Faculty of Mathematics, Physics and Informatics, Comenius University Bratislava

Economic and financial mathematics

Mathematical model of mortgage crisis

Master's thesis

Candidate: Andrea Tinajová

Advisor: Mgr. Richard Kollár, PhD.

Bratislava 2010

Matematický model hypotekárnej krízy

Diplomová práca

Andrea Tinajová

Univerzita Komenského v Bratislave
Fakulta matematiky, fyziky a informatiky

9.1.9 Aplikovaná matematika
Ekonomická a finančná matematika

Školiteľ: Mgr. Richard Kollár, PhD.

BRATISLAVA 2010

Mathematical model of mortgage crisis

Diplomová práca

Andrea Tinajová

Comenius University in Bratislava
Faculty of mathematics, physics and informatics

9.1.9 Applied mathematics
Economic and financial mathematics

Advisor: Mgr. Richard Kollár, PhD.

BRATISLAVA 2010

Acknowledgment

I would like to express special thanks to my supervisor Mgr. Richard Kollár, PhD. for motivation, support and guidance he offered throughout elaboration of this thesis.

Declaration on word of honour

I declare this thesis was written on my own, with the help provided by my supervisor and referred to literature.

....

Andrea Tinajová

Dátum kompilácie: September 1, 2010

Typeset in LaTeX

Abstrakt

TINAJOVÁ, Andrea: Matematický model hypotekárnej krízy[diplomová práca]. Univerzita Komenského v Bratislave. Fakulta matematiky, fyziky a informatiky; Katedra aplikovanej matematiky a štatistiky.

Školiteľ: Mgr. Richard Kollár, PhD. Bratislava: FMFI UK, 2010.

počet strán: 55.

V diplomovej práci zostavujeme model hypotekárnej krízy využitím reakčno - difúzneho systému rovníc a nelokálnej difúzie. Z dostupnej literatúry a článkov vyberieme hlavné činitele, ktoré mohli viesť ku vzniku krízy. Z nich vytvárame systém predpokladov, z ktorých odvodíme model správania sa investorov. Uvažujeme trh s dvomi aktívami - rizikovým a bezrizikovým. Odhadujeme výnos hypotekárnych záložných listov v rokoch 2005-2010 ako aj objem vytvorených hypoték. Numericky aplikujeme model v programe Matlab 6.5. Model kalibrujeme tak, aby bol v súlade s vývojom amerického hypotekárneho trhu v rokoch 2005-2009. Model uvažovaný v práci nebol publikovaný v žiadnej literatúre, ako aj všetky výsledky uvažované v našej práci sú vlastné.

Kľúčové slová: reakčno - difúzny systém, nelokálna difúzia, hypotekárna kríza, rozdelenie kapitálu, výnos hypotekárnych záložných listov.

Abstract

TINAJOVÁ, Andrea: Mathematical model of mortgage crisis[masters thesis]. Comenius University in Bratislava. Faculty of mathematics, physics and informatics; Department of applied matematics and statistics.

Advisor: Mgr. Richard Kollár, PhD. Bratislava: FMFI UK, 2010.

pages: 56.

In this thesis we create reaction - diffusion model of mortgage crisis in the USA in 2008. We compiled system of assumptions describing mortgage market and investors according to the literature. We estimate yield of mortgage backed securities based on primary mortgage market. We make function describing amount of capital invested in intervals with different risk. We implement and solve this system in Matlab6.5. We set the constants of model in accordance with the data on quantity of mortgages during the period 2005 - 2010. The model presented in this work have not already been published in any literature, as well as all results considered in our work are custom.

Key words: reaction - diffusion system,nonlocal diffusion, mortgage crisis, capital distribution, yield of mortgage backed securities

Contents

Chapter 1

Introduction

Economic crisis occur quiet often on financial markets. However, forecasting them is very difficult. This work is dedicated to recent mortgage crisis that emerged in the U.S. and grew up in global economic crisis.

At the beginning of this thesis we discuss very briefly mortgage market mechanisms and causes of the crisis. A key aspect is determination of fundamental assumptions characterizing behavior of investors in the market. We model this behavior using reaction - diffusion system of equations. Through input variables we describe trading, changes on investor's account as well as total amount of new originated mortgages with respect to our assumptions. Spread of information among investors is implemented by tool of nonlocal diffusion. Purchasing of assets is being modeled by logistic dynamics.

Furthermore, it is necessary to calibrate our model by choosing appropriate market data and processing them into initial conditions. We represent investors by the biggest US. funds and they risk profile by volatility of the share price. Furthermore we calculate yield of Mortgage backed securities based on yield of underlying mortgages.

It is necessary to change our model to be solvable in MATLAB 6.5. Result of this thesis should be operational model suitable for research on crisis reflecting main market mechanisms, investor's behavior as well as spread of information.

Chapter 2

US Mortgage market

At the turn of the 20th century mortgage market in the U.S. was relatively liberal. U.S. government bounded to create conditions in order to enable gaining a mortgage for as many people as possible. Apart from the traditional prime mortgages there were various kinds of mortgages such as sub - prime, non - prime mortgages, adjusted rate - with payments linked to basic interest rates (Euribor..)and generally lowered requirements for the mortgage applicants.[3]

There was a scoring of clients that was based on client's credit history and used for rating of provided mortgages. Banks were supported by refinancing mortgages with securitization.[3] Securitization meant that various mortgages were collected into the pools. These pools were transformed into mortgage backed securities (further MBS) traded on the secondary mortgage market.[9] Rating of mortgage backed securities was determined by the average rating of underlying mortgages. These MBS were mostly issued by one of three US Federal Housing Agencies. Role of the government sponsored agencies on the market was buying mortgages from commercial banks.[13]

Extension of MBS throughout the system (i.e. into all types of companies) enabled insurance products - swaps. Less risky institutions could buy investment products with lower rating, while this contract was insured by insurance company with sufficiently high rating.

In addition, there were introduced a new products on financial market - collateral debt obligations(CDO). This financial instrument allow the redistribution of payments from mortgages. Cash flow from CDO's goes to tranche holders using the cash flows produced by the CDO's assets and depends mostly on credit quality of the underlying portfolio. In the other words the possibility that investor will receive the first payment from mortgage was assessed by a higher rating. On the other side, to an investor receiving last payment was given this opportunity for less payment, signed by lower rating.[13]

The entire system, however, had several shortcomings. Firstly was assumed that house - price would increase, i.e. that the foreclosure is not unprofitable and therefore investing in mortgages is "safe". However, through increased basic interest rate clients collectively lost their ability to repay their mortgages. This led to decrease of house prices and then to insolvency of many banks. [2]Another problem was poorly adjusted scoring which caused spread of mortgage backed securities into portfolios of "secure" funds. This led to an underestimation of the risk throughout the whole system. [1][4]

Chapter 3

Dynamical model of investing

3.1 Assumptions

In this section we are creating a model describing main causes, that could lead to mortgage crisis.

These are our assumptions:

1. **Existence of 2 kinds of assets on the market**

   There are only two assets to invest on the market - one with constant risk - free lower yield and second one more risky with higher expected yield.

2. **Incorrect information about yield on more risky asset.**

   Investment is valorized with different return than investors expect by deciding where to invest.

3. **Each investor has risk profile - willingness to take a risk**

   There are many different companies on the market, f.e. pension funds as well as hedge funds. To differ these investment companies we use parameter σ. This risk - profile influent investors's behavior in the way that more risky investor buys more from risky assets.

4. **Investors can sell any MBS that they have at any time.**

5. **Constrained budget of investor**

   Investors can not sell more then they actually have, in the other words, they have constrained budget.

6. **Logistic pursuing of assets**

   We use logistic equation for description of speed of purchasing risky assets by investors. In case that risky asset is more profitable then safe asset investors purchase risky asset. Otherwise investors sell risky assets. We assume slow, careful purchasing at the beginning. Then it becomes faster as the more risky asset seems to be safe and finally slow saturation of market by high proportion of risky asset.

   This equation was firstly used by Verhulst. He derived this tool to describe the self-limiting growth of population in biology. The equation seems following:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) = rx - r\frac{x^2}{K} \tag{3.1}$$

   Where r means growth rate, maximum size of population and x size of population. This system has two stationary points: $x = K$; $x = 0$. Stationary point $x = K$ is stable, but the second stationary point $x = 0$ is unstable.

   That means that system is spreading from the first stationary point $x = 0$ modeled by the first term +rx until the second stationary point $x = K$. Then a member $-r\frac{x^2}{K}$ causes self regulation, decrease.

7. **Influence between market members**

   Investors on the market affect their decision, they compete.
   To model this behavior we use similar method, that is used to model diffusion in Physics. Diffusion in physics describes density fluctuations. Special case is the heat equation. With behalf of heat equation we can model transfer of heat in the rod. Temperature of one point in the next time step depends on temperature of two neighbors. We use this principle to model influence of investors on the market. Investors are partially copying behavior of their competitors. Local

diffusion would mean influence only two competitors with most similar risk profile (one more risky and one less risky).

Non - local diffusion is slightly different. Non local diffusion means that we consider influence of whole market to each investor. This model seems to be better approximation of the market. We would like to show, that companies operating with bigger capital have bigger impact to others market players. On the other side funds with more different risk - profile have smaller effect.

We model this behavior by non - local diffusion, or diffusion with long range effects.

8. **Volume-dependence of yield of risky asset**

   We observe dependence on whole volume of mortgages on the market, apart from payed or disappeared loans. There is a threshold by it's crossing more risky asset became less profitable then risk - free asset.

3.2 Variables and Parameters

- $r_s = r_s(t)$ yield from the US Treasury bonds (risk - free investment)
- $r_u(t) = r_u(t - \delta(\sigma))$ yield from the MBS (more risky investment)
- $r_{ur} = r_V$ real return dependent on volume of all assets on the market
- $V(t)$ volume of capital of all investors invested in mortgage debt securities / new - originated mortgage debt.
- σ [no units]is independent, measure of risk - profile of the investors $\sigma \in (0, 1)$
  σ close to 0 means low risk funds
  σ close to 1 means high risk funds
- $\kappa = \kappa(k)$ this parameter influences diffusion
- t [years]
- K [dollars] capital of 1 investor

- u [dollars] amount of money on investor's account
- a [no units]is fraction of capital in mortgage backed securities
- kapadj [no units]
- $\kappa_a b$ [no units] speed of investors' purchase
- kapdelay [no units] delay in amount of sold mortgages between the time of investor's decision an the time of investment realization
- κ [$year^{-1}$] strength of diffusion
- α [no units]
- V_{adj} = fraction of all market capital that we cover in our model

3.3 Mathematical description of the model

Let u be amount of money invested in assets with higher risk (i.e.MBS), k be amount of all capital of 1 investor.

Then rate $a = \frac{u}{k}$ means share of unsafe investment.

1. Investor's account

   Assumption (1) says that Investor can invest into two kinds of assets. Capital in portfolio of each investor is divided between these two assets: safe or risky. In each time step investor must make some decision how to change weights in his portfolio between these two assets. Let us sign amount of capital of one investor invested in MBS in time (t) as $u(t)$. Change in amount of capital in MBS is than $\triangle u$. Initial amount $u(t)$ is valorized by yield r_u in the next time step. So that we can approximate amount of MBS in time step $t + \triangle t$ as:

   $$u(t + \triangle t) = (1 + r_u \triangle t)u(t) + \triangle u \tag{3.2}$$

   The change $\triangle u$ (decision of an investor)influences:

- type of investor weighted by σ - positive, more risky you are, more capital you invest into MBS
- difference between safe and unsafe interest rate $(r_u - r_s)$ - positive, bigger yield MBS, more capital you invest into MBS
- amount of money already invested in unsafe investment $u(t)(1 - \frac{u(t)}{K(t)})$

Then the change $\triangle u$ seems following:

$$\triangle u = \sigma(r_u - r_s)\triangle t u(t)\left(1 - \frac{u(t)}{K(t)}\right) \tag{3.3}$$

Now we get u in time $t + \triangle t$

$$u(t + \triangle t) = (1 + r_u \triangle t)u(t) + \sigma(r_u - r_s)\triangle t u(t)\left(1 - \frac{u(t)}{K(t)}\right) \tag{3.4}$$

$$u_t = \frac{u(t + \triangle t) - u(t)}{\triangle t} \tag{3.5}$$

$$u_t = \frac{(1 + r_u \triangle t)u(t) + \sigma(r_u - r_s)\triangle t u(t)(1 - (\frac{u(t)}{K(t)})) - u(t)}{\triangle t} \tag{3.6}$$

so we get differential equation for amount of MBS on investor's account:

$$u_t = r_u u(t) + \sigma(r_u - r_s)u(t)\left(1 - \frac{u(t)}{K(t)}\right) \tag{3.7}$$

2. Whole Capital

$$K_t = r_u u(t) + r_s(K(t) - u(t)) \tag{3.8}$$

It seems advantageous to transform the equation for investment to equation for the proportion of risky assets in portfolio.

$$a(t) = \frac{u(t)}{K(t)} \tag{3.9}$$

Using simple quotient rule

$\left(\frac{\dot{u}}{K}\right) = \frac{\dot{u}K - u\dot{K}}{K^2} = \frac{\dot{u}}{K} - \frac{u}{K}\frac{\dot{K}}{K}$

Then

$$\left(\frac{\dot{u}}{K}\right) = \frac{r_u u}{K} + \frac{\sigma(r_u - r_s)(1 - \frac{u}{K})u}{K} - \frac{u}{K}\left[\frac{r_u u}{K} + r_s\left(1 - \frac{u}{K}\right)\right] \tag{3.10}$$

Now we note

$$\begin{aligned}
\dot{a} &= \left(\frac{\dot{u}}{K}\right)\\
\dot{a} &= r_u a + \sigma(r_u - r_s)(1-a)a - a[r_u a + r_s(1-a)]\\
\dot{a} &= r_u a + \sigma(r_u - r_s)(1-a)a - a[r_u a + r_s(1-a)]\\
\dot{a} &= (r_u - r_s a(1-a) + \sigma(r_u - r_s)(1-a)a)\\
\dot{K} &= r_s K(t) + (r_u - r_s)aK(t)
\end{aligned}$$

3.3.1 Effect of local diffusion

Local diffusion describes local effect among investors(assumption 6).

Diffusion equation seems like basic heat equation:

$$a_t = \kappa a_{\sigma\sigma} \tag{3.11}$$

We add this diffusion member into equation for proportion of risky asset.

We have constructed two equations, that we use to our model:

$$a_t = \kappa a_{\sigma\sigma} + a(1-a)[(1+\sigma)(r_{ur} - r_s)] \tag{3.12}$$

$$K_t = r_s K + (r_{ur} - r_s)aK \tag{3.13}$$

3.3.2 Effect of non - local diffusion

As we said in assumption 6, local diffusion seems that it can describe only local effects. Local effect in the sense of influence between investors with similar capital and risk profile. We do not think that each investor has only short range effect. We suppose that all the funds influent each others as well as effect on the other funds depends on size of fund.

In non - local diffusion we observe influence of all investors σ' on one investor σ

$$a_t(\sigma,t) = \kappa \int_{\sigma'} G(\sigma'-\sigma, K(\sigma'))a(\sigma',t)d\sigma' - a(\sigma,t) + reaction \tag{3.14}$$

in our model we get nonlocal diffusion as:

$$a_t(\sigma,t) = \kappa \left[\int_{\sigma'} \frac{\beta(\sigma)K(\sigma')}{e^{\alpha|(\sigma-\sigma'|)}} a(\sigma',t)d\sigma' - \kappa_{ab}a(\sigma,t)\right] + a(1-a)[(1+\sigma)(r_{ur}-r_s)] + \sigma(r_{ur}-r_{uz}) \tag{3.15}$$

3.3.3 Parametrization of equations

Finally we add constants to our model, to be able to investigate role and significance of single parts. Coefficient κ_{ab} describes importance of reaction part of equation, coefficient κ, α describes importance of diffusion part of equation. Rate r_{uz} means real yield from the investment. As we said in assumption 2. investor does not have a correct information on return. That means that his decision is made after considering return from available information, but his investment will be valorized with different yield.

$$a_t = \kappa a_{\sigma\sigma} + \kappa_{ab}a(1-a)[(1+\sigma)(r_{ur}-r_s)] + \sigma(r_{ur}-r_{uz}) \tag{3.16}$$

3.3.4 Volume of all MBS in portfolios on the market

Assumption (7) says, that yield from MBS r_u depends on amount of MBS. We determine this amount that way, that we know change on investor‘s account (4.2) As we count all the mortgages outstanding, we count only positive change in portfolio. Positive change appears, when the difference between two yields $r_u(\tau, \sigma) - r_s(\tau)$ is positive. Therefore we count maximum $max(0, r_u(\tau, \sigma) - r_s(\tau))$ instead.

$$\triangle u = \sigma(r_u - r_s)\triangle t u(t)\left(1 - \frac{u(t)}{K(t)}\right) \tag{3.17}$$

Let us sign $a = \frac{u(\tau,\sigma)}{K(\tau,\sigma)}$. Then we get:

$V = \int_0^\infty \int_0^t \sigma(r_u(\tau, \sigma) - r_s(\tau))u(\tau, \sigma)(1 - \frac{u(\tau,\sigma)}{K(\tau,\sigma)})d\tau d\sigma$

$V = \int_0^\infty \int_0^t \sigma(r_u(\tau, \sigma) - rs(\tau))a(\tau, \sigma)k(\tau, \sigma)(1 - a(\tau, \sigma))d\tau d\sigma$

For numerical purposes we will integrate the last equation after t:

$$V_t = \frac{dV}{dt} = \int_0^\infty \sigma[Max(0, r_u(t, \sigma) - r_s(t))a(t, \sigma)k(t, \sigma)(1 - a(t, \sigma))]d\sigma \tag{3.18}$$

Our local model now consists of following three equations:

3.16, 3.13 , 3.18

As well as non - local diffusion consists of these three equations:

3.15, 3.13 , 3.18

3.3.5 Initial conditions

At the beginning each investor has same proportion of capital like his risk profile.

So that initial condition for proportion is function: $a = \sigma$. An initial condition for capital is approximated from the data.

3.3.6 Boundary conditions

We need boundary conditions for least risky as well as most risky investor. For least risky investor we choose Dirichlet condition 0 risk.

We try four types of boundary conditions for the most risky investor for local diffusion - reaction system:

- $a(\sigma_{max}, t) = \sigma_{max}$

  This Dirichlet condition means that risk of most risky investor stay unchanged during the time.

- $\frac{\partial a(\sigma_{max},t)}{\partial \sigma_{max}} = 0$

  This Neumann condition means that two most risky investors has the same same proportion of whole capital in risky asset as the second most risky investor

- $\frac{\partial a(\sigma_{max},t)}{\partial \sigma_{max}} = 1$

  This Neumann condition means, that the most risky investor has still more capital in risky asset as the second most risky investor.

- $a(\sigma_{max}, t) = 1$ This Dirichlet condition means, that risk of most risky investor is still 1.

Chapter 4

Data characterization

4.1 Initial condition

4.1.1 Function describing amount of capital under different risk

In our model we are observing behavior of investors on the market. We represent investors by US funds from US mutual funds market, because each fund is represented by it's portfolio manager so that we can observe behavior of investors.

We consider in this work just the data for bond funds, since mortgage backed securities seemed similar to US treasury bonds, because of government guarantee.[13]

We specify volatility for each fund in the time: 11/23/1998 - 1/1/2005. We use this volatility as implicit risk - profile for each investor.

Up to www.ici.org the 20 largest mutual fund complexes cover 71%[**?**] of the market. We use weekly data for all these funds, because there can be some significant volatility also between single days during the week.

Up to Business week's Bond fund scoreboard 2005 [27] we choose 40 biggest bond funds in the year 2004 (net asset value to December 31st, 2004). We calculated weekly volatility from price per one share[27] as risk of each investor form the weekly data. We multiply weekly volatility by factor $\sqrt{52}$ in order to get year volatility.

We divide funds into 15 intervals up to their volatility. Then for numerical purposes we made this curve smooth and spread it by artificial points until the value 1. So that we got initial condition that can be seen on the picture.

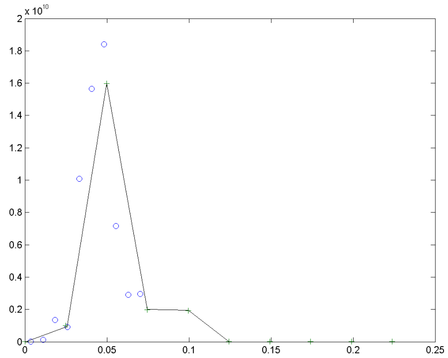

Figure 4.1: Initial condition for capital. Function describing amount of capital under different risk approximated from data. "o" represents amount of capital of investors with variance $\sigma \in interval$ for each group represented by average variance of the group, "+" is linear interpolation of "o" points.

4.1.2 Categorization of MBS based on status of payments of underwritten mortgages

Investors were oriented by purchase of Mortgage backed securities especially at the market prices of these funds as well as their historical performance. As an example, promotional material of mortgage backed securities indicated brilliant performance that was not possible.

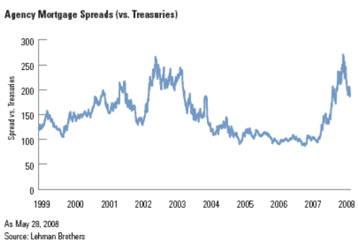

Figure 4.2: Historical performance of Agency mortgage spread vs. mortgages presented by Lehman and Brothers in May 28, 2008, less then 4 months before bancrupcy

[13]

This decision, as shown in our model was flawed because investors do not always know what is behind those securities as well as past income does not yield the future. l n Mortgage backed securities were created as an accumulation of mortgages from primary market into the 'Pool'. Their yield is derived from proceeds by the primary mortgage market less of costs from agencies. Our estimation was built on the work of A. Greenspan [4] who considered the data provided mostly by FED. However, we expand his approximation by the data on homes sold as and the data provided by Mortgage backers association about delinquent payments. We approximate mortgage security as bond with one year coupon.

We will split investment into MBS to four different categories depending on their status with respect to delinquency on payments of underwritten Mortgages as following:

- delinquent payment (D)
- ongoing foreclosures (in process) (F)
- charge - offs (CH)
- regular payments (RP)

There is brief definition of each category:

1. Delinquencies (D)

   Delinquent loans and leases are those past due thirty days or more and still accruing interest as well as those in nonaccrual status. [21]

2. Foreclosures (F)

   The legal process by which an owner's right to a property is terminated, usually due to default. Typically involves a forced sale of the property at public auction, with the proceeds being applied to the mortgage debt. [22]

3. Charge - off (CH)

   Charge - off rates are the value of loans and leases removed from the books and charged against loss reserves. Charge - off rates are annualized, net of recoveries.[24]

4. Regular payments (RP)

   Payers pay their usual downpayments including yield. We determine amount of debt outstanding that produce yield as mortgage debt outstanding reduced by foreclosures, charge offs and delinquencies. As yield we use average interest rate for 30-year mortgage.

Data resources:

1. NS: houses sold during period [14]
2. NFS: houses for sale at the end of period [14]
3. ES: existing homes sold during period [16]
4. EFS: existing homes for sale at the end of period [16]
5. NP: New Single-Family Home Prices [17]
6. EP: Existing Home Prices [18]
7. AL: Amount of loans: National Delinquency Survey for quarters: Q4 06, Q4 07, Q1 08, Q2 09, Q4 09 [23]
8. FS: foreclosures started (F) [23]
9. FI: foreclosures inventory [23]
10. 30YMY [19]

4.2 Dependence of yield of MBS on volume of new originated mortgages

4.2.1 MBS's yield determination

For our model we try to determine real return on MBS p.a. It is said that return that portfolio mangers were calculating with was overvalued and risk undervalued.

In regard of these facts, we try to determine a return of MBS directly from mortgage market data.

We consider in this work only residential - noncommercial mortgages (1-4 single family house).

1. **MBS Yield curve determination**

   We would like to determine average yield of mortgages p.a. during the crisis 2000 - 2008. Whereas Mortgage backed securities were derived from mortgages, real return of them is directly linked with the yield from mortgages.

   We calculate total yield as $\frac{GAIN-LOSS}{MDO}$

2. **Total loss caused by foreclosed buildings**

   Difference between amount of debt, that became foreclosed, and gain from foreclosure process is total loss, that must mortgage providers undergo.

   $LOSS = DF - GF$

3. **Total average debt in foreclosures**

   Multiplying $DF = MDO * F$ we get total amount of debt, that changed into foreclosure during the year.

4. **Total gain from the sale of foreclosures**

   Now we know amount of loans as well as proportion of foreclosures, so that by multiplying we can get amount of money in foreclosure. Selling a foreclosure costs commission costs (6 %) and closing costs(1,5 %) , that we estimate after http://www.forsalebyownercenter.com/tools/costofsalecomparisoncalculator.aspx.

We can approximate total loss as $GF = (1 - (0,06 + 0,015)) * AF * HPI$

5. **Amount of foreclosures**

   We can see, that amount of loans and mortgage debt outstanding are correlated with correlation 0,99 at the time of crisis. So we derived the rest of data for AL from data on mortgage backed securities, applying the year change ratio.

   We have also year data on amount of foreclosures started and inventories FS, FI in percentage of all loans, it means that proportion of foreclosures from all loans is $F = (FS + FI)$

   From these data we can get total amount of loans in foreclosures: $AF = AL * F$

6. **House price index (HPI)is average price of sold house.**

   House price index in our model is average price of house that is "for sale" at the beginning of period. We can calculate it as whole income received from sold houses divided by whole amount of houses for sale and sold in period:

   $HPI = ((ES * EP + NS * NP)/(NFS + NS + ES + EFS)$

7. **Total gain from regular mortgages**

   Firstly, we should calculate amount of regularly payed mortgage debt as total debt less of delinquent payments $MDO * (1 - F - D - CH)$. This debt is then valorized by average mortgage rate for 30 years mortgage.

   $GAIN = MDO * (1 - F - D - CH) * 30YMY$

4.2.2 Volume of mortgage debt outstanding

We would like to determine amount of all loans existing before the crisis and those that were sold during the crisis. We approximate volume of originated mortgages on the market in following way: we take amount of mortgage debt outstanding (MDO) at the beginning of the period (year 2000) and then we add amount of originations (new - originated loans)[15] every year. We get a curve in a the picture below.

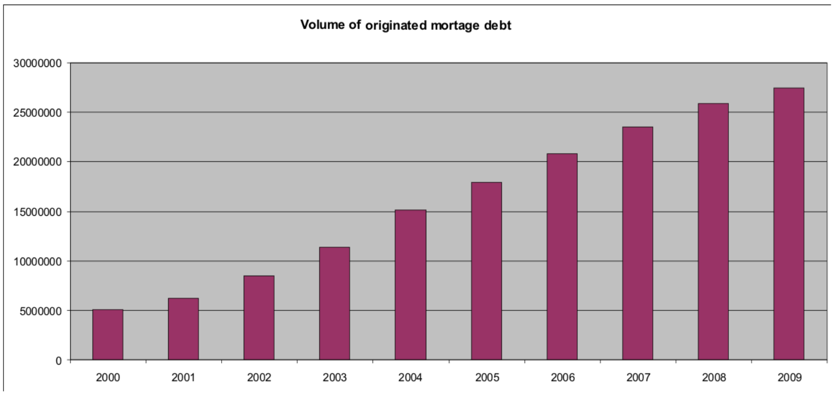

Figure 4.3: Volume of mortgages on the market

Then sum of originated mortgages increased each year. That means that time - dependence of mortgage debt is increasing function as well as time dependence of yield of MBS. So that we combine these two functions and create volume dependence of MBS yield.

4.2.3 Volume dependance of mortgage yield

Finally we match data on volume of mortgages p.a. with average yield p.a. for each year during observed period 2000 - 2009. We get curve that can be seen on the picture below.

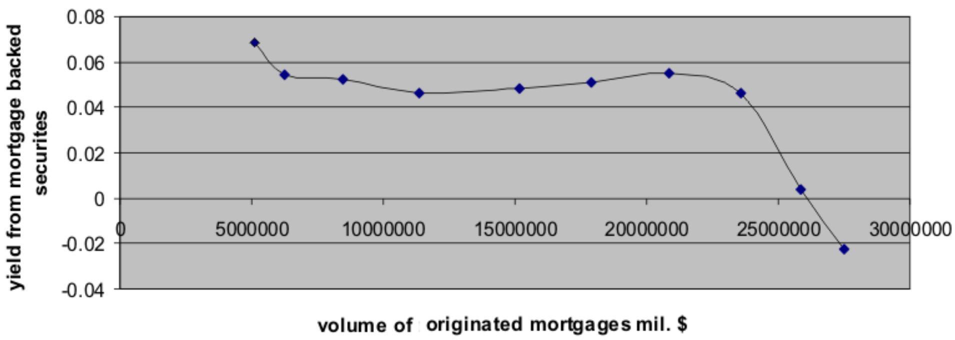

Figure 4.4: Initial condition for yield of MBS. Curve represents dependence of yield on volume of mortgages. Each point represent amount of new originated mortgage

debt on the market and yield from mortgages p.a.

For our purposes is interesting period 2004 - 2009, as we estimated volatility to the end of year 2004.

Chapter 5

Numerical Implementation

5.1 Integral of Volume

We use trapezoid rule for calculating integral determining volume of all mortgage backed securities on the market.

5.2 Solving Equations

For solving our system of 3 equations we use Matlab function ode23t that is for this type of problem most suitable. This solver solves equation y' = f(t,y) from time T0 to TFINAL with initial conditions Y0. As we have in the equation for proportion also second derivation we must use discretization in σ in order to get equation of the form y' = f(t,y)

5.3 MBS Yield

As a yield we use simple linear spline to be able to observe fundamental influence of yield decay. We use parameters for breakpoints form real yield curve. We took values for volume as well as yield for the years 2005-2009. Yield approximation can be seen on the picture:

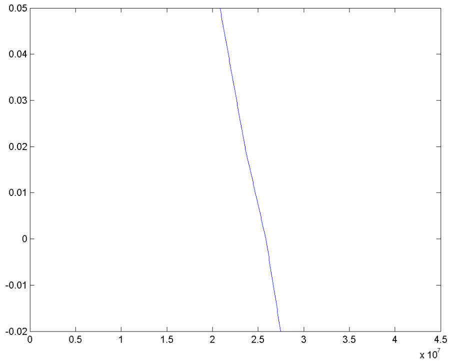

Figure 5.1: Numerical approximation of yield curve

5.4 Discretization

We use discretization in space (of investors) and in time.

symmetric mesh

Hence $x_2 - x_1 = x_3 - x_2 = h$ then

$$f''(x_2) = \frac{-f(x_1) + 2(x_2) - f(x_3)}{h^2} \tag{5.1}$$

non - symmetric mesh We use in our model non symmetric mesh characterized by single data points.

If first derivation seem following:

$$f'(y_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} f'(y_2) = \frac{f(x_3) - f(x_2)}{x_3 - x_2} \tag{5.2}$$

then we get second derivation as:

$$f''(x|2) = -f(x_1)\frac{2}{(x_2 - x_1)(x_3 - x_1)} - f(x_3)\frac{2}{(x_3 - x_2)(x_3 - x_1)} + f(x_2)\frac{2}{(x_2 - x_1)(x_3 - x_2)} \tag{5.3}$$

Into this scheme we apply boundary conditions.

non - local diffusion

Matrix of nonlocal diffusion is to be found in the programme.

Chapter 6

Results

We observe time evolution of three variables:

- Evaluation in amount of MBS by investors represented by variable a
- Evaluation in amount of capital in each fund (by each investor) through log(K)
- Volume of all originated mortgages by variable V. We try to match this data with volume of originated mortgages from section 4.2.2

6.1 Local diffusion results

We let the system with local diffusion develop for five years (2004-2009). We present results for different boundary conditions at the end of simulation for each time step. We choose time step 0.01 year. It could be rate how often make investors changes in portfolio. Most risky investor in our model has volatility $\sigma_{max} = 0.72$ (it is artificial point). We show in each case time evaluation of capital as well as proportion of risky capital.

- $a(\sigma_{max}, t) = 0$, $t \in (0, 5)$,

  This Dirichlet condition means that risk of most risky investor stay unchanged during the time.

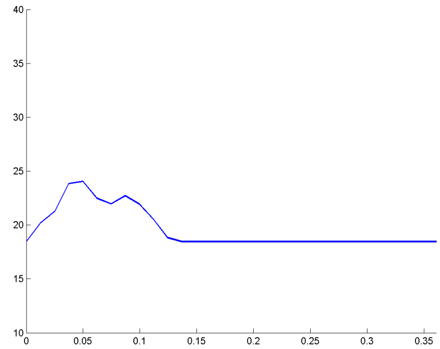

Figure 6.1: Distribution of capital by Dirichlet condition $a(\sigma, t) = 0$

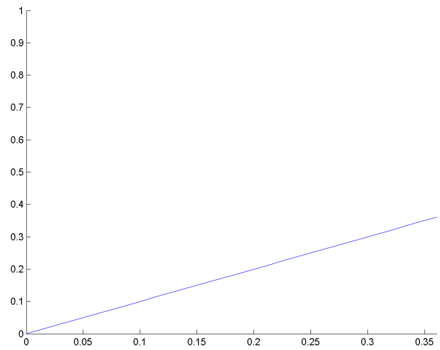

Figure 6.2: Proportion of risky asset by Dirichlet condition $a(\sigma, t) = 0$

Figure Nr. 6.1 shows that there is uniform change (increase)in amount of whole capital by each investor.

Figure Nr. 6.2 shows that there is no change in proportion of risky assets of investors, because initial condition itself is solution of the problem.

- $\frac{\partial a(\sigma_{max},t)}{\partial t}= 0$,$\sigma = max(\sigma)$,t$\in (0, 5)$

  This Neumann condition means that two most risky investors has still same proportion of capital in risky asset.

  However change in proportion of risky asset is too small to be seen in our picture. (We display only real investors - funds until risk - volatility σ=0.35) But proportion of risky assets is increased more by more risky investors. Also as Figure 6.3 shows, there is bigger increase of capital by more risky investors.

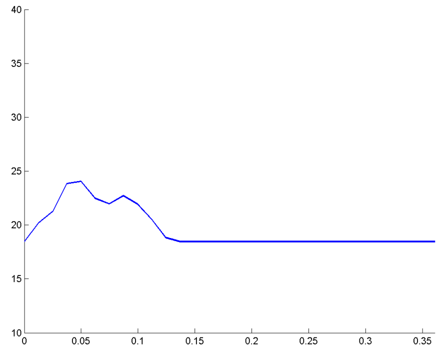

Figure 6.3: Distribution of capital by Neumann condition $\frac{\partial a(\sigma,t)}{\partial t} = 0$

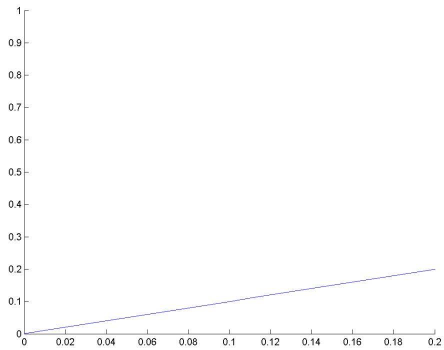

Figure 6.4: Fraction of risky asset by Neumann condition $\frac{\partial a(\sigma,t)}{\partial t}=0$

- $\frac{\partial a(\sigma_{max},t)}{\partial t} = 1,\ t \in (0,5)$

  This Neumann condition means, that the most risky investor has still more capital in risky assets then the second most risky investor.

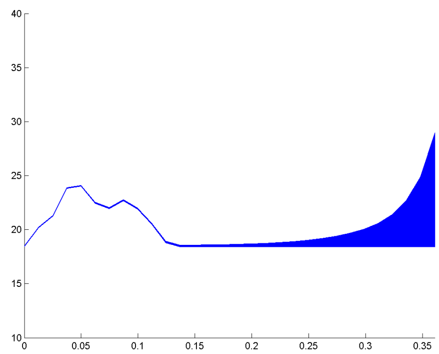

Figure 6.5: Distribution of capital by Dirichlet condition $\frac{\partial a(\sigma,t)}{\partial t} = 1$

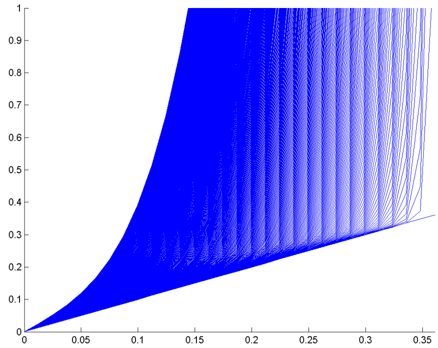

Figure 6.6: Proportion of risky asset by Neumann condition $\frac{\partial a(\sigma,t)}{\partial t} = 1$

We can see on the figures 6.5 and 6.6 big increase of risk of more risky investors as well as bigger change of risky asset by more risky investors.

- $a(\sigma_{max}, t) = 1$, $t \in (0, 5)$ This Dirichlet condition means, that risk of most risky investor is still 1.

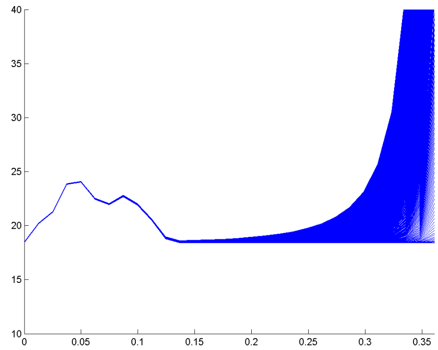

Figure 6.7: Distribution of capital by Dirichlet condition a(sigma,t) = 1

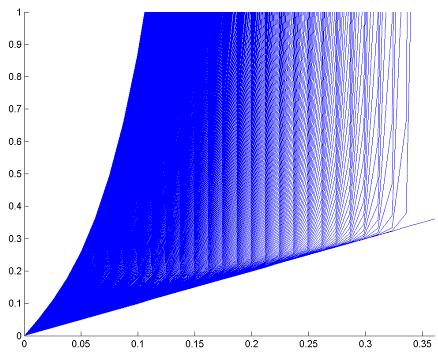

Figure 6.8: Proportion of risky asset by Neumann condition a(sigma,t) = 1

Boundary condition in this case causes increase of risk as well as whole capital. This change is faster and more significant then by the third boundary (Neumann) condition, because most risky investors are fixed.

We can see change of proportion of risky asset in portfolio as well as changes of whole capital of investors. Reaction part of equation 3.16 changes amount of capital of the investors and the diffusion part redistributes capital among investors. Boundary conditions influent redistribution of capital in the sense of direction of change on σ - axes..

6.2 Nonlocal diffusion results

6.2.1 Exclusive delay in the yield information

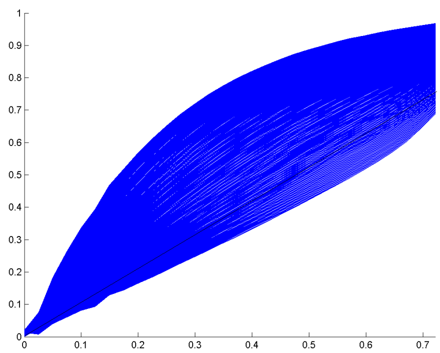

Figure 6.9: Initial and final proportion of unsafe investment in portfolio of investors

At the beginning proportion of risky assets is increasing for all investors. Proportion of risky assets of risky investors are increasing more then by less risky investors.In the other words pursuing of investment is at the beginning of our simulation increasing, then it stops. When the MBS yield reaches value V_crit and the yield start decreasing also investors sell out risky asset.

This model seems to be suitable for modeling pursuing assets by investors.

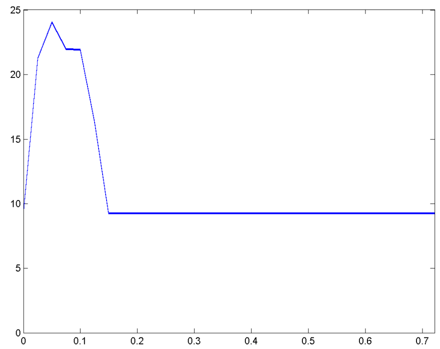

Figure 6.10: Initial and Final amount of capital by investors

In this graph we can observe capital redistribution among investors depending on their risk - profile.

6.2.2 Inclusive delay in the yield information

First of all we calibrate model coefficients in order to get historical amount of mortgages in considered period. We simulate only 5 years - 2004 - 2009 , when the crisis should be seen in our model, represented by the fact that investors stop buying assets.

Final model of crisis

Using nonlocal diffusion we can match historical data on new-originated mortgages.

First of all we calibrate model coefficients in order to get historical amount of mortgages in considered period. We simulate only 5 years - 2004 - 2009 , when the crisis have already started.

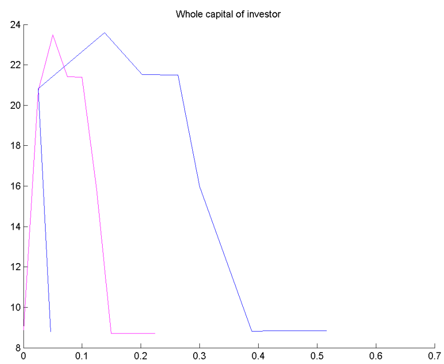

Figure 6.11: Final Distribution of capital

We can see that investors operating with most capital loose more after crisis. On the other side less risky investors increased amount of capital as well as risky investors operating with less capital at the beginning.

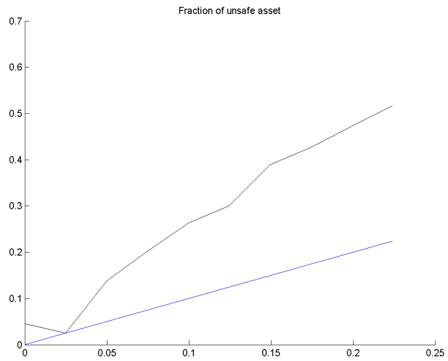

Figure 6.12: Initial and final proportion of unsafe investment in portfolio of investors

? We can see that most of investors became more risky, only part of investors operating with the largest amount of capital did not increase their risk. Investors with lowest capital, more risky became very risky.

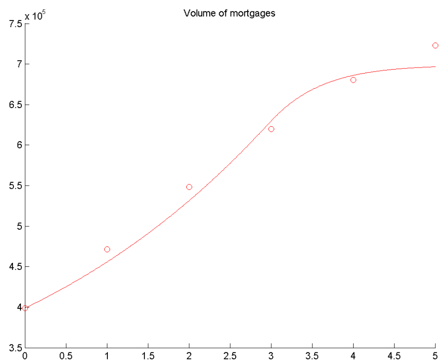

Figure 6.13: Volume of unsafe assets

Figure 6.13 shows that Investors at the beginning buy mortgages but then they stopped (as the investment become unprofitable)

6.2.3 Sensibility to chosen parameters

We observe how the final value of variables changes with parameters. For each parameter we make 10 observations for different values of parameters.

Sensibility to V delay

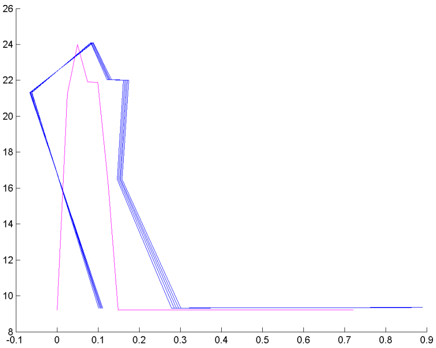

Figure 6.14: Initial and Final Distribution of capital

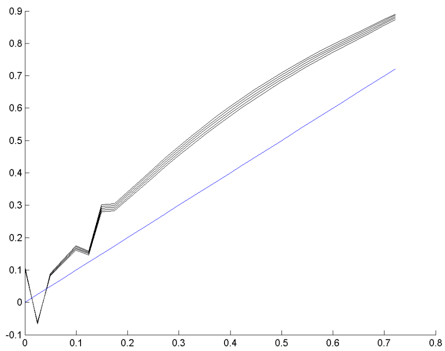

Figure 6.15: Initial and final proportion of unsafe investment in portfolio of investors

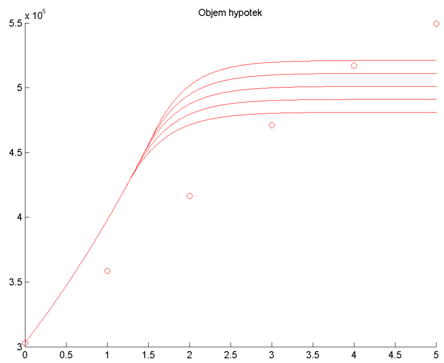

Figure 6.16: Volume of unsafe assets

By bigger delay in information investors purchase longer risky assets.

6.2.4 Sensibility to Volume of mortgages

Sensibility to V_{adj}

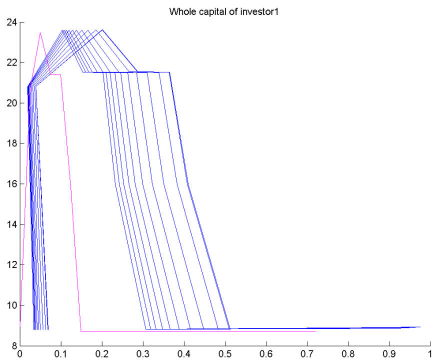

Figure 6.17: Initial and Final Distribution of capital

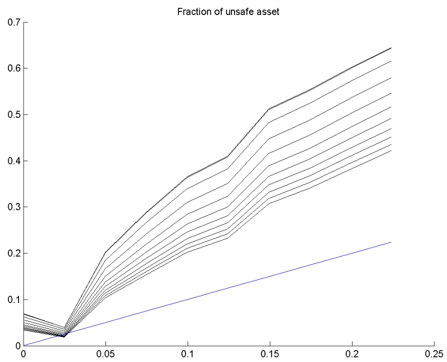

Figure 6.18: Initial and final proportion of unsafe investment in portfolio of investors

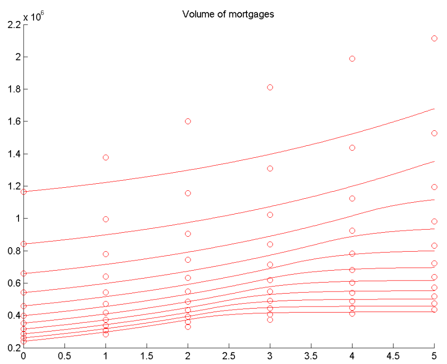

Figure 6.19: Volume of unsafe assets

$V_a dj$ means how big part of capital we observe. This parameter helps to match volume of originated mortgages in our model with market data.

6.2.5 Effects of diffusion

Change of κ

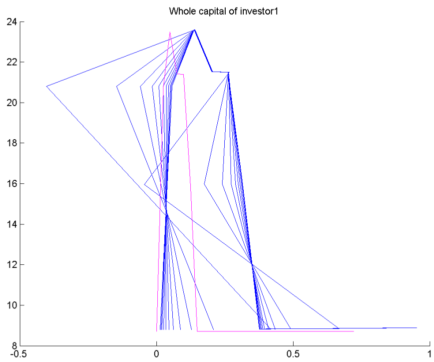

Figure 6.20: Initial and Final Distribution of capital

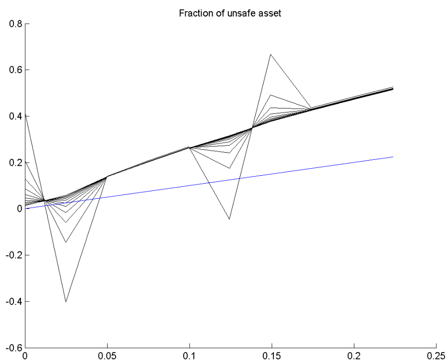

Figure 6.21: Initial and final proportion of unsafe investment in portfolio of investors

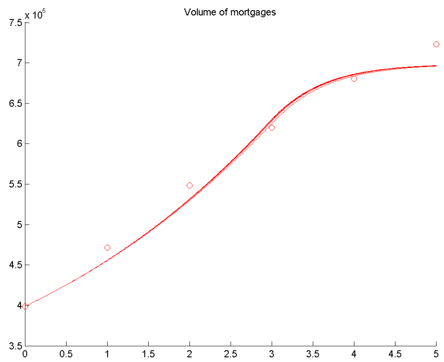

Figure 6.22: Volume of unsafe assets

We can see now, that nonlocal diffusion cause only redistribution of capital, because volume of all assets stayed unchanged in all cases. What differs is capital of investors and proportion of risky assets.

6.2.6 Sensibility to yield curve

6.2.7 Change of V_{crit}

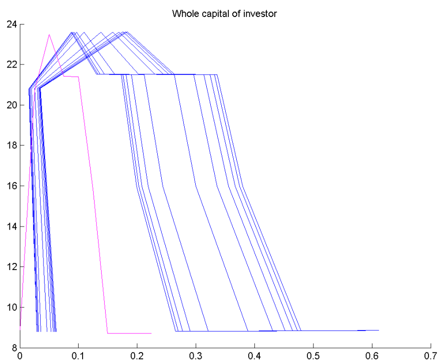

Figure 6.23: Initial and Final Distribution of capital

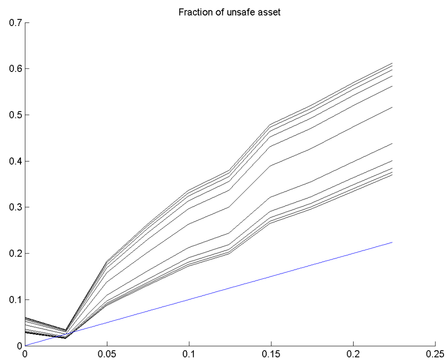

Figure 6.24: Initial and final proportion of unsafe investment in portfolio of investors

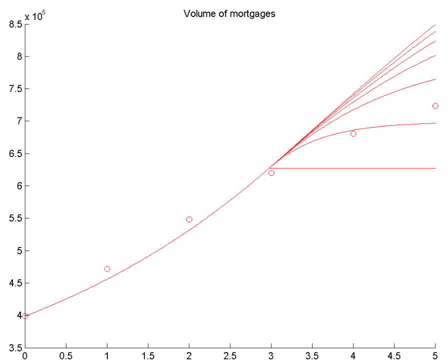

Figure 6.25: Volume of unsafe assets

We can see that by change of V_{crit} capital is moving to more risky investors, also proportion of risky assets by all investors is increasing.

6.2.8 Effects of reaction

6.2.9 Change of κ_{ab}

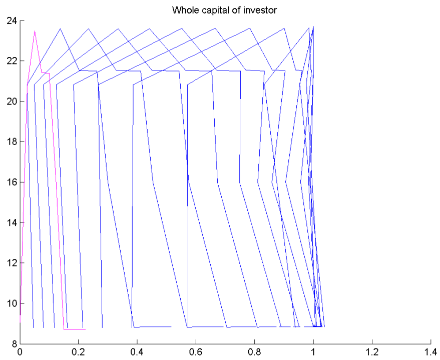

Figure 6.26: Initial and Final Distribution of capital

As shown in Figure 6.26 increase in (κ_{AB}) causes movement of capital to more risky investors.

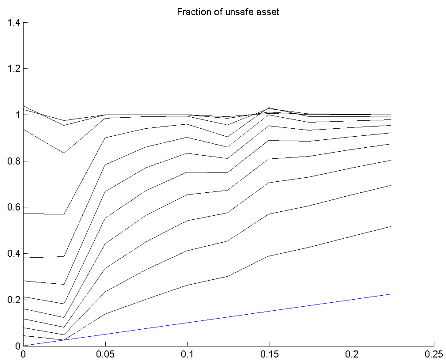

Figure 6.27: Initial and final proportion of unsafe investment in portfolio of investors

Increase in (κ_{AB}) causes increase of risk for all investors.

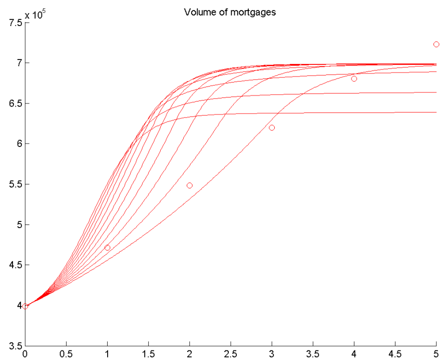

Figure 6.28: Volume of unsafe assets

We can see at Figure 6.28that increasing influence of reaction part of equation (κ_{AB}) causes increase in purchasing assets while they are profitable as well as increase in selling assets by when they became unprofitable. It should be measure of "speed" of purchasing assets.

Chapter 7

Summary and discussion

The aim of this work was to create a model of mortgage crisis using reaction - diffusion system and nonlocal diffusion. We have successfully created model of market with two kinds of assets (risky, risk - free). We estimated yield of mortgage backed securities. This model can be useful for observing market mechanisms. We can see behavior during the crisis, cooperation as well as competition among investors on the market with respect to their size and risk profile. Model can be improved using more funds data - for more fund categories, adding changeable risk - free yield and variable graphs for capital.

Bibliography

[1] BOLDIZSÁR K. *Úverová kríza v USA, jej príčiny a dôsledky* Available online: http://www.rmsfinport.sk/text/KomentareFiles/080221_Analyza%20KRIZY.pdf

[2] HEMERT O. van, DEMYANYK Y., *Understanding the Subprime Mortgage Crisis* Available online: http://www.frbsf.org/economics/conferences/0901/Demyanyk-VanHemert.pdf

[3] DIMARTINO D., DUCA V.J., *The Rise and Fall of Subprime Mortgages* Economic letter,Insights from the federal reserve bank of Dallas, Vol.2, No.11 2007 Available online: http://www.dallasfed.org/research/eclett/2007/el0711.html

[4] GREENSPAN A., *The Roots of the Mortgage Crisis*, The wall street Journal,2007 Available online: http://www.aei.org/docLib/20071214_Greenspan1.pdf

[5] VOZÁR M., *Krachy na burzách z pohľadu ekonofyziky. Diplomová práca..* Fakulta matematiky, fyziky a informatiky, Univerzita Komenského, Bratislava,2006

[6] Fannie Mae *http://www.nyse.com/about/listed/fnm.html*

[7] Freddie Mac *http://www.nyse.com/about/listed/freprz.html*

[8] *http://wikipedia.org*

[9] http://www.fanniemae.com/mbs/

[10] Introduction to Mortgage-Backed Securities (MBS) and Other Securitized Assets http://www.goldman-sachs.com/gsam/docs/instgeneral/general_materials/primer/primer_mortgages.pdf

[11] XIAOQING E.X. *What Drives the Return on Commercial Mortgage-backed Securities?*

[12] GREEN K.R., WACHTER S.M., *THE AMERICAN MORTGAGE IN HISTORICAL AND INTERNATIONAL CONTEXT* University of Pennsylvania Law School ILE INSTITUTE FOR LAW AND ECONOMICS, 2005

[13] MURPHY E.W. *Economic Analysis of a Mortgage Foreclosure Moratorium* Available online: http://fpc.state.gov/documents/organization/110095.pdf

[14] Table 6. New single family home sales, not seasonally adjusted *http://www.census.gov/const/www/newressalesindex.html*

[15] Table 1157. Mortgage Originations and Delinquency and Foreclosure Rates *http://www.census.gov/compendia/statab/2010/tables/10s/1157.xls*

[16] Table 7. Existing home sales *http://www.realtor.org/research.nsf/pages/EHSPage*

[17] Table 8. New Single-Family Home Prices: 1964–Present *http://www.census.gov/const/quarterly_sales.pdf (See Table Q6.)*

[18] Table 9. Existing Home Prices: 1969–Present *http://www.realtor.org/research.nsf/pages/EHSPage?OpenDocument*

[19] 30 years mortgage, average yield *http://www.federalreserve.gov/releases/h15/current/h15.htm*

[20] Greenspan A., Kennedy J. *Estimates of Home Mortgage Originations, Repayments, and Debt On One-to-Four-Family Residences*, Finance and

Economics Discussion Series Divisions of Research Statistics and Monetary Affairs, 2005

[21] Delinquencies *http://www.federalreserve.gov/releases/chargeoff/delallsa.htm.*The Journal of Financial and Quantitative Analysis, Vol. 27, No. 2 (Jun., 1992), pp. 151-168.

[22] Foreclosures *http://www.investorwords.com/2039/foreclosure.html*

[23] Mortgage backers association *National Delinquency survey*

[24] Charge off *www.federalreserve.gov/releases/chargeoff http://invetsopedia.org*

[25] HABERMAN R. *Applied partial differential equations with fourier series and boundary value problems*

[26] ŠEVČOVIČ D. *Parciálne diferenciálne rovnice a ich aplikácie*
ICI http://www.ici.org/pdf/fm-v18n5.pdf

[27] *http://finance.yahoo.com*

[28] Bond fund scoreboard 2005 *http://www.bussinessweek.com.*

[29] MELICHERČÍK I., OLŠAROVÁ L., ÚRADNÍČEK V. *Kapitoly z finančnej matemtiky* EPOS, 2005

[30] ŠEVČOVIČ D., STEHLÍKOVÁ B., MIKULA K. *Analytické a numerické oceňovanie finančných derivátov* Slovenská technická univerzita v Bratislave, 2009

[31] GUBA P. *Bifurkácie v dynamických systémoch* notes to lectures 2009

Chapter 8

Attachements

```
clear
format long
global S
global EXPL
global r_u
global r_s
global sigma
global N nlp
global MinKap
global V_crit
global V1
global V2
global V3
global r_loss
global alpha
global kapadj
global kappab
global kappa
global kapdelay
kapadj = 106̂; %% strength of diffusion
```

```
alpha = 25; %% smaller, more nonlocal diffusion
kappa = 2*10^(-7); %% size
kappab = 10; %% speed of buying
%%parameters set up

r_u = 0.05;
r_s = 0.02;
r_loss= -0.02;
V_adj = 50; %% how big part of capital we cover %%
V1=20829000/V_adj;
V_crit =23555000/V_adj;
V2=25861000/V_adj;
V3=27479000/V_adj;
V_init = 1.515*10^7/V_adj;
kapdelay = 5.5*10^6/V_adj;
%% time parameters
Tfinal = 5;
Tstep = 0.01;
Tspan = 0:Tstep:Tfinal;
%% type of boundary condition
EXPL=3;
%% capital
MaxKap = log(V3); %% logarithmic value log $ (only for plotting size)
MinKap = 0.01; %% min capital (not logarithmic)
%% initialization
%% initialize mesh from data
N_init = 30;
%% wanted mesh
maxVAR = 0.1;
%% original variance (will be multiplied by sqrt(52)
```

```
nodatapts = 15;
%% number of data points in funds
mesh_init = linspace(0,maxVAR,N_init);
[sigma, a0, c0] = variancia(mesh_init, nodatapts, MinKap);
%% initial condition N = length(sigma);
l0 = log(max(c0,MinKap));
u0=[a0 l0 V_init];
%% matrix for heatfun
S = difmatrixNL(sigma);
%% non - local diffusion
%solving equations
OPTIONS = odeset('RelTol',1e-4,'AbsTol',1e-6);
[t, u] = ode23t('heatfunNL', Tspan, u0, OPTIONS);
%% visualization
sigmanew=sigma(1:10)
figure(8)
hold on
plot(sigma,u(end,1:N),'k');
plot(sigma,u(1,1:N),'b');
hold off
%% Volume of mortgages
figure(3)
hold on
title('Volume of mortgages');
plot(t,u(:,2*N+1),'r')
V = [1.515 1.792 2.083 2.355 2.586 2.748];
V = V*10^7/V_adj;
T = [0 1 2 3 4 5];
plot(T,V,'ro');
hold off;
```

figure(4) hold on plot(u(1,1:N),u(1,N+1:2*N),'m');

plot(u(end,1:N),u(end,N+1:2*N));

hold off

end

%%%%%%%%%%%%%

function [f] = heatfun(t,u);

global S

global EXPL

global r_u

global r_s

global MinKap

global sigma

global N

global kapadj

global kappa

global kappab

global kapdelay

r_uu = rulin(u(2*N+1)-kapdelay);

$[bc, nlp] = boundariesNL(EXPL)$;

for k=1:N

v(k)=1/exp(u(N+k));

w(k)= exp(u(N+k));

end v(N+1) = 1/(MinKap*kapadj);

%% extension of v(:)

w(N+1) = MinKap*kapadj;

$b = (S') \setminus v'$;

B=diag(b);

V=diag(w);

Mwbc=B*S'*V-eye(N+1);

%% include BC

```
M = Mwbc(1:N,1:N);
M(1:N,N) = Mwbc(1:N,N)+bc*Mwbc(1:N,N+1);
ff(1:N) = kappa*M*u(1:N);
for i = 1:N;
%%%%%proportion
ff(i) = ff(i) + kappab*u(i)*(1-u(i))*(r_uu - r_s)*(1+sigma(i)) + nlp*Mwbc(i,N+1);
end;
for i = (N+1):(2*N) %%%%%log of capital
ff(i) = r_s + (r_uu - r_s)*u(i-N);
end;
ff(2*N+1)=volumefun(u);
%%%% volume of all MBS
f = ff';
%%%%%%%%%%%%%%%%%%
function vt=volumefun(u)
global r_u
global r_s
global sigma
global N
global kapdelay
r_uu = rulin(u(2*N+1)-kapdelay);
vt = 0;
for i = 1:N-1 aux1 = sigma(i)*max(0,(r_uu-r_s))*u(i)*(1-u(i))*exp(u(N+i));
aux2 = sigma(i+1)*max(0,(r_uu-r_s))*u(i+1)*(1-u(i+1))*exp(u(N+i+1));
vt = vt + (sigma(i+1)-sigma(i))*(aux1+aux2)/2;
end;
%%%%%%%%%%%%%%%%%
function [sg,x,y] = variancia(mesh,nodatapts,MinKap);
global kapadj
mincapital = MinKap;
```

```
pocINT=nodatapts;
p = 70; %% how many percent of capital do we cover
alpha(1) = (100-p)/p;
S=load('variance.txt');
maxvariancia=max(S(1,:));
%%%% for i = 0:pocINT
w(i+1) = i*(maxvariancia/pocINT);
end;
for i = 1:pocINT
ws(i) = (w(i)+w(i+1))/2;
end;
kapitalINT=zeros(pocINT,1);
for i=1:pocINT
for j=1:length(S)
if (S(1,j)>=w(i)) & (S(1,j)< w(i+1))
kapitalINT(i)=kapitalINT(i)+S(2,j);
end
end
end
n = pocINT;
for i = 0:n+3
db(i+1) = i*(maxvariancia/n);
end;
for i = 1:n+3 dbs(i) = (db(i)+db(i+1))/2;
end;
cp = mincapital*ones(1,n+3);
cp(1:n) = kapitalINT;

for i = 2:n+2;
cp(i) = cp(i) + alpha(1)*(0.25*cp(i-1)+0.5*cp(i)+0.25*cp(i+1));
```

```
end;
cp = max(cp, mincapital);
%%% spread of data and set up of mesh
if mesh(end)> dbs(end)
x1 = [0 dbs mesh(end)];
y1 = [mincapital cp mincapital];
else
x1 = [0 dbs];
y1 = [mincapital cp];
end;
sg = mesh;
x = mesh;
y = interp1(x1,y1,x,'linear');
sg = sqrt(52)*sg;
x = sqrt(52)*x;
y = kapadj*y;
%%%%%%%%%%%%%%%%%%%%
function r = rulin(V);
global r_u
global
r_s global
V_crit
global V1
global V2
global V3
global r_loss
if V < V1
r = r_u;
else
if V < V_crit
```

```
coef = (V-V1)/(V_crit-V1);
r = r_u*(1-coef)+r_s*coef;
else
if V < V2
coef = (V-V_crit)/(V2-V_crit);
r = r_s*(1-coef)+0*coef;
else
if V < V3
coef = (V-V2)/(V3-V2);
r = 0*(1-coef)+r_loss*coef;
else r = r_loss;
end;
end;
end;
end;
return
%%%%%%%%%%%%%%%%%%%%%%
function[M]=difmatrixNL(sigma);
global alpha
%% with boundary conditions
N = length(sigma);
% %% artificial mesh points
sigma_p=2*sigma(end)-sigma(end-1);
sigma_pp=3*sigma(end)-2*sigma(end-1);
sigma_new=[sigma sigma_p sigma_pp];
M = zeros(N+1,N+1);
% %%non symmetrical mesh
for k=1:N+1
for i =1:N+1
M(k,i)=(sigma_new(k+1)-sigma_new(k))/exp(alpha*abs(sigma_new(i)-sigma_new(k)));
```

```
end
end
return
%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function[bc,nlp]=boundariesNL(EXPL)
global sigma
global N
switch (EXPL)
case 1 %%dirichlet u = 0
bc=0;
nlp=0;
case 2 %%neumann u' = 0
bc=1;
nlp=0;
case 3 %%neumann u' = 1
bc=1;
nlp=sigma(end)-sigma(end-1);
case 4 %% dirichlet u(end) = sigma(end)
bc=0;
nlp=sigma(end);
end
```