Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics
Department of Applied Mathematics and Statistics

Mathematical analysis of a class of path-dependent options

Master's thesis

Martin Takáč

Applied Mathematics
Economic and Financial Mathematics

Supervisor:
doc. RNDr. Daniel Ševčovič, CSc.

BRATISLAVA 2010

Univerzita Komenséko v Bratislave
Fakulta Matematiky, Fyziky a Informatiky
Katedra aplikovanej matematiky a štatistiky

Matematická analýza dráhovo závislých opcií

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Abstract

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In our work we investigate Asian options. In the first part, we explore statistical properties of a time integral of the geometric Brownian motion. We approximate this integral by a lognormal distributed random variable. Then, using a suitable copula function, we price the average strike Asian option. In second part, we focus on the early exercise boundary problem for American-style Asian options. We generalize algorithm based on transformation methods to the case of an exponentially weighted arithmetic averaged Asian option and to geometric averaged Asian option.

Keywords: option pricing • American-style of Asian options • Asian options • copula • appropriation formula • exponentially weighted average • numerical valuation of the free boundary• early exercise boundary.

Abstrakt

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V našej práci sa zaoberáme Ázijskými opciami. V prvej časte skúmame štatistické vlastnosti integrálu eponenciálneho Brawnoho pohybu. Tento integrál aproximujeme log-normálym rozdelením. Použitím vhodnej copula funkcie oceňíme tzv. „average strike" Ázijskú opciu.

V druhej časti sa zameriame na problém predčasného uplatnenia Ázijskej opcie. Zovšeobecníme algoritmus, založený na transformačnej metóde, na exponenciálne vážené a geometricky vážené Ázijské opcie.

Kľúčové slová: oceňovanie opcií • Ázijské opcie s možnosťou predčasného uplatnenia • Ázijské opcie • copula • aproximatívna formula • exponenciálne vážený priemer • numerický výpočet voľnej hranice • hranica predčasného uplatnenia.

Acknowledgement/Acknowledgment

I would like to express special thanks to my supervisor doc. RNDr. Daniel Ševčovič, CSc. for all of the support and guidance he offered throughout the elaboration of this thesis.

Declaration on Word of Honour

I declare on my honour that this work is based only on my own knowledge, references and consultation with my supervisor(s).

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Martin Takáč

Contents

Introduction

Financial derivatives are financial instruments that are linked to a specific financial instrument or indicator or commodity, and which provide for market financial risk in a form that can be traded or otherwise offset in the market. Financial derivatives are used for a number of purposes including risk management, hedging, and speculation. The value of the financial derivative derives from the price of the underlying items (cf. Trewin [24]).

Historically first derivative security contracts were related to agricultural contracts. In 1973, the first U.S. options exchange has been established (Chicago Board Option Exchange). In last decades, there has been huge expansion in volume of trated financial derivatives. Between the most traded derivatives belong interest rates derivatives and options. Recently, we can observe increasing demand on exotic options. Because of non-existence of explicit formula for pricing them, one can use one from more methods. Some way how to calculate option's price is based on solve parabolic PDE another on binary trees. One of the disadvantages of these methods is the time needed to obtain result (in case of binary trees algorithm, this time can be more then thousand seconds (cf. Dai [4]) and therefore there arise some approximate formulas.

The goal of this work is derive new approximate formula for case of exponentially weighted Asian option. We also generalize numerical algorithm derived by Ševčovič & Bokes [2] for American-style Asian options.

The thesis is organized as follows. In the first chapter, we focus on basic assumptions for option pricing methodology. We discuss the basic types of options. In the second chapter we approximate time integral of geometric Brownian motion with lognormal distributed random variable and derive approximate formula. In the first part of third chapter, we introduce a new modification average, which ends up with full parabolic PDE. After dimension reduction this PDE at time close to expiry, this PDE has the same form as PDE for plain vanilla option. Notice, that this PDE is more suitable for numerical algorithm. In second part we use copula to derive an approximate formula for pricing average strike Asian options. In fourth chapter we focus on transformation method for American-style of Asian options.

Chapter 1

Financial derivatives

> The *most* we can know is in terms of probabilities.
>
> Richard Feynman

In the last decades, there was huge increase in trading financial derivatives securities in financial market. **Derivative security** is security whose value depends on the values of other more basic underlying variables, which may be the prices of traded securities, prices of commodities or stock indices, foreign currency (cf. Kwok [10]). Time evolution of stock prices of Microsoft corp. can be seen at in Figure 1.1.

There are three the most common derivatives: futures, options and swaps. Forward contract is an agreement between two parties to buy or sell an asset at certain time in the future for a predetermined price. Forward is also called as futures, if it is traded on exchanges. While in case of forward, buy or sell an asset have to be executed, but in case of option, the holder has right (but not obligation) to buy or sell an asset by a certain date for a predetermined price. Interest rate swap is an agreement between two parties to exchange interest payments for a predetermined period of time (see Kwok [10] page 351).

1.1 Options

Definition 1. *Option is a right (but not obligation) to buy or sell an asset by a certain date (called expiration date) for a predetermined price (called strike price or exercise price). The expiration date we will mark as T and strike price as X. If option can be exercised before expiration date, we say, that option has American property (or is American-style option) otherwise it is European-style option.*

Call option is an contract, which gives right to holder to *buy* an underlying asset by a certain date for strike price.

Figure 1.1: Time evolution of Microsoft corp. stock prices in 2005 – 2009 and it's trading volume. Source: www.google.com/finance.

Put option is an contract, which gives right to holder to *sell* an underlying asset by a certain date for strike price.

Simple call and put options with no special features are commonly called *plain vanilla options*. Other ones are called *exotic options*.

Payoff function for vanilla options If we hold an call option with strike price X and at expiration time T is value of the underlying asset S_T, then can occur only 2 possibilities:

1. $X > S_T$,
2. $X \leq S_T$.

In first possibility the call option gives us to right to but that asset for X USD, but we can buy it on the market for S_T, which is less. So we do not use our right and therefore our payoff will be 0 USD. But in second case, option gives us right to buy underlying assets for lower price that in market, so we can use our right and buy underlying asset for X USD and then sell it on market for S_T USD. So our payoff will be $S_T - X$ USD. Putting both cases into one equation, our payoff is $\max\{S_T - X, 0\}$. In case of put option, the idea is similar and payoff is given by $\max\{X - S_T, 0\}$.

Options trader can take different market positions, namely:

- buy call option (long call),
- sell call option (short call),

- buy put option (long put),
- sell put option (short put).

Assume for a while, that we are a writer of European call option. Let X is a strike price, S is a price of underlying asset, r is a risk less interest rate and expiration date is T years. Question is, for how much (V) we should sell that option. This price is called *option premium* or *option value*.

If we assume, that underlying asset (for example stock) follows Geometric Brownian motion

$$S_t = S_0 \exp(\mu t + \sigma W_t),$$

(where W_t is Wiener stochastic process)

and by assuming following conditions (cf. Kwok [10] page 33):

- no risk less arbitrage opportunities,
- trading takes place continuously in time,
- the risk less interest rate r is known and constant over time,
- there are no transaction costs in buying or selling the asset or the option and no taxes,
- the asset pays no dividend,
- the assets are perfectly divisible,
- there are no penalties to short selling and the full use of proceeds is permitted,

then by constructing risk less portfolio from options, assets and bonds we get following partial differential equation (PDE)

$$\frac{\partial V}{\partial \tau} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} + rV = 0 \tag{1.1}$$

which holds for option value (for details see e.g. Kwok [10], Ševčovič [20], Melicherčík[13]).

For initial condition holds:

$$V(S, 0) = \max\{S - X, 0\}. \tag{1.2}$$

Solution of (1.1) with initial condition (1.2)is

$$V(S, \tau) = SN(d_1) - Xe^{-r\tau}N(d_2), \tag{1.3}$$

where $\tau = T - t$ and

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt$$

is cumulative distribution function of normal random variable with $\mu = 0$ and $\sigma = 1$,

$$d_1 = \frac{\ln \frac{S}{X} + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \tag{1.4}$$

and

$$d_2 = d_1 - \sigma\sqrt{\tau} = \frac{\ln \frac{S}{X} + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}. \tag{1.5}$$

1.1.1 Options' types

There has been a lot of criteria, in which can be options divided. We only point out to most commons.

Types according to underlying assets:

- options on stocks,
- options on stock indices,
- options on options,
- options on foreign currency,
- options on interest rates.

Another type of criteria is if the option can be exercised only at expiration date or at only time before expiration date. Therefore we have following types:

- **American options** is an option which can be exercised at any time before expiration date,
- **European options** can be exercised only at expiration date.

If payoff depends not only on price of assets at expiration date, but depends also about value of underlying assets in the past, we say, that it is Path Dependent option. The most popular Path Dependent options are:

- Barrier options,
- Look-back options,
- Asian options.

1.1.2 Barrier options

Barrier options are considered as the simplest types of path-depend options. The payoff depends not only on the final price of the underlying asset but also on whether or not the underlying asset price has reached some barrier level B during the life of the option (cf. Kwok [10] page 246). Barrier option introduced into financial market around 1967 (cf. Ševčovič [23]). **Out** barrier options are active at the beginning. But when they reach prescribed barrier, they become null and void. This barrier is so-called **knocked-out** barrier. Another type of barrier options is so-called **in** options, which are void at the beginning and they become active if the underlying asset price attain **knock-in** barrier.
Case of out barrier options: If the price of underlying asset reached barrier then the holder of option gets some rabat $R \geq 0$. There are two possibilities: barrier is reached up front (down-and-out option) or barrier is reached from bottom (up-and-out option). In Figure 1.2 there is an example for down-and-out option. At the beginning, the price of asset is above barrier. If the asset price touch the barrier (blue line), then the option become nullified (at time $t = 0.48$). In case of green line, the option is valid until maturity.

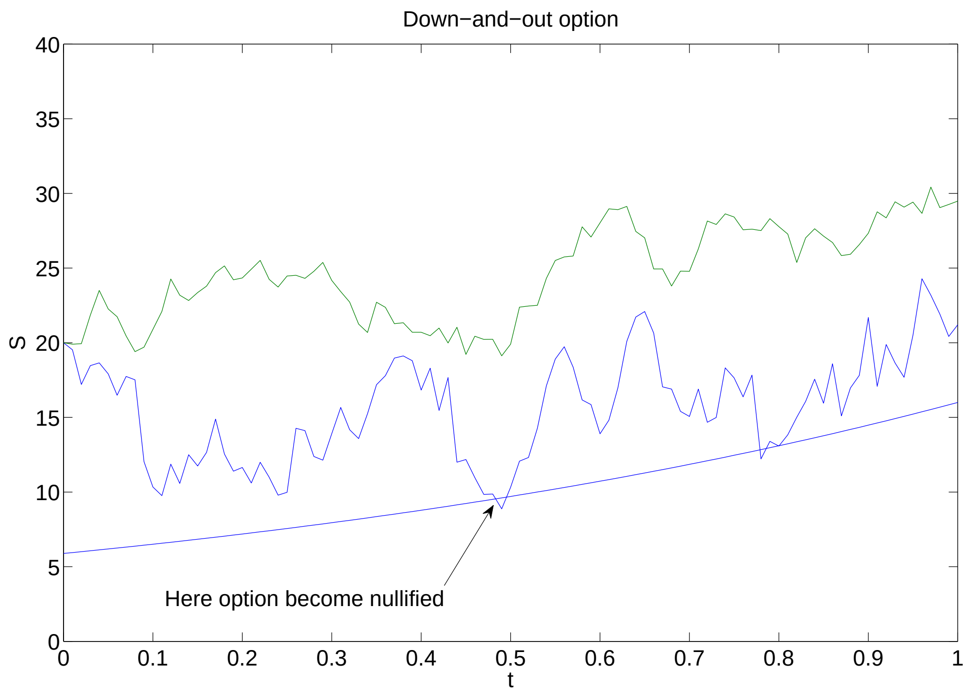

Figure 1.2: An example of exponential barrier function $B(t) = 0.8Ee^{-(T-t)}$, $E = 20$. In the case of blue line, option become nullified, whereas in case of green line, option is active until maturity.

1.1.3 Look-back options

As it is mentioned above, Look-back options is path-depend options, which payoff dependence on underlying asset price and on maximum (minimum) asset price in case of floating maximum (minimum) Look-back option. Let us only remark (cf. Ševčovič [23]) that there is only these four types of Look-back options:

1. floating maximum strike put options with payoff $V(S, M, T) = (M - S)^+$,
2. floating maximum rate call options with payoff $V(S, M, T) = (M - X)^+$,
3. floating minimum strike call options with payoff $V(S, M, T) = (S - m)^+$,
4. floating minimum strike put options with payoff $V(S, M, T) = (X - m)^+$,

where X is strike price, $M = \max_{t \in \langle 0,T \rangle} S_t$ and $m = \min_{t \in \langle 0,T \rangle} S_t$. Notice that for example of floating maximum strike call options, the payoff equals to zero, because $\forall t \in \langle 0, T \rangle : S_t \leq M_t$.

1.2 Asian options

Asian options are financial derivatives which payoff depends not only on underlying asset spot price but also on the average of these prices. These options are usually on commodities such as oil, grain, etc. Big advantage of these options is protection from speculation from big investors, which can change price of underlying asset at time close to expiry. In other words, the payoff is less sensitive at expiration on underlying spot value. In figure 1.3 is an example of development of S_t and A_t. Payoff in case of Asian call option is > 0, but in case of European call option is $= 0$. According to way how we compute average we distinguish this types of Asian options:

- arithmetically averaged options, where $A_t = \frac{1}{t} \int_0^t S_\xi \, d\xi$,
- weighted arithmetically averaged options, where $A_t = \frac{1}{\int_0^t a(\xi) d\xi} \int_0^t a(t - \xi) S_\xi \, d\xi$,
- geometric averaged options, where $\ln A_t = \frac{1}{t} \int_0^t \ln S_\xi \, d\xi$.

An example of weighted function (in case of weighted arithmetically averaged options) is exponential weight function:

$$a(\xi) = \exp(-\lambda \xi). \tag{1.6}$$

According to the way how the averaged asset price enters to the payoff diagram we can distinguish this two types:

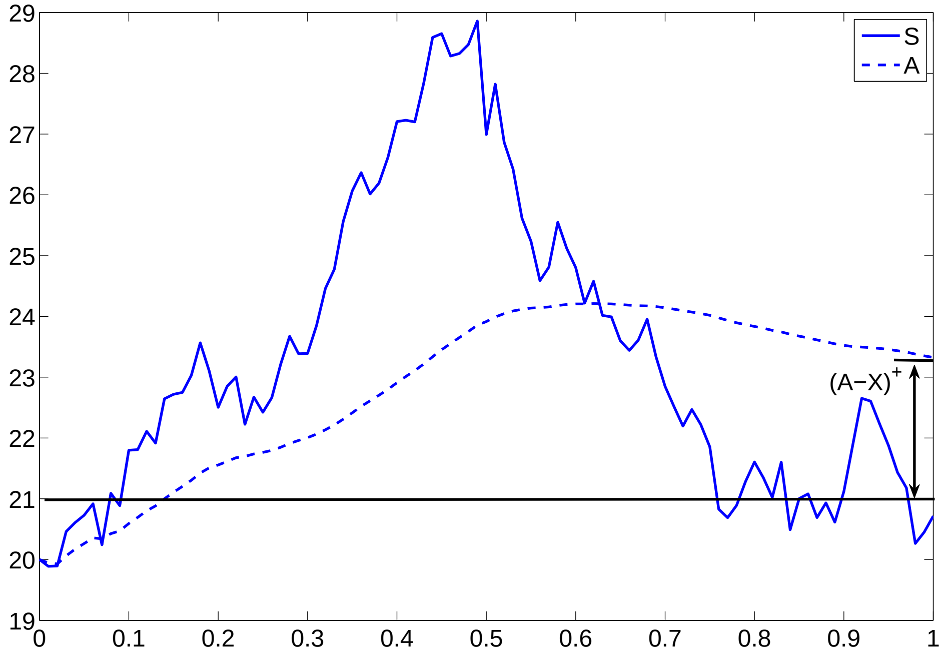

Figure 1.3: Development of price of underlying asset and corresponding arithmetic average.

- average rate call ($V(S, A, T) = \max\{0, A - X\}$), respectively put ($V(S, A, T) = \max\{0, X - A\}$),
- average strike call ($V(S, A, T) = \max\{0, S - A\}$), respectively put ($V(S, A, T) = \max\{0, A - S\}$).

In Figure 1.4 are two different development of prices of underlying asset (solid lines) and corresponding different averages (dashed lines).

Notice, that in case of geometric averaged Asian options, there exists an explicit solution for pricing this derivative (for further information see Kwok [10]).

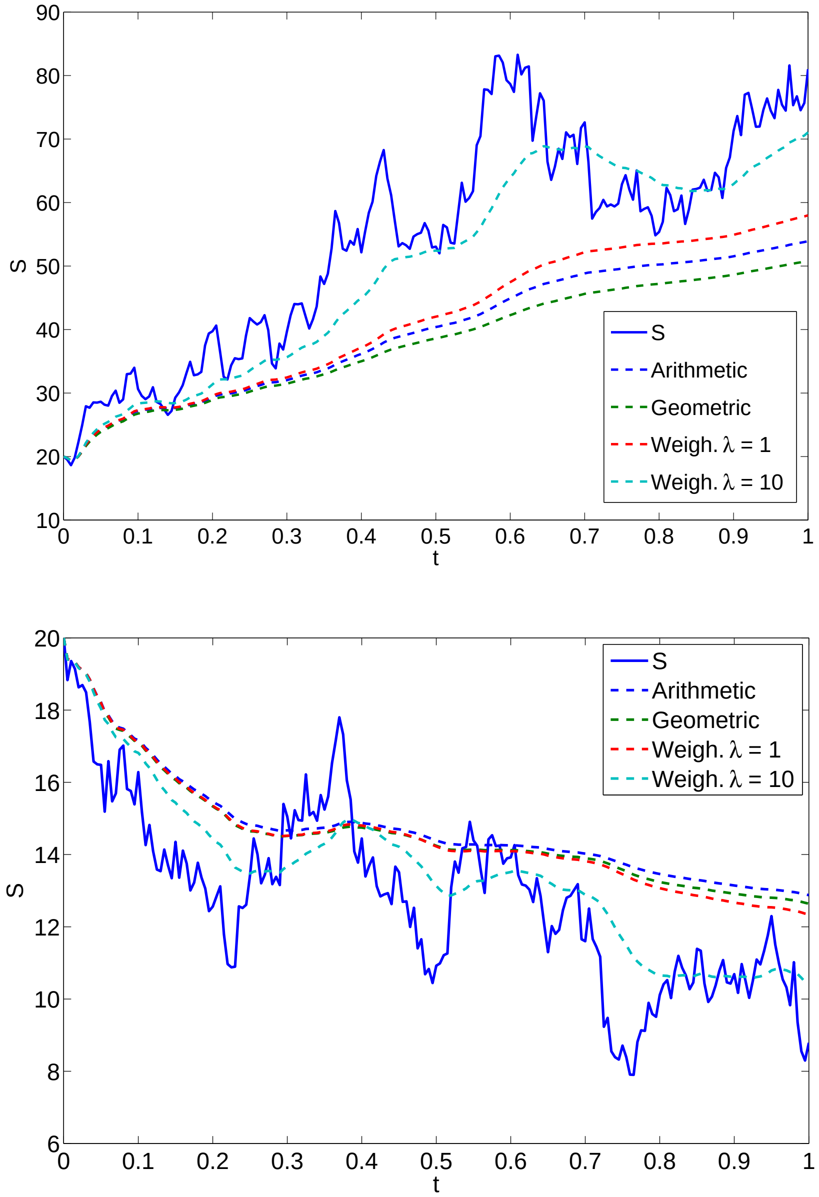

Figure 1.4: Example of development of prices of underlying asset (solid lines) and corresponding different averages (dashed lines).

Chapter 2

European-style average rate Asian options

> We use the intuition that it is easier to approximate a probability distribution than it is to approximate an arbitraty nonlinear function or transformation.
>
> Simon Julier, Jeffrey Uhlmann and Hugh Durrant-Whyte

In recent times, there is no explicit formula to calculate price of arithmetically averaged Asian option (cf. Kwok [10]). The only possibility is to solve this problem using numerical methods or using some approximate solutions. Among basic methods belongs:

- **numerical methods** - Monte Carlo simulation, solve PDE by method of finite differences, etc.,
- **analytical approach** - is usually based on appropriation of distribution of random variable A_T or $\frac{A_T}{S_T}$ and then derivation of explicit formula for pricing options,
- **estimation of lower and upper bounds for option price**.

For further information see e.g. Zhang [26].

In this chapter we derive first two moments of random variable A_T in case of arithmetic averaging and in case of weighted arithmetic averaging. Using a method of moments we estimate parameters of lognormal random variable and then we derive explicit approximated formula for pricing Asian average rate options. Let us remark, that in case of non weighted averaging is first two moments identical with those derived by Posner & Milevsky [14].

2.1 Idea of derivation

It is well known (see e.g. Melicherčík [13], Dai [4]), that price of option can be calculated as following:

$$V(S, A, 0) = e^{-rT}\mathbb{E}_Q[(A_T - X)^+], \tag{2.1}$$

where $(\xi)^+ = \max\{0, \xi\}$ and Q is technical, risk less probability measure (it's existence is guaranteed by Girsanov theorem) and $A_T = \dfrac{1}{T}\displaystyle\int_0^T S_\tau \, d\tau$.

Theorem 2 (Girsanov). *Let W_t be a Wiener process on the Wiener probability space $(\Omega, \mathcal{F}, P)$. Let $\gamma_t(\omega)$ be a measurable process adapted to the natural filtration of the Wiener process $\mathcal{F}_t^W$ and*

$$\mathbb{E}_P\left[\exp\left(\frac{1}{2}\int_0^T \gamma_t^2 dt\right)\right] < \infty.$$

Then there exists a probability measure Q on $(\Omega, \mathcal{F})$, that

- *$Q \sim P$ (probability measures Q and P are equivalent),*
- $\dfrac{dQ}{dP}(\omega) = \exp\left(-\displaystyle\int_0^T \gamma_t(\omega)dW_t(\omega) - \frac{1}{2}\int_0^T \gamma_t^2(\omega)dt\right),$
- *$\tilde{W}_t(\omega) = W_t(\omega) + \int_0^t \gamma_s(\omega)ds$ is Brownian motion on $(\Omega, \mathcal{F}, Q)$.*

For dS on probability measure P holds

$$dS_t = S_t\sigma dW_t + S_t\mu dt + \frac{1}{2}\sigma^2 S_t dt.$$

According Melicherčík [13] for risk less measure Q, it can be derived, that process $Z_t = e^{-rt}S_t$ have to be a $\mathcal{F}_t^W$-martingale (see Appendix for details). Therefore this process has to have zero drift. On risk less probability measurable Q it holds for process S_t that:

$$S_t = S_0 \exp\left(\sigma\tilde{W}_t + rt - \frac{1}{2}\sigma^2 t\right), \tag{2.2}$$

where $\tilde{W}_t$ is Wiener process on $(\Omega, \mathcal{F}, Q)$.

Apply Itō's lemma on S_t it holds

$$dS_t = Srdt + S\sigma d\tilde{W}_t. \tag{2.3}$$

We will denote W_t instead of $\tilde{W}_t$, because till now, we will use only risk less probability measure Q.

2.1.1 Binary tree

We assume, that price of underlying asset follows geometric Brownian motion

$$S(t + dt) = S(t) \cdot \exp\left[\left(r - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t\right], \tag{2.4}$$

where W_t is Wiener process, r is risk less interest rate and σ is volatility. Binary tree methodology assumes that current price S can change on next period either up to $S \cdot u$ ($u > 1$) with probability p or down to $S \cdot d$ ($d < 0$) with probability $1 - p$. In Figure 2.1 we can see two-step binary tree. Generally, for n-step binary tree, one

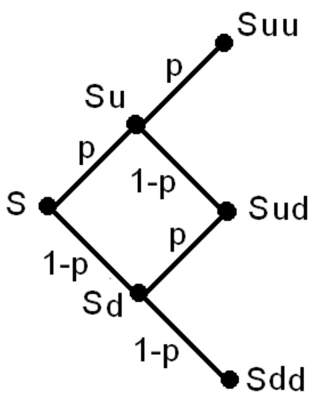

Figure 2.1: Example of two-step binary tree.

step takes time $\Delta t = \frac{T}{n}$. Then for u, d and p hold (cf. [4]):

$$p = \frac{e^{r\Delta t} - d}{u - d}, \tag{2.5}$$

$$u = e^{\sigma\sqrt{\Delta t}}, \tag{2.6}$$

$$d = \frac{1}{u}. \tag{2.7}$$

Notice that this binary tree describe evolution of asset price on risk less probability measurable Q. Our next goal is calculate $V(S, A, 0)$. To do so, we need to know distribution of A_t and then by use formula (2.1) we obtain desired result.

It is obvious, that S_t has lognormal distribution, but unfortunately sum of lognormal variables is not lognormal random variable. But if volatility of this sum is small (< 0.4) we can approximate this sum by lognormal random variable.

2.2 Calculation of moments of A_T

To estimate coefficients of A_T we can use different methods. For example maximum likelihood parameter estimation method or method of moments. We decide to use method of moments, because in our case we have no observation of data and we are able to calculate exact moments of A_T. Considering lognormal distribution (which has 2 parameters) of A_T we have to compute two moments of this random variable.

Problem of computation it's first two moments has been solved only for arithmetically averaged options for example in papers by Posner & Milevsky [14, 16, 17], but we derive them by another way and we also derive this moments for case of exponentially weighted average Asian options. At first, we use discretization of continuous process to n parts. Then pushing $n \to \infty$ we get solution. This method seems to be easier to calculate higher moments.

2.2.1 Arithmetic average

In this section we derive first two moments for case of arithmetic average.

Theorem 3. *For* $\mathbb{E}[A_T]$ *holds*

$$\mathbb{E}[A_T] = S_0 \frac{\exp(rT) - 1}{rT}. \tag{2.8}$$

Proof: At first we use discretization of continuous process S_t by discrete one and pushing limits of discretize step to zero, we proof lemma.

Let $\xi_j, j = 1, \ldots, n$ are identical independent alternative discrete random variables. This variable has value u with probability p and value d with probability $1-p$. Let $\xi_0 = 1$. Then $S_{k\Delta t} = S_0 \prod_{j=0}^{k} \xi_j$. If we denote

$$\mu = \mathbb{E}[\xi] = p(u-d) + d = \frac{e^{r\Delta t} - d}{u - d}(u-d) + d = e^{r\Delta t},$$

then

$$\mathbb{E}[A_T] = \lim_{n\to\infty} \mathbb{E}\left[\frac{1}{n+1}\sum_{i=0}^{n} S_{i\cdot\Delta t}\right] = \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} \mathbb{E}\left[S_{i\cdot\Delta t}\right]$$
$$= \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} S_0 \mathbb{E}\left[\prod_{j=0}^{i} \xi_j\right].$$

By using an independent property of ξ_j:

$$= \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} S_0 \prod_{j=0}^{i} \mathbb{E}[\xi_j] = \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} S_0 \prod_{j=1}^{i} \mu = \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} S_0 \mu^i$$

$$= \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} S_0 \mu^i = \lim_{n\to\infty} \frac{1}{n+1} S_0 \frac{1-\mu^{n+1}}{1-\mu}$$

$$= \lim_{n\to\infty} \frac{1}{n+1} S_0 \frac{-1 + e^{r\frac{T}{n}(n+1)}}{-1 + e^{r\frac{T}{n}}} = S_0 \lim_{n\to\infty} \left(-1 + e^{r\frac{T}{n}(n+1)}\right) \frac{\frac{1}{n+1}}{-1 + e^{r\frac{T}{n}}}$$

$$= S_0 \left(-1 + e^{rT}\right) \lim_{n\to\infty} \frac{-\frac{1}{(n+1)^2}}{-e^{r\frac{T}{n}} rT \frac{1}{n^2}} = S_0 \frac{-1 + e^{rT}}{rT}.$$

Comment: In Ševčovič & Bokes [2] (Lemma 3.3) and Hansen & Jørgensen [8] (Section 3.2) were studied first two moments of $x_t = \dfrac{A_T}{S_T}$.

Theorem 4. *For* $\mathbb{E}[A_T^2]$ *holds*

$$\mathbb{E}[A_T^2] = S_0^2 \frac{2}{\alpha} \left[\frac{\exp(\beta) - \exp(\alpha)}{\beta - \alpha} - \frac{\exp(\beta) - 1}{\beta} \right], \tag{2.9}$$

where $\alpha = rT$, $\beta = 2(r + \frac{1}{2}\sigma^2)T$.

Proof: Denote by $\nu = \mathbb{E}[\xi^2] = \mu(u + d) - 1$, $\eta = \mu^{-1}$, $\zeta = \nu\eta$. Then

$$(n+1)^2 S_0^{-2} \mathbb{E}[A_n^2] = \mathbb{E}\left[\sum_{i=0}^{n} S_0^{-1} S_{i\cdot\Delta t}\right]^2 = \mathbb{E}\left[\sum_{i=0}^{n} \prod_{j=0}^{i} \xi_j\right]^2$$

$$= \mathbb{E}\left[\sum_{i=0}^{n} \sum_{k=0}^{n} \prod_{j=0}^{i} \xi_j \prod_{l=0}^{k} \xi_l\right] = \sum_{i=0}^{n} \sum_{k=0}^{n} \mathbb{E}\left[\prod_{j=0}^{i} \xi_j \prod_{l=0}^{k} \xi_l\right]$$

$$= \sum_{i=0}^{n} \nu^i \left(2 \sum_{j=i+1}^{n} \mu^{j-i} + 1\right) = \sum_{i=0}^{n} \nu^i \left(2 \sum_{j=1}^{n-i} \mu^j + 1\right)$$

$$= \sum_{i=0}^{n} \nu^i \left(2\mu \frac{1 - \mu^{n-i}}{1 - \mu} + 1\right) = \sum_{i=0}^{n} \nu^i \left(\frac{2\mu}{1 - \mu} + 1 - 2\mu^{n+1} \frac{\eta^i}{1 - \mu}\right)$$

$$= \frac{1 - \nu^{n+1}}{1 - \nu} \left(\frac{2\mu}{1 - \mu} + 1\right) - 2 \frac{\mu^{n+1}}{1 - \mu} \frac{1 - \zeta^{n+1}}{1 - \zeta}.$$

After a straightforward calculations we obtain, that

$$\lim_{n\to\infty} \frac{1}{n+1} \frac{1 - \nu^{n+1}}{1 - \nu} = \frac{-1 + \exp(\beta)}{\beta}, \tag{2.10}$$

$$\lim_{n\to\infty} \frac{1}{n+1} \left(\frac{2\mu}{1 - \mu} + 1\right) = -\frac{2}{\alpha}, \tag{2.11}$$

$$\lim_{n\to\infty} \frac{1}{n+1} \frac{2\mu^{n+1}}{1 - \mu} = -\frac{2\exp(\alpha)}{\alpha}, \tag{2.12}$$

$$\lim_{n\to\infty} \frac{1}{n+1} \frac{1 - \zeta^{n+1}}{1 - \zeta} = \frac{-1 + \exp((r + \sigma^2)T)}{(r + \sigma^2)T}. \tag{2.13}$$

By summarizing above results, bring us to the result of this proof:

$$S_0^{-2} \mathbb{E}[A_T^2] = \lim_{n\to\infty} \frac{1}{(n+1)^2} \left\{ \frac{1 - \nu^{n+1}}{1 - \nu} \left(\frac{2\mu}{1 - \mu} + 1\right) - \frac{2\mu^{n+1}}{1 - \mu} \frac{1 - \zeta^{n+1}}{1 - \zeta} \right\}$$

$$= \frac{2}{\alpha}\left[\frac{\exp(\beta)-\exp(\alpha)}{\beta-\alpha} - \frac{\exp(\beta)-1}{\beta}\right].$$

Theorem 5. *For* $\mathbb{E}[S_T A_T]$ *holds*

$$\mathbb{E}[S_T A_T] = S_0^2 \frac{\exp(\beta)-\exp(\alpha)}{\beta-\alpha}, \tag{2.14}$$

where $\alpha = rT$, $\beta = 2(r + \frac{1}{2}\sigma^2)T$.

Proof: Similarly as before, we firstly discretize continuous process and again by pushing limit of $n \to \infty$ we finish the proof.

$$\mathbb{E}[S_T A_T] = \lim_{n\to\infty} \frac{1}{n+1}\mathbb{E}\left[S_T \sum_{i=0}^{n} S_0 S_{i\cdot\Delta t}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n+1}\mathbb{E}\left[S_0\left(\prod_{j=0}^{n}\xi_j\right)\sum_{i=0}^{n}\left(S_0\prod_{k=0}^{i}\xi_k\right)\right] = \lim_{n\to\infty}\frac{S_0^2}{n+1}\mathbb{E}\left[\sum_{i=0}^{n}\left(\prod_{j=0}^{n}\xi_j\prod_{k=0}^{i}\xi_k\right)\right]$$

$$= \lim_{n\to\infty}\frac{S_0^2}{n+1}\mathbb{E}\left[\sum_{i=0}^{n}\left(\prod_{j=1}^{n}\xi_j\prod_{k=1}^{i}\xi_k\right)\right] = \lim_{n\to\infty}\frac{S_0^2}{n+1}\left(\sum_{i=0}^{n}\mu^{n-i}\nu^i\right)$$

$$= \lim_{n\to\infty}\frac{S_0^2}{n+1}\mu^n\left(\sum_{i=0}^{n}\left(\frac{\nu}{\mu}\right)^i\right) = \lim_{n\to\infty}\frac{S_0^2}{n+1}\mu^n\frac{1-\nu^{n+1}\mu^{-n-1}}{1-\nu\mu^{-1}}$$

$$= \lim_{n\to\infty}\frac{S_0^2}{n+1}\frac{\mu^{n+1}-\nu^{n+1}}{\mu-\nu} = S_0^2 e^{rT}\frac{e^{(r+s^2)T}-1}{(r+s^2)T} = S_0^2\frac{\exp(\beta)-\exp(\alpha)}{\beta-\alpha}.$$

Let us remark, that this was not necessary for estimation of parameters of log-normal variable, but we will use this result in further section in Copula.

2.2.2 Weighted arithmetic averaged

In previous section we discuss case of arithmetic averaging namely $A_t = \frac{1}{t}\int_0^t S_\xi \, d\xi$. In this section we discuss more general case, namely weighted arithmetic averaging with weighted function $a(\xi)$. In option market, there exist options, which price depends only on average for last k days before expiration date.

Weighted arithmetic average can be written as

$$A_T = \frac{1}{\int_0^T a(\xi)d\xi}\int_0^T a(T-\xi)S_\xi \, d\xi.$$

There are a lot of weighted functions, for example:

- exponential weighted function $a(\xi) = \exp(-\lambda\xi)$,

- averaging ε days before expiration $a(\xi) = \begin{cases} 0, & \text{if } \xi \geq \varepsilon \\ 1, & \text{if } \xi < \varepsilon \end{cases}$.

Lemma 6 (The Itō isometry [15, 20]). *Let $\mathcal{V} = \mathcal{V}(S,T)$ be a class of functions*

$$f(t,\omega) : \langle 0, \infty) \times \Omega \to \mathbb{R}$$

such that

- *$(t,\omega) \to f(t,\omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel σ-algebra on $\langle 0, \infty)$,*
- *$f(t,\omega)$ is $\mathcal{F}_t$ adapted,*
- *$\mathbb{E}[\int_S^T f(t,\omega)^2 dt] < \infty$,*

then for all $f \in \mathcal{V}(S,T)$ holds

$$\mathbb{E}\left[\left(\int_S^T f(t,\omega)dW_t\right)^2\right] = \mathbb{E}\left[\int_S^T f^2(t,\omega)dW_t\right],$$

where W_t is Wiener process. Especially, let $\{S_\xi, \xi \geq 0\}$ is stochastic process. Then

$$\mathbb{E}\left[\left(\int_0^t S_\xi dW_\xi\right)^2\right] = \int_0^t \mathbb{E}\left[S_\xi^2\right] d\xi. \tag{2.15}$$

Theorem 7. *For $\mathbb{E}[S_t]$ holds*

$$\mathbb{E}[S_t] = S_0 e^{rt}.$$

Proof:

$$\mathbb{E}[S_T] = \lim_{n\to\infty} S_0 \mathbb{E}\left[\prod_{i=0}^n \xi_i\right] = \lim_{n\to\infty} S_0 \prod_{i=0}^n E[\xi_i] = \lim_{n\to\infty} S_0 \mu^n = S_0 e^{rT}.$$

Lemma 8 (First moment in case of weighted averaging). *For $\mathbb{E}[A_T]$ holds*

$$\mathbb{E}[A_T] = \frac{S_0}{\int_0^T a(\xi)d\xi} \int_0^T a(T-\xi)e^{r\xi}d\xi.$$

Proof:

$$\mathbb{E}[A_T] \cdot \int_0^T a(\xi)d\xi = \mathbb{E}\left[\int_0^T a(T-\xi)S_\xi d\xi\right] = \int_0^T a(T-\xi)\mathbb{E}[S_\xi]d\xi =$$

$$= \int_0^T a(T-\xi)S_0 e^{r\xi}d\xi.$$

Comment: Function $T \mapsto \displaystyle\int_0^T a(T-\xi)e^{r\xi}d\xi$ is convolution of $a(\bullet)$ and $\exp(r\bullet)$.

Lemma 9 (Second moment in case of weighted averaging). *For* $\mathbb{E}[A_T^2]$ *holds*

$$\mathbb{E}[A_T^2] = \mathbb{E}\left[\left(\int_0^T a(T-\xi)dS_\xi\right)^2\right] - 2\mathbb{E}\left[\int_0^T a(T-\xi)dS_\xi \int_0^T a(T-\xi)S_\xi\sigma dW_\xi\right]$$
$$+\mathbb{E}\left[\int_0^T (a(T-\xi)S_\xi\sigma)^2 d\xi\right].$$

Proof:

$$r^2\mathbb{E}[A_T^2] = r^2\mathbb{E}\left[\left(\int_0^T a(T-\xi)S_\xi d\xi\right)^2\right].$$

Using equation (2.3) we have

$$Sd\xi = \frac{dS_\xi - S_\xi\sigma dW_\xi}{r}. \tag{2.16}$$

Then

$$r^2\mathbb{E}\left[\left(\int_0^T a(T-\xi)S_\xi d\xi\right)^2\right] = \mathbb{E}\left[\left(\int_0^T a(T-\xi)(dS_\xi - S_\xi\sigma dW_\xi)\right)^2\right]$$
$$= \mathbb{E}\left[\left(\int_0^T a(T-\xi)dS_\xi - \int_0^T a(T-\xi)S_\xi\sigma dW_\xi)\right)^2\right] = \mathbb{E}\left[\left(\int_0^T a(T-\xi)dS_\xi\right)^2\right]$$
$$-2\mathbb{E}\left[\int_0^T a(T-\xi)dS_\xi \int_0^T a(T-\xi)S_\xi\sigma dW_\xi\right] + \mathbb{E}\left[\left(\int_0^T a(T-\xi)S_\xi\sigma dW_\xi\right)^2\right].$$

Using Itō's isometry we conclude

$$= \mathbb{E}\left[\left(\int_0^T a(T-\xi)dS_\xi\right)^2\right] - 2\mathbb{E}\left[\int_0^T a(T-\xi)dS_\xi \int_0^T a(T-\xi)S_\xi\sigma dW_\xi\right]$$
$$+\mathbb{E}\left[\int_0^T (a(T-\xi)S_\xi\sigma)^2 d\xi\right].$$

2.2.3 Exponential weighted arithmetic averaged

In this chapter we derive both moments of A_T in case of exponentially weighted function, where

$$A_T = \frac{1}{\int_0^T \exp(-\lambda\xi)d\xi}\int_0^T e^{-\lambda(T-\xi)}S_\xi\, d\xi$$

is exponentially weighted averaging with weighted function $a(\xi) = \exp(-\lambda\xi)$.

Lemma 10 (First moment in case of exponentially weighted averaging). *For* $\mathbb{E}[A_T]$ *holds*

$$\mathbb{E}[A_T] = S_0\frac{\lambda}{\lambda+r}\frac{e^{rT} - e^{-\lambda T}}{1 - e^{-\lambda T}}. \tag{2.17}$$

Proof: If we set $a(\xi) = \exp(-\lambda\xi)$ into Lemma 8, then

$$\mathbb{E}[A_T] = \frac{S_0}{\int_0^T e^{-\lambda\xi} d\xi} \int_0^T e^{-\lambda(T-\xi)} e^{r\xi} d\xi = S_0 \frac{\lambda}{\lambda + r} e^{-\lambda T} \frac{e^{(\lambda+r)T} - 1}{1 - e^{-\lambda T}}$$

$$= S_0 \frac{\lambda}{\lambda + r} \frac{e^{rT} - e^{-\lambda T}}{1 - e^{-\lambda T}}.$$

Comment: If we push limit $\lambda \to 0$ in (2.17) we obtain (2.8).

Lemma 11 (Second moment in case of exponentially weighted averaging). *For $\mathbb{E}[A_T^2]$ holds*

$$\mathbb{E}[A_T^2] = e^{-2\lambda T} \frac{S_0^2}{k^2} \frac{2}{\tilde{\alpha}} \left[\frac{\exp(\tilde{\beta}) - \exp(\tilde{\alpha})}{\tilde{\beta} - \tilde{\alpha}} - \frac{\exp(\tilde{\beta}) - 1}{\tilde{\beta}} \right], \tag{2.18}$$

where $\tilde{\alpha} = (r + \lambda)T$, $\tilde{\beta} = 2(r + \frac{1}{2}\sigma^2 + \lambda)T$, $k = \frac{1}{T} \int_0^T \exp(-\lambda\xi) d\xi$.

Proof: Let us mark $\nu = \mathbb{E}[\xi^2] = \mu(u + d) - 1$, $\eta = \pi^{-1}$, $\zeta = \varrho\eta$, $\omega = \exp(\lambda \cdot \Delta t)$, $\varrho = \omega^2 \nu$, $\pi = \omega\mu$. Then

$$e^{2\lambda T} k^2 (n+1)^2 S_0^{-2} \mathbb{E}[A_n^2] = e^{2\lambda T} \mathbb{E} \left[\sum_{i=0}^n S_0^{-1} S_{i\cdot\Delta t} e^{-\lambda(T - i\Delta t)} \right]^2 = \mathbb{E} \left[\sum_{i=0}^n \omega^i \prod_{j=0}^i \xi_j \right]^2$$

$$= \mathbb{E} \left[\sum_{i=0}^n \sum_{k=0}^n \omega^i \omega^k \prod_{j=0}^i \xi_j \prod_{l=0}^k \xi_l \right] = \sum_{i=0}^n \sum_{k=0}^n \omega^i \omega^k \mathbb{E} \left[\prod_{j=0}^i \xi_j \prod_{l=0}^k \xi_l \right]$$

$$= \sum_{i=0}^n \nu^i \omega^{2i} \left(2 \sum_{j=i+1}^n \omega^{j-i} \mu^{j-i} + 1 \right) = \sum_{i=0}^n \varrho^i \left(2 \sum_{j=1}^{n-i} \pi^j + 1 \right)$$

$$= \sum_{i=0}^n \varrho^i \left(2\pi \frac{1 - \pi^{n-i}}{1 - \pi} + 1 \right) = \sum_{i=0}^n \varrho^i \left(\frac{2\pi}{1 - \pi} + 1 - 2\pi^{n+1} \frac{\eta^i}{1 - \pi} \right)$$

$$= \frac{1 - \varrho^{n+1}}{1 - \varrho} \left(\frac{2\pi}{1 - \pi} + 1 \right) - 2 \frac{\pi^{n+1}}{1 - \pi} \frac{1 - \zeta^{n+1}}{1 - \zeta}.$$

After a straightforward calculations we conclude, that

$$\lim_{n\to\infty} \frac{1}{n+1} \frac{1 - \varrho^{n+1}}{1 - \varrho} = \frac{-1 + e^{2(\lambda + r + \frac{1}{2}\sigma^2)T}}{2(\lambda + r + \frac{1}{2}\sigma^2)T} = \frac{-1 + e^{\tilde{\beta}}}{\tilde{\beta}}, \tag{2.19}$$

$$\lim_{n\to\infty} \frac{1}{n+1} \left(\frac{2\pi}{1 - \pi} + 1 \right) = -\frac{2}{(\lambda + r)T} = -\frac{2}{\tilde{\alpha}}, \tag{2.20}$$

$$\lim_{n\to\infty} \frac{1}{n+1} \frac{2\pi^{n+1}}{1 - \pi} = -\frac{2e^{(\lambda+r)T}}{(\lambda + r)T} = -\frac{2\exp(\tilde{\alpha})}{\tilde{\alpha}}, \tag{2.21}$$

$$\lim_{n\to\infty} \frac{1}{n+1} \frac{1 - \zeta^{n+1}}{1 - \zeta} = \frac{-1 + e^{(r + \lambda + \sigma^2)T}}{(r + \lambda + \sigma^2)T}. \tag{2.22}$$

Then

$$e^{2\lambda T}k^2 S_0^{-2}\mathbb{E}[A_T^2] = \lim_{n\to\infty}\frac{1}{(n+1)^2}\left\{\frac{1-\varrho^{n+1}}{1-\varrho}\left(\frac{2\pi}{1-\pi}+1\right)-2\frac{\pi^{n+1}}{1-\pi}\frac{1-\zeta^{n+1}}{1-\zeta}\right\}$$

$$= \frac{2}{\tilde{\alpha}}\left[\frac{\exp(\tilde{\beta})-\exp(\tilde{\alpha})}{\tilde{\beta}-\tilde{\alpha}}-\frac{\exp(\tilde{\beta})-1}{\tilde{\beta}}\right].$$

Comment: Formula (2.9) is almost identical with (2.18). Only differences are that instead of parameters α, β are $\tilde{\alpha}, \tilde{\beta}$. It is obvious, that $\lim_{\lambda\to 0}\tilde{\alpha} = \alpha$ and $\lim_{\lambda\to 0}\tilde{\beta} = \beta$.

Lemma 12. *For* $\mathbb{E}[S_T A_T]$ *holds*

$$\mathbb{E}[S_T A_T] = \frac{S_0^2}{\int_0^T e^{-\lambda\xi}d\xi}\frac{e^{2(r+\frac{1}{2}s^2)T}-e^{(r+\lambda)T}}{(\lambda+r+s^2)T}. \tag{2.23}$$

Proof: Let as mark $\omega = \exp(\lambda\cdot\Delta t)$, $\varrho = \omega\nu$. Then

$$\mathbb{E}[S_T A_T]\cdot\int_0^T e^{-\lambda\xi}d\xi = \lim_{n\to\infty}\frac{1}{n+1}\mathbb{E}\left[S_T\sum_{i=0}^n S_0 e^{-\lambda T+\lambda\cdot i\cdot\Delta t}S_{i\cdot\Delta t}\right]$$

$$= \lim_{n\to\infty}\frac{e^{-\lambda T}}{n+1}\mathbb{E}\left[S_T\sum_{i=0}^n S_0\omega^i S_{i\cdot\Delta t}\right]$$

$$= \lim_{n\to\infty}\frac{e^{-\lambda T}}{n+1}\mathbb{E}\left[S_0\left(\prod_{j=0}^n\xi_j\right)\sum_{i=0}^n\omega^i\left(S_0\prod_{k=0}^i\xi_k\right)\right]$$

$$= \lim_{n\to\infty}\frac{S_0^2 e^{-\lambda T}}{n+1}\mathbb{E}\left[\sum_{i=0}^n\omega^i\left(\prod_{j=0}^n\xi_j\prod_{k=0}^i\xi_k\right)\right]$$

$$= \lim_{n\to\infty}\frac{S_0^2 e^{-\lambda T}}{n+1}\mathbb{E}\left[\sum_{i=0}^n\omega^i\left(\prod_{j=1}^n\xi_j\prod_{k=1}^i\xi_k\right)\right] = \lim_{n\to\infty}\frac{S_0^2 e^{-\lambda T}}{n+1}\left(\sum_{i=0}^n\omega^i\mu^{n-i}\nu^i\right)$$

$$= \lim_{n\to\infty}\frac{S_0^2 e^{-\lambda T}}{n+1}\mu^n\left(\sum_{i=0}^n\left(\frac{\varrho}{\mu}\right)^i\right) = \lim_{n\to\infty}\frac{S_0^2 e^{-\lambda T}}{n+1}\mu^n\frac{1-\varrho^{n+1}\mu^{-n-1}}{1-\varrho\mu^{-1}}$$

$$= \lim_{n\to\infty}\frac{S_0^2 e^{-\lambda T}}{n+1}\frac{\mu^{n+1}-\varrho^{n+1}}{\mu-\varrho} = S_0^2 e^{rT}e^{-\lambda T}\frac{e^{(\lambda+r+s^2)T}-1}{(\lambda+r+s^2)T}$$

$$= S_0^2\frac{e^{2(r+\frac{1}{2}s^2)T}-e^{(r+\lambda)T}}{(\lambda+r+s^2)T}.$$

Comment: If we calculate $\lim_{\lambda\to 0}\mathbb{E}[S_T A_T]$ in (2.23) we have (2.14).

2.3 Parameter estimation

In this section we use method of moments to estimate parameters of lognormal distribution.

Let ψ is random variable with lognormal distribution with parameters φ, χ, then probability density function is

$$f_\psi(x,\varphi,\chi) = \begin{cases} 0, & \text{if } x \le 0 \\ \frac{1}{x\chi\sqrt{2\pi}} \exp\left[-\frac{(\ln(x)-\varphi)^2}{2\chi^2}\right], & \text{if } x > 0 \end{cases} \tag{2.24}$$

and cumulative distribution function is

$$F_\psi(x,\varphi,\chi) = \begin{cases} 0, & \text{if } x \le 0 \\ \frac{1}{2} + \frac{1}{2}\text{erf}\left[\frac{\ln(x)-\varphi}{\chi\sqrt{2}}\right], & \text{if } x > 0 \end{cases}.$$

For mean and variance it holds

$$E[\psi] = e^{\varphi+\frac{1}{2}\chi^2},$$
$$Var[\psi] = \left(e^{\chi^2} - 1\right) e^{2\varphi+\chi^2}.$$

For as much as we already know two moments of A_T, we can apply methods of moments and estimate parameters φ, χ. It is well known, that

$$\begin{aligned} \varphi &= \ln(E[\psi]) - \frac{1}{2}\ln\left(1 + \frac{Var[\psi]}{(E[\psi])^2}\right), \\ \chi^2 &= \ln\left(1 + \frac{Var[\psi]}{(E[\psi])^2}\right). \end{aligned}$$

Since $Var[\psi] = E[\psi^2] - E[\psi]^2$, then

$$\varphi = \ln(E[\psi]) - \frac{1}{2}\ln E[\psi^2] + \ln(E[\psi]) = 2\ln(E[\psi]) - \frac{1}{2}\ln E[\psi^2], \tag{2.25}$$

$$\chi^2 = \ln \frac{E[\psi^2]}{(E[\psi])^2}. \tag{2.26}$$

After substitution $E[\psi]$ and $E[\psi^2]$ into (2.25) and (2.26) we have

$$\begin{aligned} \varphi &= 2\ln\left(S_0\frac{\exp(\alpha)-1}{\alpha}\right) - \frac{1}{2}\ln\left(S_0^2\frac{2}{\alpha}\left[\frac{\exp(\beta)-\exp(\alpha)}{\beta-\alpha} - \frac{\exp(\beta)-1}{\beta}\right]\right), \\ \chi^2 &= \ln\frac{S_0^2\frac{2}{\alpha}\left[\frac{\exp(\beta)-\exp(\alpha)}{\beta-\alpha} - \frac{\exp(\beta)-1}{\beta}\right]}{S_0^2\left(\frac{\exp(\alpha)-1}{\alpha}\right)^2} = \ln\frac{\frac{2}{\alpha}\left[\frac{\exp(\beta)-\exp(\alpha)}{\beta-\alpha} - \frac{\exp(\beta)-1}{\beta}\right]}{\left(\frac{\exp(\alpha)-1}{\alpha}\right)^2}. \end{aligned}$$

If we denote $\kappa = \frac{\exp(\alpha)-1}{\alpha}$ and $\theta = \frac{2}{\alpha}\left[\frac{\exp(\beta)-\exp(\alpha)}{\beta-\alpha} - \frac{\exp(\beta)-1}{\beta}\right]$, then

$$\varphi = \ln S_0 + 2\ln\kappa - \frac{1}{2}\ln\theta, \tag{2.27}$$

$$\chi^2 = \ln\theta - 2\ln\kappa, \tag{2.28}$$

where

$$\begin{aligned} \kappa &= \frac{e^{rT}-1}{rT}, \\ \theta &= \frac{2}{rT}\left[\frac{\exp(2(r+\frac{1}{2}\sigma^2)T)-\exp(rT)}{2(r+\frac{1}{2}\sigma^2)T-rT} - \frac{\exp(2(r+\frac{1}{2}\sigma^2)T)-1}{2(r+\frac{1}{2}\sigma^2)T}\right]. \end{aligned}$$

Finally we have all necessary parameters estimated and taking into account probability function of A_T (2.24) and formula for option pricing (2.1) we can derive approximate formula

$$\begin{aligned} V(S,0) &= e^{-rT}\mathbb{E}_Q[(A_T-X)^+] = e^{-rT}\int_0^\infty (x-X)^+ f_\psi(x,\varphi,\chi)\,dx \\ &= e^{-rT}\int_X^\infty (x-X)\frac{1}{x\chi\sqrt{2\pi}}\exp\left[-\frac{(\ln(x)-\varphi)^2}{2\chi^2}\right]\,dx. \end{aligned} \qquad (2.29)$$

2.4 Monte Carlo simulation

Following the idea of binary tree, we can easily generate process S_t and calculate corresponding A_t. For given parameters S, X, r, σ, T we discretize continuous process to 200 steps and generate 100,000 random paths of process S_t. Source code in MATLAB can be found in Appendix in section 7.1.1.

In Figure 2.2 are estimated densities from Monte Carlo simulation and density of lognormal random variable with parameters derived in previous section. Parameters of process is $S = 1, \sigma = 0.1, r = 0.05, T = 0.5$. In this case, the lognormal fit is satisfied. Estimation of first moment of A_T is 1.0125979 and calculated from (2.8) is 1.0126048. Estimation of second moment of A_T is 1.0270843 and calculated from (2.9) is 1.0270903. Kernel function for density estimation is following:

- normal - $k(u) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}u^2)$,
- Epanechnikov - $k(u) = \frac{3}{4}(1-u^2)$, if $|u| \leq 1$, else 0,
- box - $k(u) = \frac{1}{2}$, if $|x| \leq 1$, else 0,
- triangle - $k(u) = 1-|u|$, if $|u| \leq 1$, else 0.

More about kernel density estimation can be found in Jeffrey [12].

But, if we choose higher volatility of process S_t, $\sigma = 0.5$ and higher risk less interest rate $r = 0.15$ and longer expiration date $T = 2$ then lognormal fit is not more satisfied. See Figure 2.3. Estimation of first moment of A_T is 1.1643556 and calculated from (2.8) is 1.1661960. Estimation of second moment of A_T is 1.6313993 and calculated from (2.9) is 1.6394327. For this higher parameters, it seems to be better to use more parameters distribution, e.g. **Generalized extreme value** (GEV) distribution. If Figure 2.4, there are comparison of lognormal fit and GEV fit. One can easily see, that this GEV fits data better.

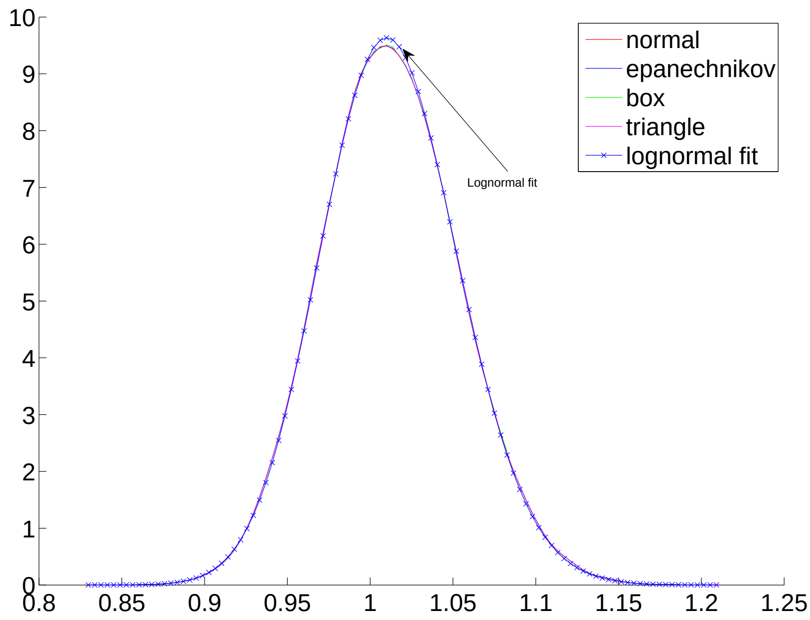

Figure 2.2: Kernel density estimation of A_T for parameters $S = 1, \sigma = 0.1, r = 0.05, T = 0.5$ and lognormal fit calculated by (2.27) and (2.28).

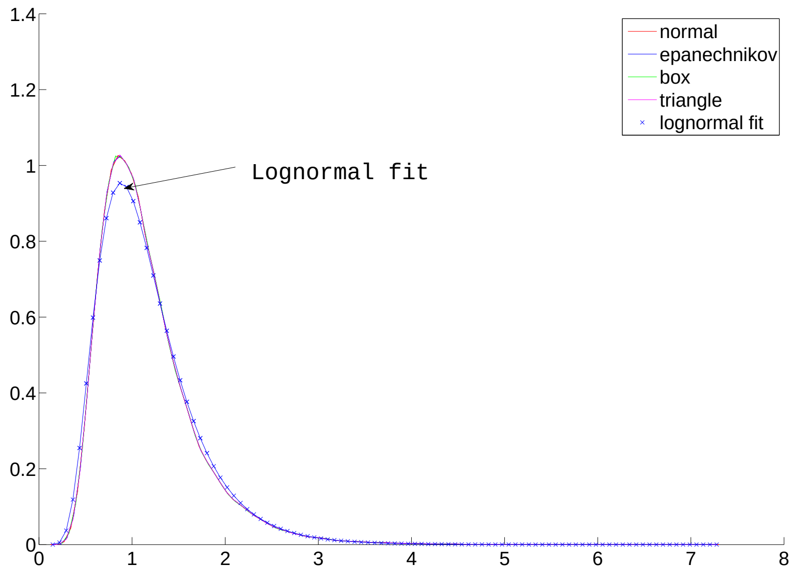

Figure 2.3: Kernel density estimation of A_T for parameters $S = 1, \sigma = 0.5, r = 0.15, T = 2$ and lognormal fit calculated by (2.27) and (2.28).

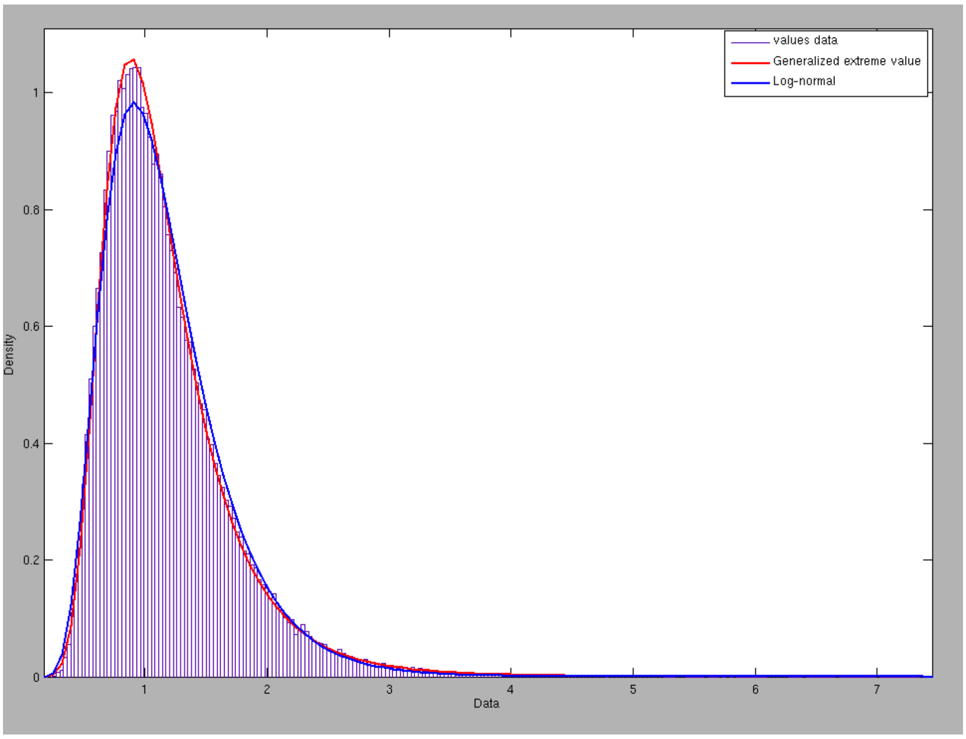

Figure 2.4: Histogram of A_T for parameters $S = 1, \sigma = 0.5, r = 0.15, T = 2$, lognormal fit (blue line) and Generalized extreme value fit (red line).

Generalized extreme value Let η is generalized extreme value distribution with parameters μ, σ, ξ, then probability density function is given by formula

$$f_\eta(x, \mu, \sigma, \xi) = \frac{1}{\sigma}\left[1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right]^{\frac{-1}{\xi-1}} \exp\left\{\left[1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right]^{\frac{-1}{\xi}}\right\}. \tag{2.30}$$

Generalized extreme value joins 3 simpler distributions (Gumbel, Frechet, Weibull). It's big advantage is, that when we fit data with GEV, we let to data decide it's own distribution. For further detail see e.g. Embrechts [6], Leadbetter [11].

2.5 Numerical results

In Table 2.1 we present numerical results from our lognormal approach and comparison with other known methods. In Figure 2.5 we plot for various interest rates dependences of option price on λ in case of exponentially weighted average. Let us remark, that for $\lambda \to \infty$ this option change to plain vanilla option. One can easily verify that

$$\begin{aligned}\lim_{\lambda\to\infty} E[A_T] &= E[S_T],\\ \lim_{\lambda\to\infty} E[A_T^2] &= E[S_T^2].\end{aligned}$$

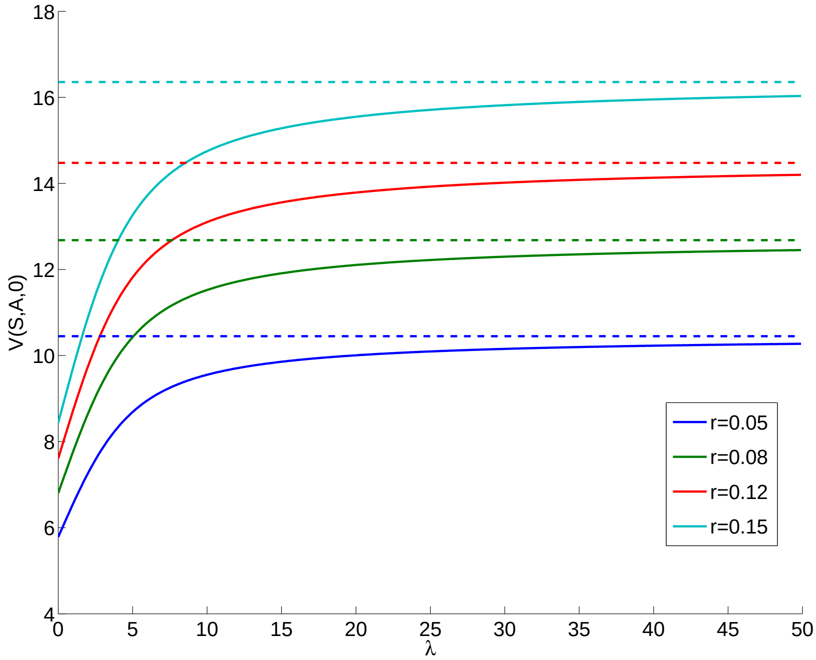

Figure 2.5: A dependence of option price on λ (solid line), corresponding vanilla plain option (dashed line). Other parameters are $S = 100, X = 100, T = 1, \sigma = 0.4$.

Table 2.1: A comparison of different methods for pricing average rate Asian options. Parameters of process S_t are $S_0 = 100, T = 1$. RS–PDE are values obtained by solve PDE (Roger a Shi) T–LB and T–UB are lower and upper bounds Thompson (2000), AA is analytical approximations Zhang [26], LN is our approach by lognormal distribution and MC is value obtained by Monte Carlo simulations.

σ	X	r	RS-PDE	T-LB	T-UB	AA	LN	MC
0.05	95	0.05	7.157	7.1777	7.1779	7.1794	7.17802	7.14323
0.05	100	0.05	2.621	2.7162	2.7162	2.7279	2.72574	2.70776
0.05	105	0.05	0.439	0.3372	0.3374	0.3257	0.34352	0.33706
0.05	95	0.09	8.823	8.8088	8.8089	8.8091	8.80888	8.87211
0.05	100	0.09	4.185	4.3082	4.3084	4.3173	4.31292	4.31100
0.05	105	0.09	1.011	0.9583	0.9585	0.9561	0.96888	0.94235
0.05	95	0.15	11.090	11.0941	11.0943	11.0941	11.09409	11.13954
0.05	100	0.15	6.777	6.7944	6.7946	6.7963	6.79500	6.78619
0.05	105	0.15	2.639	2.7444	2.7446	2.7559	2.75309	2.73807
0.10	90	0.05	11.942	11.9511	11.9523	11.9666	11.95337	11.82228
0.10	100	0.05	3.624	3.6413	3.6416	3.6725	3.64798	3.64981
0.10	110	0.05	0.359	0.3311	0.3322	0.2855	0.32426	0.34403
0.10	90	0.09	13.382	13.3852	13.3862	13.3935	13.38630	13.43389
0.10	100	0.09	4.887	4.9151	4.9154	4.9597	4.92349	4.93918
0.10	110	0.09	0.659	0.6301	0.6310	0.5840	0.62376	0.65125
0.10	90	0.15	15.398	15.3988	15.3995	15.4015	15.39906	15.48592
0.10	100	0.15	7.000	7.0277	7.0286	7.0707	7.03506	7.03506
0.10	110	0.15	1.430	1.4133	1.4143	1.3901	1.41130	1.35585
0.20	90	0.05	12.589	12.5956	12.6008	12.7837	12.62990	12.51913
0.20	100	0.05	5.760	5.7627	5.7645	5.8330	5.78310	5.78487
0.20	110	0.05	1.996	1.9892	1.9927	1.8322	1.97131	2.00315
0.20	90	0.09	13.825	13.8312	13.8373	14.0072	13.86178	13.83059
0.20	100	0.09	6.773	6.7770	6.7787	6.8915	6.80379	6.83882
0.20	110	0.09	2.551	2.5455	2.5486	2.4269	2.53478	2.57984
0.20	90	0.15	15.636	15.6416	15.6491	15.7898	15.66540	15.69377
0.20	100	0.15	8.402	8.4085	8.4105	8.5691	8.44099	8.40859
0.20	110	0.15	3.558	3.5547	3.5578	3.5098	3.55665	3.55116
0.30	90	0.05	13.951	13.9524	13.9622	14.3521	14.03825	14.24634
0.30	100	0.05	7.944	7.9444	7.9506	8.0597	7.99270	7.90126
0.30	110	0.05	4.074	4.0701	4.0787	3.8171	4.05741	4.25846
0.30	90	0.09	14.981	14.9828	14.9930	15.3963	15.06715	15.17420
0.30	100	0.09	8.827	8.8276	8.8334	9.0147	8.88593	8.78137
0.30	110	0.09	4.698	4.6949	4.7027	4.5161	4.69528	4.73203
0.30	90	0.15	16.510	16.5120	16.5239	16.9269	16.59091	16.47379
0.30	100	0.15	10.208	10.2087	10.2142	10.4856	10.27829	10.17835
0.30	110	0.15	5.731	5.7282	5.7356	5.6688	5.74817	5.74239

Chapter 3

European style average strike Asian options

In this chapter we discuss European average strike options. Payoff of these options is for call in the form

$$V(S, A, T) = \max\{S_T - A_T, 0\},$$

for put options in the form

$$V(S, A, T) = \max\{A_T - S_T, 0\},$$

where A_T is geometric or arithmetic average. Let us remark, that European style options can be exercised only at maturity. It is well know, that $V(S, A, t)$ is a solution of following PDE (cf. Kwok [20] section 6.1.1):

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} + Af\left(\frac{S}{A}, t\right) \frac{\partial V}{\partial A} - rV = 0 \tag{3.1}$$

with corresponding terminal condition. It is also well know, that in case of average strike options, one can transform (3.1) to achieve dimension reduction. By introducing auxiliary function

$$W(x, t) = \frac{1}{A} V(S, A, t), \quad \text{where } x = \frac{S}{A}, x \in \mathbb{R},$$

we obtain following PDE:

$$\frac{\partial W}{\partial t} + \frac{\sigma^2}{2} x^2 \frac{\partial^2 W}{\partial x^2} + rx \frac{\partial W}{\partial x} + f(x, t) \left(W - x \frac{\partial W}{\partial x}\right) - rW = 0 \tag{3.2}$$

which has lower dimension. Terminal condition is in the form $W(x, T) = \max\{x - 1, 0\}$ for call and $W(x, T) = \max\{1 - x, 0\}$ for put options.

3.1 Modification average

Main goal of this section is introduce modification average. Classical arithmetic average is in the form

$$A_t = \frac{1}{t}\int_0^t S_\tau d\tau \tag{3.3}$$

and for $\frac{dA_t}{dt}$ holds

$$dA_t = \frac{1}{t}S_t - \frac{1}{t^2}\int_0^t S_\tau d\tau = \frac{S_t - A_t}{t}, \tag{3.4}$$

$$\mathcal{A}_t = \frac{1}{T}\int_0^t S_\tau d\tau + \frac{T-t}{T}S_t. \tag{3.5}$$

This expression does not contain dS_t term. But if we introduce new average

$$d\mathcal{A}_t = \frac{1}{T}S_t + \frac{T-t}{T}dS_t - \frac{1}{T}S_t = \frac{T-t}{T}dS_t = \xi_t dS_t, \tag{3.6}$$

where $\xi_t = \frac{T-t}{T}$ then we obtain new PDE with nonzero terms $\frac{\partial^2 V}{\partial A^2}$, $\frac{\partial^2 V}{\partial S^2}$, $\frac{\partial^2 V}{\partial A \partial S}$. Applying Itō's lemma on $\mathcal{V} = \mathcal{V}(S, \mathcal{A}, t)$ we conclude

$$d\mathcal{V} = \frac{\partial \mathcal{V}}{\partial S}dS + \frac{\partial \mathcal{V}}{\partial \mathcal{A}}d\mathcal{A} + \frac{\partial \mathcal{V}}{\partial t}dt + \frac{1}{2}\frac{\partial^2 \mathcal{V}}{\partial S^2}(dS)^2 + \frac{1}{2}\frac{\partial^2 \mathcal{V}}{\partial \mathcal{A}^2}(d\mathcal{A})^2 + \frac{\partial^2 \mathcal{V}}{\partial \mathcal{A}\partial S}d\mathcal{A}dS.$$

After substitute $dS = \mu S dt + \sigma S dW_t$, $dA = \xi_t \mu S dt + \xi_t \sigma S dW_t$, $(dS)^2 = \sigma^2 S^2 dt$, $(dA)^2 = \xi_t^2 \sigma^2 S^2 dt$, $dAdS = \xi_t \sigma^2 S^2 dt$ we obtain

$$\begin{aligned} d\mathcal{V} &= \frac{\partial \mathcal{V}}{\partial S}(\mu S dt + \sigma S dW_t) + \frac{\partial \mathcal{V}}{\partial \mathcal{A}}\xi_t(\mu S dt + \sigma S dW_t) \\ &\quad + \frac{\partial \mathcal{V}}{\partial t}dt + \frac{1}{2}\frac{\partial^2 \mathcal{V}}{\partial S^2}\sigma^2 S^2 dt + \frac{1}{2}\frac{\partial^2 \mathcal{V}}{\partial \mathcal{A}^2}\xi_t^2 \sigma^2 S^2 dt + \frac{\partial^2 \mathcal{V}}{\partial \mathcal{A}\partial S}\xi_t \sigma^2 S^2 dt. \end{aligned} \tag{3.7}$$

Following derivation of PDE by Ševčovič [20], for function $\mathcal{V}$ has to hold

$$d\mathcal{V} - r\mathcal{V}dt - \Delta(dS - rSdt) = 0. \tag{3.8}$$

Substituting (3.7) into (3.8) we have

$$
\begin{aligned}
0 &= \frac{\partial \mathcal{V}}{\partial S}(\mu S dt + \sigma S dW_t) + \frac{\partial \mathcal{V}}{\partial \mathcal{A}}\xi_t(\mu S dt + \sigma S dW_t) \\
&\quad + \frac{\partial \mathcal{V}}{\partial t}dt + \frac{1}{2}\frac{\partial^2 \mathcal{V}}{\partial S^2}\sigma^2 S^2 dt + \frac{1}{2}\frac{\partial^2 \mathcal{V}}{\partial \mathcal{A}^2}\xi_t^2\sigma^2 S^2 dt + \frac{\partial^2 \mathcal{V}}{\partial \mathcal{A}\partial S}\xi_t\sigma^2 S^2 dt \\
&\quad - r\mathcal{V}dt - \Delta(dS - rSdt), \\
0 &= \frac{\partial \mathcal{V}}{\partial S}\mu S dt + \frac{\partial \mathcal{V}}{\partial S}\sigma S dW_t + \frac{\partial \mathcal{V}}{\partial \mathcal{A}}\xi_t\mu S dt + \frac{\partial \mathcal{V}}{\partial \mathcal{A}}\xi_t\sigma S dW_t \\
&\quad + \frac{\partial \mathcal{V}}{\partial t}dt + \frac{1}{2}\frac{\partial^2 \mathcal{V}}{\partial S^2}\sigma^2 S^2 dt + \frac{1}{2}\frac{\partial^2 \mathcal{V}}{\partial \mathcal{A}^2}\xi_t^2\sigma^2 S^2 dt + \frac{\partial^2 \mathcal{V}}{\partial \mathcal{A}\partial S}\xi_t\sigma^2 S^2 dt \\
&\quad - r\mathcal{V}dt - \Delta\mu S dt - \Delta\sigma S dW_t + \Delta r S dt, \\
0 &= \frac{\partial \mathcal{V}}{\partial S}\mu S dt + \frac{\partial \mathcal{V}}{\partial \mathcal{A}}\xi_t\mu S dt + \frac{\partial^2 \mathcal{V}}{\partial \mathcal{A}\partial S}\xi_t\sigma^2 S^2 dt + \Delta r S dt \\
&\quad + \frac{\partial \mathcal{V}}{\partial t}dt + \frac{1}{2}\frac{\partial^2 \mathcal{V}}{\partial S^2}\sigma^2 S^2 dt + \frac{1}{2}\frac{\partial^2 \mathcal{V}}{\partial \mathcal{A}^2}\xi_t^2\sigma^2 S^2 dt \\
&\quad - r\mathcal{V}dt - \Delta\mu S dt + \left(\frac{\partial \mathcal{V}}{\partial S} + \frac{\partial \mathcal{V}}{\partial \mathcal{A}}\xi_t - \Delta\right)\sigma S dW_t.
\end{aligned} \tag{3.9}
$$

In this equation there is only one source of risk dW_t. To eliminate it, we have to choose $\Delta = \frac{\partial \mathcal{V}}{\partial S} + \frac{\partial \mathcal{V}}{\partial \mathcal{A}}\xi_t$. Putting this into (3.9) we obtain

$$
\begin{aligned}
0 &= \frac{\partial \mathcal{V}}{\partial S}\mu S dt + \frac{\partial \mathcal{V}}{\partial \mathcal{A}}\xi_t\mu S dt + \frac{\partial^2 \mathcal{V}}{\partial \mathcal{A}\partial S}\xi_t\sigma^2 S^2 dt + \frac{\partial \mathcal{V}}{\partial S}rS dt \\
&\quad + \frac{\partial \mathcal{V}}{\partial \mathcal{A}}\xi_t r S dt + \frac{\partial \mathcal{V}}{\partial t}dt + \frac{1}{2}\frac{\partial^2 \mathcal{V}}{\partial S^2}\sigma^2 S^2 dt + \frac{1}{2}\frac{\partial^2 \mathcal{V}}{\partial \mathcal{A}^2}\xi_t^2\sigma^2 S^2 dt \\
&\quad - r\mathcal{V}dt - \frac{\partial \mathcal{V}}{\partial S}\mu S dt - \frac{\partial \mathcal{V}}{\partial \mathcal{A}}\xi_t\mu S dt, \\
0 &= \left(\frac{\partial^2 \mathcal{V}}{\partial \mathcal{A}\partial S}\xi_t\sigma^2 S^2 + \frac{\partial \mathcal{V}}{\partial S}rS - r\mathcal{V}\right)dt \\
&\quad + \left(\frac{\partial \mathcal{V}}{\partial \mathcal{A}}\xi_t r S + \frac{\partial \mathcal{V}}{\partial t} + \frac{1}{2}\frac{\partial^2 \mathcal{V}}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 \mathcal{V}}{\partial \mathcal{A}^2}\xi_t^2\sigma^2 S^2\right)dt.
\end{aligned}
$$

This equality has to hold for all $(S, A, t) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \langle 0, T\rangle$ and therefore

$$
r\mathcal{V} = \frac{1}{2}\frac{\partial^2 \mathcal{V}}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 \mathcal{V}}{\partial \mathcal{A}^2}\xi_t^2\sigma^2 S^2 + \frac{\partial^2 \mathcal{V}}{\partial \mathcal{A}\partial S}\xi_t\sigma^2 S^2 + \frac{\partial \mathcal{V}}{\partial S}rS + \frac{\partial \mathcal{V}}{\partial \mathcal{A}}\xi_t r S + \frac{\partial \mathcal{V}}{\partial t}. \tag{3.10}
$$

It is easy to verify, that for $\mathcal{A}_t$ holds $\mathcal{A}_0 = A_0$ and $\mathcal{A}_T = A_T$. Because both financial derivatives have identical payoff at expiry, and taking non arbitrage principle into account, we conclude, that $V(S, A, 0) = \mathcal{V}(S, \mathcal{A}, 0)$ and $V(S, A, T) = \mathcal{V}(S, \mathcal{A}, T)$. Let us remark, that $V(S_t, A_t, t) \neq \mathcal{V}(S_t, A_t, t)$, but this holds

$$
V(S_t, A_t, t) = \mathcal{V}\left(S_t, \frac{t}{T}A_t + \xi_t S_t, t\right) \quad \forall t \in \langle 0, T\rangle. \tag{3.11}
$$

Comment: Even if (3.10) contains terms $\frac{\partial^2 V}{\partial A^2}$, $\frac{\partial^2 V}{\partial S^2}$, $\frac{\partial^2 V}{\partial A \partial S}$, it shows some type of singularity. Singularity of parabolic PDE can be seen from

$$\frac{1}{2}\sigma^2 S^2 \left(\frac{\partial}{\partial S} \quad \frac{\partial}{\partial A}\right) \begin{pmatrix} 1 & \xi_t \\ \xi_t & \xi_t^2 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial S} \\ \frac{\partial}{\partial A} \end{pmatrix},$$

where determinant of this operator is equals to zero.

3.1.1 Dimension reduction

Similarly as in classical averaging, we can achieve dimension reduction by introducing auxiliary function $\mathcal{W}(x,t)$ as follows

$$\mathcal{V}(S, \mathcal{A}, t) = \mathcal{A}\mathcal{W}(x,t), \quad \text{where } x = \frac{S}{\mathcal{A}}.$$

After straightforward calculations we obtain

$$\begin{aligned} \frac{\partial \mathcal{V}}{\partial S} &= \frac{\partial \mathcal{W}}{\partial x}, & \mathcal{A}\frac{\partial^2 \mathcal{V}}{\partial S^2} &= \frac{\partial^2 \mathcal{W}}{\partial x^2}, \\ \mathcal{A}\frac{\partial^2 \mathcal{V}}{\partial S \partial \mathcal{A}} &= -\frac{\partial^2 \mathcal{W}}{\partial x^2}x, & \frac{1}{\mathcal{A}}\frac{\partial \mathcal{V}}{\partial t} &= \frac{\partial \mathcal{W}}{\partial t}, \\ \frac{\partial \mathcal{V}}{\partial \mathcal{A}} &= -\frac{\partial \mathcal{W}}{\partial x}x + \mathcal{W}, & \mathcal{A}\frac{\partial^2 \mathcal{V}}{\partial \mathcal{A}^2} &= \frac{\partial^2 \mathcal{W}}{\partial x^2}x^2. \end{aligned}$$

After substitution those into (3.10) and divide both sides by $\mathcal{A}$ and some algebraic calculations we obtain final PDE

$$\frac{\partial \mathcal{W}}{\partial t} + r\left(x - x^2\xi_t\right)\frac{\partial \mathcal{W}}{\partial x} + \frac{1}{2}\sigma^2\left(x^2 - \xi_t x^3 + \xi_t^2 x^4\right)\frac{\partial^2 \mathcal{W}}{\partial x^2} - r\left(1 - \xi_t x\right)\mathcal{W} = 0 \quad (3.12)$$

with terminal condition $\mathcal{W}(x,T) = (x-1)^+$ for call and $\mathcal{W}(x,T) = (1-x)^+$ for put option. Let us recall, that $\lim_{t \to T} \xi_t = 0$ and therefore (3.12) for $t \to T$ has form

$$\frac{\partial \mathcal{W}}{\partial t} + rx\frac{\partial \mathcal{W}}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \mathcal{W}}{\partial x^2} - r\mathcal{W} = 0. \quad (3.13)$$

This equation is identical with PDE for plain vanilla European options. Indeed, for $t \to T$, S_t has only small influence to A_t and therefore we can take A_t as constant (analogy with X in plain vanilla). We only remark, that this PDE is better for numerical scheme.

3.2 Copula

In this section we briefly explain what copulas are, some history and our application on Asian option pricing. We also use some results from previous chapter.

History According to [18] the notation of copula came from A. Sklar (1959), when he explained relation between multidimensional probability function and its lower dimensional marginal densities. At the beginning, copula was used in theory of probabilistic metric spaces, but now, it plays also important role of mathematical statistics and financial mathematic.

Concept of copula Assume, that we have some multidimensional random variable $\xi = (\xi_1, \xi_2)^T$, where ξ_1 and ξ_2 are one dimensional random variables. It is easy to see, that if we know joint cumulative density function of multidimensional random variable ξ ($F_\xi(x_1, x_2)$), we also know marginal densities of random variables ξ_1 and ξ_2. According to Anděl [1] $F_{\xi_1}(x) = F_\xi(x, \infty)$ and $F_{\xi_2}(x) = F_\xi(\infty, x)$. But on the other hand, by knowing only marginal cumulative densities F_{ξ_1} and F_{ξ_2}, we are not able to construct F_ξ. One of the reason is that from marginal densities we do not know relations between random variables ξ_1 and ξ_2 (for example, covariance).

In statistics, a copula is used as general way of formulating multivariate cumulative distribution from knowing some marginal distribution and some dependence between them. There exist many families of copulas which differ in the detail of the dependence they represent. Typical usage for copulas:

- choose an appropriate copula family,
- fit copulas parameters to match your data.

Theorem 13. *Sklar's Theorem*
Let $F \in \mathcal{F}(F_1, F_2, \ldots, F_n)$ be an n-dimensional distribution function with marginals $F_1, F_2, \ldots, F_n$. Then there exists a copula C (i.e. an n-dimensional function on $\langle 0, 1\rangle^n$ with uniform marginals) such that

$$F(x_1, x_2, \ldots, x_n) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)).$$

This theorem lets us construct multidimensional random variable by choosing the copula family.

Gaussian copula There are a lot of copula families (e.g. Archimedean copulas, Periodic copula, Gaussian copula). Let Φ_ρ is standard bivariate normal cumulative distribution function with correlation ρ, then the Gaussian copula is

$$C_\rho(u, v) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v)), \tag{3.14}$$

where $u, v \in \langle 0, 1\rangle$ and Φ is standard normal cumulative distribution function. Gaussian copula density function is given by

$$c_\rho(u, v) = \frac{d}{du\, dv} C_\rho(u, v) = \frac{\varphi_{X,Y,\rho}(\Phi^{-1}(u), \Phi^{-1}(v))}{\varphi(\Phi^{-1}(u))\varphi(\Phi^{-1}(v))}, \tag{3.15}$$

where

$$\varphi_{X,Y,\rho}(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}(x^2 - 2xy\rho + y^2)\right) \tag{3.16}$$

is joint density function of two-dimensional normal distributed random variable.

Comment: In case of two-dimensional normal distributed random variable ξ, it holds

$$f_\xi(x,y) = \frac{d}{dx\ dy} C_\rho(\Phi^{-1}(x), \Phi^{-1}(y)) = \varphi_{X,Y,\rho}(x,y).$$

On Figure 3.1 we can see 3D plot (left side) and contour plot (right side) of function $\varphi_{X,Y,\rho}(x,y)$ for different parameter $\rho = 0; 0.9; -0.9$.

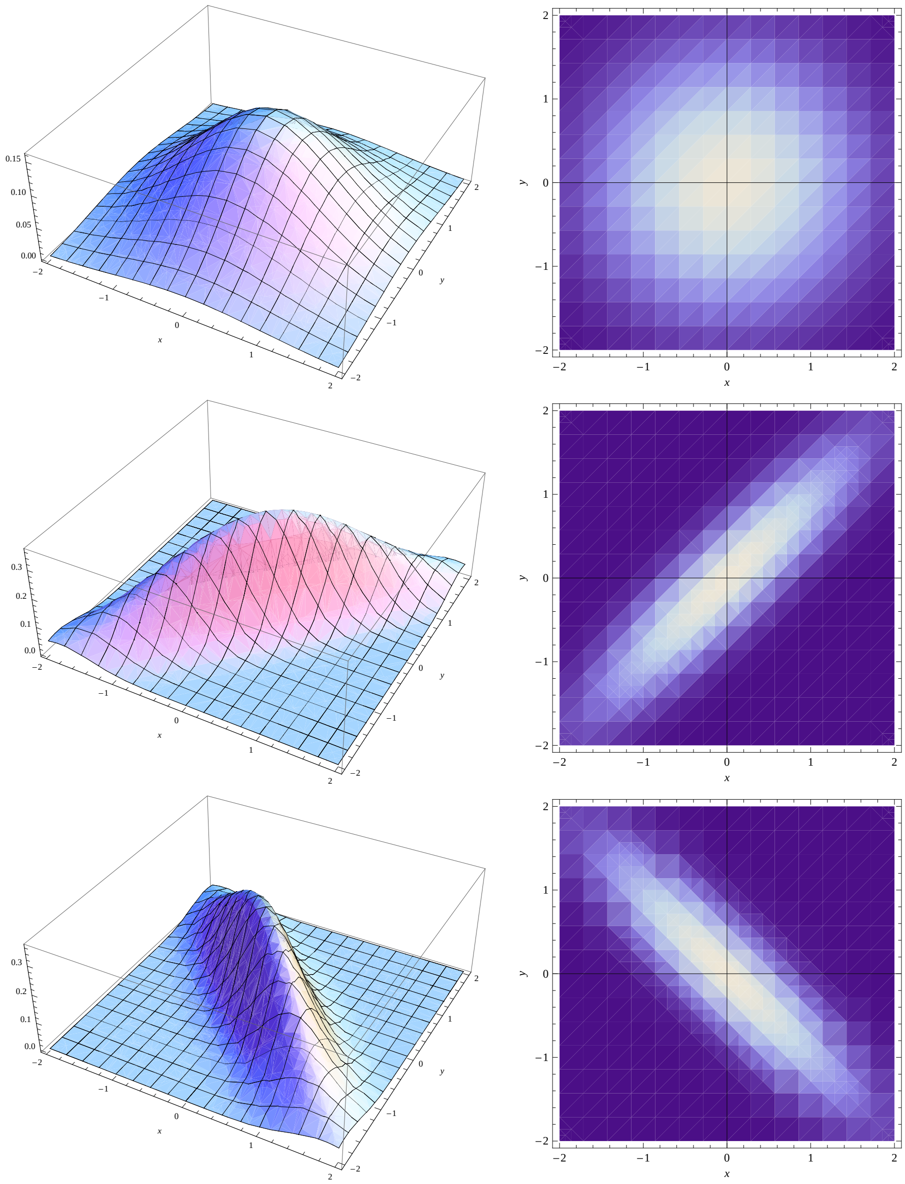

Figure 3.1: 3D plot (left) and contour (right) of function φ_ρ for different parameter $\rho = 0; 0.9; -0.9$.

3.2.1 Asian option pricing using copula

We already know, that price of average rate call option is given by

$$V(S, A, 0) = e^{-rT}\mathbb{E}_Q[(A_T - S_T)^+], \tag{3.17}$$

where Q is risk less probability measure. From Chapter 2, we know exact distribution of random variable S_T. We also know approximate distribution of A_T (lognormal distribution). But these two random variables (S_T and A_T) is not independent. At top in Figure 3.2 we present Monte Carlo simulation (parameters: $S_0 = 1, r = 0.4, \sigma = 0.6, T = 1$) of this two random variables.

At bottom we present distribution of two-dimensional random variable $\xi = (X, Y)^T$, which has independent components and have same marginal distributions as random variable above.

If $(X, Y)^T$ is two-dimensional normally distributed random variable with correlation ρ. Let $\mu_X = E[X], \sigma_X^2 = Var[X]$, $\mu_Y = E[Y], \sigma_Y^2 = Var[Y]$. Then (cf. Gao [7]) joint PDF of (e^X, e^Y) is for $(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+$

$$f(x, y; \mu_X, \sigma_X, \mu_Y, \sigma_Y, \rho) = \tag{3.18}$$
$$\frac{\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{\ln(x)-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{\ln(x)-\mu_X}{\sigma_X}\right)\left(\frac{\ln(y)-\mu_Y}{\sigma_Y}\right) + \left(\frac{\ln(y)-\mu_Y}{\sigma_Y}\right)^2\right]\right\}}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}xy},$$

otherwise 0.

So, let's go back to our problem. Recall, that $E[S_T] = S_0e^{rT}$, $Var[S_T] = S_0^2(e^{\sigma^2 T} - 1)e^{2rT}$ and therefore (see section 2.3) coefficients μ_S, σ_S are

$$\mu_S = \ln(E[S_T]) - \frac{1}{2}\ln\left(1 + \frac{Var[S_T]}{E[S_T]^2}\right), \tag{3.19}$$

$$\sigma_S = \sqrt{\ln\left(1 + \frac{Var[S_T]}{E[S_T]^2}\right)}. \tag{3.20}$$

For random variable A_T are coefficients μ_A, σ_A (see section 2.3) following:

$$\mu_A = 2\ln\left(S_0\frac{\exp(\alpha) - 1}{\alpha}\right) - \frac{1}{2}\ln\left(S_0^2\frac{2}{\alpha}\left[\frac{\exp(\beta) - \exp(\alpha)}{\beta - \alpha} - \frac{\exp(\beta) - 1}{\beta}\right]\right),$$

$$\sigma_A = \sqrt{\ln\frac{\frac{2}{\alpha}\left[\frac{\exp(\beta)-\exp(\alpha)}{\beta-\alpha} - \frac{\exp(\beta)-1}{\beta}\right]}{\left(\frac{\exp(\alpha)-1}{\alpha}\right)^2}},$$

where $\alpha = rT$, $\beta = 2(r + \frac{1}{2}\sigma^2)T$.

If we use density function (3.18), then the only one unknown parameter in $f(s, a; \mu_S, \sigma_S, \mu_A, \sigma_A, \rho)$ is ρ. But ρ cannot be explicitly estimated, but we choose ρ to fit know covariance $E(S_T, A_T)$:

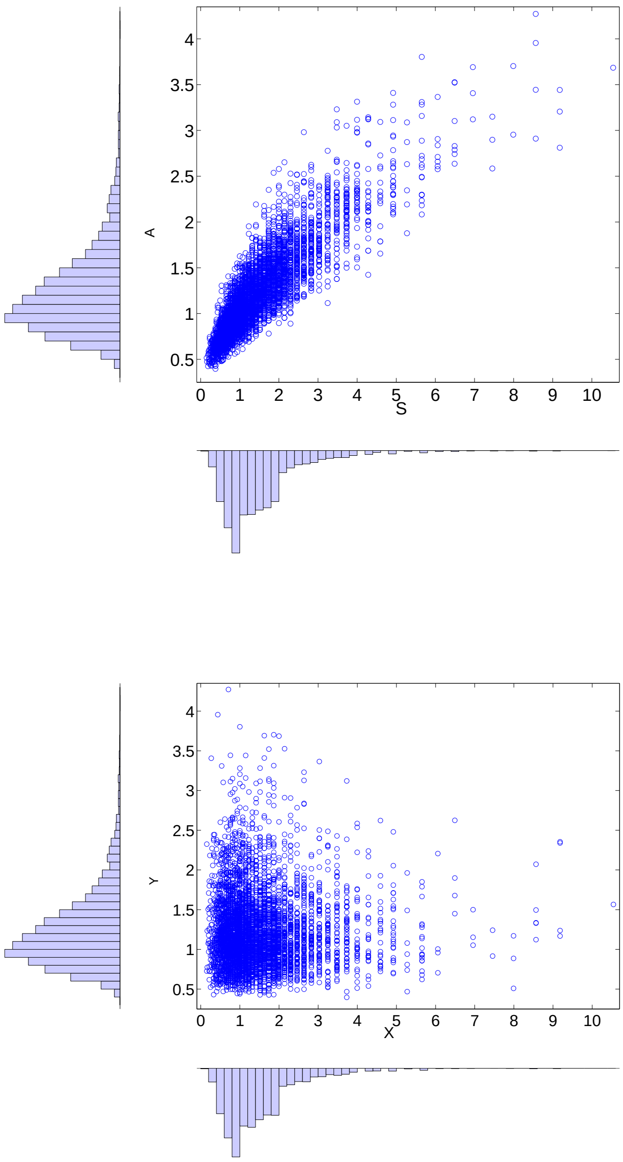

Figure 3.2: Simulation of two-dimensional random variable $(S_T, A_T)^T$ (above). At bottom, there is a distribution with independent components and the same marginal distributions.

$$\hat{\rho} = \arg \min_{\rho \in (-1,1)} \left| \int_0^\infty \int_0^\infty s \cdot a \cdot f_\rho(s,a) \; ds \; da - \zeta \right|, \tag{3.21}$$

where $\zeta = E[S_T, A_T] = S_0^2 \frac{\exp(\beta)-\exp(\alpha)}{\beta-\alpha}$ (see Lemma 5). Usually minimum is equals 0.

Finally, the price of Asian average strike Call and Put option can be approximated

$$Call = \int_0^\infty \int_0^\infty (s-a)^+ f_{\hat{\rho}}(s,a) \; da \; ds = \int_0^\infty \int_0^s (s-a) f_{\hat{\rho}}(s,a) \; da \; ds, \tag{3.22}$$

$$Put = \int_0^\infty \int_0^\infty (a-s)^+ f_{\hat{\rho}}(s,a) \; da \; ds = \int_0^\infty \int_0^a (s-a) f_{\hat{\rho}}(s,a) \; ds \; da, \tag{3.23}$$

where $f_{\hat{\rho}}(s,a) = f(s, a, \mu_S, \sigma_S, \mu_A, \sigma_A, \hat{\rho})$.

3.2.2 Numerical results

In Table 3.1 we show a comparison of our results obtained by (3.22), Hansen [8], finite difference and Monte Carlo simulations for call option. In Table 3.2 we compare that same method for Put options.

In Figure 3.3 we present dependence of $\hat{\rho}$ on different parameters. We only remark, that $\hat{\rho} = \hat{\rho}(S)$ is constant function, which gives sense, because correlation should be independent from e.g. currency in which we compute.

Table 3.1: A comparison of different method for calculates Average strike Asian call option with parameter value $S_0 = 100$. Jørgensen is calculated using approximation formula derived in Hansen [8]. Copula is calculated using (3.22).

r	T	σ	Jørgensen	Finite difference	Copula	MC
0.03	1/12	0.20	1.39	1.39	1.39153	1.391519
	1/12	0.30	2.06	2.06	2.05437	2.061377
	1/12	0.40	2.72	2.72	2.71660	2.707180
	4/12	0.20	2.91	2.91	2.90286	2.897178
	4/12	0.30	4.23	4.23	4.21707	4.174144
	4/12	0.40	5.56	5.55	5.52468	5.556474
	7/12	0.20	3.95	3.95	3.93962	3.895464
	7/12	0.30	5.70	5.69	5.66275	5.609880
	7/12	0.40	7.45	7.42	7.37095	7.437844
0.05	1/12	0.20	1.45	1.43	1.43392	1.441781
	1/12	0.30	2.10	2.10	2.09590	2.107424
	1/12	0.40	2.76	2.76	2.75734	2.736002
	4/12	0.20	3.08	3.08	3.07467	3.119073
	4/12	0.30	4.40	4.39	4.38120	4.364677
	4/12	0.40	5.72	5.71	5.68345	5.686662
	7/12	0.20	4.25	4.25	4.24205	4.250102
	7/12	0.30	5.98	5.98	5.94757	6.013364
	7/12	0.40	7.72	7.70	7.64326	7.775399
0.07	1/12	0.20	1.48	1.49	1.47710	1.467870
	1/12	0.30	2.14	2.14	2.13784	2.140409
	1/12	0.40	2.80	2.80	2.79846	2.775492
	4/12	0.20	3.26	3.26	3.25228	3.219071
	4/12	0.30	4.56	4.56	4.54900	4.578202
	4/12	0.40	5.88	5.87	5.84473	5.952404
	7/12	0.20	4.57	4.57	4.55718	4.631852
	7/12	0.30	6.27	6.27	6.24030	6.231548
	7/12	0.40	8.00	7.98	7.92107	8.018041

Table 3.2: A comparison of different method for calculates Average strike Asian put option with parameter value $S_0 = 100$. Jørgensen is calculated using approximation formula derived in Hansen [8]. Copula is calculated using (3.23).

r	T	σ	Jørgensen	Finite difference	Copula	MC
0.03	1/12	0.20	1.27	1.27	1.26663	1.270262
	1/12	0.30	1.93	1.93	1.92948	1.906286
	1/12	0.40	2.60	2.59	2.59171	2.613912
	4/12	0.20	2.41	2.41	2.40452	2.412804
	4/12	0.30	3.73	3.73	3.71873	3.753623
	4/12	0.40	5.06	5.05	5.02644	5.036301
	7/12	0.20	3.08	3.08	3.06971	3.113019
	7/12	0.30	4.83	4.82	4.79283	4.807147
	7/12	0.40	6.58	6.55	6.50104	6.598398
0.05	1/12	0.20	1.23	1.23	1.22587	1.209759
	1/12	0.30	1.89	1.89	1.88786	1.910832
	1/12	0.40	2.55	2.55	2.54929	2.560727
	4/12	0.20	2.25	2.25	2.24595	2.267949
	4/12	0.30	3.57	3.56	3.55248	3.592209
	4/12	0.40	4.89	4.88	4.85473	4.833621
	7/12	0.20	2.81	2.81	2.79779	2.853429
	7/12	0.30	4.54	4.53	4.50331	4.559098
	7/12	0.40	6.28	6.26	6.19900	6.196700
0.07	1/12	0.20	1.19	1.19	1.18600	1.177103
	1/12	0.30	1.85	1.85	1.84674	1.846956
	1/12	0.40	2.51	2.51	2.50736	2.506066
	4/12	0.20	2.10	2.10	2.09463	2.143257
	4/12	0.30	3.41	3.40	3.39136	3.397994
	4/12	0.40	4.72	4.71	4.68708	4.690242
	7/12	0.20	2.55	2.56	2.54302	2.537715
	7/12	0.30	4.26	4.26	4.22614	4.271495
	7/12	0.40	5.99	5.97	5.90691	6.029447

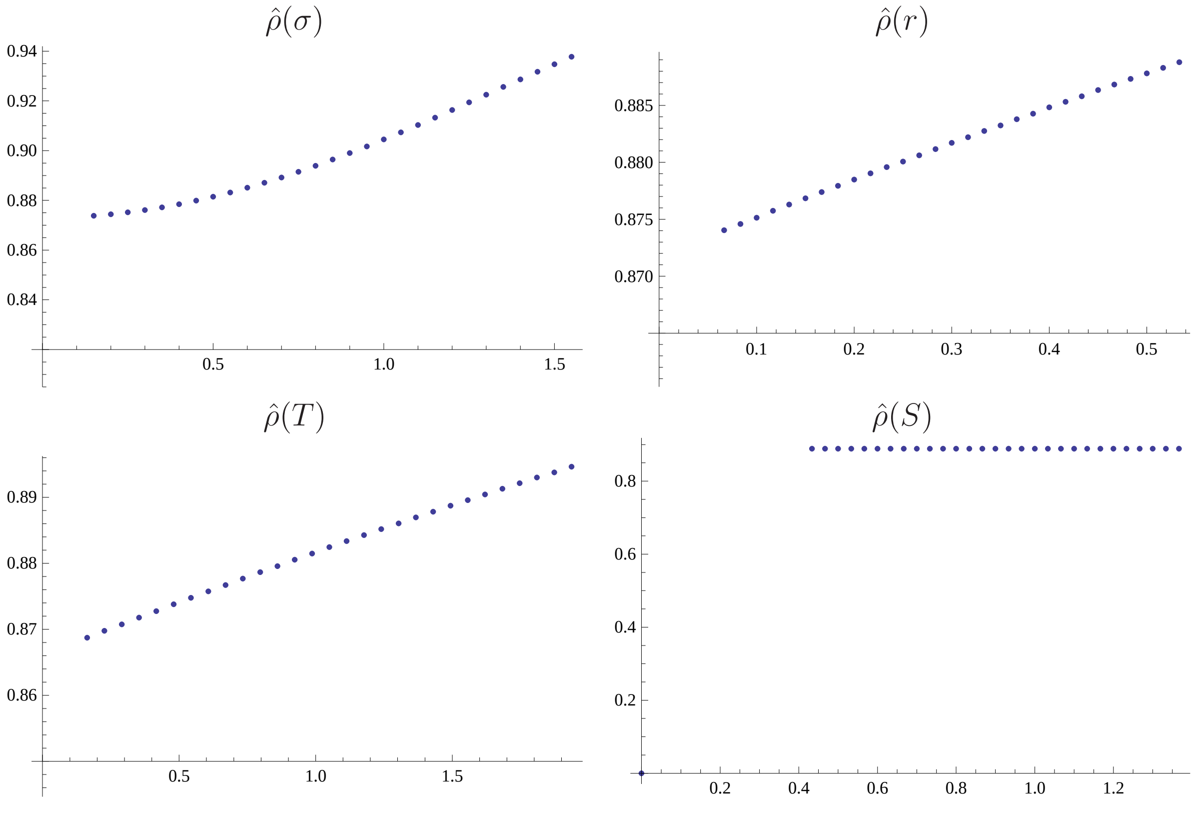

Figure 3.3: Depends $\hat{\rho}$ on σ, r, T, S.

Chapter 4

Transformation method for American-style of average strike Asian options

Finally, we make some remarks on why *linear* systems are so important. The answer is simple: bacause we can solve them!

Richard Feynman

In this Chapter we discuss a transformation methods applied to pricing Asian options. Let us recall, that American-style option is an option, which can be exercised at any time $t \in \langle 0, T \rangle$ before maturity. We focus our attention to problem of the exercise boundary and the optimal stopping time.

Transformation methods applied in options pricing problem was developed by Ševčovič et al. (cf. [21], [19], [22]).

4.1 Partial differential equation for pricing the Asian option

It is well know (cf. Ševčovič[22], Kwok [10], Dai [5]), that PDE for pricing Asian option is in the from

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r-q) S \frac{\partial V}{\partial S} + A f\left(\frac{S}{A}, t\right) \frac{\partial V}{\partial A} - rV = 0, \qquad (4.1)$$

where $S, A > 0,\ t \in (0, T)$ and

$$f(x,t) = \begin{cases} \dfrac{x-1}{t}, & \text{for arithmetic averaging,} \\ \dfrac{\lambda(x-1)}{1-exp(-\lambda t)}, & \text{for exponentially weighted arithmetic averaging,} \\ \dfrac{\ln(x)}{t}, & \text{for geometric averaging.} \end{cases} \tag{4.2}$$

For average rate Asian call option is a terminal payoff in the form

$$V(S, A, t) = \max(S - A, 0) \tag{4.3}$$

and for Asian put option is terminal payoff in the form

$$V(S, A, t) = \max(A - S, 0). \tag{4.4}$$

It is also well know (cf. Kwok [10], Dai [5]), that in our case we can achieve dimension reduction introducing new state variable $x = \dfrac{S}{A}$ and new function $W(x, \tau) = \dfrac{1}{A}V(S, A, t)$, where $\tau = T - t$.

After some computations, one get that

$$\frac{\partial V}{\partial S} = \frac{\partial W}{\partial x}, \qquad A\frac{\partial V^2}{\partial S^2} = \frac{\partial^2 W}{\partial x^2},$$
$$\frac{\partial V}{\partial A} = W - x\frac{\partial W}{\partial x}, \qquad A\frac{\partial^2 V}{\partial A^2} = x^2\frac{\partial^2 W}{\partial x^2},$$
$$\frac{\partial V}{\partial t} = -A\frac{\partial W}{\partial \tau}.$$

After putting these into (4.1) and after some calculations, we obtain following PDR for function $W(x, \tau)$:

$$\frac{\partial W}{\partial \tau} + [f(x, T-\tau) - r + q]\, x\frac{\partial W}{\partial x} - \frac{\sigma^2}{2}x^2\frac{\partial^2 W}{\partial x^2} + (r - f(x, T-\tau))W = 0, \tag{4.5}$$

where $\tau \in (0, T),\ x > 0$. Initial condition for (4.5) is

$$W(x, 0) = \frac{1}{A}V(S, A, T) = \frac{1}{A}(S - A)^+ = (x - 1)^+$$

for call options case and $W(x, 0) = (1-x)^+$, for put options, where $(\zeta)^+ = \max(\zeta, 0)$.

4.1.1 American-style of Asian call options

According to Kwok [10], the set

$$\mathcal{E} = \{(S, A, t) \in \langle 0, \infty) \times \langle 0, \infty) \times \langle 0, T), V(S, A, t) = V(S, A, T)\}$$

is exercise region. For case of call options, there exist function $S_f = S_f(A, t)$, that $\mathcal{E} = \{(S, A, t) \in \langle 0, \infty) \times \langle 0, \infty) \times \langle 0, T), S \geq S_f(A, t)\}$. We omit some technical details here (for more details see e.g. Dai & Kwok [5]), we only conclude, that the free boundary function $S_f(A, t)$ can be expressed as $S_f(A, t) = Ax_f(t)$. Spatial domain for (4.5) is

$$0 < x < \rho(\tau), \tau \in (0, T),$$

where $\rho(\tau) = x_f(T-\tau)$. From C^1 continuity of $V(S, A, t)$ at $(S_f(A, t), A, t)$ it implies, that

$$\frac{\partial V}{\partial S}(S_f(A, t), A, t) = 1 \tag{4.6}$$

and from payoff diagram, we have

$$V(S_f(A, t), A, t) = S_f(A, t) - A, \tag{4.7}$$

for $A > 0$ and $t \in (0, T)$. In terms of out new variable x, we ended by following boundary conditions for function $W(x, \tau)$:

$$W(0, \tau) = 0, \qquad W(x, \tau) = x - 1, \frac{\partial W}{\partial x}(x, \tau) = 1, \text{ at } x = \rho(\tau), \tag{4.8}$$

for $\tau \in (0, T)$ and the initial condition for $W(x, \tau)$ is

$$W(x, 0) = \max\{x - 1, 0\}, \quad \forall x > 0. \tag{4.9}$$

Let us remark, that this is a free boundary problem, because our spatial domain depends on boundary function ρ.

4.1.2 Fixed domain transformation

Following Ševčovič ([22, 2]) we introduce new variable ξ and an auxiliary function $\Pi = \Pi(\xi, \tau)$ defined as:

$$\xi = \ln\left(\frac{\rho(\tau)}{x}\right), \tag{4.10}$$

$$\Pi(\xi, \tau) = W(x, \tau) - x\frac{\partial W}{\partial x}(x, \tau). \tag{4.11}$$

After straightforward calculations we obtain:

$$\frac{\partial \Pi}{\partial \xi} = x^2 \frac{\partial^2 W}{\partial x^2}, \qquad -\frac{\partial^2 \Pi}{\partial \xi^2} - 2\frac{\partial \Pi}{\partial \xi} = x^3 \frac{\partial^3 W}{\partial x^3}, \qquad \frac{\partial \Pi}{\partial \tau} + \frac{\dot{\rho}}{\rho}\frac{\partial \Pi}{\partial \xi} = \frac{\partial W}{\partial \tau} - x\frac{\partial^2 W}{\partial x \partial \tau}.$$

If we differentiate (4.5) with respect to x and then multiply by x we have

$$
\begin{aligned}
0 &= x\frac{\partial}{\partial x}\left[\frac{\partial W}{\partial \tau} + [f(x,T-\tau) - r + q]\, x\frac{\partial W}{\partial x} - \frac{\sigma^2}{2}x^2\frac{\partial^2 W}{\partial x^2}\right] \\
&\quad + x\frac{\partial}{\partial x}\left[(r - f(x,T-\tau))W\right], \\
0 &= x\frac{\partial^2 W}{\partial \tau \partial x} + \frac{\partial f}{\partial x}x^2\frac{\partial W}{\partial x} + [f(x,T-\tau) - r + q]\, x\frac{\partial W}{\partial x} \\
&\quad + [f(x,T-\tau) - r + q]\, x^2\frac{\partial^2 W}{\partial x^2} - \sigma^2 x^2\frac{\partial^2 W}{\partial x^2} \\
&\quad - \frac{\sigma^2}{2}x^3\frac{\partial^3 W}{\partial x^3} - x\frac{\partial f}{\partial x}W + (r - f(x,T-\tau))x\frac{\partial W}{\partial x}. \qquad (4.12)
\end{aligned}
$$

Subtracting (4.12) from (4.5), we have

$$
\begin{aligned}
0 &= \frac{\partial W}{\partial \tau} + [f(x,T-\tau) - r + q]\, x\frac{\partial W}{\partial x} - \frac{\sigma^2}{2}x^2\frac{\partial^2 W}{\partial x^2} + (r - f(x,T-\tau))W \\
&\quad - x\frac{\partial^2 W}{\partial \tau \partial x} - \frac{\partial f}{\partial x}x^2\frac{\partial W}{\partial x} - [f(x,T-\tau) - r + q]\, x\frac{\partial W}{\partial x} \\
&\quad - [f(x,T-\tau) - r + q]\, x^2\frac{\partial^2 W}{\partial x^2} + \sigma^2 x^2\frac{\partial^2 W}{\partial x^2} \\
&\quad + \frac{\sigma^2}{2}x^3\frac{\partial^3 W}{\partial x^3} + x\frac{\partial f}{\partial x}W - (r - f(x,T-\tau))x\frac{\partial W}{\partial x}, \\
0 &= \frac{\partial W}{\partial \tau} - x\frac{\partial^2 W}{\partial \tau \partial x} + r\left(W - x\frac{\partial W}{\partial x}\right) + \frac{\sigma^2}{2}x^3\frac{\partial^3 W}{\partial x^3} \\
&\quad + x\frac{\partial f}{\partial x}\left(W - x\frac{\partial W}{\partial x}\right) - f(x,T-\tau)\left(W - x\frac{\partial W}{\partial x}\right) \\
&\quad x^2\frac{\partial^2 W}{\partial x^2}\left[-f(x,T-\tau) + r - q + \frac{\sigma^2}{2}\right], \\
0 &= \frac{\partial \Pi}{\partial \tau} + \left[\underbrace{\frac{\dot\rho}{\rho} - f(\rho e^{-\xi}, T-\tau) + r - q - \frac{\sigma^2}{2}}_{a(\xi,\tau)}\right]\frac{\partial \Pi}{\partial \xi} - \frac{\sigma^2}{2}\frac{\partial^2 \Pi}{\partial \xi^2} \\
&\quad + \left[\underbrace{r + x\frac{\partial f}{\partial x} - f(x,T-\tau)\bigg|_{x=\rho e^{-\xi}}}_{b(\xi,\tau)}\right]\Pi. \qquad (4.13)
\end{aligned}
$$

The initial condition for the solution $\Pi(\xi,0)$ is

$$
\Pi(\xi,0) = \begin{cases} -1, & \xi < \ln\rho(0), \\ 0, & \xi > \ln\rho(0). \end{cases} \qquad (4.14)
$$

Taking into account equations (4.8) we conclude that we have to impose Dirichlet boundary conditions for $\Pi(\xi,\tau)$ in the form

$$
\Pi(0,\tau) = -1, \qquad \Pi(\infty,\tau) = 0. \qquad (4.15)
$$

Since $\frac{\partial W}{\partial x}(\rho(\tau),\tau) = 1$ we obtain, that $\frac{\partial W}{\partial \tau}(\rho(\tau),\tau) = 0$ at $x = \rho(\tau)$. If we assume C^2 continuity of $\Pi(\xi,\tau)$ up to the boundary $\xi = 0$ we obtain for $x \to \rho(\tau)$

$$x^2 \frac{\partial^2 W}{\partial x^2}(x,\tau) \to \frac{\partial \Pi}{\partial \xi}(0,\tau), \qquad\qquad x\frac{\partial W}{\partial x} \to \rho(\tau). \tag{4.16}$$

Then passing limit $x \to \rho(\tau)$ in (4.5) we obtain following algebraic constraint between $\rho(\tau)$ and $\Pi(\xi,\tau)$

$$\begin{aligned} -(r-q)\rho(\tau) - \frac{\sigma^2}{2}\frac{\partial \Pi}{\partial \xi}(0,\tau) + r(\rho(\tau)-1) + f(\rho(\tau), T-\tau) &= 0, \\ q\rho(\tau) - r - \frac{\sigma^2}{2}\frac{\partial \Pi}{\partial \xi}(0,\tau) + f(\rho(\tau), T-\tau) &= 0. \end{aligned} \tag{4.17}$$

Notice, that this expression contains term $\frac{\partial \Pi}{\partial \xi}(0,\tau)$ and therefore this is not suitable for numerical scheme, because the whole solution is sensitive of this term. Bokes & Ševčovič [2] suggested an equivalent form. They integrate equation (4.13) with respect to $\xi \in (0,\infty)$ and taking into account boundary conditions for $\Pi(\xi,\tau)$ and $\frac{\partial \Pi}{\partial \xi}(\infty,\tau) = 0$ and using equality (4.17) they derived following differential equation:

$$\begin{aligned} 0 &= \frac{d}{d\tau}\left(\ln \rho(\tau) + \int_0^\infty \Pi(\xi,\tau)d\xi\right) + q\rho(\tau) - q - \frac{\sigma^2}{2} \\ &\quad + \int_0^\infty (r - f(\rho(\tau)e^{-\xi}, T-\tau))\Pi(\xi,\tau)d\xi. \end{aligned} \tag{4.18}$$

4.1.3 Derivation of $\rho(0^+)$

Limit of early exercise boundary at expiry for the continuous arithmetic average type of an Asian option has been derived e.q. by Dai & Kwok [5] or Bokes & Ševčovič [2, 22]. Derivation has been deduced from the smoothness of the solution of PDE. We derive the same result in another way.

Let us assume, that we are close to expiry at $t = T - \Delta t$, where $0 < \Delta t \ll 1$, current price of underlying asset is $S_{T-\Delta t}$ and current average is $A_{T-\Delta t}$. We already know, that price at time T can be either $S_T = S_{T-\Delta t}u$ with probability p or $S_T = S_{T-\Delta t}d$ with probability $1-p$. In some special cases we can also express A_T in terms of $A_{T-\Delta t}, S_{T-\Delta t}, u, p, d$.

Case of arithmetic averaging Asian call options In this case, it holds, that

$$A_T = \frac{1}{T}\int_0^T S_\tau d\tau = \frac{1}{T}\left(\int_0^{T-\Delta t} S_\tau d\tau + \int_{T-\Delta t}^T S_\tau d\tau\right) = \frac{T-\Delta t}{T}A_{T-\Delta t} + \frac{1}{T}\int_{T-\Delta t}^T S_\tau d\tau$$

$$\approx \frac{T-\Delta t}{T}A_{T-\Delta t} + \frac{\Delta t}{T}S_{T-\Delta t}\xi,$$

where ξ is alternative random variable

$$\xi = \begin{cases} u = e^{\sigma\sqrt{\Delta t}}, & \text{with probability } p, \\ d = e^{-\sigma\sqrt{\Delta t}}, & \text{with probability } 1-p \end{cases} \tag{4.19}$$

and $p = \dfrac{e^{(r-q)\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$. We exercise option at time $t = T - \Delta t$ only if expected value at time T is $\leq$ against current value. It means, that we exercise option for values $(S_{T-\Delta t}, A_{T-\Delta t})$ for which holds

$$\begin{aligned} (S_{T-\Delta t} - A_{T-\Delta t})^+ e^{r\Delta t} &= \mathbb{E}_Q (S_T - A_T)^+, \\ (S_{T-\Delta t} - A_{T-\Delta t})^+ e^{r\Delta t} &= \mathbb{E}_Q \left(S_{T-\Delta t}\xi - \frac{T-\Delta t}{T} A_{T-\Delta t} - \frac{\Delta t}{T} S_{T-\Delta t}\xi \right)^+, \\ (S_{T-\Delta t} - A_{T-\Delta t})^+ e^{r\Delta t} &= \mathbb{E}_Q \left(S_{T-\Delta t}\xi - A_{T-\Delta t} \right)^+ \frac{T-\Delta t}{T}. \end{aligned}$$

We can divide both sides by $A_{T-\Delta t}$ and we have

$$(x_{T-\Delta t} - 1)^+ e^{r\Delta t} = \mathbb{E}_Q \left(x_{T-\Delta t}\xi - 1 \right)^+ \frac{T-\Delta t}{T}.$$

American property guarantees us, that $x_{T-\Delta t} \geq 1$ then

$$\begin{aligned} (x_{T-\Delta t} - 1) e^{r\Delta t} &= \mathbb{E}_Q \left(x_{T-\Delta t}\xi - 1 \right) \frac{T-\Delta t}{T}, \\ (x_{T-\Delta t} - 1) e^{r\Delta t} &= p \left(x_{T-\Delta t} u - 1 \right) \frac{T-\Delta t}{T} + (1-p) \left(x_{T-\Delta t} d - 1 \right) \frac{T-\Delta t}{T}. \end{aligned}$$

After straightforward calculations we conclude that

$$x_{T-\Delta t} = \frac{e^{(q-r)\Delta t} \left(\Delta t + T(e^{r\Delta t} - 1) \right)}{\Delta t + T(e^{q\Delta t} - 1)}. \tag{4.20}$$

Using L' Hospital's rule in $\lim_{\Delta t \to 0} x_{T-\Delta t}$ and taking into account condition $x \geq 1$ we conclude that

$$\lim_{\tau \to 0} \rho(\tau) = \max \left\{ \frac{1 + rT}{1 + qT}, 1 \right\}. \tag{4.21}$$

Case of arithmetic averaging Asian put options Derivation in case of arithmetic Asian put option is similar with case of call option. Difference is only in payoff diagram. American property restricts us with $x_t \leq 1$ and therefore

$$\lim_{\tau \to 0} \rho(\tau) = \min \left\{ \frac{1 + rT}{1 + qT}, 1 \right\}. \tag{4.22}$$

Case of exponentially arithmetic averaging Asian call options Let us introduce auxiliary function $\Psi(\zeta) = \int_0^\zeta e^{-\lambda\xi} d\xi = \frac{1}{\lambda}(1 - \exp(-\lambda\zeta))$. Then

$$A_T = \frac{1}{\Psi(T)} \int_0^T e^{-\lambda(T-\tau)} S_\tau d\tau,$$

$$= \frac{1}{\Psi(T)} \left(\int_0^{T-\Delta t} e^{-\lambda(T-\Delta t+\Delta t-\tau)} S_\tau d\tau + \int_{T-\Delta t}^T e^{-\lambda(T-\tau)} S_\tau d\tau \right),$$

$$\approx \frac{1}{\Psi(T)} \left(e^{-\lambda\Delta t} \Psi(T-\Delta t) A_{T-\Delta t} + \Delta t S_{T-\Delta t} \xi \right).$$

Similarly as in case of arithmetic averaging

$$\begin{aligned}
(S_{T-\Delta t} - A_{T-\Delta t})^+ e^{r\Delta t} &= \mathbb{E}_Q (S_T - A_T)^+, \\
(S_{T-\Delta t} - A_{T-\Delta t})^+ e^{r\Delta t} &= \mathbb{E}_Q \left(S_{T-\Delta t}\xi \left(1 - \frac{\Delta t}{\Psi(T)}\right) - \frac{\Psi(T-\Delta t)}{\Psi(T)} e^{-\lambda\Delta t} A_{T-\Delta t} \right)^+.
\end{aligned}$$

Dividing both sides by $A_{T-\Delta t}$ and using American property

$$(x_{T-\Delta t} - 1) e^{r\Delta t} = \mathbb{E}_Q \left(x_{T-\Delta t}\xi \left(1 - \frac{\Delta t}{\Psi(T)}\right) - \frac{\Psi(T-\Delta t)}{\Psi(T)} e^{-\lambda\Delta t} \right)^+.$$

After straightforward calculations we conclude that

$$x_{T-\Delta t} = e^{-(\lambda+r-q)\Delta t} \frac{e^{\lambda\Delta t} - e^{\lambda T} - e^{(\lambda+r)\Delta t} + e^{\lambda T + (\lambda+r)\Delta t}}{(e^{\lambda T} - 1)(e^{q\Delta t} - 1) + e^{\lambda T}\lambda\Delta t}. \tag{4.23}$$

Using L' Hospital's rule in $\lim_{\Delta t \to 0} x_{T-\Delta t}$ and taking into account condition $x \geq 1$ we conclude that

$$\lim_{\tau \to 0} \rho(\tau) = \max\left\{ \frac{\lambda + r(1 - e^{-\lambda T})}{\lambda + q(1 - e^{-\lambda T})}, 1 \right\}. \tag{4.24}$$

Case of exponentially arithmetic averaging Asian put options Similarly as before, American property restricts us with $x_t \leq 1$ and therefore

$$\lim_{\tau \to 0} \rho(\tau) = \min\left\{ \frac{\lambda + r(1 - e^{-\lambda T})}{\lambda + q(1 - e^{-\lambda T})}, 1 \right\}. \tag{4.25}$$

Case of geometric averaging Asian call options Similarly as before we at first write A_T in terms $A_{T-\Delta t}, S_{T-\Delta t}$.

$$\ln A_T = \frac{1}{T} \int_0^T \ln S_\tau d\tau = \frac{1}{T} \frac{T-\Delta t}{T-\Delta t} \int_0^{T-\Delta t} \ln S_\tau d\tau + \frac{1}{T} \int_{T-\Delta t}^T \ln S_\tau d\tau$$

$$= \frac{T-\Delta t}{T} \ln A_{T-\Delta t} + \frac{1}{T} \int_{T-\Delta t}^T \ln S_\tau d\tau \approx \frac{T-\Delta t}{T} \ln A_{T-\Delta t} + \frac{\Delta t}{T} \ln(S_{T-\Delta t}\xi).$$

Therefore

$$A_T = A_{T-\Delta t}\left(1 + \frac{\Delta t}{T}\ln(x_{T-\Delta t}\xi)\right).$$

Taking payoff into account, we obtain, that it has to hold

$$\begin{aligned}(S_{T-\Delta t} - A_{T-\Delta t})^+ e^{r\Delta t} &= \mathbb{E}_Q(S_T - A_T)^+, \\ (S_{T-\Delta t} - A_{T-\Delta t})^+ e^{r\Delta t} &= \mathbb{E}_Q\left(S_{T-\Delta t}\xi - A_{T-\Delta t}\left(1 + \frac{\Delta t}{T}\ln(x_{T-\Delta t}\xi)\right)\right)^+.\end{aligned}$$

Dividing both sides by $A_{T-\Delta t}$ and using American property ($x \geq 1$)

$$\begin{aligned}(x_{T-\Delta t} - 1)e^{r\Delta t} &= \mathbb{E}_Q\left(x_{T-\Delta t}\xi - \left(1 + \frac{\Delta t}{T}\ln(x_{T-\Delta t}\xi)\right)\right), \\ (x_{T-\Delta t} - 1)e^{r\Delta t} &= x_{T-\Delta t}\mathbb{E}_Q(\xi) - 1 - \frac{\Delta t}{T}\ln(x_{T-\Delta t}) - \frac{\Delta t}{T}\mathbb{E}_Q(\ln(\xi)), \\ (x_{T-\Delta t} - 1)e^{r\Delta t} &= x_{T-\Delta t}e^{(r-q)\Delta t} - 1 - \frac{\Delta t}{T}\ln(x_{T-\Delta t}) - \frac{\Delta t}{T}\mathbb{E}_Q(\ln(\xi)), \\ (x_{T-\Delta t} - 1)(1 + r\Delta t) &= x_{T-\Delta t}(1 + (r-q)\Delta t) - 1 - \frac{\Delta t}{T}\ln(x_{T-\Delta t}) \\ &\quad -\frac{\Delta t}{T}\mathbb{E}_Q(\ln(\xi)) + o(\Delta t^2), \\ -r\Delta t + x_{T-\Delta t}q\Delta t &= -\frac{\Delta t}{T}\ln(x_{T-\Delta t}) - \frac{\Delta t}{T}\mathbb{E}_Q(\ln(\xi)) + o(\Delta t^2).\end{aligned}$$

Multiplying both sides by $\frac{T}{\Delta t}$ and using $\mathbb{E}_Q(\ln(\xi)) \to 0$, for $\Delta t \to 0$

$$\begin{aligned}-rT + x_{T-\Delta t}qT + \ln(x_{T-\Delta t}) &= \mathbb{E}_Q(\ln(\xi)) + o(\Delta t), \\ -rT + x_T qT + \ln(x_T) &= 0.\end{aligned}$$

Finally we have, that for $\lim_{\tau\to 0}\rho(\tau)$ holds

$$\lim_{\tau\to 0}\rho(\tau) = \max\{\tilde{x}, 1\}, \tag{4.26}$$

where $\tilde{x}$ is a solution of

$$xqT - rT + \ln(x) = 0. \tag{4.27}$$

Case of geometric averaging Asian put options Similarly as before, American property restricts us with $x_t \leq 1$ and therefore

$$\lim_{\tau\to 0}\rho(\tau) = \min\{\tilde{x}, 1\}, \tag{4.28}$$

where $\tilde{x}$ is a solution of (4.27).

4.2 A numerical algorithm

In this section, we use algorithm (derived by Bokes & Ševčovič in [2, 22]) for numerical calculating early exercise boundary. The original algorithm was derived for arithmetically averaged floating strike. We generalize it for geometric and exponentially weighted averaged floating strike options.

Algorithm is based on discretization. We restrict spatial domain to a finite interval $\xi \in (0, L)$, where it is sufficient to take $L \approx 2$. Let $k > 0$ is time discretization step ($k = \frac{T}{m}$) and $h = \frac{L}{n} > 0$ is spatial step. We denote Π^j time discretization of $\Pi(\xi, \tau_j)$ and $\rho^j = \rho(\tau_j)$, where $\tau_j = jk$. By Π_i^j we denote full space-time approximation of the value $\Pi(\xi_i, \tau_j)$. Then the Euler backward in time finite difference approximation of (4.13) is

$$\begin{aligned} 0 \quad = \quad & \frac{\Pi^j - \Pi^{j-1}}{k} + c^j \frac{\partial \Pi^j}{\partial \xi} - \left(\frac{\sigma^2}{2} + f(\rho^j e^{-\xi}, T - \tau) \right) \frac{\partial \Pi^j}{\partial \xi} - \frac{\sigma^2}{2} \frac{\partial^2 \Pi}{\partial \xi^2} \\ & + \left[r + x \frac{\partial f}{\partial x} - f(x, T - \tau) \bigg|_{x = \rho^j e^{-\xi}} \right] \Pi^j, \end{aligned}$$

where c^j is approximation of $c(\tau_j)$, where $c(\tau) = \frac{\dot{\rho}(\tau)}{\rho(\tau)} + r - q$. We have to apply Dirichlet boundary conditions at $\xi = 0$ and $\xi = L$ to function Π^j. Initial condition for Π^0 we use $\Pi^0 = \Pi^j(\xi, 0)$. Next we use operator splitting method to above problem introducing an auxiliary intermediate step $\Pi^{j-\frac{1}{2}}$ which splits problem into:

- Convection part

$$\frac{\Pi^{j-\frac{1}{2}} - \Pi^{j-1}}{k} + c^j \frac{\partial \Pi^{j-\frac{1}{2}}}{\partial \xi} = 0, \tag{4.29}$$

- Diffusive part

$$\begin{aligned} 0 \quad = \quad & \frac{\Pi^j - \Pi^{j-\frac{1}{2}}}{k} - \left(\frac{\sigma^2}{2} + f(\rho^j e^{-\xi}, T - \tau) \right) \frac{\partial \Pi^j}{\partial \xi} - \frac{\sigma^2}{2} \frac{\partial^2 \Pi^j}{\partial \xi^2} \\ & + \left[r + x \frac{\partial f}{\partial x} - f(x, T - \tau) \bigg|_{x = \rho^j e^{-\xi}} \right] \Pi^j. \end{aligned} \tag{4.30}$$

Solution of equation (4.29) can be approximated by explicit solution to the transport equation

$$\frac{\partial \tilde{\Pi}}{\partial \tau} + c(\tau) \frac{\partial \tilde{\Pi}}{\partial \xi} = 0,$$

for $\xi > 0$ and $\tau \in (\tau_{j-1}, \tau_j\rangle$ with initial condition $\tilde{\Pi}(\xi, \tau_{j-1}) = \Pi^{j-1}(\xi)$ and boundary condition $\tilde{\Pi}(0, \tau) = -1$. After some computations (for further details see e.g. Bokes & Ševčovič [2]) we end up with following solution:

$$\Pi_i^{j-\frac{1}{2}} = \begin{cases} \Pi^{j-1}(\nu_i), & \text{if } \nu_i = \xi_i + \ln \frac{\rho^{j-1}}{\rho^j} - (r - q)k > 0, \\ -1, & \text{otherwise.} \end{cases} \tag{4.31}$$

Applying central finite differences in order to approximate the derivative $\frac{\partial \Pi^j}{\partial \xi}$ we obtain from equation (4.30)

$$\begin{aligned} 0 &= \frac{\Pi_i^j - \Pi_i^{j-\frac{1}{2}}}{k} + \left[r + x\frac{\partial f}{\partial x} - f(x, T-\tau) \Big|_{x=\rho^j e^{-\xi_i}} \right] \Pi_i^j \\ &\quad - \left(\frac{\sigma^2}{2} + f(\rho^j e^{-\xi_i}, T-\tau) \right) \frac{\Pi_{i+1}^j - \Pi_{i-1}^j}{2h} - \frac{\sigma^2}{2} \frac{\Pi_{i+1}^j - 2\Pi_i^j + \Pi_{i-1}^j}{h^2}. \end{aligned}$$

Hence vector Π^j is a solution of a tridiagonal system of linear equations

$$\alpha_i^j \Pi_{i-1}^j + \beta_i^j \Pi_i^j + \gamma_i^j \Pi_{i+1}^j = \Pi_i^{j-\frac{1}{2}}, \tag{4.32}$$

for $i = 1, 2, \ldots, n$, where

$$\begin{aligned} \alpha_i^j(\rho^j) &= -\frac{k}{2h^2}\sigma^2 + \frac{k}{2h}\left(\frac{\sigma^2}{2} + f(\rho^j e^- \xi_i, T-\tau_j) \right), \\ \beta_i^j(\rho^j) &= 1 + b(\xi_i, T-\tau_j)k - (\alpha_i^j + \gamma_i^j), \\ \gamma_i^j(\rho^j) &= -\frac{k}{2h^2}\sigma^2 - \frac{k}{2h}\left(\frac{\sigma^2}{2} + f(\rho^j e^- \xi_i, T-\tau_j) \right), \end{aligned}$$

Boundary conditions for Π^j are $\Pi_0^j = -1, \Pi_n^j = 0$, for $j = 1, 2, \ldots, m$. Initial condition for Π^0 is given by equation (4.14). We only recall that

$$b(\xi, \tau) = r + x\frac{\partial f}{\partial x} - f(x, T-\tau) \Big|_{x=\rho e^{-\xi}}.$$

In order to determine the free boundary position we take equation (4.18) into account and after applying Euler finite difference approximation we obtain

$$\begin{aligned} \ln \rho^j &= \ln \rho^{j-1} + \int_0^\infty \Pi^{j-1}(\xi) d\xi - \int_0^\infty \Pi^j(\xi) d\xi \\ &\quad + k\left(q + \frac{\sigma^2}{2} - q\rho^{j-1} - \int_0^\infty \left(r - f(\rho^{j-1} e^{-\xi}, T-\tau_j) \right) \Pi^j d\xi \right). \end{aligned} \tag{4.33}$$

Because we can not compute $\int_0^\infty \Pi^j(\xi) d\xi$, we approximate it with trapezoid quadrature method from discrete values. If we rewrite equations into the operator form:

$$\Pi^{j-\frac{1}{2}} = \mathcal{T}(\rho_{p+1}^j), \qquad \mathcal{A}(\rho_{p+1}^j)\Pi^j = \Pi^{j-\frac{1}{2}}, \qquad \rho^j = asd\mathcal{F}(\Pi^j), \tag{4.34}$$

where $\mathcal{T}(\rho_{p+1}^j)$ is a solution of transport equation given by (4.31), $\mathcal{A}(\rho_{p+1}^j)$ is a tridiagonal matrix given by (4.32) and $\ln \mathcal{F}(\Pi^j)$ is right side of equation (4.33). System of equations (4.34) can be solved by means of successive iterations procedure. For $j \geq 1$, we use $\Pi^{j,0} = \Pi^{j-1}$, $\rho^{j,0} = \rho^{j-1}$. Then $(p+1)$-th approximation of Π^j and ρ^j is a solution of following system:

$$\rho^{j,p+1} = \mathcal{F}(\Pi^{j,p}), \tag{4.35}$$

$$\Pi^{j-\frac{1}{2},p+1} = \mathcal{T}(\rho^{j,p+1}), \tag{4.36}$$

$$\mathcal{A}(\rho_{p+1}^j)\Pi^{j,p+1} = \Pi^{j-\frac{1}{2},p+1}. \tag{4.37}$$

Table 4.1: A comparison of different interpolation methods used in operator $\mathcal{T}$.

m	interp. method	ε_ρ	ε_{Π_1}	ε_{Π_2}
200	linear	0.355520	0.008903	0.379569
	spline	0.355506	0.008234	0.378894
	cubic	0.167454	0.008234	0.378894
400	linear	0.151797	0.007341	0.190087
	spline	0.151780	0.007878	0.188898
	cubic	0.137178	0.007897	0.188898
800	linear	0.052020	0.006184	0.087705
	spline	0.052014	0.007696	0.083035
	cubic	0.049852	0.007702	0.083044
1600	linear	0.027937	0.004279	0.034823
	spline	0.027936	0.005383	0.027244
	cubic	0.010368	0.005394	0.027352

Let us note, that we use piecewise cubic Hermite interpolation to compute values $\Pi^{j-1}(\eta_i)$ from discrete values Π^{j-1} in operator $\mathcal{T}$. Comparison of numerical accuracy for different interpolation methods can be see in Table 4.1. As a benchmark we choose **piecewise cubic Hermite interpolation** and $m = 4000$. Other interpolation methods are following: **linear** - Linear interpolation; **spline** - Cubic spline interpolation; **cubic** - Piecewise cubic Hermite interpolation. As a measure of error we use $\varepsilon_\rho = ||\rho - \rho_{benchmark}||_\infty$, $\varepsilon_{\Pi_1} = ||\Pi(\cdot, 25) - \Pi_{benchmark}(\cdot, 25)||_\infty$, $\varepsilon_{\Pi_2} = ||\Pi(\cdot, 49.5) - \Pi_{benchmark}(\cdot, 49.5)||_\infty$. It is obvious that Piecewise cubic Hermite interpolation gives better results for smaller time discretization.

4.2.1 Algorithm

Input variables: $q \geq 0, L, r, \sigma, n, m, T > 0, \lambda > 0,\ p_{max} = 500, toll = 10^{-8}$
Initialization:

$$k = T/m$$

$$h = L/n$$

$$\rho^0 = \begin{cases} \max\{\frac{1+rT}{1+qT}, 1\}, & \text{for arithmetic average,} \\ \max\{\tilde{x}, 1\}, & \text{for geometric average,} \\ \max\{\frac{\lambda+r(1-e^{-\lambda T})}{\lambda+q(1-e^{-\lambda T})}, 1\}, & \text{for exponentially weighted average,} \end{cases}$$

$$\Pi^0 = \begin{cases} -1, & \xi < \ln \rho^0, \\ 0, & \xi > \ln \rho^0, \end{cases}$$

for $j = 1$ **to** m:

$$\Pi_0^j = \Pi^{j-1}$$

$$\rho_0^j = \rho^{j-1}$$

for $p = 0$ **to** p_{max}:

$$\rho_{p+1}^j = \mathcal{F}(\Pi_p^j)$$

$$\Pi_{p+1}^{j-\frac{1}{2}} = \mathcal{T}(\rho_{p+1}^j)$$

$$\mathcal{A}(\rho_{p+1}^j)\Pi_{p+1}^j = \Pi_{p+1}^{j-\frac{1}{2}}$$

if $(|\rho^j_{p+1} - \rho^j_p| < toll)$
break
endif
end
end,

where r is interest rates, q dividend yields, T expiration time, n number of spatial grid points, m time step, $L = 1.4$, λ is parameter used in exponentially weighted averaging and $\tilde{x}$ is a solution of (4.27).

4.2.2 Numerical results

4.2.2.1 Arithmetically averaged floating strike call option

In Figure 4.1 we plot $\rho(\tau)$ for different values of $\sigma = 0.1, 0.2, 0.3$. Other parameters are $T = 50, \sigma = 0.2, q = 0.04$. Let us notice, that for $\sigma \to 0$ the whole process is

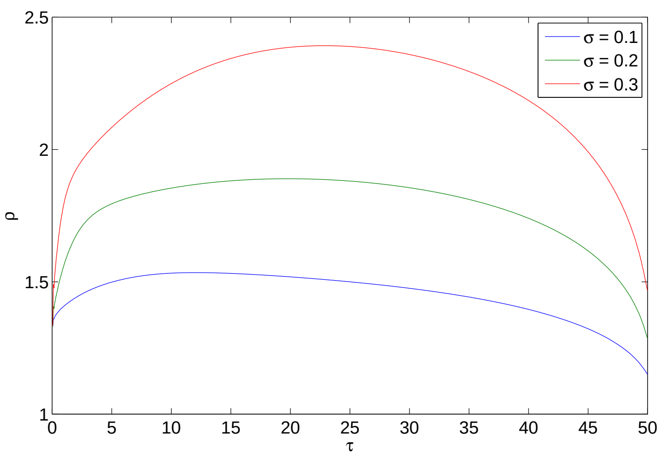

Figure 4.1: A comparison of the free boundary position for various $\sigma = 0.1, 0.2, 0.3$.

not longer stochastic but deterministic. Option price at time t is $S_t = S_0 e^{(r-q)t}$ and its arithmetic average $A_t = \frac{1}{t} S_0 \frac{e^{(r-q)t}-1}{r-q}$. In case of $q > r$, then $S_t < A_t, \forall t > 0$ and therefore corresponding option price have to be equals zero. In case of $r = q$, both S_t and A_t are equals and therefore $\rho(t) \equiv 1$. On the other hand, for case of $r > q$ then $S_t > A_t, \forall t > 0$. We exercise option at time $t^* = \arg\max_{t \in \langle 0,T \rangle} e^{-r(T-t)}(S_t - A_t)$. If we denote by $\Psi(t) = \frac{S_t - A_t}{S_0}$, then $t^* = \arg\max_{t \in \langle 0,T \rangle} S_0 e^{-rT} e^{rt} \Psi(t)$. Taylor's series of $\frac{d\Psi(t)}{dt}$ is

$$\frac{d\Psi(t)}{dt} = \sum_{n=0}^{\infty} \frac{n+1}{(n+2)n!} (r-q)^{n+1} t^n > 0, \quad \forall t \geq 0$$

and therefore $\Psi(t)$ is increasing function. As booth $\Psi(t)$ and e^{rt} are increasing and product of two increasing functions is also increasing hence $t^* = T$. In case of arithmetically averaged Asian option, we obtain from (4.17), that for $\sigma = 0$ it holds

$$\rho(\tau) = \frac{1 + r(T - \tau)}{1 + q(T - \tau)}. \tag{4.38}$$

In Figure 4.2 we plot position of free boundary for various $\sigma = 0.1, 0.05, 0.03, 0.02, 0.015, 0.01$ and for $\sigma = 0$ we use equation (4.38). We only remark, that for small values of σ, algorithm become unstable, which is obvious from Figure.

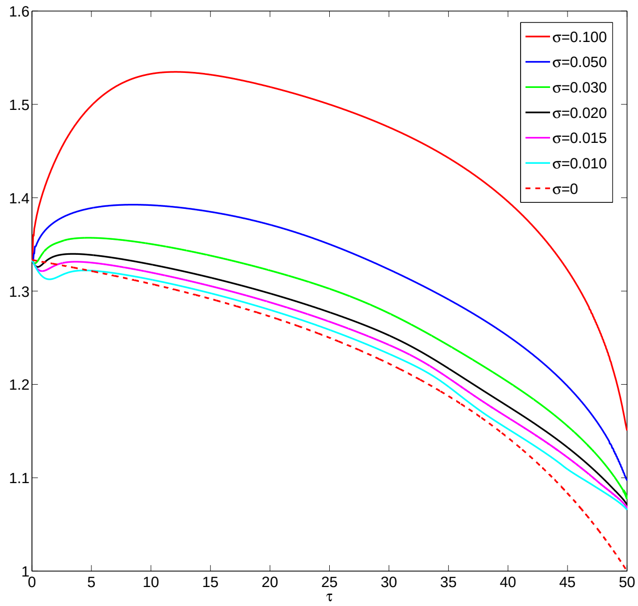

Figure 4.2: A comparison of the free boundary position for various small σ. For $\sigma = 0$ we use (4.38).

In Figure 4.3 we compare position of the free boundary position for various parameter L which introduce upper bound for spatial approximation interval. In this particular case ($r = 0.06, q = 0.04, T = 50, \sigma = 0.2$) it would be sufficient to choose $L = 3$.

In Figure 4.4 we compare the free boundary position for various interest rates $r = 0.2, 0.4, 0.6$. Other parameters are $T = 50, \sigma = 0.2, q = 0.04$. In the same Figure we also compare free boundary position computed by algorithm described above (black line), results from Ševčovič [2] (blue line) and result obtained from Dai & Kwok [5] (red dots).

In Figure 4.5 above we plot number of iteration in loop p in algorithm to achieve desired tolerance. At the middle chart we plot error and at the bottom we plot the free boundary $\rho(\tau)$. Parameters are $T = 50, \sigma = 0.2, q = 0.04, r = 0.06$.

In Figure 4.6 we plot an example of development of a spot value of underlying asset and corresponding arithmetic average (above). At bottom we plot a position of early exercise boundary and $x(t) = \dfrac{S_t}{A_t}$. We marked an exercise event.

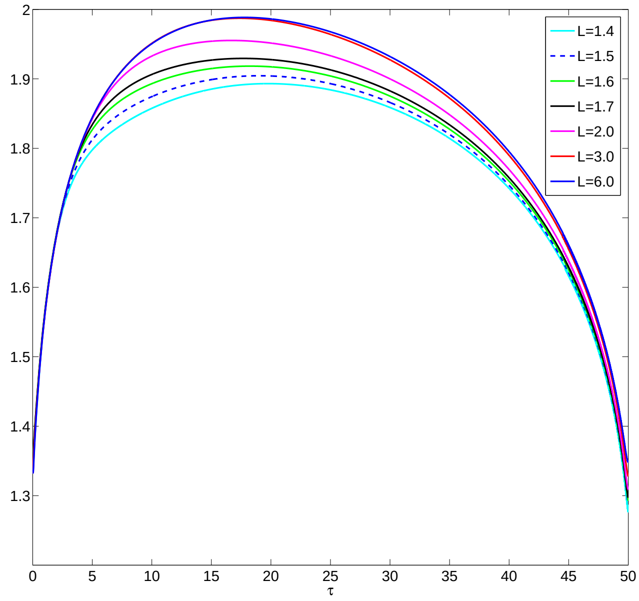

Figure 4.3: A comparison of the free boundary position for various L.

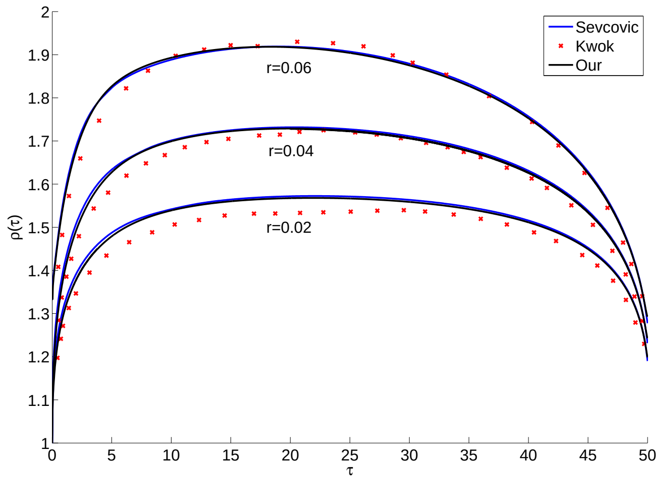

Figure 4.4: A comparison position of the free boundary position for various $r = 0.02, 0.04, 0.06$. We also compare our results with know other known methods.

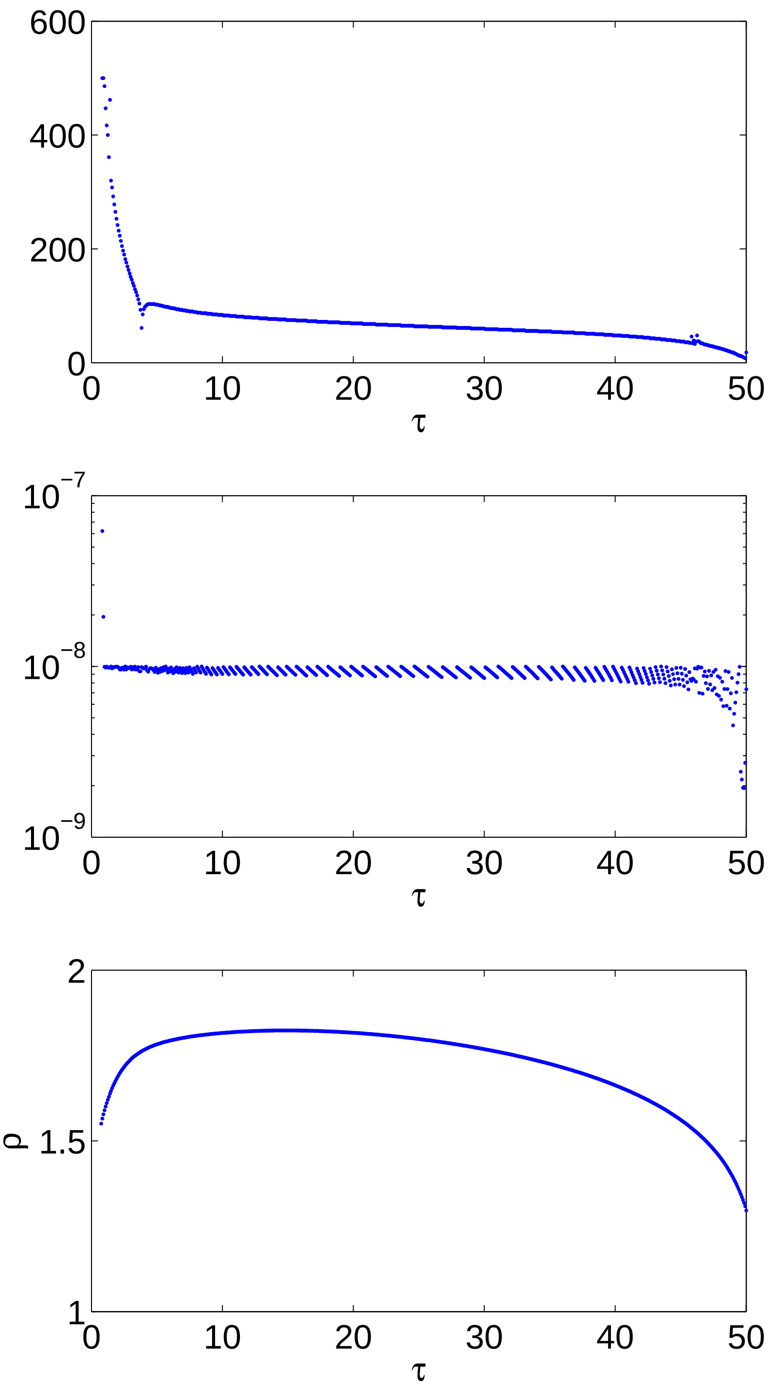

Figure 4.5: A number of iteration in loop in algorithm to achieve desired tolerance (top), corresponding error (middle) and position of free boundary position (bottom).

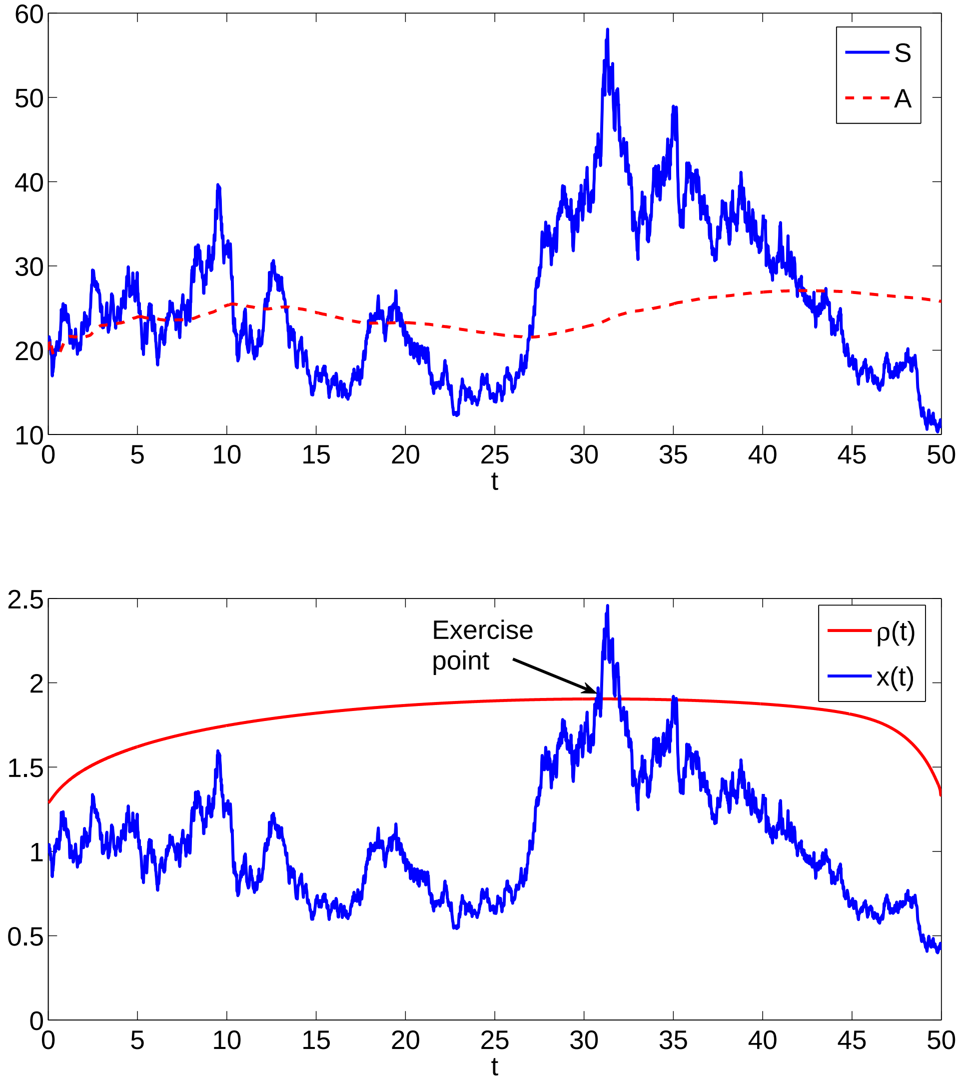

Figure 4.6: An example of development of asset price and corresponding arithmetic average (top), a position of early exercise boundary (bottom).

4.2.2.2 Geometric averaged floating strike call option

In Figure 4.7 we compare the free boundary position in case of arithmetic and geometric average. Parameters are $T = 50, \sigma = 0.2, q = 0.04, r = 0.06$.

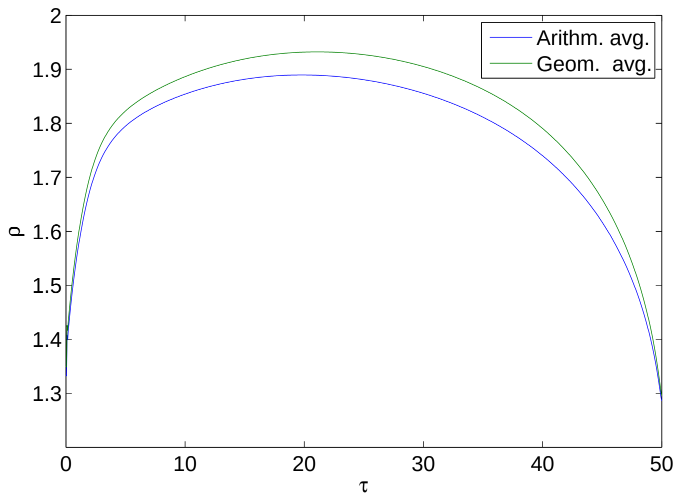

Figure 4.7: A comparison of the free boundary position for arithmetic and geometric average.

4.2.2.3 Weighted arithmetic averaged floating strike call option

In Figure 4.8 we plot free boundary position for various $\lambda = 0.001, 0.1, 0.2, 0.5, 1$. In Figure 4.9 we plot $\Pi(\xi, t)$ for various $t = 0.25, 5, 25, 45$.

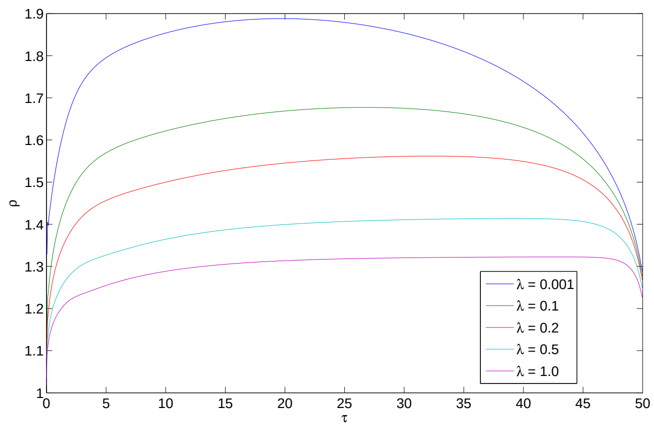

Figure 4.8: A comparison of the free boundary position for exponential weighted arithmetic averages for various $\lambda = 0.001, 0.1, 0.2, 0.5, 1$.

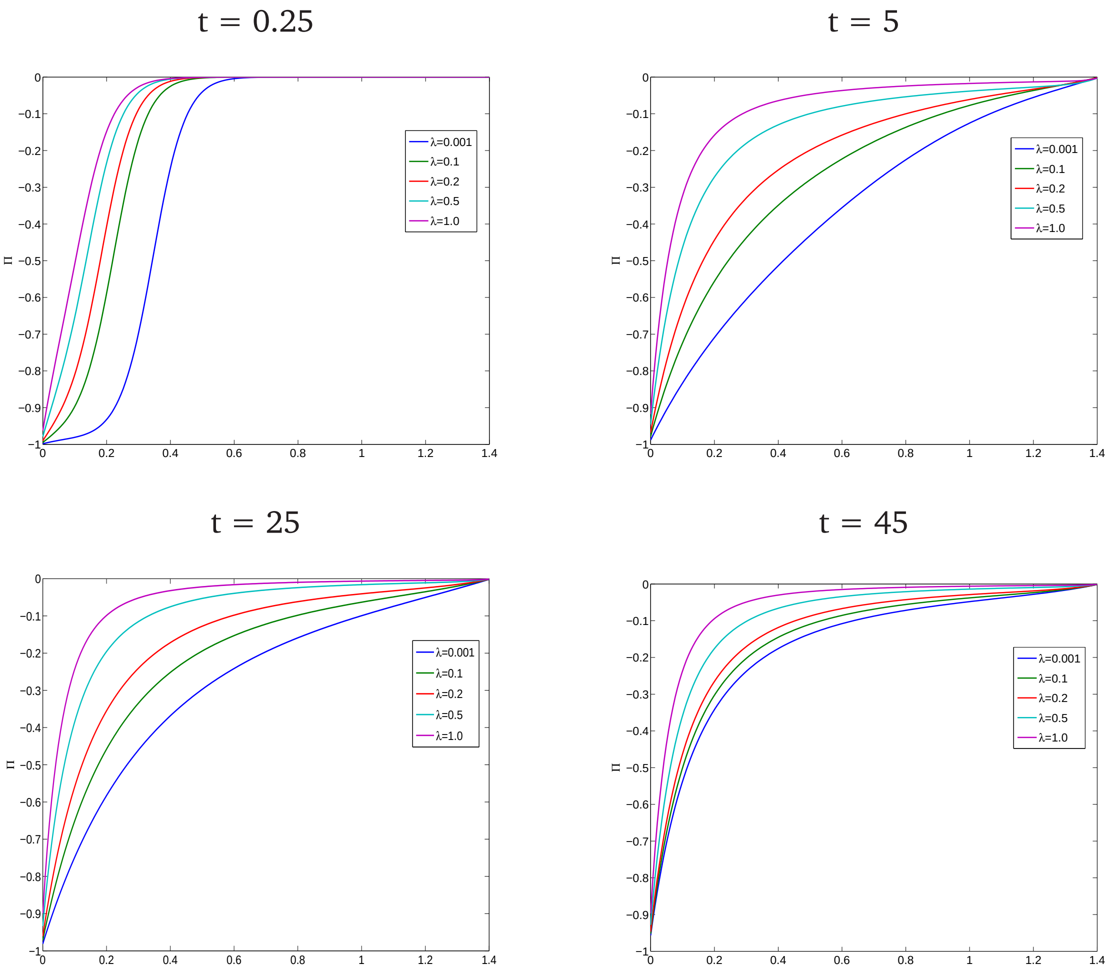

Figure 4.9: A comparison of solution $\Pi(\xi,t)$ for variance $\lambda = 0.001, 0.1, 0.2, 0.5, 1$ and $t = 0.25, 5, 25, 45$.

It is easy to verify that $\lim\limits_{\lambda\to\infty} A_t^\lambda = S_t$ and therefore it's option price is equals zero. This is also reason to $\lim_{\lambda\to\infty}\rho_\lambda \to 1$. We can estimate rate of convergence of ρ_λ to 1 using experimental order of convergence. Assuming that $||\rho_\lambda - \rho_\infty||_\infty = O(\lambda^{-\alpha_\infty})$ we can express α_∞ like

$$\alpha_\infty = \frac{\ln(\lambda_2) - \ln(\lambda_1)}{\ln(||\rho_{\lambda_1} - \rho_\infty||_\infty) - \ln(||\rho_{\lambda_2} - \rho_\infty||_\infty)}. \tag{4.39}$$

We show α_∞ in Table 4.2.

4.2.3 Time scaling

In this section, we show, that position of the free boundary depends. We already know, that $\rho_T = \rho_T(\tau; r, q, \sigma, T)$ is a solution of (4.13) and (4.18). Now, we show,

Table 4.2: Experimental order of convergence for ρ_λ.

λ	$\|\|\rho_\lambda - \rho_\infty\|\|_\infty$	α_∞
0.001	0.888104	–
0.1	0.677320	0.058835
0.2	0.561828	0.269710
0.5	0.413783	0.333796
1.0	0.320136	0.370188
2.0	0.247010	0.374115
3.0	0.212705	0.368769
4.0	0.191862	0.358490
5.0	0.177658	0.344686
10.0	0.147227	0.271064
20.0	0.113350	0.377261
30.0	0.094150	0.457702

that in case of arithmetic average for $\rho_1 = \rho_1(\tilde{\tau}; \tilde{r}, \tilde{q}, \sigma, 1)$ holds

$$\rho_T(\tau; r, q, \sigma, T) = \rho_1\left(\frac{\tau}{T}; Tr, Tq, \sqrt{T}\sigma, 1\right).$$

Proof: $\rho_1 = \rho_1(\tilde{\tau}; \tilde{r}, \tilde{q}, \sigma, 1)$ is solution of following system

$$\begin{aligned} 0 &= \frac{\partial \Pi}{\partial \tilde{\tau}} + \left[\frac{\dot{\rho}_1(\tilde{\tau})}{\rho_1(\tilde{\tau})} - \frac{\rho_1(\tilde{\tau})e^{-\xi} - 1}{1-\tilde{\tau}} + \tilde{r} - \tilde{q} - \frac{\tilde{\sigma}^2}{2}\right]\frac{\partial \Pi}{\partial \xi} - \frac{\tilde{\sigma}^2}{2}\frac{\partial^2 \Pi}{\partial \xi^2} + \left[\tilde{r} + \frac{1}{1-\tilde{\tau}}\right]\Pi, \\ 0 &= \tilde{q}\rho_1(\tilde{\tau}) - \tilde{r} - \frac{\tilde{\sigma}^2}{2}\frac{\partial \Pi}{\partial \xi}(0, \tilde{\tau}) + \frac{\rho_1(\tilde{\tau}) - 1}{1-\tilde{\tau}}. \end{aligned}$$

After substitution $\tilde{r} = rT, \tilde{q} = qT, \tilde{\sigma} = \sigma\sqrt{T}$ we obtain

$$\begin{aligned} 0 &= \frac{\partial \Pi}{\partial \tilde{\tau}} + \left[\frac{\dot{\rho}_1(\tilde{\tau})}{\rho_1(\tilde{\tau})} - \frac{\rho_1(\tilde{\tau})e^{-\xi} - 1}{1-\tilde{\tau}} + Tr - Tq - \frac{T\sigma^2}{2}\right]\frac{\partial \Pi}{\partial \xi} - \frac{T\sigma^2}{2}\frac{\partial^2 \Pi}{\partial \xi^2} \\ &\quad + \left[Tr + \frac{1}{1-\tilde{\tau}}\right]\Pi, \\ 0 &= Tq\rho_1(\tilde{\tau}) - Tr - \frac{T\sigma^2}{2}\frac{\partial \Pi}{\partial \xi}(0, \tilde{\tau}) + \frac{\rho_1(\tilde{\tau}) - 1}{1-\tilde{\tau}}. \end{aligned}$$

Now, we can substitute $T\tilde{\tau} = \tau$ and using that $\dot{\rho}_1(\tilde{\tau}) = \dot{\rho}_1(\tau/T)T$ and $\frac{\partial \Pi}{\partial \tilde{\tau}} = \frac{\partial \Pi}{\partial \tau}T$ we conclude

$$\begin{aligned} 0 &= T\frac{\partial \Pi}{\partial \tau} + \left[T\frac{\dot{\rho}_1(\tau/T)}{\rho_1(\tau/T)} - \frac{\rho_1(\tau/T)e^{-\xi} - 1}{T-\tau}T + Tr - Tq - \frac{T\sigma^2}{2}\right]\frac{\partial \Pi}{\partial \xi} - \frac{T\sigma^2}{2}\frac{\partial^2 \Pi}{\partial \xi^2} \\ &\quad + \left[Tr + \frac{1}{T-\tau}T\right]\Pi, \\ 0 &= Tq\rho_1(\tau/T) - Tr - \frac{T\sigma^2}{2}\frac{\partial \Pi}{\partial \xi}(0, \tau/T) + \frac{\rho_1(\tau/T) - 1}{T-\tau}T. \end{aligned}$$

After multiply both equations by $\frac{1}{T}$ we end up with system of equations which are identical with ones for ρ_T. Argument that both equations have the same initial and boundary conditions ends this proof.

With regards to previous argument, it is sufficient to study dependence of $\rho(T)$ on r, q, σ and not on T. In Figure 4.10 we plot dependence of $\rho(T)$ and $\dot{\rho}(T)$ on r. Other parameters are $\sigma = 0.2, q = 0.04, T = 50$. In Figure 4.11 we plot dependence

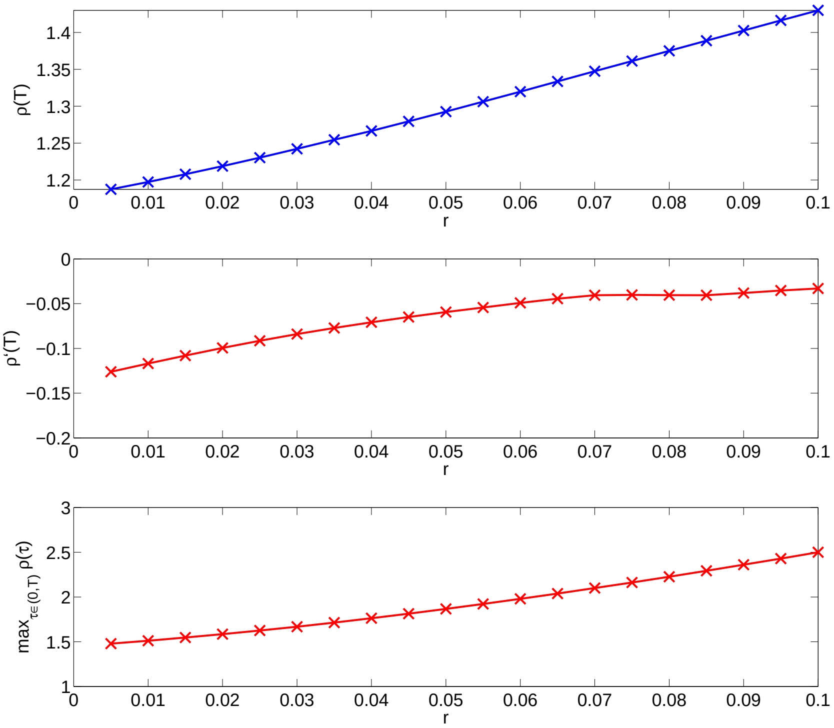

Figure 4.10: Dependece of $\rho(T)$ (top), $\rho'(T)$ (middle) and $\max_{\tau\in(0,T)} \rho(\tau)$ (bottom) on parameter r.

of $\rho(T)$ and $\dot{\rho}(T)$ on q. Other parameters are $r = 0.06, \sigma = 0.2, T = 50$. In Figure 4.12 we plot dependence of $\rho(T)$ and $\dot{\rho}(T)$ on σ. Other parameters are $r = 0.06, q = 0.04, T = 50$.

In order to find some relation between $\rho(T)$ and r, q, σ we generate 100 random vectors $(r, q, \sigma) \in \langle 0.01, 0.11\rangle \times \langle 0.01, 0.11\rangle \times \langle 0.2, 0.8\rangle$ and for $T = 50$ we compute corresponding $\rho(T)$. Using nonlinear regression we try to estimate $\rho(T) = f(\beta; r, q, \sigma)$ for various functions f. In Table 4.3 we show those functions with estimated coefficients. As an error measure we use $RSS = \Sigma(f(\beta; r_j, q_j, \sigma_j) - \rho_j(T))^2$.

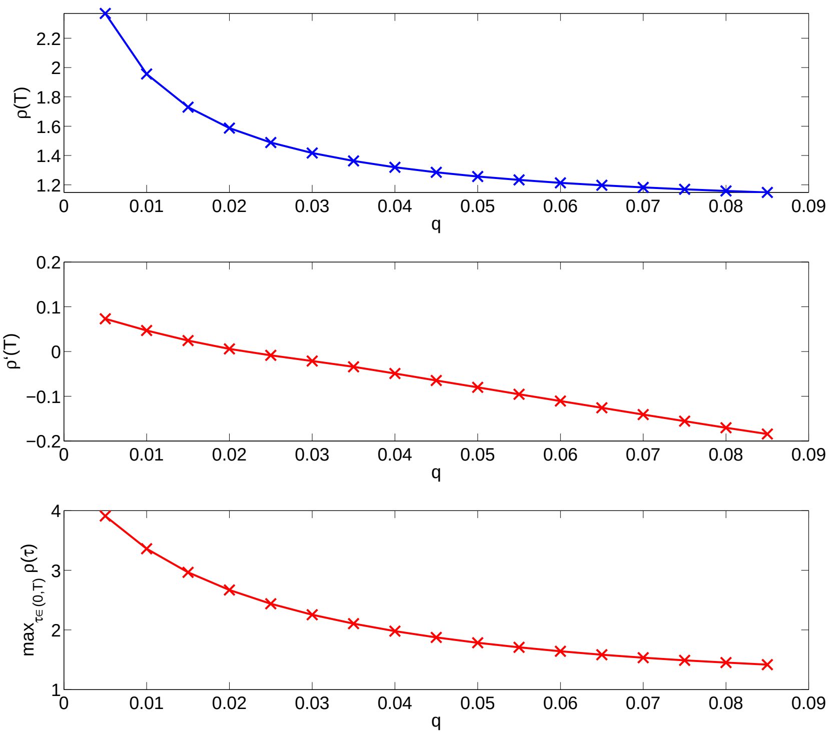

Figure 4.11: Dependece of $\rho(T)$ (top), $\rho'(T)$ (middle) and $\max_{\tau\in(0,T)} \rho(\tau)$ (bottom) on parameter q.

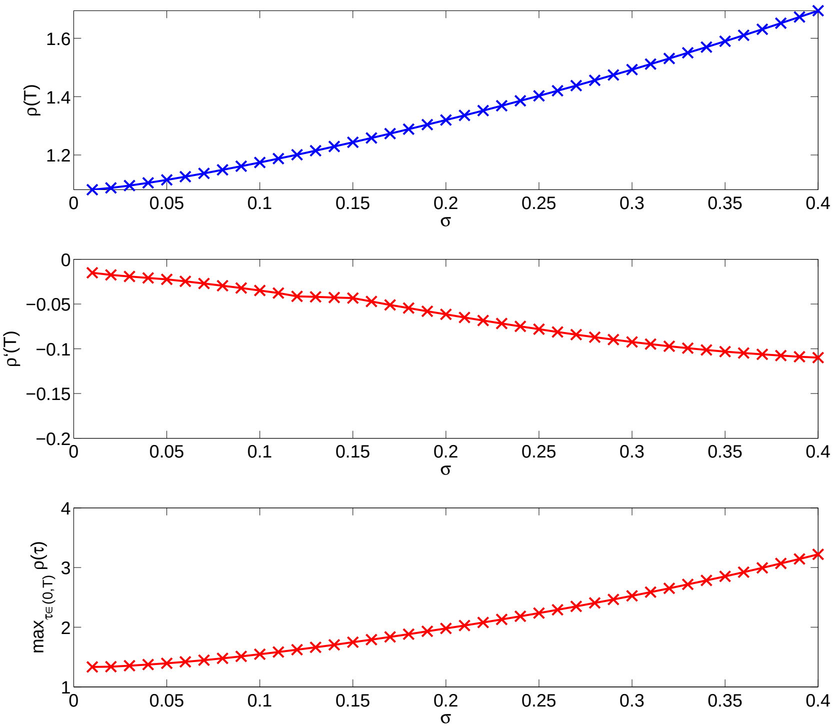

Figure 4.12: Dependece of $\rho(T)$ (top), $\rho'(T)$ (middle) and $\max_{\tau\in(0,T)}\rho(\tau)$ (bottom) on parameter σ.

Table 4.3: An nonlinear estimation of $\rho(T) = f(\beta; r, q, \sigma)$ for various functions f.

$f(\beta; r, q, \sigma)$	β_1	β_2	β_3	β_4	RSS
$b_1 + b_2 r + b_3 q + b_4 \sigma$	1.2642	1.2645	-5.8190	1.3771	0.1663
$1 + \dfrac{\sigma^2}{b_1 r + b_2 q}$	-1.0981	7.3976	–	–	0.5293
$1 + \left(\dfrac{\sigma^2}{b_1 r + b_2 q}\right)^{b_3}$	-1.5707	9.1502	0.6783	–	0.0262
$b_4 + \left(\dfrac{\sigma^2}{b_1 r + b_2 q}\right)^{b_3}$	-1.5651	9.1108	0.6727	0.9960	0.0262
$1 + \left(\dfrac{\sigma^2}{b_1 r + b_2 q}\right)^{b_3} + \dfrac{r}{q} b_4$	-0.1596	8.5215	0.7985	0.0913	0.0024

4.2.4 Look-back options

We have already discussed Look-back options in first chapter. We focus our attention on floating minimum strike call option, which payoff is $V(S, M, t) = (S_t - m_t)^+$, where $m_t = \min_{\tau \in \langle 0,t)} S_t$.

Notice, that m_t can be calculated as

$$m_t = A_t^p = \left(\frac{1}{t} \int_0^t S_\tau^p d\tau \right)^{\frac{1}{p}},$$

where $p = -\infty$. Let us remark, that in case of maximum we have to choose $p = \infty$. In Figure 4.13 we plot development of S_t and A_t^p for various $p = -1, -10, -100, -\infty$. On other hand In Figure 4.14 we plot A_t^p for $p = 1, 10, 100, \infty$.

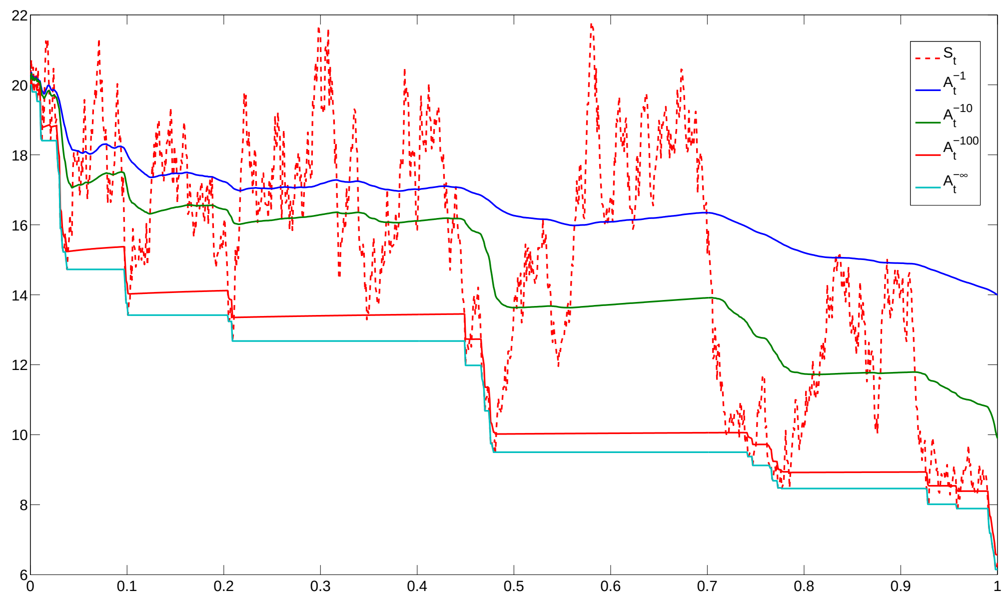

Figure 4.13: Example of development of S_t and corresponding A_t^p for various $p = -1, -10, -100, -\infty$.

4.2.4.1 American style of Look-back options

According Ševčovič [23] for differential dA_t^p holds

$$\frac{dA_t^p}{A_t^p} = f_p(x, t),$$

where $x = \dfrac{S_t}{A_t^p}$ and $f_p(x, t) = \dfrac{x^p - 1}{pt}$. Because of this property, we can use algorithm from previous section. There is only one thing, which have to be derived for this case. Namely $\rho(0^+)$.

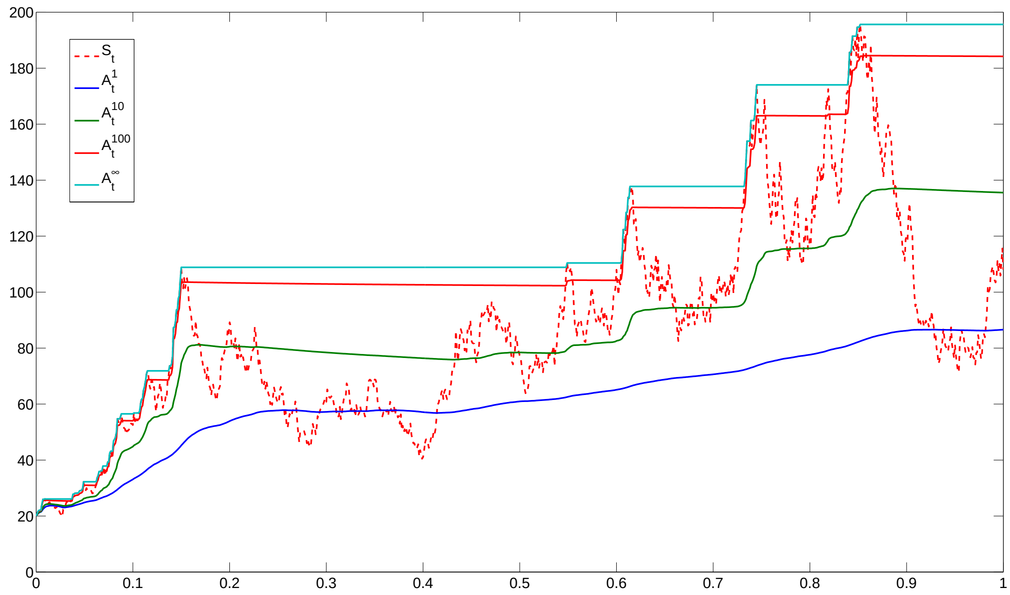

Figure 4.14: Example of development of S_t and corresponding A_t^p for various $p = 1, 10, 100, \infty$.

4.2.4.2 Derivation of $\rho(0^+)$

We derive $\rho(0^+)$ in the same way as in Asian option. In this case, it holds, that

$$A_T^p = \left(\frac{1}{T}\int_0^T S_\tau^p d\tau\right)^{\frac{1}{p}} = \left(\frac{1}{T}\int_0^{T-\Delta t} S_\tau^p d\tau + \frac{1}{T}\int_{T-\Delta t}^T S_\tau^p d\tau\right)^{\frac{1}{p}}$$

$$= \left(\frac{T-\Delta t}{T}(A_{T-\Delta t}^p)^p + \frac{1}{T}\int_{T-\Delta t}^T S_\tau^p d\tau\right)^{\frac{1}{p}} \approx \left(\frac{T-\Delta t}{T}(A_{T-\Delta t}^p)^p + \frac{\Delta t}{T}S_{T-\Delta t}^p \xi^p\right)^{\frac{1}{p}}.$$

We exercise option for values $(S_{T-\Delta t}, A_{T-\Delta t}^p)$ for which holds

$$\begin{aligned}(S_{T-\Delta t} - A_{T-\Delta t}^p)^+ e^{r\Delta t} &= \mathbb{E}_Q(S_T - A_T^p)^+,\\ (S_{T-\Delta t} - A_{T-\Delta t}^p)^+ e^{r\Delta t} &= \mathbb{E}_Q\left(S_{T-\Delta t}\xi - \left(\frac{T-\Delta t}{T}(A_{T-\Delta t}^p)^p + \frac{\Delta t}{T}S_{T-\Delta t}^p\xi^p\right)^{\frac{1}{p}}\right)^+.\end{aligned}$$

We can divide both sides by $A_{T-\Delta t}^p$ and we obtain

$$(x_{T-\Delta t} - 1)^+ e^{r\Delta t} = \mathbb{E}_Q\left(x_{T-\Delta t}\xi - \left(\frac{T-\Delta t}{T} + \frac{\Delta t}{T}x_{T-\Delta t}^p\xi^p\right)^{\frac{1}{p}}\right)^+.$$

American property guarantees us, that $x_{T-\Delta t} \geq 1$ then

$$(x_{T-\Delta t} - 1)e^{r\Delta t} = \mathbb{E}_Q\left(x_{T-\Delta t}\xi - \left(\frac{T-\Delta t}{T} + \frac{\Delta t}{T}x_{T-\Delta t}^p\xi^p\right)^{\frac{1}{p}}\right).$$

For small values Δt we obtain

$$(x_{T-\Delta t}-1)(1+r\Delta t) \quad = \quad x_{T-\Delta t}(1+(r-q)\Delta t)-\mathbb{E}_Q\left(1+\frac{\Delta t}{Tp}(x^p_{T-\Delta t}\xi^p-1)\right).$$

After straightforward calculations we conclude that for $x_{T-\Delta t}$ it has to hold

$$x_{T-\Delta t}q\Delta t - r\Delta t - \frac{\Delta t}{Tp} = -\frac{\Delta t}{Tp}x^p_{T-\Delta t}\mathbb{E}_Q(\xi^p). \tag{4.40}$$

Multiplying by $\frac{Tp}{\Delta t}$ and pushing $\Delta t \to 0$ we finally obtain the equation for x_T:

$$x_T^p + Tpx_Tq - Tpr - 1 = 0. \tag{4.41}$$

It is easy to verify, that if $p \to \infty$, this equation has following solution:

$$x_T = \begin{cases} \frac{r}{q}, & \text{if } r < q, \\ 1, & \text{otherwise.} \end{cases} \tag{4.42}$$

Recall, that we supposed, that $x \geq 1$ and therefore $\rho(0^+) = 1$. On the other hand, if $p \to -\infty$, (4.41) has following solution:

$$x_T = \begin{cases} \frac{r}{q}, & \text{if } r > q, \\ 1, & \text{otherwise.} \end{cases} \tag{4.43}$$

In this case $\rho(0^+) = \max\left\{\frac{r}{q}, 1\right\}$.

Chapter 5

Conclusion

In this master's thesis we focus on path-depend options. We derive first two moments of time integral of geometric Brownian motion in case of exponentially weighted averages. Using them, we derive appropriation formula for pricing floating rate Asian options. We compare our results with other known methods and results are very satisfactory. We also introduce new modificated average, which give more stable PDE in case of European-style Asian options then standard PDE. We derive correlation between geometric Brownian motion and it's exponentially averaged time integral. Then using appropriate copula function we derive another approximation formula for pricing European-style.

Finally, we derive by another way a position of free boundary at time close to expiry. We generalize algorithm for American-style Asian options for case of exponentially weighted Asian floating strike options. We examined sensitivity of the position of free boundary on parameters r, q, σ.

Chapter 6

Résumé

V práci študujeme Ázijské opcie. Sú to opcie ktorých payoff závisí nie len od aktuálnej hodnoty podkladového aktíva aj od jeho historického vývoja. V prípade Ázijských opcií payoff závisí od priemernej ceny. V prípad aritmetického a geometrického priemeru je známych mnoho výsledkov. Preto sme sa zaoberali aj váženým priemerovaním, konkrétne exponenciálne váženým priemerovaním.

V prvej kapitole uvádzame stručný úvod do problematiky a základné definície a prehľad vybraných opčných derivátov. Poznamenajme, že sa predpokladá, že podkladové aktívum sleduje geometrický Brownov pohyb.

V druhej kapitole odvodíme aproximatívny vzorec na výpočet ceny tzv. **averate rate** opcie. Aproximujeme integrál váženého aritmetického priemeru lognormálne rozdelenou náhodnou premennou. Odvodíme jeho prvé dva momenty a momentovou vetou odhadneme parametre danej lognormálnej náhodnej premennej. V Tabuľke 2.1 uvádzame porovnanie nami odvodeného vzorca s inými známymi výsledkami. V prípade exponenciálne váženého priemerovania uvádzame závislosť ceny opcie od váhovacieho parametra λ. Pre $\lambda \to \infty$ sa cena opcie blíži k cene európskej opcie.

V prvej časti tretej kapitoly odvádzame modifikované spriemerovanie, pre ktoré príslušná PDR je vhodnejšia pre numerickú schému a pre čas blízky k exspirácii sa redukuje na PDR totožnú s PDR pre obyčajnú Európsku opciu. V druhej časti, použitím Gausovej copuly aproximujeme cenu **averate strike** opcie. Presnosť nášho aproximatívneho vzorca opäť porovnávame so známymi výsledkami v Tabuľkách 3.1 and 3.2.

V štvrtej kapitole sa venujeme problematike Ázijských opcií s Americkou vlastnosťou. Pripomeňme, že opcie s Americkou vlastnosťou dávajú držiteľovi opcie právo uplatniť opciu v ľubovoľný čas pred maturitou. Okrem samotnej ceny opcie je dôležité vypočítať aj tzv. hranicu predčasného uplatnenia (t.j. pre každý čas určiť cenu podkladového aktíva, pri ktorej má byť opcia uplatnená). V tejto kapitole sa venujeme algoritmu navrhnutého Ševčovičom [22, 2]. Uvedený algoritmus sme zovšeobecnili na prípad geometrického priemeru a exponenciálne váženého. Zaoberali sme sa aj analýzou citlivosti polohy voľnej hranice na začiatku života opcie

od parametrov r, q, σ.

Chapter 7

Appendix

7.1 Source codes

7.1.1 Monte-Carlo for average rate Asian options

```
sigma=0.5; r=0.15; T=2; X=1; S=1; %Input parameters
n=200; m=100000; dt=T/n;
u = exp(sigma *sqrt(dt));
d = exp(-sigma *sqrt(dt));
p = (exp(r*dt)-d)/(u-d);
values=[];
for i = 1:m
  tresh = (( rand(1,n) < p )+0)*(u-d)+d;   temp=0;
  for j=1:n
     temp = (temp+1)*tresh(j);
  end
  temp = (temp + 1)/(n+1);   values(i)= temp;
end
index = S*values> X;
price = exp(-r*T)*1/m*sum( (S*values-X).*index);
```

7.1.2 Monte-Carlo for average strike Asian options

```
sigma=0.5; r=0.15; T=2; S0=1; %Input parameters
n=200; m=100000; dt=T/n;
u = exp(sigma *sqrt(dt));
d = exp(-sigma *sqrt(dt));
p = (exp(r*dt)-d)/(u-d);
S=[];A=[];
for j=1:m
    direction = (rand(n,1)<p)*2-1;
    S_Path=[];
```

```
    S_Path(1,1) = S0;
    for  i = 1:n
        if (direction(i) == 1)
            S_Path(i+1,1) = S_Path(i,1)*u;
        else
            S_Path(i+1,1) = S_Path(i,1)*d;
        end
        S(j) = S_Path(end,1);
        A(j) = mean(S_Path);
    end
end
index = S>A;
CALL = exp(-r*T)* mean( index.*(S-A)  );
```

7.1.3 Exponentially weighted average rate Asian option

```
function price = exponentiallyWeightedCall(...
                S,X,r,sigma,lambda,T)
alpha = (r+lambda)*T;
beta = 2*(r+1/2* sigma^2+ lambda)*T;
k =  1/T* ( exp(-lambda*T) - 1 )/(-lambda);
m1 = S*lambda/(lambda+r) * (exp(r*T) - ...
     exp(-lambda*T))/(1-exp(-lambda*T));
m2 =exp(-2*lambda*T)* (S/k)^2 *2/alpha * ...
    ((exp(beta)-exp(alpha))/(beta-alpha) -...
    (exp(beta)-1)/beta);
phi = 2*log(m1)-1/2*log(m2);
chi = sqrt(log(m2/m1^2));
F = @(x)lognpdf(x,phi,chi).*(x-X);
price = exp(-r*T)* quad(F,X,300);
```

7.1.4 Transformation method

asian_call.m

```
% INPUT PARAMETERS
% r - interest rate
% sigma - volatility of underlying asset
% q - divided yield
% T - expiration date
% L - space discretization parameter
% lambda - parameter lambda in case of exponentially
%          weighted averaged option
% n,m - number of steps in space and time discretization
% p_max - maximum iterations in p-loop
% toll - prescribed tolerance for p-loop
```

```
% type - type of option 1 = arithmetic, 2 = geometric
%        3 = exponentially weighted 4 = lookback option
% interpolation - type of interpolation
% look - parameter in lookback option
function [xi,tau,rhos] = asian_call(r,sigma,q,T,L,lambda,...
                         n,m,p_max,toll,type,...
                         interpolation,look)
h=L/n; k=T/m;
xi = 0:h:L;
tau = 0:k:T;
rho=rho_zero(type,1,q,r,T,lambda,look);
PI = -(xi< log(   rho   )  );
rhos(1)=rho;
for i=1:m
  rho_old = rho;
  PI_old = PI;
  for p=1:p_max
    rho_p_old=rho;
    rho = operator_F(type,PI,rho_old,PI_old,tau(i),k,...
                     sigma,q,r,xi,T,h,lambda,look);
    PI_POL = operator_T(rho,rho_old, PI_old,xi,r,q,k,...
                        h,interpolation);
    PI = operator_A(type,PI_POL,rho,tau(i),xi,r,h,k,...
                    sigma,T,lambda,look);
    if (norm(rho_p_old-rho)<toll)
      break;
    end
  end
  rhos(1+i,:)=rho;
end
```

operator_T.m

```
function x = operator_T(rho,rho_old,PI_old,xi,r,q,k,h,...
                        interpolation)
x=PI_old*0-1;
pp = interp1(xi,PI_old,interpolation,'pp');
ni = xi+log(rho_old/rho)-(r-q)*k;
x(ni>0)=ppval(pp,ni(ni>0));
```

operator_A.m

```
function x = operator_A(type,PI_POL,rho,tau,xi,r,h,k,...
                        sigma,T,lambda,look)
```

```
n=length(xi);
tmp1 = -k/(2*h^2)*sigma^2;
tmp2 = k/(2*h);
tmp3 = sigma^2/2*tmp2;
if (type==1)
  tmp4 = tmp2*(rho*exp(-xi)-1)/(T-tau);
elseif (type==2)
  tmp4 = tmp2*(log(rho)-xi)/(T-tau);
elseif (type==3)
  tmp4 = tmp2*lambda*(rho*exp(-xi) - 1)/...
          (1-exp(-lambda*(T-tau)));
else
  tmp4 = tmp2*((rho*exp(-xi)).^look-1)/((T-tau)*look);
end
alpha = tmp1+tmp3+tmp4;
gamma = tmp1-tmp3-tmp4;
PI_POL(1) = PI_POL(1)+alpha(1);
if (type ==1)
  d = 1+(r+1/(T-tau))*k-(alpha+gamma);
elseif(type==2)
  d = 1+(r+(1+xi-log(rho))/(T-tau))*k-(alpha+gamma);
elseif(type==3)
  d = 1+(r+lambda/(1-exp(-lambda*(T-tau))))*k-(alpha+gamma);
else
  d = 1+(r+1/((T-tau)*look))*k-(alpha+gamma);
end
x=gallery('tridiag',alpha(2:end),d,gamma(1:end-1))\PI_POL';
x=x';
```

operator_F.m

```
function x = operator_F(type,PI,rho_old,PI_old,t_j,k,...
                        sigma,q,r,xi,T,h,lambda,look)
DPI = PI_old-PI;
if (type==1)
    I1=(r-(rho_old*exp(-xi)-1)/(T-t_j)).*PI;
elseif (type==2)
    I1=(r-(log(rho_old) - xi )/(T-t_j)).*PI;
elseif (type==3)
    I1=(r-lambda*(rho_old*exp(-xi) - 1)/...
       (1-exp(-lambda*(T-t_j)))).*PI;
else
    I1=(r-((rho_old*exp(-xi)).^look-1)/((T-t_j)*look)).*PI;
end
tmp = log(rho_old)+(sum( DPI )-DPI(1)/2-DPI(end)/2 )*h+...
```

```
        k*(q+sigma^2/2-q*rho_old -...
        (sum(I1)-I1(1)/2-I1(end)/2)*h );
x = exp(tmp);
```

rho_zero.m

```
function x = rho_zero(type,call,q,r,T,lambda,look)
if (type==1)
    if (call==1)
        x = max((r+1/T)/(q+1/T),1);
    else
        x = min((r+1/T)/(q+1/T),1);
    end
end
if (type==2)
      f = @(x)log(x)+q*T*x-r*T;
      x = fzero(f,2);
    if (call==1)
        x = max(x,1);
    else
        x = min(x,1);
    end
end
if (type==3)
    x =( r*(1-exp(-lambda*T))+lambda)/...
        (q*(1-exp(-lambda*T))+lambda);
    if (call==1)
        x = max(x,1);
    else
        x = min(x,1);
    end
end
if (type==4)
    f = @(x)x^look-1-T*look*(r-x*q);
    x = fzero(f,2);
    if (call==1)
        x = max( x  ,1);
    else
        x = min( x  ,1);
    end
end
```

7.2 Martingale

A filtration (on $(\Omega, \mathcal{F})$) is a family $\mathcal{M} = \{\mathcal{M}\}_{t\geq 0}$ of σ-algebras $\mathcal{M}_t \subset \mathcal{F}$ such that

$$0 \leq s < t \Rightarrow \mathcal{M}_s \subset \mathcal{M}_t$$

(i.e. $\{\mathcal{M}_t\}$ is increasing). An n-dimensional stochastic process $\{M\}_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathcal{P})$ is called martingale with respect to a filtration $\{\mathcal{M}\}_{t\geq 0}$ with respect to $\mathcal{P}$ if

1. M_t is $\mathcal{M}_t$-measurable for all t,
2. $E[|M_t|] < \infty$ for all t,
3. $E[M_s|\mathcal{M}_t] = M_t$ for all $s \geq t$.

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