Comenius University, Bratislava
Faculty of Mathematics, Physics and Informatics

Master thesis

Katarína Kvašňáková

Bratislava, 2009

Comenius University, Bratislava
Faculty of Mathematics, Physics and Informatics

Mathematics of Economics end Finance

Modeling dependence structure of the stock and bond market

(Master thesis)

Katarína Kvašňáková

Supervisor:
Mgr. Pavol Jurča

Bratislava, 2009

Univerzita Komenského v Bratislave
Fakulta Matematiky, Fyziky a Informatiky

Katedra aplikovanej matematiky a štatistiky

Modelovanie závislosti medzi trhom akcií a trhom dlhopisov

(Diplomová práca)

Katarína Kvašňáková

Vedúci diplomovej práce:
Mgr. Pavol Jurča

Bratislava, 2009

I declare this thesis was written on my own, with the only help provided by my supervisor and the referred-to literature.

. . . .
Katarína Kvašňáková

Acknowledgment

I would like to express special thanks to my supervisor Mgr. Pavol Jurča for all of the support and guidance he offered throughout the elaboration of this thesis. I would also like to thank to my family for their support during my study.

Abstract

KVAŠŇÁKOVÁ, Katarína: Modeling dependence structure of the stock and bond market [Master thesis], Comenius University, Bratislava, Faculty of Mathematics, Physics and Informatics, Department of Applied Mathematics and Statistics; Supervisor: Mgr. Pavol Jurča, Bratislava, 2009, 55 pages

We investigate the dependence between stock and bond market. These two variables are expected to be strongly interdependent, but not necessarily in linear fashion. Therefore, correlation cannot capture this dependence properly. We use two other approaches, copula and multivariate GARCH models. We apply them to modeling the returns of the growth pension funds. From stock indices and discount rates we find the ones that have significant influence on returns of these funds and cluster them into one variable for the stock market and one variable for the bond market. The core of the thesis lies in investigating the dependence between these two variables by copula and multivariate GARCH models. Furthermore, we calculate Value at Risk implied by both models and compare them. Copula model produces better estimation of VaR.

Keywords: dependence, copula, MGARCH, VaR, pension funds

Abstrakt

KVAŠŇÁKOVÁ, Katarína: Kopuly [Diplomová práca], Univerzita Komenského v Bratislave, Fakulta matematiky, fyziky a informatiky, Katedra aplikovanej matematiky a štatistiky; Školiteľ: Mgr. Pavol Jurča, Bratislava, 2009, 55 strán

V diplomovej práci skúmame závislosť medzi trhom akcií a trhom dlhopisov. Je očakávateľné, že medzi trhmi existuje určitá súvislosť, no nie nutne v lineárnej forme. Preto korelácia nemusí byť schopná zachytiť tento vzťah správnym spôsobom. My použijeme dva iné prístupy, kopula modely a mnohorozmerné GARCH modely. Modely aplikujeme na modelovanie výnosov rastových fondov v penzijných spoločnostiach. Z vybraných diskontných sadzieb, akciových a swapových indexov vyberieme tie, ktoré majú signifikantný vplyv na výnosy rastových fondov a tie zlúčime do jednej premennej pre trh akcií a jednej premennej pre trh dlhopisov. Jadro práce tvorí skúmanie štruktúry závislosti týchto dvoch premenných pomocou kopula modelov a mnohorozmerných GARCH modelov. Okrem toho vypočítame VaR pre dané modely. Testy ukazujú, že kopula modely dávajú lepšie odhady VaR.

Kľúčové slová: závislosť, kopula, mnohorozmerný GARCH, VaR, penzijné fondy

Contents

List of Tables

List of Figures

1 Introduction

Correlation is typically used as a natural and easily understandable measure of dependence. It is one of the critical input of financial management. We desire assets not to move with each other in order to protect our investment's portfolio. Stronger dependence increases our loss, therefore we do not like to put our eggs in one basket. A limitation of correlation is that it assumes the elliptical family of distributions that Normal or Student distribution belong to. Unfortunately, many recent papers show that the financial returns are not normally distributed. Furthermore, they tend to exhibit asymmetric dependence.[1] There are two approaches how to handle these problems.

The first approach uses copula theory to model the joint distribution. A copula links together marginal distributions to form a multivariate distribution. All the univariate information is contained in the marginal distribution functions, while the dependence is fully captured by the copula. Copula modeling allows the variables to have different dependence in different quantiles. Loosely speaking, dependence during the crisis can be stronger or weaker comparing to the boom. In contrast, the usual correlation coefficient is not sufficient to describe the dependence structure of copula unless the joint distribution is elliptical.

Another approach to handle changing covariance is called multivariate GARCH models. They are natural extension of the univariate GARCH models pioneered by Engle. The models are based on the modeling conditional covariance matrix by using the past information. Their drawback is that they assume a joint normal distributions for the innovations.

We apply the models in portfolio management and risk evaluation. Our focus is on the pension area. Since 2005 a new law has enabled to save money for pension in pension funds. We look at the pension management companies in Slovakia and compute the risk they face investing in the stock and bond markets. We investigate the dependency between the stock and bond markets and compare the copula and multivariate GARCH approach. Furthermore, we calculate Value at Risk, which both a pension company and a pension saver are interested in. The pension company cannot go to the high risk because it has to guarantee some level of return. If such level of return is not reached, the company will be sanctioned by the National Bank of Slovakia. From the view of the pension saver, he is interested in the risk that his money, his future pension, faces.

[1] cf. Longin and Solnik (1995) and Patton (2003).

The remainder of the thesis is organized as follows. Chapter 2 introduces the reader to the theoretical background for bivariate copula case. It presents mathematical and probabilistic principles on which the empirical part is built and explains the process of copula estimation. It discusses different approaches for different parts of the estimation. Chapter 3 gives a brief overview of univariate and multivariate GARCH models. Chapter 4 introduces the concept of Value of Risk. Chapter 5 presents applications of copula models comparing to multivariate GARCH models. It focuses on investments in pension funds and investigate their risk by modeling dependence between the stock and bond markets. Chapter 6 concludes.

2 Copulas

A copula is a function that links together marginal distribution functions[1] to form a joint distribution. The study of copulas originates with Sklar (1959). After a long time of silence, last ten years were very rich in using copulas in economics and finance. For instance, see the work of Patton (2003), Nelsen (1998) and Cherubini et al. (2004).
The focus of this section is to give the theoretical background for their modeling. First we define copulas and mention the most important theorem in copula theory. Then we discuss different copula families. Finally, we explain different methods of copula estimation.

2.1 Basic definitions

We define copulas as a certain class of functions with specific properties. We restrict the attention to the bivariate case because that is the object of our empirical work. However, the generalization to the multivariate case is straightforward.

Definition 2.1 *A two-dimensional copula is a function* $C : [0,1] \times [0,1] \to [0,1]$ *with the following properties:*

1. *For every* u, v *in* $[0,1]$,

$$C(u,0) = C(0,v) = 0$$

$$C(u,1) = u \quad \text{and} \quad C(1,v) = v.$$

2. *For every* u_1, u_2, v_1, v_2 *in* $[0,1]$ *such that* $u_1 \leq u_2$ *and* $v_1 \leq v_2$,

$$C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0.$$

The definition presents simple and expected conditions. We can look at copula as a joint distribution function with margins U, V that are uniform on $[0,1]$. The value $C(u,v)$ is the probability that values of random variables U and V fall below their $(100u)^{th}$ quantile and $(100v)^{th}$ quantile. The first property says that any of the random variables U, V is less than zero quantile with a zero probability and the probability that the random variable U

[1] In the following text we shall also used the expression margins or marginals.

is less than $(100u)^{th}$ quantile, while V can gain any value, is u and vice versa. The second property says that the probability that U is between $(100u_1)^{th}$ quantile and $(100u_2)^{th}$ quantile and V is between $(100v_1)^{th}$ quantile and $(100v_2)^{th}$ quantile has to be nonnegative. The most important result of copula theory is the Sklar's theorem.

Theorem 2.1 (Sklar's theorem.) *Let H be a joint distribution function with margins F and G. Then there exists a copula C such that for all x,y in $\mathbb{R}$,*

$$H(x, y) = C(F(x), G(y)). \tag{2.1}$$

If F and G are continuous, then C is unique. Conversely, if C is a copula and F and G are distribution functions, then the function H defined by (2.1) is a joint distribution function with margins F and G.

By Sklar's theorem, all distribution functions can be expressed by copulas. The proof is based on the properties of copulas and distribution functions and can be found in [13].

Random variables can be defined not only by their distribution functions, but by their densities as well. Denote the density of bivariate random variable by $h(x, y)$ and copula density by $c(u, v)$ where

$$c(u, v) = \frac{\partial^2 C(u, v)}{\partial u \partial v}.$$

We can derive the following expression

$$h(x, y) = c(F(x), G(y))f(x)g(y), \tag{2.2}$$

where $f(x), g(y)$ are marginal densities.

The expression (2.2) gives us another view of what a copula presents. By copula density we are able to express joint density using the marginal densities. In estimating copulas we issue from this relationship.

2.2 Copula families

There are number of parametric copula specifications. We look at them with respect to their behavior in tails that we are interested in. Tail dependence is a measure for couples of random variables that capture their behavior on tails. It is defined as

$$\begin{array}{ll} \text{Left tail dependence} & \lambda_L = \lim\limits_{v \to 0^+} \Pr(U < v | V < v) = \lim\limits_{v \to 0^+} \Pr(V < v | U < v), \\ \text{Right tail dependence} & \lambda_R = \lim\limits_{v \to 1^-} \Pr(U > v | V > v) = \lim\limits_{v \to 1^-} \Pr(V > v | U > v), \end{array}$$

Copula family	Distribution function
Normal copula	$C_N(u,v;\rho) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v))$
Clayton copula	$C_C(u,v;\alpha) = \max\ [(u^{-\alpha} + v^{-\alpha} - 1)^{-\frac{1}{\alpha}}, 0]$
Gumbel copula	$C_G(u,v;\alpha) = \exp\ \{-[(-\ln\ u)^\alpha + (-\ln\ v)^\alpha]^{\frac{1}{\alpha}}\}$
Rotated Gumbel copula	$C_R(u,v;\alpha) = u + v - 1 + C_G(1-u, 1-v;\alpha)$

Table 2.1: Distribution functions of different copulas

where $\Pr(X)$ means the probability of X. Let us remark that U, V are not the values of random variables, but their quantiles. Therefore, $\Pr(U < v|V < v) = \Pr(V < v|U < v)$ and tail dependence is properly defined.

Tail dependence describes the limiting probability that one margin exceeds a certain threshold given that the other margin has already exceeded that threshold. Loosely speaking, it tells us if the extreme events tend to occur in the same time. We can easily derive formula for copula left tail dependence

$$\lambda_L = \lim_{v\to 0} \frac{\Pr(U < v \wedge V < v)}{\Pr(V < v)} = \lim_{v\to 0} \frac{C(v,v)}{C(1,v)} = \lim_{v\to 0} \frac{C(v,v)}{v}$$

and right tail dependence

$$\begin{aligned} \lambda &= \lim_{v\to 1} \frac{\Pr(U > v \wedge V > v)}{\Pr(V > v)} = \lim_{v\to 1} \frac{C(1,1) - C(v,1) - C(1,v) + C(v,v)}{1 - C(1,v)} \\ &= \lim_{v\to 1} \frac{1 - 2v + C(v,v)}{1-v}. \end{aligned}$$

Existing literature typically documents that extreme events are more correlated than ordinary ones. We will study this fact by focusing on four types of copulas. We use Normal, Clayton, Gumbel and Rotated Gumbel copula. They are defined by formulas given in Table 2.1. Note that Φ is a cumulative distribution function of standard normal distribution and Φ_ρ is a cumulative distribution function of multivariate normal distribution with correlation matrix ρ.

The considered copula families cover all different behaviors in tails. Normal copula has zero tail dependence for both tails. Its parameter ρ varies from -1 to 1 and it is equal to correlation for normal distributed marginals. Gumbel copula represents an asymmetric copula with non-zero upper tail dependence, i.e. extreme positive events occur together. Its parameter α is from $(1, \infty)$. The higher α is the more dependence is between the variables. To capture non-zero lower tail dependence we consider Clayton and Rotated

Gumbel copulas. The parameter α of Clayton copula lies in $(-1, \infty)\backslash\{0\}$ and $(1, \infty)$ for Rotated Gumbel copula. For both, the higher the parameter is the more dependence there is between variables. We plot copula densities of the used copulas in Figure 2.1. We can see high values in low quantiles for copulas where lower tail dependence is high and in high quantiles for copulas where upper tail dependence is high. Besides parametric copula

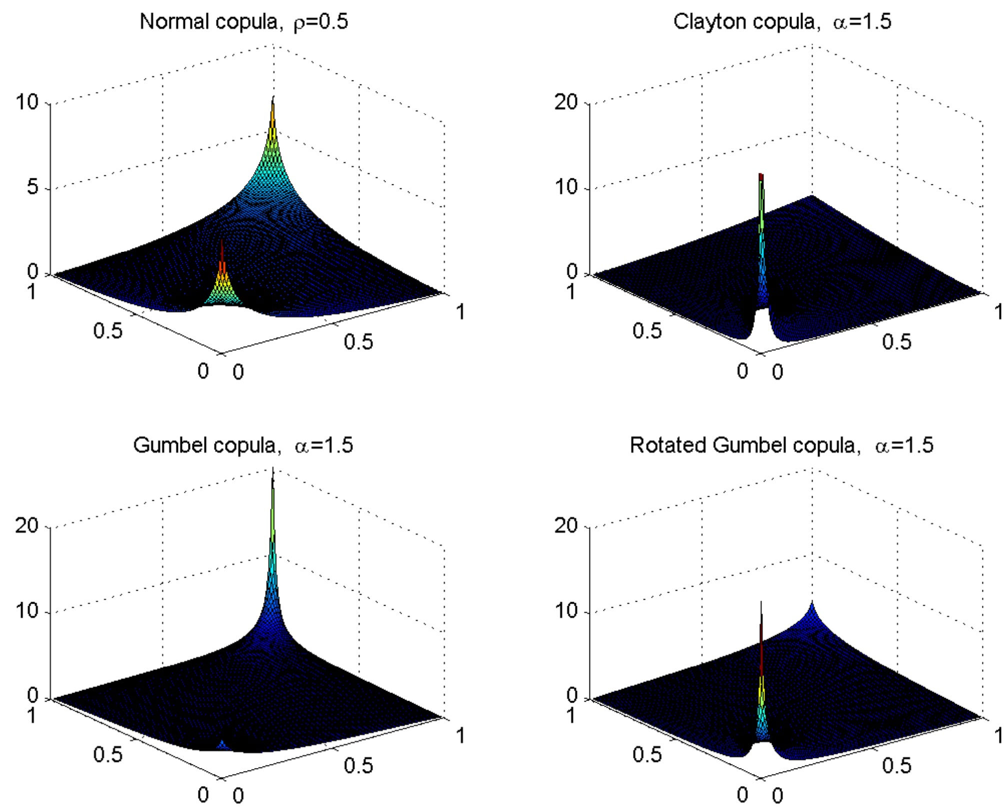

Figure 2.1: Copula densities

families we establish one non-parametric copula.

Definition 2.2 *Let $V = \{x_n, y_n\}_{n=1}^N$ be the sample data set. Then the empirical copula is the following function*

$$C\left(\frac{i}{N}, \frac{j}{N}\right) = \frac{\#\{(x, y) \in V : x < x_i, y < y_j\}}{N} \qquad i, j \in \mathbb{N} : 0 < i, j \leq N,$$

where $\#$ means a number of (x, y) that fulfills the given property, and empirical copula *density with precision k*

$$c\left(\frac{i}{k},\frac{j}{k}\right) = C\left(\frac{i}{k},\frac{j}{k}\right) - C\left(\frac{i-1}{k},\frac{j}{k}\right) - C\left(\frac{i}{k},\frac{j-1}{k}\right) + C\left(\frac{i-1}{k},\frac{j-1}{k}\right)$$

$$i,j \in \mathbb{N} : 1 < i,j \leq N.$$

They are analogous to the empirical cumulative distribution function and histogram which we know by univariate random variables. This approach allows us to analyze dependence of random variables without any previous assumptions. By inspection the graph of empirical copula density we get the intuition which parametric copula family fits the data best.

2.3 Process of copula modeling

There are several issues one needs to consider when estimating copulas. First, we seek for the best estimation of joint density by a considered copula. Second, after estimating the parameters of different copulas, we look for the one that fits the data best.

2.3.1 Estimating parameters of copula and marginals

In the first step, our goal is to determine the parameters of the copula that describes the given density the best. Estimation is based on Maximum Likelihood Method. Maximum Likelihood Estimation is widely used and favored because of its feasibility and good asymptotic properties. By estimating the parameters we issue from the relationship between density of copula and density of the given data (2.2).

Let $\{x_n, y_n\}_{n=1}^N$ be the sample data matrix. Thus, the log likelihood function

$$l(\theta) = \sum_{n=1}^{N} \ln c(F(x_n), G(y_n); \theta) + \sum_{n=1}^{N} \ln f(x_n; \theta) + \sum_{n=1}^{N} \ln g(y_n; \theta),$$

where θ is the set of all parameters of both the copula and the marginals. The maximum likelihood estimator is obtained as

$$\hat{\theta} = \arg\max_{\theta} l(\theta)$$

The direct estimation of all parameters could be very computationally demanding. However, the representation of the log likelihood function allows us to split the copula estimation in two steps. First identifying the marginals and second estimating the copula

parameter. There are two basic methods using this approach: Interference for Margins (IFM), a fully parametric approach and Canonical Maximum Likelihood (CML), a semi-parametric approach.
We analyze the details for each approach separately below.

IFM estimator

IFM is a fully parametric method. We need not only a copula to be parametric, but marginal densities as well. It means, we have to assume that marginals issue from some specific distribution family. Let θ_C represents the parameter of a copula and θ_X, resp. θ_Y represent the parameters of marginal distributions. Thus, the log likelihood function can be re-expressed

$$l(\theta_C, \theta_X, \theta_Y) = \sum_{n=1}^{N} \ln c(F(x_n; \theta_X), G(y_n; \theta_Y); \theta_C) + \sum_{n=1}^{N} \ln f(x_n; \theta_X) + \sum_{n=1}^{N} \ln g(y_n; \theta_Y). \quad (2.3)$$

Now we could just maximize the function over all parameters, but the optimization over more parameters is more computationally demanding. To avoid it, the log likelihood function is split into parts where parameters are estimated separately. In the first step, we estimate parameters of marginal densities (θ_X, θ_Y) maximizing the second part of (2.3)

$$\hat{\theta}_X = \arg\max_{\theta_X} \sum_{n=1}^{N} \ln f(x_n; \theta_X) \quad \text{and} \quad \hat{\theta}_Y = \arg\max_{\theta_Y} \sum_{n=1}^{N} \ln f(y_n; \theta_Y)$$

The second step uses the estimated parameters of marginal distributions to maximize the log likelihood function and estimate the copula parameter

$$\hat{\theta}_C = \arg\max_{\theta_C} \sum_{n=1}^{N} \ln c(F(x_n; \hat{\theta}_X), G(y_n; \hat{\theta}_Y); \theta_C).$$

Note that copula density depends not only the parameter θ_C, but on marginals parameters θ_X and θ_Y as well. By splitting the original log likelihood function into two parts, this property is ignored because of less computationally demanding estimation. As consequence the estimator $(\hat{\theta}_C, \hat{\theta}_X, \hat{\theta}_Y)$ is not efficient in general, but we regard it as a good trade off between difficulty of the approach and efficiency of the estimator.

CML estimator

CML is a semi-parametric method with two steps. First, we estimate marginal distributions using empirical cumulative distribution (a non-parametric step). Second, we use Canonical Maximum Likelihood to estimate joint density by copula (a parametric step). We put estimated distribution functions from the first stage to maximum likelihood function which we want to maximize over the parameter of the considered copula

$$\hat{\theta}_C = \arg\max_{\theta_C} \sum_{n=1}^{N} \ln c(\hat{F}(x_n), \hat{G}(y_n), \theta_C),$$

where $\hat{F}(x)$, $\hat{G}(y)$ are empirical distribution functions. Comparing the methods, CML and IFM approaches differ in the estimation of marginal distributions in the first step. The second step is the same.

2.3.2 Copula selection

After estimating joint distribution with different copulas we are mostly interested in the copula which can estimate the joint distribution of data in the *best* way. We use several criteria to select the model.

The first group of criteria is based on Maximum Likelihood function: Akaike information criterion (AIC) and Bayes Information Criterion (BIC). They are closely connected to the maximum likelihood method and it is very natural to consider them. We prefer the copula with the minimal value of AIC or BIC. The criteria are defined by formulas

$$\begin{aligned} AIC &= -2\ln L(\hat{\theta}) + 2q, \\ BIC &= -2\ln L(\hat{\theta}) + q\ln(N), \end{aligned}$$

where $\hat{\theta}$ is the vector of estimators, q is number of estimated parameters and N is number of observations in our dataset. It is important to notice, that in our case, when we consider only one-dimension parameter copulas, both criteria AIC and BIC always select the same copula. That is why we will consider just AIC in our empirical work.

The second group of criteria compares empirical joint distribution to parametric distribution of estimated copula. We use Kolmogorov-Smirnov and Anderson-Darling statistics given by

$$KS = \max_i |H_m(x_i, y_i; \hat{\theta}) - H_e(x_i, y_i)|, \tag{2.4}$$

$$AD = \max_i \frac{|H_m(x_i, y_i; \hat{\theta}) - H_e(x_i, y_i)|}{\sqrt{H_e(x_i, y_i)(1 - H_e(x_i, y_i))}}, \tag{2.5}$$

where H_e is an empirical joint distribution and H_m is a joint distribution estimated from the model. Anderson-Darling statistics, compared to Kolmogorov-Smirnov statistics, takes the differences $|H_m(x_i, y_i; \hat{\theta}) - H_e(x_i, y_i)|$ into consideration with higher weights in the middle of distribution than on the tails (denumerator in (2.5)). The best copula is a copula with the minimal statistics.

2.3.3 Simulating the criteria

There is no theory which criterion is the best. When estimating copulas on data[2] different criteria preferred different copulas as the most appropriate. These results motivate us to analyze the robustness of copula models using different criteria. We use the following approach for simulations.

Consider the formula for copula density (2.2). First, we choose the marginal distributions. Second, we compute the corresponding values of copula density and calculate joint density. There is no uncertainty so far. We generate pairs of values from the calculating joint density. We put the noise to data by adding the error term with distribution $N(0, \sigma^2)$. Third we estimate the parameters for all copulas by the IFM method. We are interested in probability, with which the criterion selects the generating copula.

In our simulations we use marginals from normal distributions. We consider all four copulas. We compute the joint density for each copula and generate 1000 data from it. Then we add the error term with standard deviation 0.01 and fit all for copulas. We select the best according all criteria and compare to the generating one. We repeat this technique 1000 times and get the probability of success for all three criteria. We report results in Table 2.2.

Results from simulations find the AIC criterion the best. We decided to use this criterion

Criterion	Probability
AIC/BIC	**87%**
Kolmogorov-Smirnov	72%
Anderson-Darling	68%

Table 2.2: Probability of selecting the generating copula

by choosing the best copula family in our empirical work. Furthermore, this criterion is in coherence with the method how the parameters of copulas and marginals were estimated, by Maximum Likelihood Method.

[2] More about data in Chapter 5

3 Multivariate GARCH

The main feature of the GARCH framework is the modeling of second moments that are connected to uncertainty. In this chapter we present functional forms of a general model and its simplification to more parsimonious models which we use in our empirical work. We discuss their properties and estimation of parameters.

3.1 History and basic models

Studying financial time series came up with the problem that variance was not time invariant. Traditional time series tools such as Autoregressive Moving Average (ARMA) models have been extended to analogous models for the variance. Engle (1982) suggested that the unobservable second moments could be modeled together with the first moments. The second moment depends on the elements in the information set in autoregressive manners. This model was called the Autoregressive Conditional Heteroscedasticity (ARCH) model. Conditional implies a dependence on the past observations and autoregressive describes a mechanism how past observations are incorporated into the present. Heteroscedasticity is the other expression for time-varying variance.
Bollerslav (1986) generalized the Engle's model by adding past conditional variances in the current conditional variance equation to get the famous Generalized ARCH (GARCH) model.[1] We present here the precise definition of the model.

Definition 3.1 (Univariate GARCH.) *Consider the univariate, serially uncorrelated, zero mean process u_t. The process u_t is said to follow a generalized autoregressive conditionally heteroscedastic process of order p and q, GARCH(p,q), if the conditional distribution of u_t given its past $\Sigma_{t-1} := \{u_{t-1}, u_{t-2}, \ldots\}$, has zero mean and the conditional variance is*

$$\begin{aligned} \sigma^2_{t|t-1} : &= Var(u_t|\Sigma_{t-1}) = E(u_t^2|\Sigma_{t-1}) \\ &= \alpha_0 + \alpha_1 u^2_{t-1} + \ldots + \alpha_q u^2_{t-q} + \beta_1 \sigma^2_{t-1|t-2} + \ldots + \beta_p \sigma^2_{t-p|t-p-1}, \end{aligned}$$

that is, $u_t|\Sigma_{t-1} \sim (0, \sigma^2_t|t-1)$.

GARCH models are now widely adopted in applications. They become very popular due to very intuitive idea behind the model. Furthermore, the models are tractable and easy

[1]GARCH, GARCH process and GARCH model mean the same

to estimate. Many different classes of models were derived from the basic GARCH model. The main advantage of GARCH models is that they take into account the fat tail behavior and volatility clustering, a tendency of small observation to be followed by other small observations and vice versa. Both properties are very common in financial time series.
The extension for multivariate case followed shortly. Multivariate GARCH can capture relationship between multiple series. It is obvious that an increase in the volatility in one series may make the other series more volatile as well. The multivariate case is the direct generalization of univariate case.

Definition 3.2 (Multivariate GARCH.) *Suppose that $u_t = (u_{1t}, \ldots, u_{Nt})$ is a N–dimensional zero mean, serially uncorrelated process which can be represented as*

$$u_t = H_{t|t-1}^{1/2} \epsilon_t,$$

where ϵ_t is a N-dimensional i.i.d white noise, $\epsilon_t \sim$ i.i.d. $(0, I_N)$, and $H_{t|t-1}$ is the conditional covariance positive definite matrix of u_t given the past information. They represent a multivariate GARCH process, MGARCH(p,q), if

$$\mathrm{vech}(H_{t|t-1}) = C_0 + \sum_{j=1}^{q} a_j \mathrm{vech}(u_{t-j} u'_{t-j}) + \sum_{j=1}^{p} G_j \mathrm{vech}(H'_{t-j|t-j-1}), \tag{3.1}$$

where vech denotes the half-vectorization operator which stacks the columns of a square matrix from the diagonal downwards in a vector, C_0 is $\frac{1}{2}N(N+1)$-dimensional vector of constants, A_j's and G_j's are $\frac{1}{2}N(N+1) \times \frac{1}{2}N(N+1)$ coefficient matrices. The system of equations (3.1) defines the parametrization that is called vec representation.

To illustrate in the bivariate ($N = 2$) case, MGARCH(1,1) *vec* model in matrix notation is simply

$$\begin{pmatrix} \sigma_{11,t|t-1} \\ \sigma_{12,t|t-1} \\ \sigma_{22,t|t-1} \end{pmatrix} = \begin{pmatrix} c_{10} \\ c_{20} \\ c_{30} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} u_{1,t-1}^2 \\ u_{1,t-1} u_{2,t-1} \\ u_{2,t-1}^2 \end{pmatrix}$$
$$+ \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \begin{pmatrix} \sigma_{11,t-1|t-2} \\ \sigma_{12,t-1|t-2} \\ \sigma_{22,t-1|t-2} \end{pmatrix}.$$

From the example it is obvious that even the simple model has a fair numbers of parameters which makes it difficult to handle. The challenge for scientist is to avoid too many

parameters to keep them feasible but on the other hand to maintain enough flexibility in behavior of H_t.
The second problem is whether the model guarantees the positive semidefiniteness of conditional covariance matrix H_t in the time. We require H_t to be positive semidefinite for all values of ϵ_t in the sample space. This restriction can be difficult to check in the *vec* representation. The problem was solved by Engle and Kroner (1995). They suggested a new representation of MGARCH model, BEKK representation.[2]

Definition 3.3 (BEKK representation.) *MGARCH(p,q) model written as*

$$H_t = C_0^{\star'} C_0^\star + \sum_{k=1}^{K} \sum_{i=1}^{q} A_{ik}^{\star'} \epsilon_{t-1} \epsilon_{t-1}^{'} A_{ik}^\star + \sum_{k=1}^{K} \sum_{i=1}^{p} G_{ik}^{\star'} H_{t-1} G_{ik}^\star,$$

where $C_0^\star$ is a upper triangular $N \times N$ matrix, $A_{ik}^\star$'s and $G_{ik}^\star$'s are $N \times N$ coefficient matrices, is called the BEKK representation.

This is a convenient representation for analysis definiteness of the system. It is clear that the BEKK representation guarantees the positive semidefiniteness of H_t under very weak conditions. Engle and Kroner discussed and proofed them for any MGARCH model given by BEKK representation. For $K = 1$, positivity of H_0 and regularity at least one of the C_0 or $G_{i1}^\star$ is sufficient. Furthermore, Engle and Kroner showed the relationship between *vec* and BEKK representation. Positivity conditions for original *vec* representation was derived by Gourieroux (1997) but they are not so intuitive and easy to impose in estimation.
The assumption of normality in residuals give rise to a likelihood function. Maximizing the log likelihood function we get the parameters of models. We do not discuss the details of the Maximum Likelihood Estimation for MGARCH models as we did in previous chapter about copula estimation. Whereas we had to program estimation of copula parameters, many softwares have already included estimation of MGARCH parameters based on Maximum Likelihood Estimation.

3.2 Models

In our empirical work we use three models – basic Constant Conditional Correlation (CCC) model and two special forms of bivariate GARCH(1,1) models, diagonal *vec* model and diagonal BEKK model.
CCC model will be our benchmark. It captures the dependence structure only by univariate

[2]The acronym BEKK is the abbreviation of the names Baba, Engle, Kraft and Kroner.

GARCH models. Conditional means that the variance of each series is modeled by GARCH model and constant implies that the correlation is time invariant. Let h_{it} is univariate GARCH for each series i. Then CCC model is

$$
\begin{aligned}
H_t &= D_t R D_t, \\
D_t &= \operatorname{diag}(\sqrt{\mathrm{h_{it}}}), \\
R &= (\rho_{ij}),
\end{aligned}
$$

where R is a $N \times N$ correlation matrix and is estimated as a simple unconditional correlation matrix.

The second model is diagonal *vec* model. Matrices A and G from the *vec* representation are assumed to be diagonal. Loosely speaking, each variance h_{iit} depends only on its own past squared error $\epsilon^2_{i,t-1}$ and its own lag $h_{ii,t-1}$ and covariance h_{ijt} depends only on its own past cross-product of errors $\epsilon_{i,t-1}\epsilon_{i,t-2}$ and its own lag $h_{ij,t-1}$. This seems an intuitively plausible restriction because information about variances is usually hidden in squared residuals and there is some ability of past squared residuals to forecast future variances. A similar explanations can be made for covariances. As a revenue for the restriction the number of parameters is reduced from 21 to 9 for the bivariate case. The matrix notation of bivariate GARCH(1,1) *vec* model is equivalent to the system of equations

$$
\begin{aligned}
h_{11t} &= c_1 + a_{11}\epsilon^2_{1,t-1} + g_{11}h_{11,t-1} \\
h_{21t} &= c_2 + a_{22}\epsilon_{1,t-1}\epsilon_{1,t-1} + g_{22}h_{21,t-1} \\
h_{22t} &= c_3 + a_{33}\epsilon^2_{2,t-1} + g_{33}h_{22,t-1}
\end{aligned}
$$

The third model is a bivariate diagonal BEKK(1,1,1) model. It takes $A^\star_k$ and $G^\star_k$ in the BEKK representation as diagonal matrices. The model is simplified to the system

$$
\begin{aligned}
h_{11t} &= \omega_{11} + a^{\star 2}_{11}\epsilon^2_{1,t-1} + g^{\star 2}_{11}h_{11,t-1} \\
h_{21t} &= \omega_{21} + a^\star_{11}a^\star_{22}\epsilon_{1,t-1}\epsilon_{2,t-1} + g^\star_{11}g^\star_{22}h_{21,t-1} \\
h_{22t} &= \omega_{22} + a^{\star 2}_{22}\epsilon^2_{2,t-1} + g^{\star 2}_{22}h_{22,t-1}
\end{aligned}
$$

Comparing the structure of the model to the previous *vec* model we see the same linear structure. However, there are some additional constraints on parameters. By adding the restrictions

$$
a_{22} = \sqrt{a_{11}a_{33}}, \quad g_{22} = \sqrt{g_{11}g_{33}} \quad \text{and} \quad a_{11}, a_{33}, g_{11}, g_{33} \geq 0
$$

to the bivariate GARCH(1,1) *vec* model we get BEKK(1,1,1) model. The constraints restrict the flexibility of the model as trade-off for less parameters. We need to estimate only 7 parameters in this model.

Note that while the first CCC model specifies the conditional correlation in addition to the variances, *vec* and BEKK model specifies the conditional covariance in addition to the variances.

4 Value at Risk

Each investor is interested in the risk his portfolio faces. Currently, portfolio risk is measured in terms of its "value at risk". Consider a portfolio Z of two assets X and Y. Let x be the return of X and y be the return of Y. Denote β the weight of X in the portfolio. Thus, the return of the portfolio is $z = \beta x + (1-\beta)y$. The Value at Risk for a given confidence level θ is the threshold below which the return falls with the probability θ. Formally, we have

$$\Pr(Z \leq \mathrm{VaR}_z) = \theta.$$

4.1 Calculating VaR for models

In the first group of models, without copula modeling, we assume the normality of the residuals. The portfolio Z is a combination of two normal distributions. Thus, it itself is normal. Let us use the notation from the MGARCH models that H is the covariance matrix of X and Y. By matrix notation, the variance $D(Z)$ of Z is

$$Z = \begin{pmatrix} \beta & 1-\beta \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \longrightarrow D(Z) = \begin{pmatrix} \beta & 1-\beta \end{pmatrix} \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} \beta \\ 1-\beta \end{pmatrix}$$

When we know the variance of the normal distributed portfolio, more exactly, its residuals from MGARCH, we can express the Value at risk for a confidence level $\theta = 0.05$ as

$$\mathrm{VaR}_z = 1,65\sqrt{\beta^2 H_{11} + (1-\beta)^2 H_{22} + 2\beta(1-\beta)H_{12}}, \tag{4.1}$$

where $-1,65$ is the 5^{th} quantile of the normalized normal distribution. By deriving we use the symmetry of covariance matrix $H_{12} = H_{21}$, i.e $cov(X,Y) = cov(Y,X)$. Let us remark that the sign of VaR is positive because VaR is known as an amount of loss.
On the other hand, copula methods do not require normal distribution. By deriving the formula for VaR we have to be more general. Using the expression (2.2) for the copula density, the probability distribution of the portfolio return is given

$$\Pr(Z \leq z) = \Pr(\beta X + (1-\beta)Y \leq z) = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\frac{1}{\beta}z - \frac{1-\beta}{\beta}y} c(F(x), G(y)) f(x) dx \right\} g(y) dy.$$

There is no close formula for calculating VaR for specific copulas. Monte Carlo simulations are used to the calculations. Here we present the example of simulations.

We choose the standardized normal distributions for marginals and the weights $(\beta, 1-\beta) = (0.5, 0.5)$ of the portfolio. Then we simulate values from the distribution functions of different copulas. We use the different level of dependence which is measured by Spearman's coefficient.[1] For each copula and each level of dependence, we simulated 1 000 000 scenarios. Thus, VaR was the value below which 5% of simulations lied. In Figure 4.1 we compare VaR for different copulas with the increasing measure of dependence. The relationship among

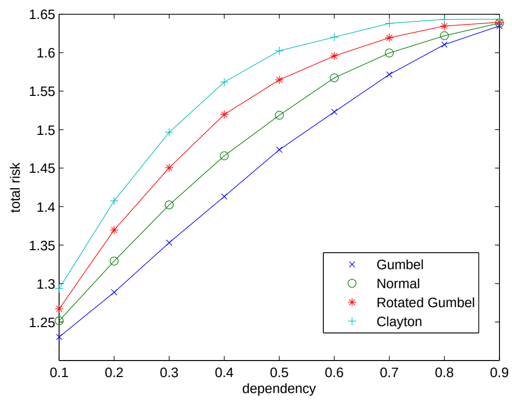

Figure 4.1: VaR for different copula families

the VaRs for different copula families coincides with the intuition. The loss is bigger for copulas with non-zero lower tail dependence, Clayton and Rotated Gumbel copulas. The dependence is stronger in lower quantiles what resumes in higher potential loss. On the other hand, VaR for Gumbel copula is smaller because dependence is focused in the upper quantiles what VaR does not capture.

4.2 Tests of VaR accuracy

We are interested if our VaR calculated from the models performs properly. We examine the adequacy of calculated VaR by Kupiec's POF test and Christoffersen's test.

[1] Spearman's coefficient is a nonparametric measure of dependence, that takes into account only the order of data. It is defined as

$$r = 1 - \frac{6 \sum d_i^2}{n^3 - n},$$

where n is the number of pairs and d_i is the difference in the rank of the pair i. There is the direct relationships between the copula parameter and Spearman's coefficient for each copula family.

Define a hit as

$$\text{hit}_t = I(r_t < -\text{VaR}_t),$$

where r_t is a return of a pension fund and $I(.)$ is 1 when the expression in brackets is true and 0 otherwise. If the return of a pension fund is below VaR, i.e. $\text{hit}_t = 1$, we say that VaR is violated. Kupiec's POF test is based on the idea that the ratio of the number of violations to the number of observations should be around the confidence level which VaR is calculated on. Formally, it assumes that hit_t are independent random variables from binomial distribution with parameters (N, p), where N is a number of observations and p is a confidence level of VaR. Kupiec's test statistics takes the form

$$\text{LR}_{\text{POF}} = -2\ln\frac{p^K(1-p)^{N-K}}{\hat{p}^K(1-\hat{p})^{N-K}},$$

where $\hat{p}$ is the percentage of violations and K is a number of violations. Note that K equals to the sum of hit_t and $\hat{p}$ is the ratio of the number of violations K to the number of observations N. The statistics has the asymptotic χ^2 distribution with one degree of freedom.

Christoffersen improved the Kupiec's test by considering the position of violations. While Kupiec tests only unconditional values of hit_t, Christoffersen takes into an account that violations should be unpredictable and independent. To Kupiec's POF statistics he adds the statistics testing the independence of violations

$$\text{LR}_{\text{CH}} = \text{LR}_{\text{POF}} + \text{LR}_{\text{ind}},$$

where

$$\text{LR}_{\text{ind}} = 2\big((1-p_{01})^{T_{00}}p_{01}^{T_{01}}(1-p_{11})^{T_{10}}p_{11}^{T_{11}} - (1-p)^{T_{00}+T_{01}}p^{T_{10}+T_{11}}\big)$$

Statistics LR_{ind} indicates the independence of violations. A variable T_{ij} is a number of observations that $\text{hit}_t = j$ and $\text{hit}_{t-1} = i$. A variable p_{ij} is a probability that $\text{hit}_t = j$ and $\text{hit}_{t-1} = i$ and is calculated as

$$p_{01} = \frac{T_{01}}{T_{00}+T_{01}}, \quad p_{11} = \frac{T_{11}}{T_{10}+T_{11}} \quad \text{and} \quad p = \frac{T_{01}+T_{11}}{N}.$$

The statistics has the asymptotic χ^2 distribution with two degrees of freedom. In both test, we reject the zero hypothesis about the VaR accuracy if the statistics exceeds the corresponding critical value, i.e if the p-value is below the given level (usually 0.05).

5 Empirical work

Since January 2005 we have a new pension system in Slovakia. It is based on three pillars now. The first pillar is the old but reformed pay-as-you-go system, the second is a new fully funded pillar and is based on the principle of savings and the third one consists of various forms voluntary pension and life insurances. The reform of the pension scheme in 2005 has brought a fundamental change – an option to save for a pension on a private pension account in a Pension Funds Management Company (PFMC). If you enter the second pillar, what is voluntary, a half of your obligatory contributions for old-age insurance is saved in the PFMC of your choice. Each PFMC manages three different types of funds, among which you can choose according to your risk aversion. Our motivation is to model the yield, particularly the risk PFMC faces in the individual PFMCs in Slovakia. We focus on the most risky pension fund, the growth pension fund, where the most of the population is. In Table 5.1 we name all PFMC operating in Slovakia with the names of their progress funds.[1]

In this chapter we apply the theory from previous chapters to model the returns of

Pension Funds Management Company	Progress fund	Market share
AEGON d.s.s., a.s.	Vital	10%
Allianz - Slovenská d.s.s., a.s.	Progress	31%
AXA d.s.s., a.s.	Rastový	28%
ČSOB d.s.s., a.s.	Prosperita	6%
ING d.s.s., a.s.	Dynamika	11%
VÚB Generali d. s. s., a.s.	Profit	15 %

Table 5.1: Progress funds in Slovakia

progressive pension funds. We estimate VaR for each model.

5.1 Data

The structure of the portfolio for the progress funds consists of two main components: investing into stocks, which represents the riskier part of the portfolio, and investing into

[1]Shares on the market are from [12], as on 30^{th} June 2008.

bonds, which is the safer investment. The law allows to invest up to 80% of assets into stocks. Reality in PFMCs is much more below 80% that reflects prudence and carefulness of PFMCs in the beginning of the system. Furthermore, safer investment is adequate in the time of financial crisis which the world faces now.

We model the stock market by different indices from Europe, the USA and Japan. We do not work with the index prices because comparing the prices might be tricky. Instead, we use returns that represent relative changes more suitable for comparison. Let P_t is the price of the stock at time t, then the return r_t^S of the stock at time t is

$$r_t^S = \frac{P_t}{P_{t-1}} - 1.$$

The bond market we model by discount rates for zero-coupon bonds with different time to maturity for Slovak crown, American dollar and Euro. To have consistent data for model we need to work with relative changes of bond prices. We show that using the differences of discount rates is desirable. Let R_t be the discount rate at time t of the zero-coupon with the maturity T (in years) and B_t the price of the bond. Thus, the return r_t^B of the bond is

$$\begin{aligned} r_t^B &= \frac{B_t}{B_{t-1}} - 1 = \frac{\left(\frac{1}{1+TR_t}\right)^{\frac{1}{T}}}{\left(\frac{1}{1+TR_{t-1}}\right)^{\frac{1}{T}}} - 1 = \frac{(1+TR_{t-1})^{\frac{1}{T}}}{(1+TR_t)^{\frac{1}{T}}} - 1 \sim \frac{1+R_{t-1}}{1+R_t} - 1 \\ &= \frac{R_{t-1} - R_t}{1+R_t} \sim R_{t-1} - R_t, \end{aligned}$$

where we neglect the higher-order terms in approximation.

Besides the discount rates, we use iTtraxx Europe indices to take the credit risk into consideration. They are calculated from credit default swaps (CDS).[2] We use iTraxx Europe, that consists of CDSs on the most traded companies in Europe and iTraxx Europe Senior that consists of CDSs on senior financial companies. ITraxx indices are a kind of rates (not prices), therefore we also use differences for our models.[3]

The yield in pension fund is expressed by a pension unit (PU). The pension unit represents the value of one slovak crown deposited by the establishment of the pension fund. Fees for management of pension funds are not included in the pension unit. PFMC reduces the number of pension units instead. Thus, the pension unit is the measure of net yields and it reflects the investment strategy without deformation by fees. Similarly to the stocks we

[2] CDS is a credit derivative, where the buyer pays a premium and, in return, he receives a money when an underlying financial instrument defaults.

[3] List of all data with their symbols and definitions is in Table A.1 in Appendix A.

work with the returns of the pension unit

$$r_t^{PU} = \frac{P_t}{P_{t-1}} - 1,$$

where P_t is the price of the pension unit at time t. We use daily data from the first working day of January 2008 to the last working day of December 2008 for all time series. In total, we have a sample of 249 observations for each series. Descriptive statistics for all data are attached in Appendix A (see Tables A.2, A.3, A.4).

5.2 Basic method for VaR calculation

A simple approach of modeling VaR is based on historical simulations. We get VaR from the information about empirical distribution of past values. We calculate VaR at 5% confidence level as 5% empirical quantile from the past data. The method is very simple and that is why it is very popular. Its drawback is that it adapts to changes very slowly and therefore it is very imprecise. In Figure 5.1 we see VaR calculated by this method for two pension funds. We calculated VaR as 5% quantile from the last 100 values. We see how slow it reacts in time of higher variance. There are many violations of VaR and furthermore, they are not independent, but they cluster together. Both Kupiec's and Christoffersen's tests reject the hypothesis that the model performs properly. Christoffersen's test that takes independence into consideration rejects it more strongly. P-values for both tests are given in Table 5.2. At 5% confidence level VaR performs properly only for CSOB pension fund and only according to Kupiec's test. These results give us a motivation for finding a better method for modeling VaR.

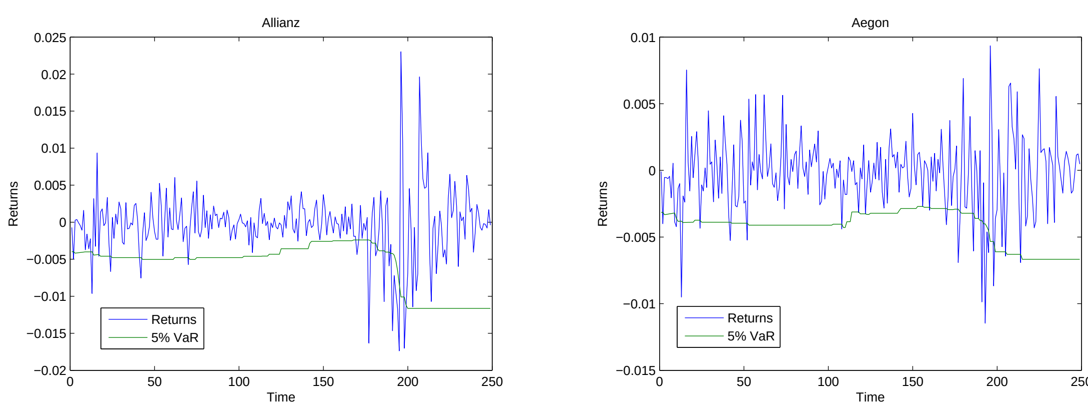

Figure 5.1: VaR based on historical simulations

PFMC	AEGON	ALLIANZ	AXA	CSOB	ING	VUB
Kupiec's test	0.0428	0.0058	0.0027	0.1292	0.0118	0.0118
Christoffersen's test	0.0002	0.0097	0.0025	0.0008	0.0147	0.0048

Table 5.2: P-values from tests of VaR accuracy for historical simulations

5.3 VaR based on copula and MGARCH models

Our aim is to model the pension unit by a more sophisticated model. We do not use the past pension unit values but we find the structure of the portfolio and model the dependence between its two important parts, the stock and bond market.
In this section we demonstrate the whole process of estimation. Our goal is to model the returns of the pension unit returns by returns from the stock and bond market. All steps we perform are summed up in Figure 5.2. First, from a huge amount of stocks and indices we

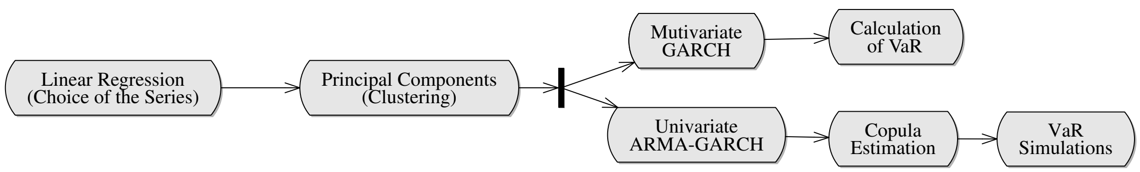

Figure 5.2: Estimation scheme

find the ones that have significant influence on the pension unit. We choose the significant indices by simple linear regression on the pension unit. Second, we cluster all information about stocks into one variable to reduce the number of dimensions. In this step we use the technique of principal components which is based on projecting the multivariate space into one dimension. Similarly, we cluster all significant discount rates into one variable. Third, we model the dependence between the market variable and the bond variable by two different approaches - copula models and MGARCH model. We compare MGARCH based on time variant correlation to copula models that are able to capture heavy tails. Fourth, we calculate VaR for both models.
Linear regression, principal components analysis and estimation of MGARCH we did in Eviews and ARMA-GARCH models we estimated in Matlab. All copula estimations and simulations we did by ourselves in Matlab.
In the following subsections we discuss the estimation process in details. We demonstrate

the whole process for the progress fund of Allianz, the PFMC with the largest market share. Individual steps of modeling for the funds of other companies we do not present here. We summarize them in Section 5.4.

5.3.1 Linear regression

At the beginning we have a group of indices and discount rates which we may use in the pension unit modeling. Month reports that PFMC has to publish give us a brief overview about the geographical areas where PFMC invests. According to the report of Allianz, they invest into stocks mostly from Western Europe and the USA, less from Central Europe and Japan. The bonds are usually in Slovak currency or EUR with average duration slightly over 1. By using linear regression we estimate the whole set of series with pension unit as a dependent variable. Mathematically speaking, our linear regression model takes the form

$$r_t^{PU} = c_0 + \mathbf{c}^T \mathbf{r}_t^S + \mathbf{d}^T \mathbf{r}_t^B + \epsilon_t, \qquad (5.1)$$

where the vector $\mathbf{r}_t^S$ represent the returns of different stocks, the vector $\mathbf{r}_t^B$ returns of different bond or iTraxx indices, α is a constant, β, γ are coefficient vectors and ϵ_t is a disturbance term. Note that we can interpret the coefficients as weights of the individual indices and bonds in the pension fund portfolio.

In estimation process we gradually exclude the series that are insignificant at 5% confidence level. For Allianz, the chosen stocks and discount rates are given in Table 5.3. According

Series	BUX	CAC40	DAX	NIKKEI	ITRAXX	EUR6MZ
Coefficient	0.011	0.025	0.055	0.021	0.436	0.415
P-value	0.000	0.000	0.000	0.000	0.037	0.005

Table 5.3: Significant variables from linear regression

to the linear regression, we choose CAC40 and DAX indices that represent the region of Western Europe, BUX index from Central Europe and NIKKEI index that represents the market of Japan. The bond market is significantly influenced by 6 month EUR discount rate and iTraxx index. The constant was not significant.

5.3.2 Principal components

In this stage of the modeling, our goal is to reduce the data to get a good one-dimensional measure for stock markets and a good one dimensional measure for the bond market. One of the most common techniques is Principal Components (PC) analysis. PC method is a linear transformation of the variables. To reduce the dimension it uses the high covariance of the variables as the redundant information. On the other hand, it captures the information in data with high variance. Mathematically speaking, we find another base of the vector space. The transformed variables, called principal components, have diagonal covariance matrix. The principal component that belongs to the highest eigenvalue of transformation market, called the first principal component, carries the most information. This vector represents the information from the data in one dimension the best.

We use significant stocks with estimated weights from the linear regression to get one series representing the stock market. The weights adjust the proportion of variance that influences the pension unit. In other words, if some stock has a lower weight on the pension unit, we have to take it into consideration. The whole process can be show in the scheme

$$\mathbf{c}^T\mathbf{r}_t^S \longrightarrow \quad \text{transformation} \quad \longrightarrow \quad PC_t^S.$$

Similarly for significant discount rates representing the bond market we get the vector

$$\mathbf{d}^T\mathbf{r}_t^B \longrightarrow \quad \text{transformation} \quad \longrightarrow \quad PC_t^B.$$

To sum up, from original huge number of variables we get two series, PC_t^S for the stock market and PC_t^B for the bond variable. The model 5.1 from the beginning is simplified

$$r_t^{PU} = c_0 + cPC_t^S + dPC_t^B + \epsilon_t, \tag{5.2}$$

where the coefficients c_0, c, d are estimated by linear regression again and represents the weights of stocks and bonds in the pension fund portfolio.

Note that from the eigenvalues of the transformation matrix we know how much information is in the individual principal components. From the proportion of the eigenvalue belonging to the first principal component we get how much information we keep. For Allianz, the first principal component contains 72.73% information for the stock variable and 68.15% for the bond variable.

5.3.3 Estimation of the margins

By using the PC analysis we have one time series for the stock market and one series for the bond market. For copula modeling, we assume that inputs are from the same distribution

and independent. In many papers was documented that there is some persistence in data and that is why we need to filter them. We use univariate ARMA-GARCH models.
ARMA-GARCH models consist from the ARMA part

$$y_t = c + \sum \phi_i y_{t-i} + u_t + \sum \theta_j u_{t-j}, \tag{5.3}$$

where y_t is one of our variables (PC_t^S or PC_t^B), c is a constant, ϕ_i, θ_j are coefficients and u_t are innovations. The GARCH part captures heteroscedasticity in y_t and innovations by formula

$$\sigma_t^2 = \alpha_0 + \sum \alpha_i u_{t-i}^2 + \sum \beta_j \sigma_{t-j}^2,$$

where σ_t^2 is a conditional variance of u_t.
Estimation of ARMA-GARCH model consists of several steps. At first we test if we need the ARMA-GARCH specification or our series is just a white noise. We use Ljung-Box test based on the autocorrelation plot. Its statistics is defined by

$$Q_{LB} = n(n-2) \sum_{j=1}^{k} \frac{\rho_j^2}{n-j},$$

where n is a number of observations, ρ_j is autocorrelation for the lag j and k is a number of coefficients to test autocorrelation. The statistics follows asymptotically the χ^2 distribution with k degrees of freedom.
By calculating the Ljung-Box Q-statistics we reject or accept the hypothesis whether a group of autocorrelations equals zero, what is equivalent to white noise. Second, we fit the best ARMA model to eliminate autocorrelation in the series. We choose the best model according to the BIC criterion and by using Ljung-Box Q-statistics we check whether there is still autocorrelation in residuals. If not, we look at the autocorrelation in squared residuals which indicates the heteroscedasticity and necessity of GARCH model. Finally, we fit the GARCH model to the residuals from the estimated ARMA model. After all these steps we get ARMA-GARCH model for each series. Standardized residuals are used for copula estimation and the estimated conditional variances by calculating the VaR. The whole process of margins estimation is summarized in Figure 5.3. We document particular steps for stock market variable. The Ljung-Box Q-statistics (see Table 5.4) does not reject the zero hypothesis for all k at 5% confidence level. The results are not convincing, the test would do reject the zero hypothesis at 10% confidence level. That encourages us to be a little bit suspicious and check the performance of some ARMA models. However, there are

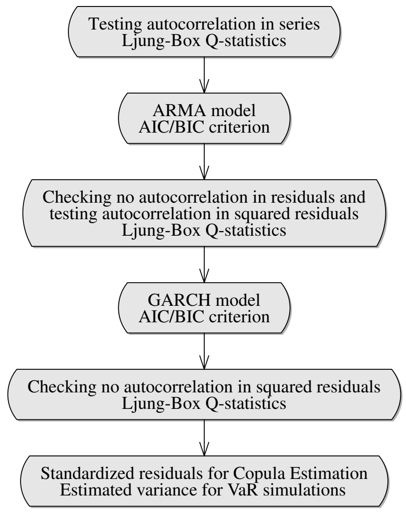

Figure 5.3: Estimation of the margins

Up to k lags	1	2	3	5	10	20
P-value	0.668	0.163	0.414	0.095	0.073	0.051

Table 5.4: P-values from Ljung-Box Q-statistics test

no significant regressors in ARMA models. We do not have significant autocorrelation in residuals but there is still a strong correlation in the squared residuals. Ljung-Box test for the squared residuals has p-values under 0.0001 for all k. To eliminate heteroscedasticity we estimate GARCH models. The AIC criterion chooses GARCH(1,1) without a constant. The coefficients of the best ARMA-GARCH model are summed up in Table 5.5. Finally,

Model	α_1	β_1
Coefficient	0.144	0.845
P-value	0.000	0.000

Table 5.5: Estimated GARCH(1,1) model for stock market

Ljung-Box test checks the autocorrelation in the residuals and squared residuals. The test confirms that there is no more autocorrelation. Furthermore, we check the normality of residuals. Both tests verify that the model is well-specified.

The same technique is used on the series for bond market. The same GARCH(1,1) model without a constant is the most appropriate and the coefficients are given in Table 5.6.
The parameters of GARCH models determine the dynamics of the series. Looking at estimated values for both series we see quite high values of the parameter β_1. This fact says about high persistence of the process, i.e. high persistence of the returns for both stock and bond market.

Model	α_1	β_1
Coefficient	0.026	0.813
P-value	0.003	0.000

Table 5.6: Estimated GARCH(1,1) model for bond market

5.3.4 Estimation of the copula and VaR simulations

We already know the behavior of the both series for the stock market and for the bond market. Now we proceed to the most important part of estimation. From ARMA-GARCH model we estimated standardized residuals

$$su_t = \frac{\hat{u}_t}{\hat{\sigma}_t},$$

that have the standardized normal distribution for all times t. Therefore, we can model the dependence of them by a copula, that assumes the same distribution over the time. First, we estimate the parameters of each copula family which were mentioned in Chapter 2 - Normal, Clayton, Gumbel and Rotated Gumbel copula. According to the AIC criterion we choose the copula for our model.
By using the mentioned technique, Rotated Gumbel copula with the parameter $\alpha = 1.4933$ was estimated for Allianz.
After estimating the copula, VaR was simulated. We proceed backwards. We simulate the pairs of quantiles (q_1, q_2) for the chosen copula with the estimated parameters from the copula function

$$C(q_1, q_2) \longrightarrow (q_1, q_2)$$

Then, we find the values u_t^B, u_t^S that belongs to the generated quantiles from the normal distribution $N \sim (0, \hat{\sigma}_t^2)$. By using the ARMA models given by (5.3) for PC^S and PC^B we get $\hat{PC}_t^S$ and $\hat{PC}_t^B$. The estimated return for pension fund we get from the model (5.2)

$$\hat{r}_t^{PU} = \hat{\gamma}_0 + \hat{\gamma}\hat{PC}_t^S + \hat{\delta}\hat{PC}_t^B.$$

For each time and for each pension fund we simulate 100 000 scenarios. VaR was the 5% quantile of the simulation distribution. The estimated Var for Allianz is shown in Figure B.2 in Appendix B.

5.3.5 MGARCH models

We compare the copula-based model to multivariate GARCH models. The first two steps, choice of significant variables and clustering them in two dimensions are the same as for copula models. Then, we apply three bivariate MGARCH models described in Chapter 3. We estimate all parameters by the maximum likelihood method and get the conditional covariance matrix for each time. From estimated covariance matrix of the stock and bond variable we calculate VaR according the formula (4.1). Note that by multivariate model we did not model the mean of the series because in univariate models for all funds coefficients of ARMA model were insignificant (including a constant).

5.4 Results for all pension funds

The first step, identification of the significant indices and discount rates, corresponds with our expectation. Particularly choice of indices for the stock market follows the month reports of pension funds. Table 5.7 presents the significant variables for individual markets. The results reflect the fact that PFMCs invest mainly on markets of Central Europe (Hungarian BUX, Czech PX, polish WIG) and Western Europe (German DAX, French CAC40, British FTSE100) and American (DJIA) partly on Japan markets (NIKKEI). On the other hand, the choice of significant variables for the bond market gives us more amazing results. Very few discount rates are significant. Although month reports indicate especially investment into Slovak bonds, there are no significant SKK discount rates. We see that the information about the bond market is hidden especially in iTraxx indices. This fact is not so surprising, when we realize that the discount rates did not move so much in 2008 and credit risk was a more important factor. In Figure B.1 in Appendix B we report the percentage structure of each pension fund. Usually the portfolio consists of $10\% - 15\%$ of stocks and $70\% - 90\%$ is explained by bond variables. The rest that is missed to 100% is the information which we are not able to capture by our data.

By proceeding the principal components method we loose some information as a trade off for reducing the dimension to bivariate problem. In Table 5.8 we report the percentage of information we keep by taking only the first component. Usually we keep $65\% - 75\%$

PFMC	Stock market	Bond market
AEGON	BUX, DAX, DJIA, PX, NIKKEI	ITRAXXSEN, EUR6MZ
ALLIANZ	BUX, CAC40, DAX, NIKKEI	ITRAXX, EUR6MZ
AXA	BUX, CAC40, DAX, FTSE100, PX, NIKKEI	ITRAXX, EUR6MZ
CSOB	CAC40, DJIA, PX, NIKKEI	ITRAXX
ING	BUX, DAX, DJIA, PX, NIKKEI, WIG	ITRAXX, USD1YZ
VUB	CAC40, DAX, DJIA, PX, NIKKEI	ITRAXX

Table 5.7: Structure of markets

PFMC	AEGON	ALLIANZ	AXA	CSOB	ING	VUB
Stock variable	70.75%	72.73%	74.47%	68.25%	64.82%	71.74%
Bond variable	67.01%	68.15%	67.93%	100%	66.22%	100%

Table 5.8: Information in the first principal component

of information that we consider as sufficient for our next modeling. Note that there is only one significant variable for CSOB and VUB for the bond market, so we keep the all information about them.

Next step of modeling are models of dependence. We compare the results for copula models and multivariate GARCH models. Copula modeling confirms the hypothesis about the stronger relationship in left tails. In bad times the dependence between the bond and stock market is higher. This hypothesis verifies the results documented in Table 5.9. For five of six pension funds AIC criterion chooses Rotated Gumbel copula as the most appropriate copula. Note that Rotated Gumbel copula has non-zero left tail dependence. Furthermore, we see a strong dependence between the bond and stock market for all pension funds. The parameter α is pretty high. However, we are not able to compare it to well-known correlation, because there is no direct relationship between α and correlation. It depends on the distributions of margins that vary over time. By contrast multivariate GARCH models assume multivariate normal distributions of innovations and they do not allow another form of dependence. However, they enable correlation to vary over time. In Figure 5.4 the correlation between bond and stock component part estimated by *vec* model for all pension funds is shown. Comparing the graph to the returns of pension funds (see Figures B.2 and B.3 in Appendix B) we see that the correlation is higher in less turbulent time and low

PFMC	AEGON	ALLIANZ	AXA	CSOB	ING	VUB
Copula	Rot. Gum.	Rot. Gum.	Rot. Gum.	Rot. Gum.	Rot. Gum.	Normal
α	1.5536	1.4933	1.5993	1.6421	1.6548	0.6085

Note: Paramater α lies in $(-1, 1)$ for Normal copula and $(-1, \infty)\backslash\{0\}$ for Rotated Gumbel copula.

Table 5.9: Copula estimation

in time of high volatility. In other words, we observe that in the time of uncertainty the dependence between the stock returns and bond returns is weaker. Otherwise, estimation

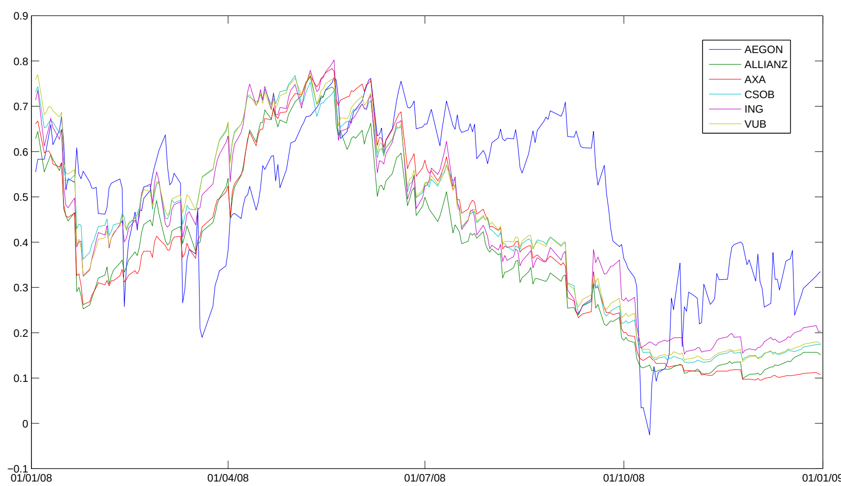

Figure 5.4: Time-varying correlation estimated by *vec* model

of MGARCH models gives us expectable results (see Tables B.1,B.2, B.3 in Appendix B). Both variances and covariances have high values of gs (for BEKK and *vec* model) or βs (for CCC model) that says about high persistence of the series. The highest values are estimated for BEKK model where the coefficients are really close to one. On the other hand the parameters as, resp. αs are pretty low. They reflect a slow tendency of the process to react on the new informations (innovations, disturbances). They are higher for BEKK model but that result might be caused by the relationship for covariance, where the coefficient by the innovation term is equal to the product $a_{11}a_{22}$. To get this coefficient not to small we need a_{11} and a_{22} be higher. Significant high coefficient g_{22} for covariance

equation from *vec* model documented, that there is also persistence in covariance.
The last and the most important part of our estimation is to determine Value at Risk. VaR for all models are reported in Figures B.2 and B.3 in Appendix B. Looking at the graphs we do not see big differences in estimating VaR. All models are able to capture non-constant volatility. Comparing the individual models we see that in the end of the year BEKK model is not able to adapt to lower volatility so fast. That is the consequence of high persistence resulted from the really high b values. Visual inspection does not give us more useful information for comparison of models. We accomplish the tests described in Chapter 4. Both Kupiec's and Christoffersen's tests give much more better results than the method of historical simulations. In Table B.4 and B.5 we report the results for both test. The results are quite satisfactory. The tests mostly verify the accuracy of models for calculating VaR. However, there is a quite high number of violations for Allianz and Axa for MGARCH models and that is why their statistics are not so optimistic. Another interesting fact to note is the performance of copula models. They perform clearly better results for calculating VaR comparing to MGARCH models. According to the results it seems to be more relevant to model the dependency of markets by the appropriate structure constant in time (by copula) than the time variant correlation (by MGARCH).

6 Conclusion

In this master thesis different bivariate models of bond and stock market were described and applied to estimate VaR for pension funds.
Before applying the models we investigated the robustness of criteria for the choice of the best copula. We compared criteria based on maximum likelihood to the criteria based on empirical distributions. According to the results by our simulations, we concluded the AIC criterion was the most appropriate.
In empirical section of this thesis, we proposed several models. The basic model based on historical simulations did not give satisfactory results. Therefore, we decided not to model the returns of pension funds by its past values, but by the returns on the stock and bond market. We accomplished copula and MGARCH models. Results from copula estimation verified the hypothesis about stronger left tail dependence between the markets. For five of six pension funds, the AIC criterion chose the Rotated Gumbel copula that has higher correlation in low quantiles. It implies the higher dependence of stocks and bonds in times of crisis.
Results of MGARCH models documented the high persistence in both covariance and variance of the markets. However, they are quite conservative in VaR calculations. There are many violations of VaR. The comparison of VaR based on copula and on multivariate GARCH models shows that the copula model is often more accurate. This is true whether the criterion is Kupiec's test or Christoffersen's test.
We can conclude that both copula and MGARCH models give a good estimation of VaR, much more better than the basic approach of historical simulations. Note that our estimation would be more precise if we had details on asset allocation of each portfolio. Such allocation is known by the funds managers and can effectively increase the performance of the models suggested in the thesis.

Bibliography

[1] Canela, Miguel A. and Collazo, Eduardo P., 2006, Modeling Dependence in Latin American Markets using Copula Functions, Working paper, Universitat de Barcelona.

[2] Cherubini, U., E. Luciano and W. Vecchiato, 2004, *Copula Methods in Finance*, John Wiley & Sons, England.

[3] Chollete, L., V. de la Pena and C.-C. Lu, 2005, Comovement of International Financial Markets, mimeo, Norwegian School of Economics and Business.

[4] Diebold, Francis X. and Nerlove, Marc, 1989, The Dynamics of Exchange Rate Volatility: A Multivariate Latent Factor ARCH Model, Journal of Applied Econometrics, **4**, pp. 1-21.

[5] Engle, Robert F. and Kroner, Kenneth F., 1995, Multivariate Simultaneous Generalized ARCH, Econometric Theory, **11**, pp. 122-150.

[6] Engle, Robert F., 2002, Dynamic Conditional Correlation: A Simple Class of Multivariate Generalized Autoregressive Conditional Heteroscedasticity Models, Journal of Business and Economic Statistics, **20**, pp. 339-350.

[7] Frees, Edward W. and E. A. Valdez, Understanding Relationships Using Copulas, Working paper, University of Wisconsin-Madison, Madison.

[8] Goorbergh, Rob, A Copula-Based Autoregressive Conditional Dependence Model of International Stock Markets, Working Paper, De Nederlandsche Bank.

[9] Harrell, Frank E., Jr., 2001, *Regression Modeling Strategies*, Springer, New York.

[10] Longin, F., and B. Solnik, 2001, Extreme Correlation of International Equity Markets, Journal of Finance, **56**, pp. 649–76.

[11] Lütkepohl, Helmut, 2005, *New Introduction to Multiple Time Series Analysis*, Springer, Berlin.

[12] National Bank of Slovakia, 2008, Analýzy slovenského finančného sektora, Bratislava.

[13] Nelsen, Roger B., 1998, *An Introduction to Copulas*, Springer, New York.

[14] Patton, Andrew J., 2003, Modeling Asymmetric Exchange Rate Dependence, Working paper, University of California, San Diego.

[15] Rank, Jörn, 2007, *Copulas: From Theory to Application in Finance*, Risk Books, UK.

[16] Sun W., S. Rachev, F. Fabozzi and Kalev, P., 2009, A new approach to modeling co-movement of international equity markets: evidence of unconditional copula-based simulation of tail dependence, Empirical Economics, **36**, pp. 201-229.

A Data

BUX	stock index of large companies trading on the Budapest Stock Exchange
CAC40	stock index of the 40 most significant values among the 100 highest market caps on the Paris Bourse
DAX	stock index of the 30 major German companies trading on the Frankfurt Stock Exchange
DJESTOXX50	Dow Jones EURO STOXX 50, stock index of Eurozone stocks designed by STOXX Ltd
DJIA	Dow Jones Industrial Average, stock index of 30 largest and most widely held public companies in the United States
FTSE100	stock index of the 100 most highly capitalized UK companies listed on the London Stock Exchange
NASDAQ	stock index of the largest companies listed on the NASDAQ stock exchange
NIKKEI	stock index of large companies on the Tokyo Stock Exchange
PX	stock index of large companies trading on the Prague Stock Exchange
SAP500	stock index of 500 large cap common stocks actively traded in the United States
WIG	stock index of large companies on the Warsaw Stock Exchange
EUR1YZ	one year EUR discount rate
EUR2YZ	two year EUR discount rate
EUR3MZ	three month EUR discount rate
EUR6MZ	six month EUR discount rate
EURTNZ	overnight EUR discount rate
SKK1YZ	one year SKK discount rate
SKK2YZ	two year SKK discount rate
SKK3MZ	three month SKK discount rate
SKK6MZ	six month SKK discount rate
SKKTNZ	overnight SKK discount rate
USD1YZ	one year USD discount rate
USD2YZ	two year USD discount rate
USD3MZ	three month USD discount rate
USD6MZ	six month USD discount rate
USDTNZ	overnight USD discount rate
ITRAXX	credit default swap index for the most traded companies in Europe
ITRAXXSEN	credit default swap index for the senior financial companies in Europe
RF AEGON	pension unit for the progress fund, AEGON d.s.s.
RF ALLIANZ	pension unit for the progress fund, ALLIANZ - Slovenská d.s.s.
RF AXA	pension unit for the progress fund, AXA d.s.s.
RF CSOB	pension unit for the progress fund, ČSOB d.s.s.
RF ING	pension unit for the progress fund, ING d.s.s.
RF VUB	pension unit for the progress fund, VÚB Generali d.s.s.

Table A.1: List of Symbols

Stock	BUX	DAX	DJIA	CAC40	NASDAQ	PX	NIKKEI	SAP500	FTSE100	DJESTO	WIG
Minimum	-0.1281	-0.0723	-0.0787	-0.0910	-0.1052	-0.1494	-0.1140	-0.0903	-0.1004	-0.0790	-0.1339
Mean	-0.0024	-0.0018	-0.0013	-0.0018	-0.0017	-0.0025	-0.0017	-0.0015	-0.0011	-0.0019	-0.0015
Median	-0.0020	-0.0015	-0.0006	-0.0000	-0.0016	-0.0019	0.0000	-0.0003	-0.0003	-0.0020	-0.0030
Maximum	0.1629	0.1141	0.1108	0.1447	0.1259	0.1173	0.1416	0.1154	0.1103	0.1253	0.1553
Std. Dev.	0.0324	0.0241	0.0238	0.0272	0.0267	0.0301	0.0289	0.0251	0.0265	0.0263	0.0457
Skewness	0.3587	0.7673	0.4505	0.6750	0.3186	-0.2651	-0.0182	0.1035	0.3761	0.6994	0.1953
Kurtosis	8.9944	8.6797	7.2016	8.1453	6.6951	8.3126	7.1511	7.9573	6.8183	7.1802	4.7644
Jarque-Bera	378.14	359.13	191.58	293.58	145.87	295.74	178.79	255.41	157.13	201.60	33.88
Probability	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Observ.	249	249	249	249	249	249	249	249	249	249	249

Table A.2: Descriptive statistics for stocks

PFMC	AEGON	ALLIANZ	AXA	CSOB	ING	VUB
Minimum	-0.0173	-0.0114	-0.0113	-0.0159	-0.0117	-0.0123
Mean	-0.0004	-0.0002	-0.0002	-0.0003	-0.0002	-0.0003
Median	-0.0002	0.0000	-0.0001	-0.0001	0.0000	-0.0001
Maximum	0.0230	0.0093	0.0094	0.0090	0.0079	0.0079
Std. Dev.	0.0044	0.0028	0.0029	0.0027	0.0027	0.0026
Skewness	0.0767	-0.2247	-0.2616	-1.1621	-0.6177	-0.3997
Kurtosis	9.4370	4.9469	5.0052	8.9107	5.9489	5.1210
Jarque-Bera	430.14	41.42	44.55	418.52	106.06	53.30
Probability	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Observations	249	249	249	249	249	249

Table A.3: Descriptive statistics for pension units

	EUR1YZ	EUR2YZ	EUR3MZ	EUR6MZ	EURTNZ	SKK1YZ	SKK2YZ	SKK3MZ	SKK6MZ
Minimum	-0.0068	-0.0037	-0.0056	-0.0105	-0.0090	-0.0082	-0.0020	-0.0053	-0.0057
Mean	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
Median	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000
Maximum	0.0022	0.0022	0.0033	0.0027	0.0087	0.0065	0.0021	0.0066	0.0062
Std. Dev.	0.0008	0.0008	0.0007	0.0009	0.0019	0.0016	0.0008	0.0014	0.0016
Skewness	-2.6425	-0.5524	-1.4982	-7.2816	-0.0477	-0.4653	0.0065	0.0690	0.1211
Kurtosis	26.0735	5.7060	23.0943	94.3716	9.2356	7.1314	3.2672	6.8950	5.5811
Jarque-Bera	5813.31	88.63	4282.37	88818.83	403.50	186.07	0.74	157.60	69.73
Probability	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.6900	0.0000	0.0000
Observations	249	249	249	249	249	249	249	249	249

	SKKTNZ	USD1YZ	USD2YZ	USD3MZ	USD6MZ	USDTNZ	ITRAXX	ITRAXXSEN
Minimum	-0.0179	-0.0121	-0.0041	-0.0183	-0.0150	-0.0199	-0.0055	-0.0042
Mean	0.0001	0.0001	0.0001	0.0001	0.0001	0.0002	0.0000	0.0000
Median	0.0001	0.0001	0.0001	0.0001	0.0001	0.0000	-0.0001	-0.0001
Maximum	0.0182	0.0122	0.0035	0.0092	0.0150	0.0381	0.0088	0.0032
Std. Dev.	0.0039	0.0018	0.0011	0.0024	0.0021	0.0044	0.0009	0.0007
Skewness	-0.1419	-0.2533	-0.1530	-1.6838	-0.4994	2.5076	2.6536	-0.0717
Kurtosis	7.8774	30.5764	4.3866	19.2370	32.2951	29.3982	36.1635	10.5639
Jarque-Bera	247.65	7892.38	20.92	2852.94	8914.21	7490.94	11702.83	593.79
Probability	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Observations	249	249	249	249	249	249	249	249

Table A.4: Descriptive statistics for discount rates and iTraxx indices

B Models

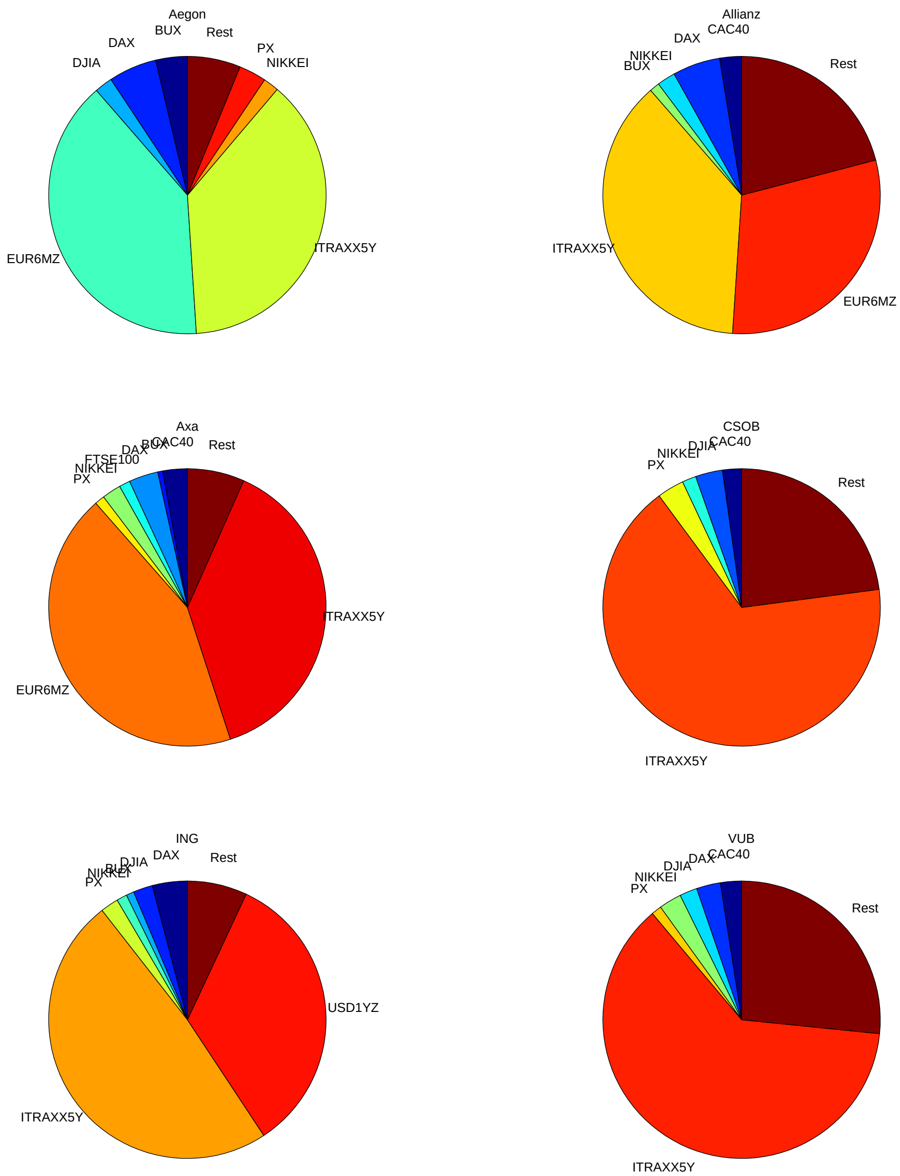

Figure B.1: Portfolio structure

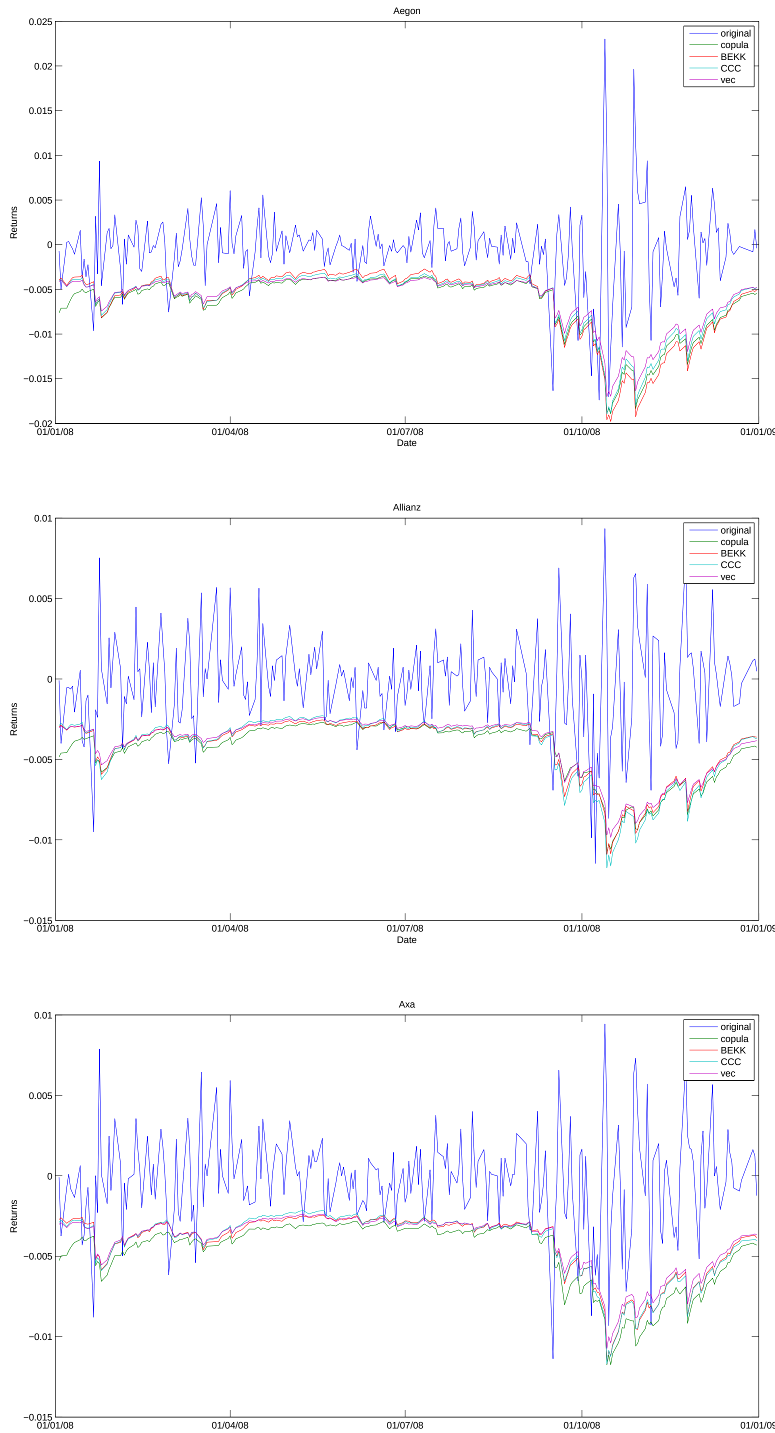

Figure B.2: VaR for pension funds(1)

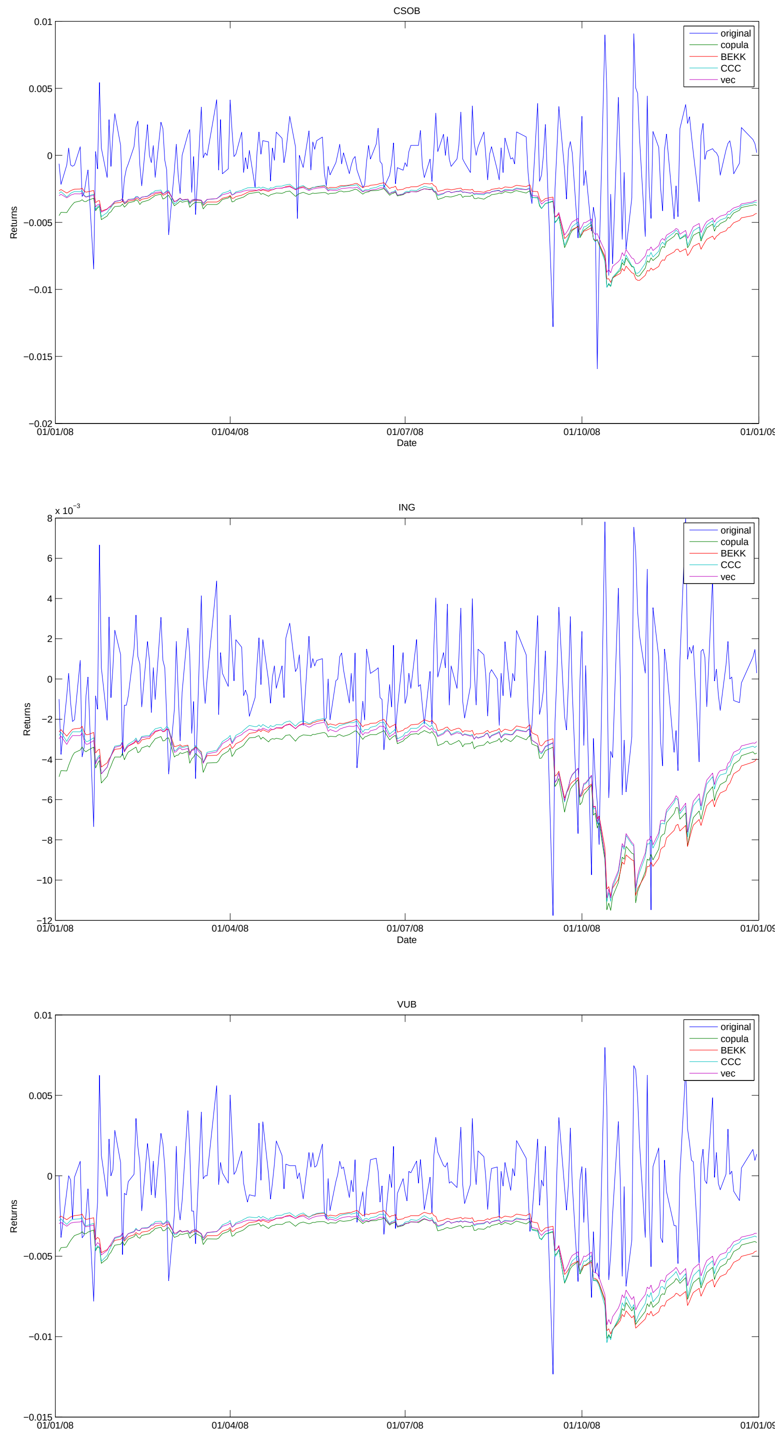

Figure B.3: VaR for pension funds(2)

PFMC	AEGON	ALLIANZ	AXA	CSOB	ING	VUB
α_{0stock}	0.000	0.000	0.000	0.000	0.000	0.000
	(0.105)	(0.073)	(0.089)	(0.109)	(0.119)	(0.071)
α_{1stock}	0.174	0.153	0.158	0.139	0.157	0.140
	(0.001)	(0.002)	(0.002)	(0.003)	(0.005)	(0.003)
β_{1stock}	0.817	0.831	0.825	0.836	0.823	0.836
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
α_{0bond}	0.000	0.000	0.000	0.000	0.000	0.000
	(0.011)	(0.189)	(0.928)	(0.144)	(0.049)	(0.130)
α_{1bond}	0.131	0.012	0.005	0.012	0.012	0.012
	(0.005)	(0.000)	(0.046)	(0.000)	(0.000)	(0.000)
β_{1bond}	0.833	0.930	0.915	0.930	0.933	0.930
	0.005	(0.000)	(0.046)	(0.000)	(0.000)	(0.000)
ρ	0.527	0.438	0.457	0.487	0.482	0.497
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Log lik.	2050.8	1984.5	1935.3	2001.8	1981.1	1963.4

Note: P-values are reported in brackets below the coefficients' estimates.

Table B.1: CCC model

PFMC	AEGON	ALLIANZ	AXA	CSOB	ING	VUB
ω_{11}	0.002	0.002	0.003	0.002	0.002	0.002
	(0.122)	(0.048)	(0.040)	(0.056)	(0.070)	(0.029)
ω_{22}	0.000	0.000	0.000	0.000	0.000	0.000
	(0.606)	(0.641)	(0.741)	(0.838)	(0.752)	(0.598)
ω_{12}	0.000	0.000	0.000	0.000	0.000	0.000
	(0.008)	(0.247)	(0.175)	(0.154)	(0.257)	(0.323)
a_{11}	0.429	0.350	0.362	0.317	0.348	0.317
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
a_{22}	0.338	0.106	0.120	0.123	0.124	0.119
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
b_{11}	0.921	0.945	0.942	0.954	0.944	0.954
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
b_{22}	0.917	0.999	0.997	0.997	0.997	0.998
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Log lik.	2052.6	1986.2	1938.7	2006.4	1983.5	1967.9

Note: P-values are reported in brackets below the coefficients' estimates.

Table B.2: BEKK model

PFMC	AEGON	ALLIANZ	AXA	CSOB	ING	VUB
c_1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	(0.1184)	(0.0326)	(0.0475)	(0.1180)	(0.0803)	(0.0449)
a_{11}	0.1622	0.1554	0.1616	0.1439	0.1622	0.1452
	(0.0008)	(0.0003)	(0.0002)	(0.0003)	(0.0008)	(0.0003)
g_{11}	0.8272	0.8337	0.8279	0.8394	0.8272	0.8396
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
c_2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
a_{22}	0.1718	0.1201	0.1291	0.1263	0.1249	0.1254
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
g_{22}	0.8494	0.9023	0.8980	0.8986	0.8981	0.8994
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
c_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	(0.0253)	(0.0433)	(0.0187)	(0.0000)	(0.0685)	(0.0000)
a_{33}	0.1116	0.0910	0.0893	0.0109	0.0931	0.0109
	(0.0182)	(0.0002)	(0.0015)	(0.0001)	(0.0023)	(0.0001)
g_{33}	0.8202	0.9241	0.9256	0.9399	0.9112	0.9399
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
Log lik.	2068.825	1996.810	2051.294	2012.493	1987.355	1971.920

Note: P-values are reported in brackets below the coefficients' estimates.

Table B.3: VEC model

PFMC	AEGON	ALLIANZ	AXA	CSOB	ING	VUB
Copula	**0.6674**	**0.4719**	**0.8953**	**0.6674**	**0.3219**	**0.8738**
BEKK	0.2089	0.0428	0.2089	0.4719	0.0428	0.1292
VEC	0.3219	0.0428	0.0761	0.6583	**0.3219**	0.2089
CCC	**0.6674**	0.0428	0.0428	0.6583	0.0761	0.4719

Table B.4: P-values from Kupiec's test

PFMC	AEGON	ALLIANZ	AXA	CSOB	ING	VUB
Copula	**0.7233**	**0.4331**	**0.5384**	0.1948	0.4808	**0.9174**
BEKK	0.4477	0.1220	0.4477	0.1235	0.1099	0.3023
VEC	0.4092	0.1220	0.1220	**0.4220**	**0.6118**	0.4477
CCC	0.5471	0.1220	0.1890	**0.4220**	0.1890	0.7679

Table B.5: P-values from Christoffersen's test