390 and \$420). The underlying stock is GOOGLE, expiration date of the options is Dec 16, 2006. We start the trading on July 14, 2006, i.e. 22 weeks before maturity.[1] The initial stock value is \$403.5, i.e. one of the options is in the money and the other is out the money. We change the portfolio structure every 4 weeks, rehedging once a week in between.

4.1 Stock price

We shall model the stock price process by exponential normal inverse Gaussian process. This means that the increments of log returns are NIG distributed (cf. Chapter 3). For the estimation of NIG parameters we use daily historical prices of Google stock from the time period Aug 19, 2004 - July 13, 2006 (source: *http://finance.yahoo.com*). We divide the log returns into 667 bins (covering the interval $(-1, 1)$), construct the corresponding histogram of log returns and calculate the empirical probabilities of individual bins. Based on these data, we calculate the first four moments of log returns (each

[1] We have chosen older data in order to avoid possible abnormalities caused by recently increased volatility of stock prices.

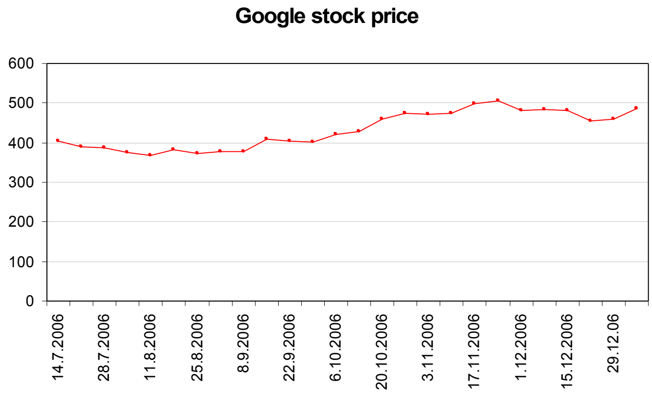

Figure 4.1: GOOGLE stock price in the period July 14, 2006 - Dec 29, 2006

bin being represented by its middle value) and solve the following system of equations:

$$
\begin{aligned}
\mu + \frac{\beta\delta}{\gamma} &= E(X), \\
\frac{\alpha^2\delta}{\gamma^3} &= E(X - E(X))^2, \\
\frac{3\beta}{\alpha\sqrt{\delta\gamma}} &= \frac{E(X - E(X))^3}{(E(X - E(X))^2)^{3/2}}, \\
\frac{3\alpha^2(\alpha^2 + 4\beta^2)}{\alpha^2\delta\gamma} &= \frac{E(X - E(X))^4}{(E(X - E(X))^2)^2} - 3, \\
\gamma &= \sqrt{(\alpha^2 - \beta^2)}.
\end{aligned}
$$

Thus we obtain the estimates of NIG parameters and the corresponding density function. Since we have chosen to rehedge our portfolio every week, we must re-scale the distribution to weekly log returns distribution. We do so by fixing new parameters $(\alpha, \beta, \tilde{\delta}, \tilde{\mu}) = (\alpha, \beta, 5\delta, 5\mu)$.

In our model we allow only discrete values of log returns, aggregating the real log returns into several bins (25) of the same range. We wish to choose the bins in such a way that we do not loose too many data. It is therefore important that we determine the lower and upper boundary of the interval containing the most of the log returns. We set the cut-offs to 0.5% and 95.5% quantiles, leaving 99% of the data in hand. To find the quantiles, we may use numerical integration of the density function. First, we find the roots of the equation $f(x) = 10^{-9}$, neglecting the points with smaller values of density function. Denote the roots by x_{low}, x_{high}. Then we numerically integrate the density function and then we approximate the integral of the NIG density function using the trapezoidal rule. This employs the approximation

$$\int_a^b f(x)dx \approx (b-a)\frac{f(a)+f(b)}{2}.$$

The algorithm for finding the proper boundaries is as follows:

```
int = 0;
q = 0.005;                                                        % quantile
while int < q
    int = int + (nigpdf(low+dx,est)+nigpdf(low,est))*dx/2;
    low = low + dx;
end
int = 0;
while int < q
    int = int + (nigpdf(high,est)+nigpdf(high-dx,est))*dx/2;
    high = high - dx;
end
```

Once we have determined the boundaries of the interval we shall work with, we are ready to fix the bins. It is essential that we set their width uniformly so that the decision tree recombines and the number of possible states at time t does not grow exponentially (cf. [1] for a better image). Thus we obtain the following probability distribution of log returns:

4.2 Globally Optimal Trading Strategy

The program then runs in two embedded loops. The main loop includes the decision-making concerning the structure of portfolio. Every 4 weeks we check the actual portfolio structure (adjusted to unit initial wealth and

r	-20.99%	-19.47%	-17.95%	-16.43%	-14.91%	-13.38%	-11.86%
P	0.0027	0.004	0.0058	0.0085	0.0122	0.0173	0.0241
r	-10.34%	-8.82%	-7.30%	-5.77%	-4.25%	-2.73%	-1.21%
P	0.0329	0.0438	0.0563	0.0696	0.0821	0.0917	0.0963
r	0.31%	1.84%	3.36%	4.88%	6.40%	7.92%	9.45%
P	0.0948	0.0872	0.0749	0.0603	0.0457	0.0329	0.0227
r	10.97%	12.49%	14.01%	15.53%			
P	0.015	0.0097	0.006	0.0037			

Table 4.1: Approximation of the probability distribution of log returns

risk tolerance) and maximize the utility function described in Chapter 2 $\left(\text{i.e. we minimize } \frac{f(\eta)}{(G_0/\gamma_U(G_0))^2}\right)$. To do so, we need to find the risk-neutral measure Q and calculate the Σ matrix consisting of individual hedging errors of $\frac{H_i + H_j}{4}$, $\frac{H_i - H_j}{4}$. The errors are computed recursively in a tree using (1.6), (1.8) and (1.10). The ask and bid prices of the options are taken from the website *http://www.poweropt.com*. Then we calculate the optimal values of η adjusted to the real value of our portfolio and our risk tolerance. We take the trading costs into consideration by setting $\tilde{G}_0 = G_0 + \eta^+ C_{bid} - \eta^- C_{ask}$. With the new structure of portfolio and new initial wealth $\tilde{G}_0$, we perform dynamically optimal hedging - this is the second, embedded loop. We calculate the locally optimal hedging strategy (1.6) recursively in a tree again and then, depending on the actual value of portfolio, calculate the corresponding dynamically optimal hedge (1.19) and adjust the portfolio value. The final portfolio value after 4 weeks of hedging is then set to be the new initial value of portfolio in the next portfolio rebalancing.

The initial amounts of options at the very beginning of trading are set to 0 and we assume $G_0 = 100, V^* = 200$, i.e. the investor started with initial wealth 100, desired to reach final wealth 200 and had a zero option portfolio in the beginning.

4.3 Results

For a detailed investigation of the results of our historical simulation, see Appendix A.1.

The historical simulation shows that rebalancing the portfolio throughout hedging is not useless. It provides a tool for amending the errors which may occur from pure hedging. Even if we follow the dynamically optimal strategy derived in [2], unexpected market behavior might bring considerable losses. This is a good opportunity to emphasise the importance of good estimation of data distribution. It is necessary that we have a good image of common behavior of the stock. In the case of unexpected long–term changes in stock price movement the hedging tools might prove helpless. Consider a stock with $E(X) < 0$. The dynamically optimal strategy (1.19) "assumes" that the stock price is likely to decrease. Hence, it performs the corresponding adjustment of locally optimal strategy ξ_t. However, if the stock price increases significantly, this adjustment brings more damage than use.

While testing different initial set-ups of our strategy, we noticed that the problem is quite sensitive to the value of $\tilde{\lambda}$ parameter. Setting $\tilde{\lambda} = 0$ instead of the original value $\tilde{\lambda} = -3.106$ brought much better results (the largest deviation $G_t - V_t$ throughout the whole hedging time was -8.84, the final realized value $G_T - V_T = -1.69$ and the expected squared hedging error in the last rebalancing period was $E(G_T - V_T)^2 = 1.48$). Very similar results were obtained by setting $\tilde{\lambda} = 1.1$. This is an interesting observation. However, one cannot draw conclusions from a single simulation. For a more robust result, we would have to perform much more simulations and evaluate their results.

We simultaneously performed a simulation of globally optimal hedging without rebalancing, using the same data. The results are charted in Appendix A.2. During the first four periods the two strategies showed very similar results, but the last 6 weeks of trading brought significant turbulence in the hedging errors. This is a wonderful opportunity to observe that rebalancing the portfolio might bring similar situations "back to normal", whereas pure hedging strategy may experience problems in handling large misalingments $V_t - G_t$. The following chart depicts the development of the value $G_t - V_t$ for both strategies.

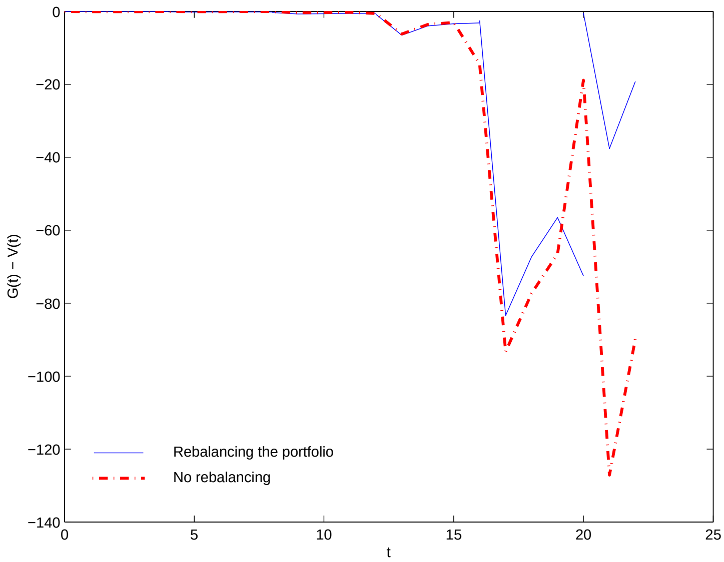

Figure 4.2: Comparison of the performance of option rebalancing strategy and globally optimal hedging strategy with no rebalancing

Chapter 5

Conclusion

We have presented a solution to the problem of maximizing the utility function by means of rebalancing the option portfolio and optimally hedging the portfolio at the same time. We have shown that for a portfolio of several options, the hedging error $\varepsilon_0^2(H\alpha)$ can be easily rewritten as a quadratic form expression, thus allowing us to maximize the utility function with greater ease. We have derived the formulation of the problem in the form of a quadratic programming problem. In the end we have performed a single historical simulation demonstrating our expectations. The results show that following the strategy derived in this thesis may be very useful indeed, especially in the cases when sudden or unexpected stock price movements bring large deviations from the desired portfolio value.

Appendix A

Charts

A.1 Globally Optimal Hedging Strategy With Intermediate Portfolio Rebalancing

Time to maturity (in weeks):	22
Initial portfolio value (before trading):	100
Relative local risk aversion:	1
Stock price:	403.5

strike price	ρ**(initial)**	ζ	ρ**(optimal)**
390	0	0.013832	1.3832
420	0	0.011186	1.1186

Portfolio Value after trading options:	199.9392
$E(G_T - V_T)^2$ **before trading options:**	10000
$E(G_T - V_T)^2$ **after trading options:**	2.5008

Hedging

t	G_t	ξ_t	V_t	φ_t	$G(t) - V(t)$
0	199.9392	-0.00010938	200.0023	-0.00064379	-0.063105
1	199.9472	-8.1626e-005	200.0299	-0.00078136	-0.082627
2	199.9487	-9.3698e-005	200.0219	-0.00071332	-0.073167
3	199.9586	-8.4645e-005	200.0251	-0.00064745	-0.066458
4	199.9621	0	200.0225	-0.00051173	-0.060427

Time to maturity (in weeks):	18
Initial portfolio value (before trading):	199.9621
Relative local risk aversion:	5270.9897
Stock price:	368.5

strike price	ρ**(initial)**	ζ	ρ**(optimal)**
390	1.3832	12.4702	1.8562
420	1.1186	-22.6154	0.2607

Portfolio Value after trading options:	199.8883
$E(G_T - V_T)^2$ **before trading options:**	2.1971
$E(G_T - V_T)^2$ **after trading options:**	2.1055

Hedging

t	G_t	ξ_t	V_t	φ_t	$G(t) - V(t)$
4	199.8883	-0.00017249	200.003	-0.0011312	-0.11464
5	199.8711	-0.00017448	200.0457	-0.0016344	-0.17458
6	199.8874	-9.2821e-005	200.0462	-0.0014203	-0.15875
7	199.8797	-6.8034e-005	200.0245	-0.0012787	-0.14478
8	199.8807	0	200.0181	-0.0011491	-0.13742

Time to maturity (in weeks):	14
Initial portfolio value (before trading):	199.8807
Relative local risk aversion:	1675.5466
Stock price:	377.85

strike price	ρ**(initial)**	ζ	ρ**(optimal)**
390	1.8562	2.1752	2.1157
420	0.2607	-4.6011	-0.28818

Portfolio Value after trading options:	199.7205
$E(G_T - V_T)^2$ **before trading options:**	6.4463
$E(G_T - V_T)^2$ **after trading options:**	6.3126

Hedging

t	G_t	ξ_t	V_t	φ_t	$G(t) - V(t)$
8	199.7205	-0.0012189	200.0213	-0.0035883	-0.30078
9	199.6007	-0.0011054	200.3255	-0.0068146	-0.72475
10	199.6416	-0.00075335	200.2972	-0.0059178	-0.65559
11	199.6527	-0.00023218	200.2061	-0.004592	-0.55345
12	199.5673	0	200.0664	-0.0039315	-0.49908

Time to maturity (in weeks):	10
Initial portfolio value (before trading):	199.5673
Relative local risk aversion:	461.1953
Stock price:	420.5

strike price	ρ**(initial)**	ζ	ρ**(optimal)**
390	2.1157	-0.69901	1.8133
420	-0.28818	1.1999	0.23105

Portfolio Value after trading options:	199.7158
$E(G_T - V_T)^2$ **before trading options:**	46.1346
$E(G_T - V_T)^2$ **after trading options:**	45.0361

Hedging

t	G_t	ξ_t	V_t	φ_t	$G(t) - V(t)$
12	199.7158	-0.023107	200.5004	-0.028334	-0.78453
13	199.5185	-0.0063053	206.063	-0.049907	-6.5444
14	197.7448	-0.0023399	201.7594	-0.029087	-4.0146
15	197.2791	-0.0017482	200.7071	-0.024587	-3.428
16	197.3627	0	200.5683	-0.021357	-3.2056

Time to maturity (in weeks):	6
Initial portfolio value (before trading):	197.3627
Relative local risk aversion:	74.8348
Stock price:	471.8

strike price	ρ**(initial)**	ζ	ρ**(optimal)**
390	1.8133	0.11331	2.1121
420	0.23105	-0.074262	0.035193

Portfolio Value after trading options:	211.5106
$E(G_T - V_T)^2$ **before trading options:**	502.5987
$E(G_T - V_T)^2$ **after trading options:**	321.8503

Hedging

t	G_t	ξ_t	V_t	φ_t	$G(t) - V(t)$
16	211.5106	-0.42705	214.0041	-0.44268	-2.4935
17	210.734	-0.24936	294.1367	-0.77223	-83.4027
18	190.3664	-0.16972	258.3426	-0.59588	-67.9762
19	186.633	-0.30084	244.0421	-0.66075	-57.4091
20	202.6233	0	274.3847	-0.44989	-71.7614

Time to maturity (in weeks):	2
Initial portfolio value (before trading):	202.6233
Relative local risk aversion:	-77.2406
Stock price:	403.78

strike price	ρ**(initial)**	ζ	ρ**(optimal)**
390	2.1121	0.69772	0.28178
420	0.035193	-0.27469	0.75577

Portfolio Value after trading options:	201.3271
$E(G_T - V_T)^2$ **before trading options:**	310.2366
$E(G_T - V_T)^2$ **after trading options:**	30.5198

Hedging

t	G_t	ξ_t	V_t	φ_t	$G(t) - V(t)$
20	201.3271	-0.14224	201.7938	-0.14529	-0.4667
21	201.6014	-0.020768	239.9159	-0.27134	-38.3145
22	195.6012	0	215.4541	-0.12983	-19.8529

A.2 Globally Optimal Hedging Strategy Without Rebalancing

ρ
1.3832
1.1186

Hedging

t	G_t	ξ_t	V_t	φ_t	$G(t) - V(t)$
0	199.9392	-0.00010938	200.0023	-0.00064379	-0.063105
1	199.9472	-8.1626e-005	200.0299	-0.00078136	-0.082627
2	199.9487	-9.3698e-005	200.0219	-0.00071332	-0.073167
3	199.9586	-8.4645e-005	200.0251	-0.00064745	-0.066458
4	199.9621	-0.00015735	200.0027	-0.00049684	-0.040599
5	199.9545	-0.00015826	200.0421	-0.00089066	-0.087583
6	199.9634	-8.3092e-005	200.0423	-0.00074291	-0.078904
7	199.9594	-6.0306e-005	200.0222	-0.0005852	-0.062769
8	199.9599	-0.0010737	200.0183	-0.001534	-0.058434
9	199.9098	-0.0009601	200.2928	-0.0039776	-0.38305
10	199.9337	-0.00064069	200.2639	-0.0032417	-0.33018
11	199.9398	-0.00019041	200.1793	-0.002077	-0.23949
12	199.9011	-0.022119	200.4678	-0.025895	-0.56665
13	199.7217	-0.005801	205.9254	-0.047133	-6.2037
14	198.0461	-0.0020891	201.652	-0.026113	-3.6059
15	197.6279	-0.001532	200.6438	-0.021625	-3.0159
16	197.7014	-0.39254	212.1614	-0.48319	-14.4599
17	196.8447	-0.21529	290.1451	-0.80021	-93.3004
18	175.6668	-0.13911	253.0072	-0.62398	-77.3404
19	171.7545	-0.24547	238.2475	-0.66233	-66.493
20	187.7828	-0.47655	206.6326	-0.59982	-18.8498
21	188.9566	-0.097178	316.0836	-0.92856	-127.127
22	168.5223	0	258.3758	-0.58762	-89.8536

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