COMENIUS UNIVERSITY IN BRATISLAVA
Faculty of Mathematics, Physics and Informatics

Maximum principle for infinite horizon discrete time optimal control problems

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Maximum principle for infinite horizon discrete time optimal control problems

Master's Thesis

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Department of Applied Mathematics and Statistics
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Princíp maxima pre diskrétne úlohy optimálneho riadenia na nekonečnom horizonte

Diplomová práca

Bc. Jakub Beran

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Declaration

I declare on my honour that this thesis was written on my own, with the only help provided by my supervisor and the referred-to literature.

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Jakub Beran

Abstract

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Thesis Title: Maximum principle for infinite horizon discrete time optimal control problems
Institution: Comenius University in Bratislava
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The aim of this thesis is to prove a necessary condition of optimality of maximum principle type for infinite horizon discrete time optimal control problems by functional analysis techniques. At the beginning we study a problem with linear state equation. Thereafter we extend the results to the problem with general state equation and at the end of our work we consider a problem with constraints on the control variable.

Keywords: optimal control, infinite horizon, discrete time, dual space, functional analysis, cone, adjoint equation, adjoint variable, Pontryagin's maximum principle

Abstrakt

Autor: Bc. Jakub Beran
Názov práce: Princíp maxima pre diskrétne úlohy optimálneho riadenia na nekonečnom horizonte
Škola: Univerzita Komenského v Bratislave
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Cieľom tejto práce je odvodiť nutné podmienky optimality pre diskrétnu úlohu optimálneho riadenia na nekonečnom horizonte za použitia nástrojov funkcionálnej analýzy. V úvode práce študujeme problém optimálneho riadenia s lineárnou stavovou rovnicou. Následne rozšírime výsledky na problém so všeobecnou stavovou rovnicou a prácu ukončíme problémom s ohraničeniami na riadiacu premennú.

Kľúčové slová: optimálne riadenie, nekonečný horizont, diskrétny čas, duálny priestor, funkcionálna analýza, kužeľ, adjungovaná rovnica, adjungovaná premenná, Pontryaginov princíp maxima

Contents

Introduction

Even though the first optimal control problem was introduced in the seventeenth century, the optimal control theory is considered to be invented in the late 1950s by Pontryagin and his group [13]. Since then the use of optimal control tools has had an increasing tendency and has become a standard optimization method. The reason is that the optimal control problems are widespread in many fields, for example in economics (especially in the growth theory and in the game theory) or in physics.
Consequently, the underlying theory has been generalized and extended in a variety of ways. Nowadays there are basically two solution methods to the optimal control problems. The first is the dynamic programming introduced by Bellman, which we are not concerned with in this thesis. The second is known as the Pontryagin maximum principle.
Pontryagin maximum principle is a local necessary condition of optimality of variational type. That is, the control is tested to optimality against its small admissible variations satisfying the constraints. To this end, in the basic finite-horizon case the effect of such variations to the response at the terminal time T of the problem is analyzed. Moreover, necessary conditions express the fact that variations do not improve the cost function.
In this work we focus on the Pontryagin maximum principle from the infinite-horizon discrete-time point of view. In the book of Pontryagin et al. [13] there is a short section dealing with the infinite horizon problem. It is studied as a limit case of finite horizon problem for $T \to \infty$. Because of changing horizon the variations have to be transferred to a fixed time independent of T. Because of the invertible dynamics of the continuous time problem this is possible. Unfortunately, this approach cannot by applied to the discrete-time framework. A systematic study of the discrete-time framework has been initiated by Boltyanskii [4]. However Boltyanskii's results are mainly concerning the finite-horizon case. Interesting results regarding the infinite-horizon case can be found in [2], where the dynamics in the form

$$\begin{aligned} x_0 &= \bar{x} \\ x_t &= F_t(x_t, u_t) \ \ \forall t \in \mathbb{N}_0 \end{aligned}$$

is considered and $F_t \in C^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$. Then the approach is the one of Pontryagin et al. The problem is solved in three steps. At first, it is reduced to the finite case. The reduced problem is solved and thereafter expanded to the infinite-horizon. In the last step one needs to extend the finite solution by limit transition. If we denote $(\hat{x}_t, \hat{u}_t)$ as the tested solution at time t, it is necessary to derive y_t from the following linearized system

$$y_{t+1} - D_{x_t}F_t(\hat{x}_t, \hat{u}_t)y_t - D_{u_t}F_t(\hat{x}_t, \hat{u}_t)v_t = z_t.$$

Clearly, the solution could be written as follows

$$y_t = D_{x_t}F_t(\hat{x}_t, \hat{u}_t)^{-1}[y_{t+1} - z_t - D_{u_t}F_t(\hat{x}_t, \hat{u}_t)v_t],$$

but the operator

$$D_{x_t}F(\hat{x}_t, \hat{u}_t) \tag{1}$$

has to be regular. Hence the major disadvantage of this approach is that one needs a matrix of the linearized state equation with respect to the state variable to be regular.
Our goal is to avoid the necessity of the regularity condition. Therefore we use different approach that is motivated by Blot and Hayek [3], where the infinite problem is solved directly instead of the reduction to the finite case. In contrary to the previous papers, the methods of functional analysis and the general theorem of Ioffe-Tihomirov [10] are employed, in this one. It is supposed that the state and also the control variable is from the space of all bounded sequences, i.e.

$$(x_0, x_1, \dots) = \mathbf{x} \in l_\infty^n = \{\{w_t\}_{t\in\mathbb{N}_0} :\ w_t \in \mathbb{R}^n\ \forall t \in \mathbb{N}_0\ \wedge\ \sup_{t\in\mathbb{N}_0}|w_t| < \infty\}$$

and $(u_0, u_1, \dots) = \mathbf{u} \in l_\infty^m$. Under these assumptions the problem of maximization the cost function with the state equation

$$\begin{aligned} J(\mathbf{x}, \mathbf{u}) &= \sum_{t=0}^{\infty} \beta^t f(x_t, u_t) \ \rightarrow\ \max, \quad f \in C^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}), \\ x_{t+1} &= F(x_t, u_t) \quad t \in \mathbb{N}_0, \quad F \in C^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n), \\ x_0 &= \bar{x}, \end{aligned}$$

could be worked out directly using functional analysis. Nevertheless the condition of regularity has to be replaced by the condition

$$\sup_{t\in\mathbb{N}_0}|D_{x_t}F_t(\hat{x}_t, \hat{u}_t)| < 1. \tag{2}$$

The original aim of this thesis was to establish the Pontryagin maximum principle without restrictive conditions (1) or (2). To this end we attempted to employ an independent approach based upon the closed range theorem.
To get an idea of the power of our approach we studied the problem without discount first. As shown in [1] or [15] it is useful as a limit case from which interesting conclusions can be drawn for $\beta < 1$ near 1. Then of course a suitable kind of decrease to zero of the terms $f(x_t, u_t)$ has to be assumed, in order to have the cost function J finite. For technical reasons we have chosen the space l_1, where

$$l_1^k = \{\{w_t\}_{t\in\mathbb{N}_0} :\ w_t \in \mathbb{R}^k\ \forall t \in \mathbb{N}_0\ \wedge\ \sum_{t=0}^{\infty}|w_t| < \infty\},$$

mainly because its dual can be interpreted as l_∞. Another motivation comes from [1] and [15] where it is showed that the space l_1 can be replaced by a shifted space of sequences converging to $(x_\infty, u_\infty) \neq (0, 0)$ for which $f(x_\infty, u_\infty) \neq 0$. It is known that such a point is an equilibrium point of the problem (i.e (x_∞, u_∞) is a time independent solution of the Pontryagin maximum principle).

We consider our work as an initial optimal control problem solving method that could be extended to a wider class of problems. For example we can study a cost function with a discount rate. Subsequently, since $l_1 \subset l_\infty$, we can extend the domain of the state and control variable to the space l_∞.
The work itself is organized as follows. The first chapter introduces the necessary theory as definitions, notations and theorems that are used in the further chapter, especially those from the functional analysis. The second chapter is then divided into three sections. In the first, we explore the properties of the solution of the optimal control problem with linear state equation and show that in this case there is no need to use regular matrices. We also derive the adjoint equation and show that the adjoint variable is contained in the dual space. The purpose of the second section is to study the problem with general state equation. In the third part, we impose constraints on the control variable and derive the Pontryagin maximum principle.

Chapter 1

Preliminaries

The aim of this chapter is to provide the reader with a basic introduction to all necessary concepts, definitions and theorems used throughout the present thesis. The main part of the thesis is focused on the work with infinite sequences and vectors, therefore the most important purpose of this part is to give the insight to functional analysis and related branches of mathematics. We run through metric spaces, Banach spaces and useful theorems. We mention the important theorems connected with differentiability and explain the used notation, as well. The examples are also a significant part of this chapter. In them we prove all necessary properties of variables and functions, which we subsequently use in the further chapter.

1.1 Functional Analysis

1.1.1 Metric Spaces

We begin with the notion of a metric space; that is, a set where a notion of distance (called a metric) between elements of the set is defined.

Definition 1. Let $X \neq \emptyset$ be a set and $d : X \times X \to \mathbb{R}$ be a real function such that for any $x, y, z \in X$ one has

1. $d(x, y) \geq 0$
2. $d(x, y) = 0 \Leftrightarrow x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, y) \leq d(x, z) + d(z, y)$

The pair (X, d) is a *metric space* and a function d is called a *metric*.

The most common metric space is the Euclidean space. It's metric, the Euclidean metric, defines the distance between two points as the length of the straight line segment connecting them. However, in this thesis, we use two other spaces, the l_1^k and the l_∞^k, together with their metrics.

Example 1. Let l_1^k be a set of all sequences $\{x_t\}_{t=0}^{\infty}$, $x_t \in \mathbb{R}^k$ such that

$$\sum_{t=0}^{\infty} |x_t| < \infty,$$

where $|\cdot|$ is a norm in the space $\mathbb{R}^k$, $k \in \mathbb{N}$. Let us define

$$\rho(\mathbf{x}, \mathbf{y}) = \sum_{t=0}^{\infty} |x_t - y_t| \quad \forall \mathbf{x}, \mathbf{y} \in l_1^k.$$

Note that all variables, which represent infinite dimensional vectors or sequences are written in bold, whereas the other variables are written in regular font. To prove that (l_1^k, ρ) is a metric space, we have to show that the four conditions in the definition 1 are fulfilled. We prove just the fourth condition, because the proof of the first three is trivial.

$$\begin{aligned}
\rho(\mathbf{x}, \mathbf{y}) &= \sum_{t=0}^{\infty} |x_t - y_t| = \sum_{t=0}^{\infty} |(x_t - z_t) - (z_t - y_t)| \leq \sum_{t=0}^{\infty} [|x_t - z_t| + |z_t - y_t|] \\
&= \sum_{t=0}^{\infty} |x_t - z_t| + \sum_{t=0}^{\infty} |z_t - y_t| = \rho(\mathbf{x}, \mathbf{z}) + \rho(\mathbf{z}, \mathbf{y}).
\end{aligned}$$

Example 2. Let l_∞^k be a set of all bounded sequences $\{x_t\}_{t=0}^{\infty}$, $x_t \in \mathbb{R}^k$, so

$$\sup_{t \in \mathbb{N}_0} |x_t| < \infty,$$

where $|\cdot|$ is a norm in the space $\mathbb{R}^k$, $k \in \mathbb{N}$. Let us define

$$\rho(\mathbf{x}, \mathbf{y}) = \sup_{k \in \mathbb{N}_0} |x_t - y_t| \quad \forall \mathbf{x}, \mathbf{y} \in l_\infty^k.$$

In this case (l_∞^k, ρ) is again a metric space. The proof of first three conditions is trivial, while the proof of the fourth is analogous to the previous case.

Definition 2. Let (X, d) be a metric space. A *Cauchy's sequence* is a sequence $\{x^n\}_{n=0}^{\infty}$, $x^n \in X$ for all $n \in \mathbb{N}_0$, such that for every $\varepsilon > 0$ there is N_ε such that $d(x^n, x^m) < \varepsilon$ for all $n, m > N_\varepsilon$.

Definition 3. A metric space (X, d) is *complete*, if every Cauchy's sequence in this space converges.

Definition 4. A subset U of a metric space (X, d) is called *closed*, if whenever $x^n \in U$ and $x^n \to z$, then $z \in U$.

1.1.2 Banach Spaces

To define the Banach space the notions of the norm and the normed space are very important. At first, we recall their's definitions and then we show the connection between the normed space and the metric space.

Definition 5. Let $X \neq \emptyset$ be a vector space and $\|\cdot\| : X \times X \to \mathbb{R}$ be a real function such that for any $x, y \in X$ one has

(a) $\|x\| \geq 0$
(b) $\|x\| = 0 \Leftrightarrow x = 0$
(c) $\|\lambda x\| = |\lambda|\|x\|$
(d) $\|x + y\| \leq \|x\| + \|y\|$

The function $\|\cdot\|$ is called a *norm* and the pair $(X, \|\cdot\|)$ is called a *normed vector space.*

Proposition 1.1.1. *Let $(X, \|\cdot\|)$ be a normed vector space and define $d(x, y) = \|x, y\|$ for all $x, y \in X$. Then the corresponding space (X, d) is a metric space.*

Proof. The only non-obvious verification is the fourth condition of the metric. By the property (d) of the norm, we have

$$d(x, y) = \|x - y\| = \|(x - z) + (z - y)\| \leq \|x - z\| + \|z - y\| = d(x, z) + d(z, y)$$

□

Finally we can introduce the notion of the Banach space.

Definition 6. Let $(X, \|\cdot\|)$ be a normed vector space. If the corresponding metric space (X, d) is complete we say $(X, \|\cdot\|)$ is a *Banach space* (in further work, we will leave out the symbol $\|\cdot\|$ in the notation of the Banach space).

Example 3. The spaces l_1^k and l_∞^k are normed vector spaces, if we introduce the norms

$$\begin{aligned} \|\mathbf{x}\| &= \sum_{t=0}^{\infty} |x_t|, \ \forall \mathbf{x} \in l_1^k, \\ \|\mathbf{x}\| &= \sup_{t \in \mathbb{N}_0} |x_t|, \ \forall \mathbf{x} \in l_\infty^k. \end{aligned}$$

In addition to that, they are Banach spaces (the proof can be found in [9]).

1.1.3 Operators

The core of functional analysis is formed by the study of Banach spaces and the linear functionals acting upon these spaces. We generally call these linear functionals operators. In this part we briefly summarize all necessary definitions and theorems associated with operators.
Let X, Y be normed vector spaces and $T : X \to Y$ a map between them. There are three common notations connected with this map

$$\begin{aligned} \mathcal{D}(T) &= \{x \in X : \ T(x) \in Y\} && (\textit{The Domain}) \\ \mathcal{N}(T) &= \{x \in X : \ T(x) = 0\} && (\textit{The Null Space}) \\ \mathcal{R}(T) &= \{y \in Y : \ \exists x \in X, \ T(x) = y\} && (\textit{The Range}) \end{aligned}$$

Definition 7. Let X, Y be normed vector spaces and $T : X \to Y$ a map between them. Let $x \in X$. We call a map *continuous in* x, if

$$\forall \varepsilon > 0 \ \exists \delta > 0 : \|x - z\| < \delta \Rightarrow \|T(x) - T(z)\| < \varepsilon$$

The map is *continuous* (on X), if it is continuous in all $x \in X$.

Definition 8. Let X, Y be normed vector spaces and $T : X \to Y$ be a map between them. This map is called *linear* if one has

$$\begin{aligned} T(x + y) &= T(x) + T(y) \\ T(\alpha x) &= \alpha T(x), \end{aligned}$$

for all $x, y \in X$ and $\alpha \in R$. A linear map is often called a *linear operator.*

In further work, the notions of boundedness and closeness of an operator will be very important. We can also link these notions with the notion of continuity.

Definition 9. Let X, Y be normed vector spaces. A linear map $T : X \to Y$ is *bounded* if there exists a constant $M > 0$ such that $\|T(x)\| \leq M\|x\|$ for all $x \in X$.

Definition 10. Let X, Y be normed vector spaces. A linear map $T : X \to Y$ is *closed* if for every sequence $\{x_n\}_{n \in \mathbb{N}}$ in X converging to $x \in X$ such that $\lim_{n \to \infty} T(x_n) = y \in Y$ one has $T(x) = y$.

Proposition 1.1.2. *Let X, Y be normed vector spaces and T be a linear mapping from X to Y. This map is continuous, if and only if it is bounded.*

Proof. The proof can be found in [9]. □

Proposition 1.1.3. *Let X, Y be normed vector spaces and T be a linear mapping from X to Y. If this map is continuous, then it is closed.*

Proof. The proof can be found in [12]. □

Besides the norm of a vector, we can define the norm of an operator, as well.

Definition 11. Let X, Y be normed vector spaces and $T : X \to Y$ be a map between them. We define the *norm* of the map T as

$$\|T\| = \sup_{\|x\|=1} \|Tx\|.$$

The linear space of all bounded linear operators from X to Y together with the norm $\|\cdot\|$, is denoted by $\mathcal{B}(X, Y)$.

Remark 1. We can prove that this is a norm. The proof of conditions (a)-(c) is trivial, hence we prove just the last one.

$$\|T + S\| = \sup_{\|x\|=1} \|(T + S)x\| \leq \sup_{\|x\|=1} \|Tx\| + \sup_{\|x\|=1} \|Sx\| = \|T\| + \|S\|$$

Remark 2. Note that $\|T\|$ is the smallest number M that satisfies the condition of boundedness in 9. So if $\|T\| < \infty$, then the operator T is bounded.

In further chapter, we use mainly two kinds of operators and now we prove their boundedness.

Example 4. An operator $\boldsymbol{\sigma} : l_1^k \to l_1^k$, such that $\boldsymbol{\sigma}(x_0, x_1, \dots) = (x_1, x_2, \dots)$ is called a *shift operator* and one has

$$\begin{aligned} \|\boldsymbol{\sigma}\| &= \sup_{\|\mathbf{x}\|=1} \|\boldsymbol{\sigma}\mathbf{x}\| = \sup_{\|\mathbf{x}\|=1} \|\boldsymbol{\sigma}(x_0, x_1, \dots)\| = \sup_{\|\mathbf{x}\|=1} \|(x_1, x_2, \dots)\| = \sup_{\|\mathbf{x}\|=1} \{\sum_{t=1}^{\infty} |x_t|\} \\ &= \sup_{\|\mathbf{x}\|=1} \{\sum_{t=0}^{\infty} |x_i| - |x_0|\} \leq \sup_{\|\mathbf{x}\|=1} \{\sum_{t=0}^{\infty} |x_i|\} = \sup_{\|\mathbf{x}\|=1} \|\mathbf{x}\| = 1 < \infty. \end{aligned}$$

Example 5. Similarly for a general linear operator $\mathbf{N} = (N_0, N_1, \dots)$ on l_1^k such that $|N_t| \leq M < \infty \ \forall \in \mathbb{N}_0$ one has

$$\begin{aligned} \|\mathbf{N}\| &= \sup_{\|\mathbf{x}\|=1} \|\mathbf{N}\mathbf{x}\| = \sup_{\|\mathbf{x}\|=1} \sum_{t=0}^{\infty} |N_t x_t| \leq \sup_{\|\mathbf{x}\|=1} \sum_{t=0}^{\infty} |N_t||x_t| \\ &\leq \sup_{\|\mathbf{x}\|=1} \sum_{t=0}^{\infty} M|x_t| = M \sup_{\|\mathbf{x}\|=1} \|\mathbf{x}\| = M < \infty. \end{aligned} \tag{1.1}$$

Definition 12. An operator $T \in \mathcal{B}(X, Y)$ is called a *linear isomorphism* (or just *isomorphism*), if it is one to one, onto Y and $T^{-1} \in \mathcal{B}(X, Y)$.

Proposition 1.1.4. *Let $T : X \to Y$ be a linear map between vector spaces and C be a closed complement of $\mathcal{N}(T)$ in X. Then the map $T : C \to \mathcal{R}(T)$ is an isomorphism. Furthermore, this map is also called a restriction of a map T to C and denoted by $T|_C$.*

Proof. A map is injective, if and only if

$$\mathcal{N}(T|_C) = \{0\}.$$

Since C is an complement of $\mathcal{N}(T)$ one has $\mathcal{N}(T) \cap C = \{0\}$ and hence

$$\mathcal{N}(T|_C) = \mathcal{N}(T) \cap C = \{0\}.$$

On the other hand we have to show that our map is surjective, as well. This is equivalent to the equation

$$\mathcal{R}(T|_C) = \mathcal{R}(T).$$

As the space X could be splitted into the null space of the map T and it's complement one has

$$\mathcal{R}(T|_C) = T(C) = T(C \oplus \mathcal{N}(T)) = T(X) = \mathcal{R}(T).$$

Summing up, the operator $T|_C$ is an isomorphism. □

1.1.4 Dual Spaces

To obtain an adjoint variable we need to "switch" from primal to dual space. As we are working with Banach spaces we need to define the concept of the dual space on them. In this section we also show that if we introduce a dual space on the space l_1, then we get a space of bounded sequences - l_∞.

Definition 13. Let X be a normed vector space. The linear operator $x^* : X \to R$ is called a *linear functional* and we define $x^*(x) =< x^*, x >$ and $\|x^*\| = \sup_{\|x\|=1} x^*(x)$. The space of all continuous linear functionals from X to R is called a *dual space* of X and is denoted by X^*.

Proposition 1.1.5. $(l_1^k)^* = l_\infty^k$ *in the sense that for every* $\mathbf{x}^*$ *there is an unique* $\{d_t\}_{t=0}^\infty = \mathbf{d} \in l_\infty^k$ *such that*

$$\mathbf{x}^*(\mathbf{x}) = \sum_{t=0}^{\infty} d_t x_t \quad \forall \mathbf{x} \in l_1^k.$$

Proof. We show a proof for $k = 1$. However, it is analogous to extend the proof for any finite k. At first, we distinguish the two norms used in this proof. Let $\|\cdot\|_1$ be a norm associated with l_1 and $\|\cdot\|_\infty$ be a norm associated with l_∞. Let $\{\mathbf{e}_t\}_{t=0}^\infty$ be a basis in l_1 (standard unit vectors). Hence each $\mathbf{x} \in l_1$ can be rewritten as

$$\mathbf{x} = \sum_{t=0}^{\infty} a_t \mathbf{e}_t.$$

Let $\mathbf{x}^* \in l_1^*$ be given . We define $d_t = \mathbf{x}^*(\mathbf{e}_t)$ for all $t \in \mathbb{N}_0$. By $\|e_t\|_1 = 1$ we have

$$|d_t| = |\mathbf{x}^*(\mathbf{e}_t)| \leq \|\mathbf{x}^*\| \|\mathbf{e}_t\|_1 = \|\mathbf{x}^*\|.$$

If we define $\mathbf{d} = \{d_t\}_{t=0}^\infty$, according to the proposition (1.1.2) we have

$$\|\mathbf{d}\|_\infty = \sup_{t \in \mathbb{N}_0} |d_t| \leq \|\mathbf{x}^*\| < \infty$$

that is $\mathbf{d} \in l_\infty$. Conversely we construct a linear functional on l_1 with elements in l_∞ if $\mathbf{d} \in l_\infty$ is given. Consider a linear functional h defined by

$$h(\mathbf{x}) = \sum_{t=0}^{\infty} d_t x_t \quad \forall \mathbf{x} \in l_1.$$

Linearity of this functional is clear, we prove boundedness

$$|h(\mathbf{x})| \leq \sum_{t=0}^{\infty} |d_t x_t| \leq \sup_{t \in \mathbb{N}_0} |d_t| \sum_{t=0}^{\infty} |x_t| = \|\mathbf{d}\|_\infty \|\mathbf{x}\|_1$$

and this implies

$$\|h\| = \sup_{\|\mathbf{x}\|=1} \|h(\mathbf{x})\| \leq \|\mathbf{d}\|_\infty < \infty.$$

It is clear that $h(\mathbf{e}_t) = d_t$. Hence $h \in l_1^*$. □

Definition 14. Let X, Y be Banach spaces and $T \in \mathcal{B}(X, Y)$. We define the *dual operator* (also called *adjoint operator*) $T^* \in \mathcal{B}(Y^*, X^*)$ for $y^* \in Y^*$ by

$$T^*(y^*)(x) = y^*(T(x)) \ \forall x \in X.$$

Proposition 1.1.6. *Let X, Y be Banach spaces and $T \in \mathcal{B}(X, Y)$. Then $\|T^*\| = \|T\|$.*

Proof. The proof can be found in [9]. □

Definition 15. The Banach space X is called reflexive, if $(X^*)^* = X$.

1.1.5 Quotient Spaces

The aim of this section is to introduce the First Isomorphism Theorem, which we use in our work. Therefore we need to define the notion of the quotient space.

Definition 16. Let Y be a closed subspace of a vector space X. Then the *coset* $\overline{x}$ relative to Y is any of the sets

$$x + Y = \{x + y, y \in Y\}$$

where $x \in X$. The space of all cosets is denoted by X/Y.

Remark 3. It is easy to check that the space X/Y together with the addition ($\overline{v + w} = \overline{v} + \overline{w}$) and the scalar multiplication ($\alpha\overline{v} = \overline{\alpha v}$) is linear space. Moreover, we obtain a normed vector space, if we add a norm $\|\overline{x}\| = \inf\{\|x\|, x \in \overline{x}\}$ to our space X/Y.

Definition 17. Let X be a Banach space and Y be it's closed subspace. The space X/Y together with a norm $\|\overline{x}\| = \inf\{\|x\|, x \in \overline{x}\}$ is called a *quotient space* of X with respect to Y.

Proposition 1.1.7. *Let X be a Banach space and Y be it's closed subspace. Then X/Y is a Banach space.*

Proof. The proof can be found in [9]. □

Theorem 1.1.1. ***(The First Isomorphism Theorem)*** *Let $T : X \to Y$ be a linear map between vector spaces. Then*

a) $\mathcal{R}(T)$ is a subspace of Y
b) $\mathcal{N}(T)$ is a subspace of X
c) $V/\mathcal{N}(T)$ is isomorphic to $\mathcal{R}(T)$

Proof. In fact, this theorem is a corollary of the First Isomorphism Theorem for groups and it's proof can be found in [11]. □

1.2 Differential Calculus

To obtain a maximum principle we need to differentiate our functions. When working with one dimensional real functions it is enough to use a basic notion of differentiability. The situation becomes much more complicated, if we use functions defined on a Banach spaces. In this situation, the concept of Fréchet differentiability have to be put in. Therefore we define the directional derivative, Gâteaux differentiability and subsequently the Fréchet differentiability, too. All three kinds of differentiability are used in the example. Another very important part of this section is the Implicit Function Theorem that is used in the further chapter.

Definition 18. Let X, Y be Banach spaces, $U \subset X$ open and $J : U \to Y$. Let $x \in U$ and $h \in X$. The *directional derivative* of a function $J(x)$ along a vector h is the function defined by the limit

$$\partial_h J(x) = \lim_{\tau \to 0} \frac{1}{\tau}[J(x + \tau h) - J(x)] \tag{1.2}$$

if this limit exists.

Definition 19. Let X, Y be Banach spaces, $U \subset X$ open, $J : U \to Y$ and $x \in U$. Let us assume that there exists $\partial_h J(x)$ for all h. If the map $h \to \partial_h J(x)$ is linear and bounded, then the function J is *Gâteaux differentiable* in x. The *Gâteaux differential* is thus defined by

$$dJ(x)h = \partial_h J(x).$$

Definition 20. Let X, Y be Banach spaces, $U \subset X$ open, $J : U \to Y$ and $x \in U$. The map J is *Fréchet differentiable* in x, if there exists a linear bounded operator $DJ(x)$ such that

$$\lim_{|h| \to 0} \frac{1}{|h|}[J(x + h) - J(x) - DJ(x)h] = 0.$$

The following proposition plays a significant role in the proof of Fréchet differentiability of the cost function.

Lema 1.2.1. ***(Hadamard's lema)*** *Let X, Y be Banach spaces, $f : U \to Y$ be a Gâteaux differentiable mapping. If $x + \vartheta h \in U$ for $0 \le \vartheta \le 1$ then one has*

$$f(x+h) - f(x) = \int_0^1 df(x+\vartheta h) h d\vartheta = \left[\int_0^1 df(x+\vartheta h) d\vartheta\right] h \tag{1.3}$$

where the integral is in the sense of Riemann.

Proof. The proof can be found in [6]. □

Example 6. In this thesis we study the optimal control problem with a cost function $J(\mathbf{x}, \mathbf{u})$ in a form

$$J(\mathbf{x}, \mathbf{u}) = \sum_{t=0}^{\infty} f(x_t, u_t), \tag{1.4}$$

where $f \in C^1(X \times U, \mathbb{R})$, $X \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ open. We would like to show that $J(\mathbf{x}, \mathbf{u})$ is of class C^1 on l_1^{n+m}. In this example we show that the cost function is differentiable (in Fréchet sense). At first we show the Gâteaux differentiability of the cost function and consequently the Fréchet differentiability.

Gâteaux Differentiability: Let $(\mathbf{x}, \mathbf{u})$ be such that $x_t \in X$ and $u_t \in U$. Since $\mathbf{x} \in l_1^n$ and $\mathbf{u} \in l_1^m$, each term of the sequences $\{x_t\}_{t=0}^{\infty}$ and $\{u_t\}_{t=0}^{\infty}$ has to be bounded (otherwise the conditions $\sum_{t=0}^{\infty} |x_t| < \infty$ and $\sum_{t=0}^{\infty} |u_t| < \infty$ are not satisfied). Hence there are compact sets X_0 and U_0 such that $x_t \in \text{int } X_0$, $u_t \in \text{int } U_0$. As a continuous function on a compact set is bounded one has

$$|D_x f| < M, |D_u f| < M \quad \text{on} \quad X_0 \times U_0 \tag{1.5}$$

and $D_x f, D_u f$ are uniformly continuous on $X_0 \times U_0$, as well.
For the sake of simplicity let $(\mathbf{x}, \mathbf{u}) = \mathbf{z}$ and $(x_t, u_t) = z_t$. At first we prove that for $\mathbf{z} \in l_1^{n+m}$ there exists $\partial_{\mathbf{h}} J(\mathbf{z})$ for all $\mathbf{h} \in l_1^{n+m}$. Let $\mathbf{h} = (h_0, h_1, \dots)$ where $h_t \in \mathbb{R}^{n+m}$ for all $t \in \mathbb{N}_0$. Then we have

$$\begin{aligned}
\partial_{\mathbf{h}} J(\mathbf{z}) &= \lim_{\tau \to 0} \frac{1}{\tau}[J(\mathbf{z} + \tau \mathbf{h}) - J(\mathbf{z})] \\
&= \lim_{\tau \to 0} \frac{1}{\tau} \left[\sum_{t=0}^{\infty} f(z_t + \tau h_t) - \sum_{t=0}^{\infty} f(z_t)\right] \\
&= \lim_{\tau \to 0} \sum_{t=0}^{\infty} \frac{1}{\tau} \left[f(z_t + \tau h_t) - f(z_t)\right].
\end{aligned}$$

To proceed we have to check that we are allowed to interchange the summation and limit. Therefore we need to prove the absolute convergence of the terms in sum. For τ sufficiently small one has $(z_t + \xi \tau h_t) \in X_0 \times U_0$ so using Hadamard's lemma (see Lemma 1.2.1) we obtain

$$\begin{aligned}\sum_{t=0}^{\infty}|f(z_t+\tau h_t)-f(z_t)| &= \sum_{t=0}^{\infty}|(z_t+\tau h_t-z_t)\int_0^1 D_z f(z_t+\vartheta(z_t+\tau h_t-z_t))\mathrm{d}\vartheta| \\ &\leq \sum_{t=0}^{\infty}|\tau h_t||\int_0^1 D_z f(z_t+\vartheta\tau h_t)\mathrm{d}\vartheta| \leq \sum_{t=0}^{\infty}|\tau h_t||\int_0^1 2M\mathrm{d}\vartheta| \\ &\leq \sum_{t=0}^{\infty}|\tau||h_t|2M = 2M|\tau|\|\mathbf{h}\| < \infty.\end{aligned}$$

Therefore

$$\begin{aligned}\partial_{\mathbf{h}}J(\mathbf{z}) &= \sum_{t=0}^{\infty}\lim_{\tau\to 0}\frac{1}{\tau}\left[f(z_t+\tau h_t)-f(z_t)\right] \\ &= \sum_{t=0}^{\infty}\partial_{h_t}f(z_t) = \sum_{t=0}^{\infty}D_z f(z_t)h_t. \end{aligned} \tag{1.6}$$

Since $f \in C^1$, $D_z f$ exists, so $\partial_{\mathbf{h}}J(\mathbf{z})$ exists.
Further we prove that the function $\mathbf{h} \to \partial_{\mathbf{h}}J(\mathbf{x},\mathbf{u})$ is linear and bounded. The directional derivative is always homogenous, hence remains to prove additivity and boundedness.

$$\begin{aligned}\partial_{\mathbf{h}+\mathbf{g}}J(\mathbf{z}) &= \sum_{t=0}^{\infty}D_z f(z_t)(h_t+g_t) = \sum_{t=0}^{\infty}D_z f(z_t)h_t + \sum_{t=0}^{\infty}D_z f(z_t)g_t \\ &= \partial_{\mathbf{h}}J(\mathbf{z}) + \partial_{\mathbf{g}}J(\mathbf{z}).\end{aligned}$$

Now we prove boundedness of $\partial_{\mathbf{h}}J(\mathbf{z})$ for all $\mathbf{h} \in l_1^{n+m}$. By (1.5) we can denote $|D_z f| \leq |D_x f| + |D_u f| < 2M$. Thus

$$\begin{aligned}\|\partial_{\mathbf{h}}J(\mathbf{z})\| &= \|\sum_{t=0}^{\infty}D_z f(z_t)h_t\| \leq \sum_{t=0}^{\infty}|D_z f(z_t)||h_t| \\ &\leq \sum_{t=0}^{\infty}2M|h_t| = 2M\|\mathbf{h}\| < \infty.\end{aligned}$$

Fréchet Differentiability: To prove Fréchet differentiability we employ the next proposition (by [6]).

Proposition 1.2.1. *Let X, Y be Banach spaces, $U \subset X$ open, $J : U \to Y$ and $x \in U$. If J is Gâteaux differentiable and the Gâteaux derivative is continuous on a neighborhood V of x, then J is Fréchet differentiable at x.*

According to 1.2.1 we have to prove the continuity of Gâteaux derivative. This means to prove that for all $\mathbf{z} \in l_1^{n+m}$ and for all $\varepsilon > 0$ there exists $\delta > 0$, such that if $\mathbf{w} \in l_1^{n+m}$ and $\|\mathbf{z}-\mathbf{w}\| < \delta$ then for all $\mathbf{h} \in l_1^{n+m}$

$$\|\partial_{\mathbf{h}}J(\mathbf{z}) - \partial_{\mathbf{h}}J(\mathbf{w})\| \leq \varepsilon\|\mathbf{h}\|.$$

By (1.6) this is equivalent to

$$\|\sum_{t=0}^{\infty} D_z f(z_t)h_t - \sum_{t=0}^{\infty} D_z f(w_t)h_t\| = \|\sum_{t=0}^{\infty}[D_z f(z_t) - D_z f(w_t)]h_t\| \leq \varepsilon\|\mathbf{h}\|.$$

We know that $D_z f$ is uniformly continuous on $X_0 \times U_0$. Taking into account that

$$|z_t - w_t| \leq \sum_{t=0}^{\infty} |z_t - w_t| = \|\mathbf{z} - \mathbf{w}\| < \delta$$

then we have that for a given ε there exists a $\delta > 0$ such that for all $|z_t - w_t| < \delta$

$$|D_z f(z_t) - D_z f(w_t)| \leq \varepsilon \text{ on } X_0 \times U_0.$$

Hence

$$\begin{aligned} \|\sum_{t=0}^{\infty}[D_z f(z_t) - D_z f(w_t)]h_t\| &\leq \sum_{t=0}^{\infty} |D_z f(z_t) - D_z f(w_t)||h_t| \\ &\leq \sum_{t=0}^{\infty} \varepsilon|h_t| = \varepsilon \sum_{t=0}^{\infty} |h_t| \leq \varepsilon\|\mathbf{h}\|. \end{aligned}$$

Theorem 1.2.1. ***(The Implicit Function Theorem)*** *Let X, Y, Z be Banach spaces, $U \subset X$, $V \subset Y$ open, $\Phi : U \times V \to Z$ be C^r, $0 < r \leq \infty$, $(x_0, y_0) \in U \times V$, $\Phi(x_0, y_0) = 0$. Let us assume that $D_y\Phi(x_0, y_0)$ has a continuous inverse operator. Then there exists a neighbourhood $U_1 \times V_1 \subset U \times V$ of (x_0, y_0) and a function $\varphi \in C^r(U_1, V_1)$ such that $\varphi(x_0) = y_0$ and $\Phi(x, y) = 0$ for $(x, y) \in U_1 \times V_1$ if and only if $y = \varphi(x)$. Furthermore one has*

$$D\varphi(x_0) = -[D_y\Phi(x_0, y_0)]^{-1} D_x\Phi(x_0, y_0). \tag{1.7}$$

Proof. The proof can be found in [6]. □

1.3 Cones

The notion of a cone is very important, when speaking about restrictions on the admissible spaces of the state and of the control variable. Therefore in this section we define the cone and also all theorems that we use in the following chapter.

Definition 21. Let X be a Banach space. The subspace $\mathcal{K} \subset X$ is called a *cone* if for all $x \in \mathcal{K}$ one has $\alpha x \in \mathcal{K}$ for all $\alpha \geq 0$.

Definition 22. The cone $\mathcal{K}$ is called *convex* if

$$\forall x, x' \in \mathcal{K},\ \forall \alpha \in (0, 1) :\ \alpha x + (1 - \alpha)x' \in \mathcal{K}.$$

Definition 23. For a convex cone $\mathcal{K}$ the *dual cone* is defined as

$$\mathcal{K}^* = \{x^* \in X^* : x^*(x) =< x^*, x > \geq 0, \forall x \in \mathcal{K}\}.$$

Remark 4. The dual cone $\mathcal{K}^*$ is clearly convex and closed, regardless $\mathcal{K}$ is closed or not.

Definition 24. For a dual cone $\mathcal{K}^*$ the *normal cone* (also called *polar*) is defined as

$$\mathcal{K}^\circ = \{x^* \in X^* : x^*(x) =< x^*, x > \leq 0, \forall x \in \mathcal{K}\} \; (= -\mathcal{K}^*).$$

The following theorem is by [8].

Proposition 1.3.1. *Suppose that $\mathcal{K}_1$ and $\mathcal{K}_2$ are two closed convex subsets of a Banach space X and $\mathcal{K}_1 \cap \operatorname{int} \mathcal{K}_2 \neq \emptyset$ where $\operatorname{int} \mathcal{K}_2$ denotes the interior of the set $\mathcal{K}_2$. Then one has*

$$(\mathcal{K}_1 \cap \mathcal{K}_2)^\circ = \mathcal{K}_1^\circ + \mathcal{K}_2^\circ.$$

Proposition 1.3.2. *Let $A : X \to Y$ be a linear operator. Then*

$$\mathcal{K} = \{x : Ax = 0, x \in X\}$$

is a closed convex cone.

Proof. Let $x, x' \in X$. Then $Ax = Ax' = 0$ and for all $\alpha \in (0,1)$ and $\beta \geq 0$ one has

$$\begin{aligned} A(\alpha x) + A((1-\alpha)x') = \alpha Ax + (1-\alpha)Ax' &= 0 \\ A(\beta x) = \beta Ax &= 0 \end{aligned}$$

Hence $\mathcal{K}$ is a convex cone. Since the linear operator A maps $\mathcal{K}$ onto a closed set, the cone $\mathcal{K}$ has to be closed. □

1.4 Optimal Control Theory

The core of this work is based on the study of the infinite horizon optimal control problem. In this section we would like to present the basic discrete time optimal control problem, it's adjoint equation, adjoin variable and maximum principle. All mentioned notions will be subsequently used in the second chapter.

$$\begin{aligned} J(x,u) &= \sum_{t=0}^{\infty} f(x_t, u_t) \;\to\; \max, && (1.8) \\ x_{t+1} &= F(x_t, u_t) \quad t \in \mathbb{N}_0, && (1.9) \\ x_0 &= \bar{x}, \\ u_t &\in U_t = \{u : s_t(u) \leq 0\} \quad t \in \mathbb{N}_0, && (1.10) \\ \lim_{t\to\infty} x_t &\in C, && (1.11) \end{aligned}$$

where $f \in C^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R})$, $F \in C^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ and $s_t \in C^1(\mathbb{R}^m, \mathbb{R}^{m_t})$ for all $t \in \mathbb{N}_0$. The problem consists of the cost function (1.8), of the state equation (1.9) and we have some restrictions on the state variable (1.11) and on the control variable (1.10), as well. This basic problem helps us to demonstrate the necessary notions. In the following chapter we use various modifications of this problem.

Definition 25. Denote $I_t(u_t)$ as the set of all $k \in \{1, \dots, m_t\}$ for which $s_t^k(u_t) = 0$ (to be called an *active constraints set*). The optimal control problem fulfills the *regularity condition* in $\mathbf{u} = (u_0, u_1, \dots)$, if for all $t \in \mathbb{N}_0$ the vectors

$$\frac{\mathrm{d}s_t^k}{\mathrm{d}u_t}(u_t),\ k \in I_t(u_t) \tag{1.12}$$

are linearly independent.

Theorem 1.4.1. *Let $(\hat{\mathbf{x}}, \hat{\mathbf{u}}) = (\hat{x}_0, \hat{x}_1, \dots; \hat{u}_0, \hat{u}_1, \dots)$ be optimal response/control pair for our problem and let the regularity condition be fulfilled in $\hat{\mathbf{u}}$. Then there exists $\{\psi_t\}_{t=1}^{\infty}$ (called adjoint variables), such that the following equations hold*

$$\psi_t = \left(\frac{\partial f}{\partial x_t}(\hat{x}_t, \hat{u}_t)\right)^T + \left(\frac{\partial F}{\partial x_t}(\hat{x}_t, \hat{u}_t)\right)^T \psi_{t+1} \qquad \forall t \in \mathbb{N}_0, \tag{1.13}$$

where the equation (1.13) is called the adjoint equation.

Proof. The proof can be found in [7]. □

Theorem 1.4.2. *Let the conditions of the theorem 1.4.1 be fulfilled. Furthermore, let the function F be linear in u, the function f be concave in control variable and the sets U_t be convex. Then there exists a sequence of adjoint variables $\{\psi_t\}_{t=1}^{\infty}$ that solves the adjoint equations (1.13) and*

$$f(\hat{x}_t, \hat{u}_t) + F(\hat{x}_t, \hat{u}_t)^T \psi_{t+1} = \max_{u_t \in U_t} (f(\hat{x}_t, u_t) + F(\hat{x}_t, u_t)^T \psi_{t+1}) \ \ \forall t \in \mathbb{N}_0, \tag{1.14}$$

where the equation (1.14) is called the maximum principle.

Proof. The proof can be found in [7]. □

Chapter 2

Maximum Principle

This chapter is divided into three sections. In the first we derive the necessary conditions of optimality for the optimal control problem with linear state equation. In the second we study the optimal control problem with general state equation. Finally, the necessary conditions of optimality for a constrained optimal control problem are derived at the end of this chapter.

2.1 Linear Problem

In this section our aim is to derive the necessary conditions of optimality for the following optimal control problem. Find $(\hat{\mathbf{x}}, \hat{\mathbf{u}})$ among pairs $(\mathbf{x}, \mathbf{u}) \in l_1^n \times l_1^m$ satisfying the equations

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + d \quad \forall t \in \mathbb{N}_0, \\ x_0 &= \bar{x}, \end{aligned}$$

which maximizes the function

$$J(\mathbf{x}, \mathbf{u}) = \sum_{t=0}^{\infty} f(x_t, u_t), \tag{2.1}$$

where $f \in C^1(X \times U, \mathbb{R})$, $X \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ open, $x_t \in \operatorname{int} X$ and $u_t \in \operatorname{int} U$ for all $t \in \mathbb{N}_0$. Let $(\hat{\mathbf{x}}, \hat{\mathbf{u}})$ be maximum. We call a pair $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in l_1^{n+m}$ *admissible* if for some $\varepsilon_0 > 0$ and for all $\varepsilon \in (0, \varepsilon_0)$ we have

$$\begin{aligned} \hat{x}_0 + \varepsilon\xi_0 &= \bar{x} \\ \hat{x}_{t+1} + \varepsilon\xi_{t+1} &= A(\hat{x}_t + \varepsilon\xi_t) + B(\hat{u}_t + \varepsilon\eta_t) + d \quad \forall t \in \mathbb{N}_0. \end{aligned}$$

We can rewrite these equations as $\xi_0 = 0$ and

$$\begin{aligned} \hat{x}_{t+1} + \varepsilon\xi_{t+1} &= A\hat{x}_t + B\hat{u}_t + d + A(\varepsilon\xi_t) + B(\varepsilon\eta_t) \\ \varepsilon\xi_{t+1} &= A(\varepsilon\xi_t) + B(\varepsilon\eta_t) \\ \xi_{t+1} &= A\xi_t + B\eta_t \quad \forall t \in \mathbb{N}_0. \end{aligned}$$

We have already shown that J is Fréchet differentiable (see Example 6). Then J can not increase along any admissible vector from the maximum. Therefore

$$\partial_{(\boldsymbol{\xi},\boldsymbol{\eta})} J(\hat{\mathbf{x}}, \hat{\mathbf{u}}) = \frac{\partial}{\partial\varepsilon} J(\hat{\mathbf{x}} + \varepsilon\boldsymbol{\xi}, \hat{\mathbf{u}} + \varepsilon\boldsymbol{\eta})|_{\varepsilon=0} \leq 0.$$

If $(\boldsymbol{\xi}, \boldsymbol{\eta})$ is admissible, then also $(-\boldsymbol{\xi}, -\boldsymbol{\eta})$ is admissible and we have

$$\partial_{(-\boldsymbol{\xi},-\boldsymbol{\eta})} J(\hat{\mathbf{x}}, \hat{\mathbf{u}}) = -\partial_{(\boldsymbol{\xi},\boldsymbol{\eta})} J(\hat{\mathbf{x}}, \hat{\mathbf{u}}) \leq 0.$$

Summing up if $(\hat{\mathbf{x}}, \hat{\mathbf{u}})$ is a maximum, then for every admissible vector $(\boldsymbol{\xi}, \boldsymbol{\eta})$ we have

$$\frac{\partial}{\partial \varepsilon} J(\hat{\mathbf{x}} + \varepsilon \boldsymbol{\xi}, \hat{\mathbf{u}} + \varepsilon \boldsymbol{\eta})|_{\varepsilon=0} = \sum_{t=0}^{\infty} [D_x f(\hat{x}_t, \hat{u}_t)\xi_t + D_u f(\hat{x}_t, \hat{u}_t)\eta_t] = 0.$$

In other words

$$DJ(\hat{\mathbf{x}}, \hat{\mathbf{u}})(\boldsymbol{\xi}, \boldsymbol{\eta}) = 0 \tag{2.2}$$

for all $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in l_1^{n+m}$ such that $\xi_0 = 0$ and

$$(\boldsymbol{\sigma} - \mathbf{A})\boldsymbol{\xi} - \mathbf{B}\boldsymbol{\eta} = \mathbf{0}, \tag{2.3}$$

where $\mathbf{A} = (A, A, \dots)$, $\mathbf{B} = (B, B, \dots)$ and $\boldsymbol{\sigma}$ is defined as in the Example 4. Moreover, since $|A| < M_A < \infty$ and $|B| < M_B < \infty$ (we can use the same argumentation as in the Example 6) these operators are according to the examples 4 and 5 bounded.

2.1.1 Necessary Conditions of Optimality

For a further work we use the following modification of the closed range theorem (for a complete theorem and proof see [14])

Proposition 2.1.1. *Let X, Y be Banach spaces, $T \in \mathcal{B}(X, Y)$ be a closed operator. Assume that $\mathcal{R}(A^*)$ is closed. Then $< v, x >= 0$ for all $x \in \mathcal{N}(A)$ if and only if $v \in \mathcal{R}(A^*)$.*

The equation $\xi_0 = 0$ is represented by the operator $(I_{n\times n}, 0_{n\times n})$. This operator is clearly closed and it's dual has a closed range, so it is sufficient to study the operator $(\boldsymbol{\sigma} - \mathbf{A}, -\mathbf{B})$. Hence we would like to use the proposition with $X = l_1^{n+m}$, $Y = l_1^m$ and $T = (\boldsymbol{\sigma} - \mathbf{A}, -\mathbf{B})$ to obtain the adjoint equation. Since in our case the operator $(\boldsymbol{\sigma} - \mathbf{A}, -\mathbf{B})$ is linear and bounded, it is continuous (Proposition 1.1.2) and thus it is closed (Proposition 1.1.3). It remains to prove that the set

$$\mathcal{R}((\boldsymbol{\sigma} - \mathbf{A}, -\mathbf{B})^*) = \{(\mathbf{z}, \mathbf{w}) \in l_\infty^{n+m} = (l_1^{n+m})^* |\ \exists \boldsymbol{\psi} \in l_\infty^n = (l_1^n)^* : (\mathbf{z}, \mathbf{w}) = (\boldsymbol{\sigma} - \mathbf{A}, -\mathbf{B})^* \boldsymbol{\psi}\} \tag{2.4}$$

is closed (according to the Proposition 1.1.5 we use the fact that $l_\infty^k = (l_1^k)^*$). This means to prove (see Definition 4) that if

$$(\mathbf{z}^0, \mathbf{w}^0), (\mathbf{z}^1, \mathbf{w}^1), \dots \in \mathcal{R}((\boldsymbol{\sigma} - \mathbf{A}, -\mathbf{B})^*) \quad \wedge \quad \lim_{k\to\infty} (\mathbf{z}^k, \mathbf{w}^k) = (\mathbf{z}, \mathbf{w}),$$

then also $(\mathbf{z}, \mathbf{w}) \in \mathcal{R}((\boldsymbol{\sigma} - \mathbf{A}, -\mathbf{B})^*)$. Let us denote the t-th component of $(\mathbf{z}^k, \mathbf{w}^k)$ as (z_t^k, w_t^k). Then we can rewrite the equation in (2.4) as follows

$$\begin{aligned} z_t &= \psi_{t-1} - A^*\psi_t \ \forall t \in \mathbb{N} \quad \text{and} \quad z_0 = -A^*\psi_0, \\ w_t &= -B^*\psi_t \ \forall t \in \mathbb{N} \quad \text{and} \quad w_0 = -B^*\psi_0, \end{aligned} \tag{2.5}$$

where $z_t \in \mathbb{R}^n$ and $w_t \in \mathbb{R}^m$ for all $t \in \mathbb{N}_0$. In order to establish the existence of $\boldsymbol{\psi}$ that fulfills the equations (2.5) along with $(\mathbf{z}, \mathbf{w})$ we construct a sequence $\{\boldsymbol{\psi}^{k_l}\}_{l=0}^{\infty}$ that fulfills the equations (2.5) along with $\{(\mathbf{z}^{k_l}, \mathbf{w}^{k_l})\}_{l=0}^{\infty}$ and all terms $\psi_t^{k_l}$ have a limit, which we denote ψ_t. We do this by a diagonalization procedure.
At first, let us define the map $C : \mathbb{R}^n \to \mathbb{R}^{n+m}$ as $Cx = (A^*x, B^*x)\ \forall x \in \mathbb{R}^n$. According to the First Isomorphism Theorem (see 1.1.1) there exists an isomorphism between $\mathcal{R}(C)$ and $\mathbb{R}^n/\mathcal{N}(C)$. To the quotient space $\mathbb{R}^n/\mathcal{N}(C)$ there exists a linear space Q, such that $Q + \mathcal{N}(C) = \mathbb{R}^n$ (this space consists of the representants of all equivalence classes in our quotient space) . Then

$$\tilde{C} : Q \to \mathcal{R}(C)$$

is this isomorphism. Note that if A^* and B^* are regular maps, then $C \equiv \tilde{C}$.
We know that for every $(\mathbf{z}^k, \mathbf{w}^k)$ there exists $\boldsymbol{\psi}^k$ such that the equations (2.5) holds. We know that $\tilde{C}$ is an isomorphism. Therefore for each (z_t^k, w_t^k) there exists exactly one $\psi_t^k \in Q$ that the desired equations are satisfied. Let $\tilde{C}^{-1}$ be the map that fulfills

$$\psi_t^k = -\tilde{C}^{-1}(z_t^k - \psi_{t-1}^k, w_t^k) \quad \forall k \in \mathbb{N}_0, t \in \mathbb{N}, \tag{2.6}$$
$$\psi_0^k = -\tilde{C}^{-1}(z_0^k, w_0^k) \quad \forall k \in \mathbb{N}_0. \tag{2.7}$$

Now we construct the subsequence $\{\boldsymbol{\psi}^{k_l}\}_{l=0}^{\infty}$ of $\{\boldsymbol{\psi}^k\}_{k=0}^{\infty}$ that converges in each component to $\boldsymbol{\psi}$.
Since $\lim_{k\to\infty}(\mathbf{z}^k, \mathbf{w}^k) = (\mathbf{z}, \mathbf{w})$, then $\{(\mathbf{z}^k, \mathbf{w}^k)\}_{k=0}^{\infty}$ must converge in each component and hence each of this component is bounded. Let L be such a constant that $|w_i^k|, |z_i^k| \leq |(z_i^k, w_i^k)| \leq L \quad \forall i,\ k \in \mathbb{N}_0$. As $\tilde{C}$ is regular, $\tilde{C}^{-1}$ is bounded. By (2.7) the sequence $\{\psi_0^k\}_{k=0}^{\infty}$ is bounded and because $\psi_0^k \in \mathbb{R}^n\ \forall k$, there exists such a subsequence $\{k_l^0\}_{l=0}^{\infty}$ of $\mathbb{N}_0$ that $\{\psi_0^{k_l^0}\}_{l=0}^{\infty}$ is convergent. Let $\psi_0 = \lim_{l\to\infty} \psi_0^{k_l^0}$.
Since every subsequence of a convergent sequence is convergent, $\{\mathbf{z}^{k_l^0}, \mathbf{w}^{k_l^0}\}_{l=0}^{\infty}$ is also convergent. As $\{\psi_0^{k_l^0}\}_{l=0}^{\infty}$ is convergent, the sequence $\{\psi_1^{k_l^0}\}_{l=0}^{\infty}$, where $\psi_1^{k_l^0} = -(\tilde{C})^{-1}(z_1^{k_l^0} - \psi_0^{k_l^0}, w_1^{k_l^0})$ is bounded ($|\psi_1^{k_l^0}| = |-\tilde{C}^{-1}(z_1^{k_l^0} - \psi_0^{k_l^0}, w_1^{k_l^0})| \leq |\tilde{C}^{-1}|(2L + |\psi_0^{k_l^0}|)$). Therefore we can choose a sequence $\{k_l^1\}_{l=0}^{\infty} \subset \{k_l^0\}_{l=0}^{\infty}$, such that the subsequence $\{\psi_1^{k_l^1}\}_{l=0}^{\infty}$ of $\{\psi_1^{k_l^0}\}_{l=0}^{\infty}$ is convergent. Let $\psi_1 = \lim_{l\to\infty} \psi_1^{k_l^1}$. As $\{k_l^1\}_{l=0}^{\infty} \subset \{k_l^0\}_{l=0}^{\infty}$ also $\lim_{l\to\infty} \psi_0^{k_l^1} = \psi_0$.
We can proceed in this way and by mathematical induction in $n+1$ st step we construct a subsequence $\{k_l^n\}_{l=0}^{\infty}$ of $\{k_l^{n-1}\}_{l=0}^{\infty}$ such that the subsequence $\{\psi_n^{k_l^n}\}_{l=0}^{\infty}$ of $\{\psi_n^{k_l^{n-1}}\}_{l=0}^{\infty}$ is convergent and $\psi_n = \lim_{l\to\infty} \psi_n^{k_l^n}$ (such a subsequence exists, because the sequence $\{\psi_n^{k_l^{n-1}}\}_{l=0}^{\infty}$ is bounded ($|\psi_n^{k_l^{n-1}}| = |-\tilde{C}^{-1}(z_n^{k_l^{n-1}} - \psi_{n-1}^{k_l^{n-1}}, w_n^{k_l^{n-1}})| \leq |\tilde{C}^{-1}|(2L + |\psi_{n-1}^{k_l^{n-1}}|)$ and $\{\psi_{n-1}^{k_l^{n-1}}\}_{l=0}^{\infty}$ is by the induction hypothesis convergent and thus bounded).
As $\{k_l^n\}_{l=0}^{\infty} \subset \{k_l^{n-1}\}_{l=0}^{\infty} \subset \cdots \subset \{k_l^0\}_{l=0}^{\infty}$ also $\lim_{l\to\infty} \psi_i^{k_l^n} = \psi_i \quad \forall i \leq n$.
But this is still not a subsequence convergent in all components. To get this, we have to write our subsequences into an infinite array

$$\begin{array}{cccc} \psi_0^{k_0^0} & \psi_0^{k_1^0} & \psi_0^{k_2^0} & \cdots \\ \psi_1^{k_0^1} & \psi_1^{k_1^1} & \psi_1^{k_2^1} & \cdots \\ \psi_2^{k_0^2} & \psi_2^{k_1^2} & \psi_2^{k_2^2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

and take the entries, which lie on the diagonal, i.e. $\{k_l^l\}_{l=0}^{\infty}$. Clearly the sequence $\{k_l^l\}_{l=n}^{\infty}$ is a subsequence of $\{k_l^n\}_{l=0}^{\infty}$ and thus $\{\psi_n^{k_l^l}\}_{l=0}^{\infty}$ converge to ψ_n (the first n terms in the sequence are not important in the sense of convergence). Hence the sequence $\{\boldsymbol{\psi}^{k_l^l}\}_{l=0}^{\infty}$ is convergent termwise with limits $\psi_t = \lim_{l\to\infty} \psi_t^{k_l^l}$ for all $t \in \mathbb{N}_0$.
It remains to prove that $\boldsymbol{\psi}$, we have just constructed, satisfies the equations (2.5) along with $(\mathbf{z}, \mathbf{w}) = \lim_{k\to\infty}(\mathbf{z}^k, \mathbf{w}^k)$.
At first since $-\tilde{C}^{-1}(z_0^{k_l^l}, w_0^{k_l^l}) = \psi_0^{k_l^l}\ \forall l \in \mathbb{N}_0$, $(\tilde{C})^{-1}$ is regular (thus it represents a continuous operator) and both sequences are convergent we have

$$(z_0, w_0) = \lim_{l\to\infty}(z_0^{k_l^l}, w_0^{k_l^l}) = -\lim_{l\to\infty} \tilde{C}\psi_0^{k_l^l} = -\tilde{C}\lim_{l\to\infty}\psi_0^{k_l^l} = -\tilde{C}\psi_0.$$

For $i \geq 1$ we have to prove that the equation

$$\psi_i = -\tilde{C}^{-1}(z_i - \psi_{i-1}, w_i)$$

holds, if for every $l \geq 1$ we have $\psi_i^{k_l^l} = -\tilde{C}^{-1}(z_i^{k_l^l} - \psi_{i-1}^{k_l^l}, w_i^{k_l^l})$. The matrix $\tilde{C}^{-1} = (\tilde{A}, \tilde{B})^{-1}$ is regular (bounded and thus continuous) therefore we have

$$\begin{aligned}
(z_i, w_i) &= \lim_{l\to\infty}(z_i^{k_l^l}, w_i^{k_l^l}) = \lim_{l\to\infty}(\psi_{i-1}^{k_l^l} - \tilde{A}\psi_i^{k_l^l}, -\tilde{B}\psi_i^{k_l^l}) \\
&= (\lim_{l\to\infty}\psi_{i-1}^{k_l^l} - \tilde{A}\lim_{l\to\infty}\psi_i^{k_l^l}, -\tilde{B}\lim_{l\to\infty}\psi_i^{k_l^l}) \\
&= (\psi_{i-1} - \tilde{A}\psi_i, -\tilde{B}\psi_i).
\end{aligned}$$

According to the Proposition (2.1.1), for all $(\boldsymbol{\xi}, \boldsymbol{\eta})$ such that

$$(\boldsymbol{\sigma} - \mathbf{A}, -\mathbf{B})(\boldsymbol{\xi}, \boldsymbol{\eta})^T = (\boldsymbol{\sigma} - \mathbf{A})\boldsymbol{\xi} - \mathbf{B}\boldsymbol{\eta} = \mathbf{0},$$

$DJ(\hat{\mathbf{x}}, \hat{\mathbf{u}})(\boldsymbol{\xi}, \boldsymbol{\eta}) = 0$ if and only if $DJ(\hat{\mathbf{x}}, \hat{\mathbf{u}}) \in \mathcal{R}((\boldsymbol{\sigma} - \mathbf{A}, -\mathbf{B})^*)$ i.e.[1]

$$\exists\ \boldsymbol{\psi} \in l_\infty^n = (l_1^n)^* : \quad DJ(\hat{\mathbf{x}}, \hat{\mathbf{u}}) = (\boldsymbol{\sigma} - \mathbf{A}, -\mathbf{B})^*\boldsymbol{\psi}. \tag{2.8}$$

This relation implies

$$D_x f(\hat{x}_t, \hat{u}_t) = \psi_{t-1} - A^*\psi_t \quad \forall t \in \mathbb{N}, \tag{2.9}$$

$$D_u f(\hat{x}_t, \hat{u}_t) = -B^*\psi_t \quad \forall t \in \mathbb{N}_0, \tag{2.10}$$

where (2.9) is the adjoint equation of our problem (2.1). Moreover, if the function f is concave in u_t for a fixed x_t, then the equation (2.10) is sufficient for a maximum principle in the form

$$f(\hat{x}_t, \hat{u}_t) + (A\hat{x}_t + B\hat{u}_t)^T\psi_t = \max_{u_t\in U_t}(f(\hat{x}_t, u_t) + (A\hat{x}_t + Bu_t)^T\psi_t) \quad \forall t \in \mathbb{N}_0.$$

[1] We can omit the equation $\xi_0 = 0$ indeed. Since it is represented by the operator $(I_{n\times n}, 0_{n\times m})$, the relations should be properly written as follows. For all $((\xi_0, \eta_0), (\boldsymbol{\xi}, \boldsymbol{\eta}))$ such that $\xi_0 = 0$ and $(\boldsymbol{\sigma} - \mathbf{A})\boldsymbol{\xi} - \mathbf{B}\boldsymbol{\eta} = \mathbf{0}$ one has $(0,0)(\xi_0, \eta_0)^T + DJ(\hat{\mathbf{x}}, \hat{\mathbf{u}})(\boldsymbol{\xi}, \boldsymbol{\eta}) = DJ(\hat{\mathbf{x}}, \hat{\mathbf{u}})(\boldsymbol{\xi}, \boldsymbol{\eta}) = 0$, if and only if $((0,0), DJ(\hat{\mathbf{x}}, \hat{\mathbf{u}})) \in \mathcal{R}(((I_{n\times n}, 0_{n\times m}), (\boldsymbol{\sigma} - \mathbf{A}, -\mathbf{B}))^*)$. So we should have

$$D_x f(\hat{x}_t, \hat{u}_t) = \phi_t - A^*\phi_{t+1}\ \forall t \in \mathbb{N} \quad \text{and} \quad D_x f(\hat{x}_0, \hat{u}_0) = -A^*\phi_1,\ 0 = I_{n\times n}^*\phi_0 = \phi_0,$$
$$D_u f(\hat{x}_t, \hat{u}_t) = -B^*\phi_{t+1}\ \forall t \in \mathbb{N}_0 \quad \text{and} \quad 0 = 0_{n\times n}^*\phi_0.$$

However if we put $\psi_t = \phi_{t+1}$ these equations implies (2.9) and (2.10).

2.2 General Dynamics

In this part we study the generalized problem. This means among the pairs $(\mathbf{x}, \mathbf{u})$, $\mathbf{x} \in l_1^n$, $\mathbf{u} \in l_1^m$ to find the extreme (maximum or minimum) of the function

$$J(\mathbf{x}, \mathbf{u}) = \sum_{t=0}^{\infty} f(x_t, u_t), \tag{2.11}$$

that satisfies the state and the initial equation

$$\begin{aligned} x_{t+1} &= F(x_t, u_t) \quad \forall t \in \mathbb{N}_0, && (2.12) \\ x_0 &= \bar{x}. && (2.13) \end{aligned}$$

As before $f \in C^1(X \times U, \mathbb{R})$, $X \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ be open, $x_t \in \operatorname{int} X$ and $u_t \in \operatorname{int} U$ for all $t \in \mathbb{N}_0$. In addition, we assume that $F \in C^1(X \times U, \mathbb{R})$. Further we consider only the case of maximum, but the reasoning for the minimum is analogous.

2.2.1 Necessary Conditions of Optimality

Let us assume that $(\hat{\mathbf{x}}, \hat{\mathbf{u}})$ is an optimal response/control pair and let us denote

$$\begin{aligned} D_x F(\hat{x}_t, \hat{u}_t) &= A_t \quad \forall t \in \mathbb{N}_0, \\ D_u F(\hat{x}_t, \hat{u}_t) &= B_t \quad \forall t \in \mathbb{N}_0, \\ (A_0, A_1, \dots) &= \tilde{\mathbf{A}}, \\ (B_0, B_1, \dots) &= \tilde{\mathbf{B}}. \end{aligned}$$

The cost function cannot increase along any admissible perturbation curve starting at $(\hat{\mathbf{x}}, \hat{\mathbf{u}})$. Hence if in any direction there exists an admissible perturbation curve, we can use this fact to derive the necessary condition of optimality. We define these directions in the following definition.

Definition 26. We call a pair $(\boldsymbol{\xi}, \boldsymbol{\eta}) = (\xi_0, \xi_1, \dots; \eta_0, \eta_1, \dots) \in l_1^n \times l_1^m$ *admissible*, if there exist $\varepsilon_0 > 0$ and differentiable curves

$$\begin{aligned} \mathbf{p}(\varepsilon) &= \{p_t(\varepsilon)\}_{t=0}^{\infty}, \quad p_t : [0, \varepsilon_0) \to \mathbb{R}^n \ \forall t \in \mathbb{N}_0, \\ \mathbf{q}(\varepsilon) &= \{q_t(\varepsilon)\}_{t=0}^{\infty}, \quad q_t : [0, \varepsilon_0) \to \mathbb{R}^m \ \forall t \in \mathbb{N}_0 \end{aligned}$$

such that the following conditions hold

i) $\mathbf{p}(0) = \mathbf{q}(0) = 0$ (i.e. the curves are starting from $(\hat{\mathbf{x}}, \hat{\mathbf{u}})$)

ii) $\mathbf{p}'(0) = \boldsymbol{\xi}$ and $\mathbf{q}'(0) = \boldsymbol{\eta}$ (i.e. the initial directions are given)

iii) for each $\varepsilon \in [0, \varepsilon_0)$ we have $p_0(\varepsilon) = 0$ and

$$\hat{x}_{t+1} + p_{t+1}(\varepsilon) = F(\hat{x}_t + p_t(\varepsilon), \hat{u}_t + q_t(\varepsilon)) \quad \forall t \in \mathbb{N}_0 \tag{2.14}$$

(i.e. the initial condition (2.13) and the state equation (2.12) hold) *iv)* for each $\varepsilon \in [0, \varepsilon_0)$ we have $(\mathbf{p}(\varepsilon) + \hat{\mathbf{x}}) \in l_1^n$ and $(\mathbf{q}(\varepsilon) + \hat{\mathbf{u}}) \in l_1^m$.

In the following proposition we show that if for a given vector $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in l_1^{n+m}$ we have $\xi_0 = 0$ and

$$\begin{aligned} \xi_{t+1} &= \frac{\partial}{\partial \varepsilon} F(\hat{x}_t + \varepsilon \xi_t, \hat{u}_t + \varepsilon \eta_t)|_{\varepsilon=0} \\ &= D_x F(\hat{x}_t, \hat{u}_t)\xi_t + D_u F(\hat{x}_t, \hat{u}_t)\eta_t \\ &= A_t \xi_t + B_t \eta_t \quad \forall t \in \mathbb{N}_0 \end{aligned} \tag{2.15}$$

and some conditions are fulfilled, then this vector is admissible. For the sake of simplicity we denote $\iota_0 = (I_{n\times n}, 0_{n\times m})$.

Proposition 2.2.1. *Let us assume that $\mathcal{N}(\iota_0, (\boldsymbol{\sigma} - \tilde{\mathbf{A}}, -\tilde{\mathbf{B}}))$ has a closed complement. Then each vector $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathcal{N}(\iota_0, (\boldsymbol{\sigma} - \tilde{\mathbf{A}}, -\tilde{\mathbf{B}}))$ is admissible.*

Proof. We prove the proposition by employing the implicit function theorem (see Theorem 1.2.1) with X, Y, Z and (x_0, y_0) defined below. Let us denote $X = \mathbb{R}$, Y as the closed complement of $\mathcal{N}(\iota_0, (\boldsymbol{\sigma} - \tilde{\mathbf{A}}, -\tilde{\mathbf{B}}))$ and $Z = l_1^n$. Note that each $(\mathbf{x}, \mathbf{u}) \in l_1^n \times l_1^m$ can be uniquely rewritten as

$$(\mathbf{x}, \mathbf{u}) = (\boldsymbol{\xi}, \boldsymbol{\eta}) + (\mathbf{v}, \mathbf{w}),$$

where $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathcal{N}(\iota_0, (\boldsymbol{\sigma} - \tilde{\mathbf{A}}, -\tilde{\mathbf{B}}))$ and $(\mathbf{v}, \mathbf{w}) \in Y$. To prove the proposition we fix $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathcal{N}(\iota_0, (\boldsymbol{\sigma} - \tilde{\mathbf{A}}, -\tilde{\mathbf{B}}))$ and construct the curves $\mathbf{p}$ and $\mathbf{q}$, complying with the conditions of the definition 26, in the form

$$\begin{aligned} \mathbf{p}(\varepsilon) &= \varepsilon \boldsymbol{\xi} + \mathbf{v}(\varepsilon), \quad \mathbf{v} : [0, \varepsilon_0) \to Y, \\ \mathbf{q}(\varepsilon) &= \varepsilon \boldsymbol{\eta} + \mathbf{w}(\varepsilon), \quad \mathbf{w} : [0, \varepsilon_0) \to Y. \end{aligned}$$

Let us further define the function $(\Phi_0, \Phi) : X \times Y \to Z$ as

$$\begin{aligned} \Phi_0(\varepsilon, (\mathbf{v}, \mathbf{w})) &= (\varepsilon \xi_0 + v_0 + \hat{x}_0) - \bar{x}, \\ \Phi(\varepsilon, (\mathbf{v}, \mathbf{w})) &= \boldsymbol{\sigma}(\varepsilon \boldsymbol{\xi} + \mathbf{v} + \hat{\mathbf{x}}) - \mathbf{F}(\varepsilon \boldsymbol{\xi} + \mathbf{v} + \hat{\mathbf{x}}, \varepsilon \boldsymbol{\eta} + \mathbf{w} + \hat{\mathbf{u}}), \end{aligned}$$

where $\boldsymbol{\sigma}$ is the shift operator and $\mathbf{F} = (F, F, \dots)$. The optimal pair $(\hat{\mathbf{x}}, \hat{\mathbf{u}})$ fulfills the constraints (2.13) and (2.12), hence

$$\begin{aligned} \Phi_0(0, (\mathbf{0}, \mathbf{0})) &= \hat{x}_0 - \bar{x} = 0, \\ \Phi(0, (\mathbf{0}, \mathbf{0})) &= \boldsymbol{\sigma}\hat{\mathbf{x}} - \mathbf{F}(\hat{\mathbf{x}}, \hat{\mathbf{u}}) = \mathbf{0} \end{aligned} \tag{2.16}$$

and we set $(x_0, y_0) = (0, (\mathbf{0}, \mathbf{0}))$. Since $F \in C^1$ it follows that the function $(\Phi_0, \Phi) \in C^1$ (in the Fréchet sense). According to (2.16) in order to use the implicit function theorem it remains to show that the operator

$$D_{(\mathbf{v},\mathbf{w})}(\Phi_0, \Phi)(0, (\mathbf{0}, \mathbf{0})) \tag{2.17}$$

has a continuous inverse. Clearly $D_{(\mathbf{v},\mathbf{w})}\Phi_0(0, (\mathbf{0}, \mathbf{0})) = (I_{n\times n}, 0_{n\times m})|_Y$ and

$$
\begin{aligned}
D_{(\mathbf{v},\mathbf{w})}\Phi(0,(\mathbf{0},\mathbf{0})) &= \\
&= (\boldsymbol{\sigma} - D_x\mathbf{F}(\hat{\mathbf{x}},\hat{\mathbf{u}}), -D_u\mathbf{F}(\hat{\mathbf{x}},\hat{\mathbf{u}}))|_Y \\
&= (\boldsymbol{\sigma} - (D_xF(\hat{x}_0,\hat{u}_0), D_xF(\hat{x}_1,\hat{u}_1),\dots), -(D_uF(\hat{x}_0,\hat{u}_0), D_uF(\hat{x}_1,\hat{u}_1),\dots))|_Y \\
&= (\boldsymbol{\sigma} - (A_0,A_1,\dots), -(B_0,B_1,\dots))|_Y = (\boldsymbol{\sigma} - \tilde{\mathbf{A}}, -\tilde{\mathbf{B}})|_Y.
\end{aligned}
$$

Hence

$$D_{(\mathbf{v},\mathbf{w})}(\Phi_0,\Phi)(0,(\mathbf{0},\mathbf{0})) = (\iota_0,(\boldsymbol{\sigma} - \tilde{\mathbf{A}}, -\tilde{\mathbf{B}}))|_Y.$$

By the Proposition 1.1.4, the restriction of a map to a closed complement of it's null space is an isomorphism. So $D_{(\mathbf{v},\mathbf{w})}(\Phi_0,\Phi)(0,(\mathbf{0},\mathbf{0}))$ is an isomorphism and hence it has a continuous inverse operator. The implicit function theorem yields that there exist a neighbourhood $X_0 \times Y_0 \subset X \times Y$ of $(0,(\mathbf{0},\mathbf{0}))$ and a differentiable function $\boldsymbol{\varphi} : X_0 \to Y_0$, $\boldsymbol{\varphi} = (\mathbf{v},\mathbf{w})$ such that

$$\boldsymbol{\varphi}(\varepsilon) = (\mathbf{v}(\varepsilon),\mathbf{w}(\varepsilon)),$$

if and only if

$$\boldsymbol{\varphi}(0) = (\mathbf{0},\mathbf{0}),\ \ \Phi_0(\varepsilon,(\mathbf{v}(\varepsilon),\mathbf{w}(\varepsilon))) = 0 \ \text{ and } \ \Phi(\varepsilon,(\mathbf{v}(\varepsilon),\mathbf{w}(\varepsilon))) = 0.$$

These equations prove the properties *i)* and *iii)* of the definition 26. Since for all $\varepsilon \in X_0$

$$(\mathbf{v}(\varepsilon),\mathbf{w}(\varepsilon)) \in Y_0 \subset l_1^n \times l_1^m$$

we have that $\mathbf{v}(\varepsilon) \in l_1^n$ and $\mathbf{w}(\varepsilon) \in l_1^m$. From the linearity of the space l_1^k for $k \in \mathbb{N}$ it follows

$$
\begin{aligned}
\mathbf{p}(\varepsilon) + \hat{\mathbf{x}} = \mathbf{v}(\varepsilon) + \varepsilon\boldsymbol{\xi} + \hat{\mathbf{x}} &\in l_1^n, \\
\mathbf{q}(\varepsilon) + \hat{\mathbf{u}} = \mathbf{w}(\varepsilon) + \varepsilon\boldsymbol{\eta} + \hat{\mathbf{u}} &\in l_1^m,
\end{aligned}
$$

which is exactly the condition *iv)* in the definition 26. Now we show that also the condition *ii)* is fulfilled. To this end we use the equation (1.7). We have already computed the directional derivative $D_{(\mathbf{v},\mathbf{w})}(\Phi_0,\Phi)$. Therefore we need to compute

$$
\begin{aligned}
D_\varepsilon\Phi_0(0,(\mathbf{0},\mathbf{0})) &= \xi_0 = 0, \\
D_\varepsilon\Phi(0,(\mathbf{0},\mathbf{0})) &= [\frac{\partial}{\partial\varepsilon}\boldsymbol{\sigma}(\varepsilon\boldsymbol{\xi} + \mathbf{v}(\varepsilon) + \hat{\mathbf{x}}) \\
&\quad - D_x\mathbf{F}(\varepsilon\boldsymbol{\xi} + \mathbf{v}(\varepsilon) + \hat{\mathbf{x}}, \varepsilon\boldsymbol{\eta} + \mathbf{w}(\varepsilon) + \hat{\mathbf{u}})\frac{\partial}{\partial\varepsilon}(\varepsilon\boldsymbol{\xi} + \mathbf{v}(\varepsilon) + \hat{\mathbf{x}}) \\
&\quad - D_u\mathbf{F}(\varepsilon\boldsymbol{\xi} + \mathbf{v}(\varepsilon) + \hat{\mathbf{x}}, \varepsilon\boldsymbol{\eta} + \mathbf{w}(\varepsilon) + \hat{\mathbf{u}})\frac{\partial}{\partial\varepsilon}(\varepsilon\boldsymbol{\eta} + \mathbf{w}(\varepsilon) + \hat{\mathbf{u}})]|_{(\varepsilon,(\mathbf{v},\mathbf{w}))=(0,(\mathbf{0},\mathbf{0}))} \\
&= \boldsymbol{\sigma}\boldsymbol{\xi} - \tilde{\mathbf{A}}\boldsymbol{\xi} - \tilde{\mathbf{B}}\boldsymbol{\eta} = \mathbf{0},
\end{aligned}
$$

because the vector $(\boldsymbol{\xi},\boldsymbol{\eta})$ is in the null space of the map $(\iota_0,(\sigma - \tilde{\mathbf{A}}, -\tilde{\mathbf{B}}))$. Summing up, one has

$$\boldsymbol{\varphi}'(0,(\mathbf{0},\mathbf{0})) = -[D_{(\mathbf{v},\mathbf{w})}(\Phi_0,\Phi)(0,(\mathbf{0},\mathbf{0}))]^{-1}[D_\varepsilon(\Phi_0,\Phi)(0,(\mathbf{0},\mathbf{0}))] = \mathbf{0}$$

and hence

$$
\begin{aligned}
\mathbf{v}'(0) = \mathbf{0} \quad &\Rightarrow \quad \mathbf{p}'(0) = \boldsymbol{\xi} + \mathbf{v}'(0) = \boldsymbol{\xi}, \\
\mathbf{w}'(0) = \mathbf{0} \quad &\Rightarrow \quad \mathbf{q}'(0) = \boldsymbol{\eta} + \mathbf{w}'(0) = \boldsymbol{\eta}.
\end{aligned}
$$

This completes the proof. $\square$

In each of the following examples we derive a condition under which the set $\mathcal{N}(\iota_0, (\boldsymbol{\sigma} - \tilde{\mathbf{A}}, -\tilde{\mathbf{B}}))$ has a closed complement. Therefore in each example we define a closed set Y and then we show that this set is a sought complement. To this end we prove that for a given $\mathbf{z} \in l_1^n$ the pair $(\mathbf{v}, \mathbf{w}) \in Y$ such that

$$v_0 = z_0 \tag{2.18}$$

$$v_{t+1} - A_t v_t - B_t w_t = z_{t+1} \quad \forall t \in \mathbb{N}_0 \tag{2.19}$$

is uniquely defined in the set Y and also that $\mathcal{N}(\iota_0, (\boldsymbol{\sigma} - \tilde{\mathbf{A}}, -\tilde{\mathbf{B}})) \cap Y = \mathbf{0}$ (so there exists an isomorphism between Y and l_1^n).

Example 7. Let us denote

$$C(t,s) = \prod_{j=s}^{t-1} A_j \quad \text{and} \quad C(t,t) = I_{n\times n} \; \forall t,s \in \mathbb{N}_0, \; t > s.$$

The closed set $l_1^n \times \{\mathbf{0}\}$ is a closed complement of $\mathcal{N}(\iota_0, (\boldsymbol{\sigma} - \tilde{\mathbf{A}}, -\tilde{\mathbf{B}}))$ if

$$\exists M \in [1,\infty) \;\wedge\; \exists \lambda < 1 : \quad |C(t,s)| \le M\lambda^{t-s} \; \forall t,s \in \mathbb{N}_0, \; t \ge s. \tag{2.20}$$

The equations (2.19) and (2.18) turn in this case into

$$v_0 = z_0,$$
$$v_{t+1} - A_t v_t = z_{t+1} \quad \forall t \in \mathbb{N}_0.$$

Now we prove that if $\mathbf{z} \in l_1^n$, then also $\mathbf{v} \in l_1^n$ and it is uniquely defined. We have

$$\begin{aligned}
&v_0 = z_0\\
&v_1 = A_0 v_0 + z_1 = A_0 z_0 + z_1\\
&v_2 = A_1 v_1 + z_2 = A_1 A_0 z_0 + A_1 z_1 + z_2\\
&\quad\Rightarrow\\
&v_t = A_{t-1} v_{t-1} + z_t = \cdots = \sum_{i=0}^{t-1} \prod_{j=i}^{t-1} A_j z_i + z_t = \sum_{i=0}^{t} C(t,i) z_i.
\end{aligned}$$

So the expression of $\mathbf{v}$ is unique for a given $\mathbf{z} \in l_1^n$ and $\mathbf{w} = \mathbf{0}$. Now we show that $\mathbf{v} \in l_1^n$. We can bound the norm of the t-th term

$$|v_t| = |\sum_{i=0}^{t} C(t,i) z_i| \le \sum_{i=0}^{t} |z_i| \cdot |C(t,i)| \le M \sum_{i=0}^{t} |z_i| \lambda^{t-i}$$

and compute the norm of $\mathbf{v}$

$$\begin{aligned}
\|\mathbf{v}\| = \sum_{t=0}^{\infty} |v_t| &\le \sum_{t=0}^{\infty} M \sum_{i=0}^{t} |z_i| \lambda^{t-i} = M \sum_{i=0}^{\infty} |z_i| \sum_{t=i}^{\infty} \lambda^{t-i}\\
&= M \sum_{i=0}^{\infty} |z_i| \sum_{t=0}^{\infty} \lambda^{t} = \frac{M}{1-\lambda} \sum_{i=0}^{\infty} |z_i| = \frac{M}{1-\lambda} \|\mathbf{z}\|.
\end{aligned}$$

We know that $\|\mathbf{z}\| < \infty$ (as $\mathbf{z} \in l_1^n$) and also $M/(1-\lambda) < \infty$ (as $\lambda < 1$ and $M < \infty$). Hence $\|\mathbf{v}\| < \infty$ and thus $\mathbf{v} \in l_1^n$. Furthermore, from the construction of $\mathbf{v}$ it is clear that $(\mathbf{v}, \mathbf{0}) \in \mathcal{N}(\iota_0, (\boldsymbol{\sigma} - \tilde{\mathbf{A}}, -\tilde{\mathbf{B}}))$, if and only if $\mathbf{v} = 0$.

Remark 5. Note that the condition (2.20) is rather abstract and there are some simpler conditions of which this one is a consequence. For example let

$$\sup_{t\in\mathbb{N}_0} |D_xF(\hat{x}_t, \hat{u}_t)| = \sup_{t\in\mathbb{N}_0} |A_t| = \lambda < 1. \tag{2.21}$$

Then we can put $M = 1$ and the general condition is fulfilled since

$$|C(t,s)| = |\prod_{j=s}^{t-1} A_j| \le \prod_{j=s}^{t-1} |A_j| \le \lambda^{t-s}.$$

Note that for (2.20) to hold it is also sufficient that

$$\sup_{t\in(\mathbb{N}_0\setminus K)} |A_t| = \lambda < 1 \quad \text{and} \quad \sup_{t\in K} |A_t| = \kappa < \infty, \tag{2.22}$$

where $K \subset \mathbb{N}_0$ is a set with $k < \infty$ elements. Then one has

$$|C(t,s)| \le \prod_{j=s}^{t-1} |A_j| \le \kappa^k \lambda^{t-s-k} = \frac{\kappa^k}{\lambda^k}\lambda^{t-s}.$$

Since $k, \kappa < \infty$ and $\lambda > 0$ (if $\lambda = 0$ it is sufficient to put $M = \kappa^k$), $M = (\kappa/\lambda)^k < \infty$.
Now we mention even simpler condition on the matrices A_t that implies (2.20). Seeing that $\hat{\mathbf{x}} \in l_1^n$ and $\hat{\mathbf{u}} \in l_1^m$ one has $\lim_{t\to\infty} \hat{x}_t = \lim_{t\to\infty} \hat{u}_t = 0$. Let us denote $A_\infty = D_xF(0,0)$. Since $F \in C^1$ one has $\lim_{t\to\infty} A_t = A_\infty$. Suppose

$$\max\{|\lambda| : \ \lambda \in sp(A_\infty)\} = \bar{\lambda} < 1,$$

where $sp(A_\infty)$ denotes the spectrum of the matrix A_∞. So we have

$$|A_\infty^k| < C\mu^k$$

for some $0 < \mu < 1$ and $C > 0$. From the continuity of D_xF for any $\varepsilon > 0$ there must exists $T \in \mathbb{N}_0$ such that for every $t > T$ we have

$$A_t = A_\infty + H_t$$

and $|H_t| < \varepsilon$. Then for a sufficiently large $s > T$ we have

$$C(t,s) = \prod_{j=s}^{t-1} A_j = \prod_{j=s}^{t-1}(A_\infty + H_j) = \prod_{j=s}^{t-1} H_j + \sum_{i=s}^{t-1}\left(\prod_{j=s}^{i-1} H_j\right) A_\infty \left(\prod_{j=i+1}^{t-1} H_j\right) + \cdots + A_\infty^{t-s}.$$

Hence

$$|C(t,s)| = |\prod_{j=s}^{t-1} A_j| < C\sum_{j=0}^{t-s} \tbinom{t-s}{j}\mu^j C^{t-s-j}\varepsilon^{t-s-j} = C(\mu + C\varepsilon)^{t-s}.$$

So if we choose ε such that $(\mu + C\varepsilon) < 1$ the condition (2.20) is fulfilled for all $t \ge s > T$. The remaining matrices $(t < T)$ can be replaced by the constant M similarly as in the previous case.

Example 8. A_t are regular for all $t \in \mathbb{N}_0$, $\sup_{t\in\mathbb{N}_0} \|D_x F(\hat{x}_t, \hat{u}_t)^{-1}\| = \sup_{t\in\mathbb{N}_0} \|A_t^{-1}\| = \lambda < 1$ In this case we again define the complement of $\mathcal{N}(\iota_0, (\boldsymbol{\sigma} - \tilde{\mathbf{A}}, -\tilde{\mathbf{B}}))$ as the closed set $l_1^n \times \{\mathbf{0}\}$. Hence the equations (2.19) and (2.18) turn into

$$\begin{aligned} & v_0 = z_0, \\ v_{t+1} - A_t v_t = z_{t+1} \;\Rightarrow\; & v_t = A_t^{-1}(v_{t+1} - z_{t+1}). \end{aligned} \tag{2.23}$$

We show that for all $\mathbf{z} \in l_1^n$ there is an unique solution in the form

$$v_t = -\sum_{i=t}^{\infty} \prod_{j=t}^{i} A_j^{-1} z_{i+1}.$$

If we substitute $\mathbf{v}$ in this form into the equation mentioned above we obtain

$$\begin{aligned} A_t^{-1}(v_{t+1} - z_{t+1}) &= -A_t^{-1} \sum_{i=t+1}^{\infty} \prod_{j=t+1}^{i} A_j^{-1} z_{i+1} - A_t^{-1} z_{t+1} \\ &= -\sum_{i=t+1}^{\infty} \prod_{j=t}^{i} A_j^{-1} z_{i+1} - A_t^{-1} z_{t+1} = -\sum_{i=t}^{\infty} \prod_{j=t}^{i} A_j^{-1} z_{i+1}. \end{aligned}$$

Since the left and the right side of the equation (2.23) are for this solution equal, this is indeed a solution to this equation. Now we prove that this solution is in the defined set l_1^n.

$$\begin{aligned} \|\mathbf{v}\| &= \sum_{t=0}^{\infty} |v_t| = \sum_{t=0}^{\infty} |\sum_{i=t}^{\infty} \prod_{j=t}^{i} A_j^{-1} z_{i+1}| \leq \sum_{t=0}^{\infty} \sum_{i=t}^{\infty} |z_{i+1}| \lambda^{i-t+1} = \sum_{i=0}^{\infty} |z_{i+1}| \sum_{t=0}^{i} \lambda^{i-t+1} \\ &= \sum_{i=1}^{\infty} |z_i| \sum_{t=1}^{i+1} \lambda^t \leq \frac{\lambda - \lambda^{i+2}}{1-\lambda} \sum_{i=0}^{\infty} |z_i| < \frac{\lambda}{1-\lambda} \|\mathbf{z}\| < \infty, \end{aligned}$$

thus $\mathbf{v} \in l_1^n$. Furthermore, from the form of $\mathbf{v}$ it is clear that for a given $\mathbf{z} = \mathbf{0}$ we have $\mathbf{v} = \mathbf{0}$. Therefore $(l_1^n \times \mathbf{0}) \cap \mathcal{N}(\iota_0, (\boldsymbol{\sigma} - \tilde{\mathbf{A}}, -\tilde{\mathbf{B}})) = (\mathbf{0}, \mathbf{0})$. Summing up, the set $l_1^n \times \mathbf{0}$ is a closed complement of $\mathcal{N}(\iota_0, (\boldsymbol{\sigma} - \tilde{\mathbf{A}}, -\tilde{\mathbf{B}}))$.

Example 9. In the case $m = n$ we can introduce similar condition on the matrices B_t. Let B_t be regular for all $t \in \mathbb{N}_0$ and $\sup_{t\in\mathbb{N}_0} |B_t^{-1}| = M_B < \infty$. Then we can define the closed complement of $\mathcal{N}(\iota_0, (\boldsymbol{\sigma} - \tilde{\mathbf{A}}, -\tilde{\mathbf{B}}))$ as $\{\mathbf{0}\} \times l_1^n$. If we apply this fact to the equation (2.19) we get

$$z_{t+1} = -B_t w_t.$$

This equation yields $w_t = -B_t^{-1} z_{t+1}$ (so $\mathbf{w}$ is unique for a given $\mathbf{z}$ and $\mathbf{v} = 0$) and one has

$$\|\mathbf{w}\| = \sum_{t=0}^{\infty} |w_t| \leq \sum_{t=0}^{\infty} |B_t^{-1}| \cdot |z_{t+1}| \leq M_B \|\mathbf{z}\|.$$

Hence $\mathbf{w} \in l_1^n$ if $\mathbf{z} \in l_1^n$. Moreover, if $\mathbf{z} = \mathbf{0}$, then $\mathbf{w} = \mathbf{0}$.

Now we are ready to derive the necessary conditions of optimality. If $(\boldsymbol{\xi}, \boldsymbol{\eta})$ is admissible, then the optimality of $(\hat{\mathbf{x}}, \hat{\mathbf{u}})$ implies

$$\begin{aligned} \frac{\partial}{\partial \varepsilon} J(\hat{\mathbf{x}} + \mathbf{p}(\varepsilon), \hat{\mathbf{u}} + \mathbf{q}(\varepsilon))|_{\varepsilon=0} &= \sum_{t=0}^{\infty} [D_x f(x_t, u_t) p_t'(\varepsilon) + D_u f(x_t, u_t) q_t'(\varepsilon)]|_{\varepsilon=0} \\ &= \sum_{t=0}^{\infty} [D_x f(x_t, u_t)\xi_t + D_u f(x_t, u_t)\eta_t] \leq 0. \end{aligned} \tag{2.24}$$

We have derived that if $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathcal{N}(\iota_0, (\boldsymbol{\sigma} - \tilde{\mathbf{A}}, -\tilde{\mathbf{B}}))$ (and any of the condition stated in the examples is fulfilled), then $(\boldsymbol{\xi}, \boldsymbol{\eta})$ is admissible and (2.24) holds. It is clear that in this case also $(-\boldsymbol{\xi}, -\boldsymbol{\eta}) \in \mathcal{N}(\iota_0, (\boldsymbol{\sigma} - \tilde{\mathbf{A}}, -\tilde{\mathbf{B}}))$ and the necessary conditions hold, as well. Therefore

$$D_x f(x_t, u_t)(-\xi_t) + D_u f(x_t, u_t)(-\eta_t) = -[D_x f(x_t, u_t)\xi_t + D_u f(x_t, u_t)\eta_t] \leq 0.$$

Summing up if $(\hat{\mathbf{x}}, \hat{\mathbf{u}})$ is an optimal pair and some conditions on $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ are satisfied, then for all $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in l_1^{n+m}$ that satisfies $\xi_0 = 0$ and

$$\xi_{t+1} = A_t \xi_t + B_t \eta_t, \quad \forall t \in \mathbb{N}_0$$

one has

$$\sum_{t=0}^{\infty} [D_x f(x_t, u_t)\xi_t + D_u f(x_t, u_t)\eta_t] = 0.$$

Note that according to the example 6 $|A_t| < \infty$ and $|B_t| < \infty$ for all $t \in \mathbb{N}_0$ and hence $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ are bounded (see Example 5). So we can use an analogous method to prove that $\mathcal{R}((\iota_0, (\boldsymbol{\sigma} - \tilde{\mathbf{A}}, -\tilde{\mathbf{B}}))^*)$ is closed as in the section 2.1.1. Thus we are allowed to employ the Proposition 2.1.1 and one has

$$DJ(\hat{\mathbf{x}}, \hat{\mathbf{u}})(\boldsymbol{\xi}, \boldsymbol{\eta}) = 0 \;\; \forall (\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathcal{N}(\iota_0, (\boldsymbol{\sigma} - \tilde{\mathbf{A}}, -\tilde{\mathbf{B}})) \;\Leftrightarrow\; DJ(\hat{\mathbf{x}}, \hat{\mathbf{u}}) \in \mathcal{R}((\iota_0, (\boldsymbol{\sigma} - \tilde{\mathbf{A}}, -\tilde{\mathbf{B}}))^*).$$

Hence there exists $\boldsymbol{\psi} \in l_\infty^n$ such that

$$\begin{aligned} D_x f_t(\hat{x}_t, \hat{u}_t) &= \psi_{t-1} - A_t^* \psi_t && \forall t \in \mathbb{N}, \\ D_u f_t(\hat{x}_t, \hat{u}_t) &= -B_t^* \psi_t && \forall t \in \mathbb{N}_0. \end{aligned}$$

2.3 Restrictions on the Control Variable

In this section we extend our problem from the previous one. We do this by imposing restrictions on the set of feasible control variables (this set is denoted by the letter U). So our infinite horizon optimal control problem, which we study, can be rewritten as follows

$$\begin{aligned} J(\mathbf{x}, \mathbf{u}) &= \sum_{t=0}^{\infty} f(x_t, u_t) \to \max, \\ x_{t+1} &= F(x_t, u_t) \quad \forall t \in \mathbb{N}_0, \\ x_0 &= \bar{x}, \\ u_t &= U_t = \{u \in U : s_t(u) \leq 0\} \quad \forall t \in \mathbb{N}_0. \end{aligned}$$

As before $f \in C^1(X \times U, \mathbb{R})$, $X \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ be open, $x_t \in \text{int } X$, $u_t \in \text{int } U$ for all $t \in \mathbb{N}_0$ and $F \in C^1(X \times U, \mathbb{R}^m)$. Moreover, the set U_t is a closed convex subset of U for all $t \in \mathbb{N}_0$ and $s_t \in C^1(U, \mathbb{R}^{m_t})$ for all $t \in \mathbb{N}_0$. As before, we consider only the case of maximum, but the reasoning for minimum is analogous.

2.3.1 Necessary Conditions of Optimality

Because we study a different problem in comparison to the previous cases, we need to adjust the definition of an admissible pair. Afterwards, we derive the necessary conditions of optimality.

Let $(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t)$ be an optimal control/response pair. Recall the Definition 25. Let in $\hat{\mathbf{u}}$ the regularity conditions be fulfilled and let us denote

$$\tilde{s}_t(\hat{u}_t) = \{s_t^k(\hat{u}_t)\}_{k \in I_t(\hat{u}_t)} \ \forall t \in \mathbb{N}_0.$$

The definition of the set of all admissible directions concerning the set U_t is splitted in two parts. At first we define the variation cone and then we show that for all vectors from this cone there exists an admissible perturbation curve in the set U_t.

Definition 27. Let $U_t \subset \mathbb{R}^m$ be a set and $\hat{u}_t \in U_t$. The *variation cone* $\delta U_t(\hat{u}_t)$ of the set U_t at $\hat{u}_t$ is defined as

$$\delta U_t(\hat{u}_t) = \{\eta \in \mathbb{R}^m : D_u \tilde{s}_t(\hat{u}_t)\eta \leq 0\}.$$

Proposition 2.3.1. *$\delta U_t(\hat{u}_t)$ is a cone with vertex at zero. Moreover, if U_t is a closed convex set, then $\delta U_t(\hat{u}_t)$ is a closed convex cone.*

Proof. The proof is trivial and is omitted. □

In the following proposition we prove that in each direction $\eta \in \delta U_t(\hat{u}_t)$ there exists an admissible perturbation curve.

Proposition 2.3.2. *For all $\eta \in \delta U_t(\hat{u}_t)$ there exists $\varepsilon_0 > 0$ and a differentiable curve $p : [0, \varepsilon_0) \to U_t$ such that $p(0) = \hat{u}_t$ and $p'(0) = \eta$.*

Proof. It is clear that if $\hat{u}_t \in \text{int } U_t$ then $I_t(\hat{u}_t) = \emptyset$ and $\delta U_t(\hat{u}_t) = \mathbb{R}^m$. Hence in each direction η there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in [0, \varepsilon_0)$ we have $(\hat{u}_t + \varepsilon\eta) \in U_t$. So $p(\varepsilon) = \hat{u}_t + \varepsilon\eta$ satisfies the required conditions. Now let $I_t(\hat{u}_t) \neq \emptyset$. If $D_u \tilde{s}_t(\hat{u}_t)\eta < 0$ the function $p(\varepsilon) = \hat{u}_t + \varepsilon\eta$ fulfills the conditions, because $\tilde{s}_t(p(0)) = \tilde{s}_t(\hat{u}_t) = 0$ and the function $\tilde{s}_t$ is decreasing at $\hat{u}_t$

$$D_\varepsilon \tilde{s}_t(p(\varepsilon))|_{\varepsilon=0} = D_u \tilde{s}_t(\hat{u}_t)\eta < 0.$$

Therefore there exists ε_0 such that

$$\tilde{s}_t(p(\varepsilon)) \leq 0 \ \forall \varepsilon \in [0, \varepsilon_0).$$

It remains to find p and ε_0 in the case, where the set

$$\tilde{I}_t(\hat{u}_t) = \{k \in I_t(\hat{u}_t) : D_u s_t^k(\hat{u}_t)\eta = 0\}$$

is nonempty and the vector η is tangent to the set

$$\tilde{U}_t = \{u \in U_t : \bar{s}_t(u) = 0\}, \text{ where } \bar{s}_t = \{s_t^k\}_{k \in \tilde{I}_t(\hat{u}_t)}.$$

We do this employing the Implicit Function Theorem (see Theorem 1.2.1) with $X = \mathbb{R}$, Y as a closed complement to $\mathcal{N}(D_u \bar{s}_t(\hat{u}_t))$, $Z = \mathbb{R}^{|\tilde{I}_t(\hat{u}_t)|}$ and the function Φ defined as

$$\Phi(\varepsilon, u) = \bar{s}_t(\hat{u}_t + u + \varepsilon\eta)$$

and we find the function p in the form

$$p(\varepsilon) = \varphi(\varepsilon) + \eta\varepsilon + \hat{u}_t, \quad \varphi : [0, \varepsilon_0) \to Y.$$

If one sets $(x_0, y_0) = (0, 0)$ one has

$$\Phi(0, 0) = \bar{s}_t(\hat{u}_t) = 0.$$

We are allowed to use the implicit function theorem, if the operator

$$D_u\Phi(0, 0)|_Y = [D_u\bar{s}_t(\hat{u}_t + u + \varepsilon\eta)|_{(0,0)}]|_Y = D_u\bar{s}_t(\hat{u}_t)|_Y$$

has a continuous inverse, which is according to Proposition 1.1.4 satisfied (the closeness is fulfilled, since we can define Y as $D_u\bar{s}_t(\hat{u}_t)$ and each of the cases is finite). Hence there exists a neighbourhood $X_0 \times Y_0 \subset X \times Y$ of $(0, 0)$ and a differentiable function $\varphi : X_0 \to Y_0$ such that

$$\varphi(\varepsilon) = u(\varepsilon),$$

if and only if

$$\varphi(0) = 0 \text{ and } \Phi(\varepsilon, u(\varepsilon)) = \bar{s}_t(\hat{u}_t + u(\varepsilon) + \varepsilon\eta) = 0.$$

As a result $p(\varepsilon) \in U_t$ for all $\varepsilon \in [0, \varepsilon_0)$, p is differentiable and $p(0) = \hat{u}_t + \varphi(0) = \hat{u}_t$. In addition, as

$$D_\varepsilon\Phi(0, 0) = [D_u\bar{s}_t(\hat{u}_t + u + \varepsilon\eta)\eta|_{(0,0)}] = D_u\bar{s}_t(\hat{u}_t)\eta = 0$$

one has

$$p'(0) = \varphi'(0) + \eta = -[D_u\Phi(0, 0)|_Y]^{-1}D_\varepsilon\Phi(0, 0) + \eta = \eta,$$

what was to be proven. □

The notation introduced in the previous chapter can be extended to our case. Let us denote

$$\begin{aligned} L_t &= \delta U_t(\hat{u}_t) \quad \forall t \in \mathbb{N}_0, \\ \mathbf{L} &= l_1^n \times (l_1^m \cap (L_0 \times L_1 \times L_2 \times \cdots)), \\ \mathbf{K} &= \{(\boldsymbol{\xi}, \boldsymbol{\eta}) \in l_1^{n+m} : \xi_0 = 0 \ \wedge \ \boldsymbol{\sigma\xi} - \tilde{\mathbf{A}}\boldsymbol{\xi} - \tilde{\mathbf{B}}\boldsymbol{\eta} = \mathbf{0}\}. \end{aligned}$$

Since the state equation holds as in the previous case, the necessary condition for admissibility of the pair $(\boldsymbol{\xi}, \boldsymbol{\eta})$ is

$$(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathcal{N}(\iota_0, (\boldsymbol{\sigma} - \tilde{\mathbf{A}}, -\tilde{\mathbf{B}})) = \mathbf{K}$$

and we have to assume that $\mathbf{K}$ has a closed complement. As the value of the admissible perturbation must lie within the set U_t for all $t \in \mathbb{N}_0$, the admissible pair $(\boldsymbol{\xi}, \boldsymbol{\eta})$ must lie within the variation cone $\mathbf{L}$. Therefore $(\boldsymbol{\xi}, \boldsymbol{\eta})$ is admissible, if $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbf{K} \cap \mathbf{L}$. Thus the cost function cannot decrease along any admissible direction from the optimal pair $(\hat{\mathbf{x}}, \hat{\mathbf{u}})$

$$DJ(\hat{\mathbf{x}}, \hat{\mathbf{u}})(\boldsymbol{\xi}, \boldsymbol{\eta}) \leq 0 \;\; \forall (\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbf{K} \cap \mathbf{L} \;\Rightarrow\; DJ(\hat{\mathbf{x}}, \hat{\mathbf{u}}) \in (\mathbf{K} \cap \mathbf{L})^{\circ}.$$

Since $\delta U_t(\hat{u}_t)$ is closed and convex for all $t \in \mathbb{N}_0$ (Proposition 2.3.1) $\mathbf{L}$ is clearly convex and closed. In addition to that, $\mathbf{K}$ is a closed convex cone (see Propositions 1.3.2). Now we would like to use the Proposition 1.3.1 to decompose the set $(\mathbf{K} \cap \mathbf{L})^{\circ}$. Hence we have to check, if the condition $\mathbf{K} \cap \operatorname{int} \mathbf{L} \neq \emptyset$ is satisfied. In the following example we introduce a condition on the matrices $\mathbf{A}$ and $\mathbf{B}$, when this is fulfilled.

Example 10. At first let us show that $[\operatorname{int}\,(L_0 \times L_1 \times L_2 \times \cdots)] \neq \emptyset$. To this end let us denote $D_t = D_u \tilde{s}_t(\hat{u}_t)$ for all $t \in \mathbb{N}_0$ and $|I_t(\hat{u}_t)| = \tilde{m}_t$. Since the regularity conditions are fulfilled in $\hat{u}_t$, the matrix D_t is an $\tilde{m}_t \times m$ matrix with rank $\tilde{m}_t$ and linearly independent rows. We want to prove that

$$\forall t \in \mathbb{N}_0 \;\; \exists \eta_t \in \mathbb{R}^m : \quad D_t \eta_t < 0.$$

Consider the vector

$$D_t^T v, \text{ where } v = (v_1, \ldots, v_{\tilde{m}_t}) \in \mathbb{R}^{\tilde{m}_t}, \; v_i < 0 \;\forall i = 1, \ldots, \tilde{m}_t.$$

Then the linear independence yields $D_t D_t^T v < 0 \;\forall t \in \mathbb{N}_0$, hence $[\operatorname{int}\,(L_0 \times L_1 \times L_2 \times \cdots)] \neq \emptyset$. Now we show that the set $l_1^m \cap \operatorname{int}\,(L_0 \times L_1 \times L_2 \times \cdots)$ is nonempty. Consider a sequence of positive numbers $\{\varepsilon_t\}_{t \in \mathbb{N}_0} = \boldsymbol{\varepsilon} \in l_1$ and let us define

$$\delta_t = \frac{\varepsilon_t}{|\eta_t|} \;\forall t \in \mathbb{N}_0.$$

Clearly $\delta_t > 0$, so if $\eta_t \in \operatorname{int} L_t$, then also $\delta_t \eta_t \in \operatorname{int} L_t$ and one has

$$\|\boldsymbol{\delta}\eta\| = \sum_{t=0}^{\infty} |\delta_t \eta_t| = \sum_{t=0}^{\infty} \delta_t |\eta_t| = \sum_{t=0}^{\infty} \varepsilon_t = \|\boldsymbol{\varepsilon}\| < \infty.$$

It remains to show that for some $\boldsymbol{\eta} \in [l_1^m \cap \operatorname{int}\,(L_0 \times L_1 \times L_2 \times \cdots)]$ there exists $\boldsymbol{\xi} \in l_1^m$ such that $(\boldsymbol{\xi}, \boldsymbol{\eta})$ is a solution to the equations

$$\begin{aligned} \xi_0 &= 0, \\ \xi_{t+1} &= A_t \xi_t + B_t \eta_t \;\; \forall t \in \mathbb{N}_0. \end{aligned}$$

This could be written as

$$\begin{aligned} \xi_1 &= A_0 \xi_0 + B_0 \eta_0 = B_0 \eta_0, \\ \xi_2 &= A_1 \xi_1 + B_1 \eta_1 = A_1 B_0 \eta_0 + B_1 \eta_1, \\ \xi_3 &= A_2 \xi_2 + B_2 \eta_2 = A_2 A_1 B_0 \eta_0 + A_2 B_1 \eta_1 + B_2 \eta_2, \\ &\Rightarrow \\ \xi_t &= \cdots = \sum_{i=0}^{t-2} \left(\prod_{j=i+1}^{t-1} A_j \right) B_i \eta_i + B_{t-1} \eta_{t-1} = \sum_{i=0}^{t-1} C(t, i+1) B_i \eta_i. \end{aligned}$$

where $C(t, i+1)$ is defined as in the Example 7. Consider the operator $\tilde{\mathbf{A}}$ such that

$$\exists M \in [1, \infty) \ \wedge \ \exists \lambda < 1 : \quad |C(t,s)| \leq M\lambda^{t-s} \ \forall t, s \in \mathbb{N}_0, \ t \geq s$$

(this is fulfilled for example if one has $\max\{|\lambda| : \ \lambda \in sp(A_\infty)\} = \bar{\lambda} < 1$, see Remark 5). It follows that

$$|\xi_t| = |\sum_{i=0}^{t-1} C(t, i+1) B_i \eta_i| \leq \sum_{i=0}^{t-1} |B_i||\eta_i| \cdot |C(t, i+1)| \leq M \cdot M_B \sum_{i=0}^{t-1} |\eta_i| \lambda^{t-i-1},$$

where M_B is defined as the smallest constant such that $\sup_{t \in \mathbb{N}_0} |B_t| \leq M_B < \infty$ (such a constant exists according to the Example 6). Therefore if $\boldsymbol{\xi}$ fulfills the linearized state equation we obtain

$$\|\boldsymbol{\xi}\| \leq M \cdot M_B \sum_{t=1}^{\infty} \sum_{i=0}^{t-1} |\eta_i| \lambda^{t-i-1} = M \cdot M_B \sum_{i=0}^{\infty} |\eta_i| \sum_{t=0}^{\infty} \lambda^t = \frac{M \cdot M_B}{1-\lambda} \|\boldsymbol{\eta}\| < \infty.$$

Hence for all $\boldsymbol{\eta} \in [l_1^m \cap \mathrm{int}\ (L_0 \times L_1 \times L_2 \times \cdots)]$ there exists $\boldsymbol{\xi} \in l_1^n$ such that the linearized state equation is fulfilled and therefore $\mathbf{K} \cap \mathrm{int}\ \mathbf{L} \neq \emptyset$.

Finally, we can employ the Proposition 1.3.1 and we obtain

$$DJ(\hat{\mathbf{x}}, \hat{\mathbf{u}}) \in (\mathbf{K} \cap \mathbf{L})^\circ = \mathbf{K}^\circ + \mathbf{L}^\circ.$$

Hence there exist $\mathbf{p} \in \mathbf{K}^\circ$ and $\mathbf{q} \in \mathbf{L}^\circ$ such that

$$DJ(\hat{\mathbf{x}}, \hat{\mathbf{u}}) = \mathbf{p} + \mathbf{q}.$$

Now we rewrite the variable $\mathbf{p}$ using the following proposition.

Proposition 2.3.3. *Let $A : X \to Y$ be a linear operator and let $\mathcal{R}(A^*)$ be closed. Then one has*

$$-\mathcal{K}^* = \mathcal{K}^\circ = \mathcal{R}(A^*),$$

where the cone $\mathcal{K}$ is defined as in Proposition 1.3.2.

Proof. We would like to prove that $\eta \in \mathcal{K}^\circ$, if and only if $\eta \in \mathcal{R}(A^*)$. Since

$$\mathcal{K} = \{x : Ax = 0, \ x \in X\} = \mathcal{N}(A),$$

this can be rewritten as

$$< \eta, x > \leq 0 \ \forall x \in \mathcal{N}(A) \ \Leftrightarrow \ \eta \in \mathcal{R}(A).$$

Let us consider $x \in \mathcal{N}(A)$. Then also $(-x) \in \mathcal{N}(A)$ (because $Ax = A(-x) = 0$) and hence

$$< \eta, x > \leq 0 \ \forall x \in \mathcal{N}(A) \ \Leftrightarrow < \eta, -x > = - < \eta, x > \leq 0 \ \forall x \in \mathcal{N}(A).$$

Summing up, we would like to prove that if $\mathcal{R}(A^*)$ is closed, then

$$< \eta, x > = 0 \ \forall x \in \mathcal{N}(A) \ \Leftrightarrow \ \eta \in \mathcal{R}(A),$$

what is exactly the claim in the Proposition 2.1.1. □

By this proposition we have $\mathbf{p} \in \mathbf{K}^\circ = \mathcal{R}((\iota_0, (\boldsymbol{\sigma} - \tilde{\mathbf{A}}, -\tilde{\mathbf{B}}))^*)$. Hence there exists $\boldsymbol{\phi} \in (l_1^n)^* = l_\infty^n$ (see Proposition 1.1.5) such that $\mathbf{p} = (\iota_0, (\boldsymbol{\sigma} - \tilde{\mathbf{A}}, -\tilde{\mathbf{B}}))^* \boldsymbol{\phi}$. Therefore

$$DJ(\hat{\mathbf{x}}, \hat{\mathbf{u}}) = (\iota_0, (\boldsymbol{\sigma} - \tilde{\mathbf{A}}, -\tilde{\mathbf{B}}))^* \boldsymbol{\phi} + \mathbf{q}.$$

If we put $\boldsymbol{\sigma\phi} = \boldsymbol{\psi}$ we can rewrite these equation with regard to the previous sections as

$$D_x f(\hat{x}_t, \hat{u}_t) = \psi_{t-1} - A_t^* \psi_t \quad \forall t \in \mathbb{N}, \tag{2.25}$$
$$D_u f(\hat{x}_t, \hat{u}_t) = -B_t^* \psi_t + q_t \quad \forall t \in \mathbb{N}_0, \tag{2.26}$$

where $q_t \in [\delta U_t(\hat{u}_t)]^\circ$ for all $t \in \mathbb{N}_0$. In the following theorem we summarize the Pontryagin maximum principle according to the assumptions in the Proposition 1.4.2.

Proposition 2.3.4. *Let us suppose that U_t are convex, the function F is linear in u_t for a fixed x_t (i.e. $F(x_t, u_t) = F_0(x_t) + Bu_t$) and the function f is concave in u_t for a fixed x_t and for all $t \in \mathbb{N}_0$. Then the obtained equation (2.26) is sufficient for the existence of a maximum and therefore one has*

$$f(\hat{x}_t, \hat{u}_t) + F(\hat{x}_t, \hat{u}_t)^T \psi_t = \max_{u_t \in U_t} (f(\hat{x}_t, u_t) + F(\hat{x}_t, u_t)^T \psi_t) \quad \forall t \in \mathbb{N}_0.$$

The last equation is the so-called Pontryagin's maximum principle.

Conclusion

In this thesis we focused on the discrete-time infinite-horizon optimal control problem. In the first chapter we summarized the essential theory that was consequently used in the second chapter. In the second chapter of this work we successively studied the problems with linear state equation, with general state equation and with constraints.
For the problem with linear state equation we derived the adjoint equation and the necessary conditions of the Pontryagin maximum principle. We managed to do this without any further condition on the matrices of the linearized dynamics.
The adjoint equation was derived for the problem with general dynamics, as well. However we did not succeed to get rid of the restrictive conditions on our matrices. Therefore we introduced some sufficient conditions such that under any of them the deductions hold.
The derivations in the third section were combined with the notion of a cone and we obtained the adjoint equation and the necessary conditions of the Pontryagin maximum principle under a sufficient condition again.
In conclusion, the problem itself turned out to be much more difficult than anticipated. Even in this case we were not able to dispose of the restrictive conditions, albeit we managed to weaken them. Research is still under way. For this reason we did not include the perhaps more widely studied discounted problems in this work.

Resumé

V tejto práci sa zaoberáme možnosťou rozšírenia nutných podmienok optimality pre diskrétne úlohy optimálneho riadenia na nekonečnom horizonte na úlohy s nie regulárnou maticou dynamickej linearizácie. Za týmto účelom používame nástroje funkcionálnej analýzy. Motivácia pre tento prístup vychádza z článku [3], v ktorom bol podobný postup využitý. Výhodou oproti klasickému prístupu, ako je použitý napríklad v [13], je fakt, že namiesto štúdia konečnorozmerného príkladu a následného prechodu k jeho nekonečnorozmernej verzii, sa priamo zaoberáme nekonečnorozmernou verziou príkladu. Z toho dôvodu predpokladáme, že premenná riadenia $\mathbf{u} = (u_0, u_1, \dots)$, ako aj stavová premenná $\mathbf{x} = (x_0, x_1, \dots)$, patria do priestoru l_1^k, kde k označuje rozmer príslušnej premennej v jednotlivých časových vrstvách a priestor l_1^k je definovaný ako

$$l_1^k = \{\{w_t\}_{t\in\mathbb{N}_0} : \ w_t \in \mathbb{R}^k \ \forall t \in \mathbb{N}_0 \ \wedge \ \sum_{t=0}^{\infty} |w_t| < \infty\}.$$

Na začiatku práce skúmame nasledovný problém optimálneho riadenia

$$\begin{aligned} J(\mathbf{x},\mathbf{u}) &= \sum_{t=0}^{\infty} f(x_t,u_t) \to \max, && (2.27)\\ x_{t+1} &= Ax_t + Bu_t \quad \forall t \in \mathbb{N}_0, && (2.28)\\ x_0 &= \bar{x}, && (2.29) \end{aligned}$$

kde $f \in C^1(X \times U, \mathbb{R})$, $X \subset \mathbb{R}^n$ a $U \subset \mathbb{R}^m$ sú otvorené množiny, $x_t \in \operatorname{int} X$, $u_t \in \operatorname{int} U$ pre všetky $t \in \mathbb{N}_0$. Za predpokladu, že dvojica $(\hat{\mathbf{x}}, \hat{\mathbf{u}})$ je optimálna, odvodíme adjungovanú premennú ako prvok priestoru l_∞^n, ako aj adjungovanú rovnicu v tvare

$$D_x f_t(\hat{x}_t, \hat{u}_t) = \psi_{t-1} - A^*\psi_t \ \ \forall t \in \mathbb{N}.$$

Navyše pri tomto postupe nepotrebujeme dodať žiadne dodatočné predpoklady na matice A a B, teda ani predpoklad regularity.

V ďalšej časti práce skúmame zovšeobecnený problém optimálneho riadenia, v ktorom nahradíme stavovú rovnicu (2.28) rovnicou

$$x_{t+1} = F(x_t, u_t) \quad \forall t \in \mathbb{N}_0.$$

Ak má byť dvojica $(\hat{\mathbf{x}}, \hat{\mathbf{u}})$ optimálna, tak účelová funkcia musí klesať v každom smere, v ktorom existuje perturbácia ležiaca v prípustnej množine a začínajúca v bode $(\hat{\mathbf{x}}, \hat{\mathbf{u}})$. Ak

označíme

$$\begin{aligned} D_xF(\hat{x}_t,\hat{u}_t) &= A_t \quad \forall t\in\mathbb{N}_0,\\ D_uF(\hat{x}_t,\hat{u}_t) &= B_t \quad \forall t\in\mathbb{N}_0,\\ (A_0,A_1,\dots) &= \tilde{\mathbf{A}},\\ (B_0,B_1,\dots) &= \tilde{\mathbf{B}},\\ (I_{n\times n},0_{n\times m}) &= \iota_0, \end{aligned}$$

a $\boldsymbol{\sigma}$ ako operátor posunu, tak podmienky, za ktorých existuje takýto smer sumarizuje nasledujúca veta.

Veta 2.3.1. *Predpokladajme, že jadro zobrazenia $(\iota_0,(\boldsymbol{\sigma}-\tilde{\mathbf{A}},-\tilde{\mathbf{B}}))$ má uzavretý komplement. Potom v každom smere $(\boldsymbol{\xi},\boldsymbol{\eta})\in\mathcal{N}(\iota_0,(\boldsymbol{\sigma}-\tilde{\mathbf{A}},-\tilde{\mathbf{B}}))$ existuje prípustná perturbácia.*

Uzavretosť komplementu uvedeného v predchádzajúcej vete dokážeme v ľubovoľnom z nasledujúcich prípadov.

- $\max\{|\lambda| :\ \lambda\in sp(A_\infty)\}<1$
- A_t regulárne a $\sup_{t\in\mathbb{N}_0}|A_t^{-1}|<1$
- $m=n$, B_t regulárne a $\sup_{t\in\mathbb{N}_0}|B_t^{-1}|<\infty$

Pomocou vety 2.3.1 potom vyjadríme nutné podmienky optimality

$$\begin{aligned} D_xf_t(\hat{x}_t,\hat{u}_t) &= \psi_{t-1}-A_t^*\psi_t & \forall t\in\mathbb{N},\\ D_uf_t(\hat{x}_t,\hat{u}_t) &= -B_t^*\psi_t & \forall t\in\mathbb{N}_0, \end{aligned}$$

kde prvá rovnica opäť vyjadruje tzv. adjungovanú rovnicu.
V poslednej časti študujeme predošlý problém rozšírený o ohraničenia na stavovú premennú v tvare

$$u_t\in U_t=\{u\in\mathbb{R}^m : s_t(u)\le 0\},\quad \forall t\in\mathbb{N}_0,$$

kde $s_t\in C^1(\mathbb{R}^m,\mathbb{R}^{m_t})$ pre všetky $t\in\mathbb{N}_0$. Navyše množina U_t je konvexná uzavretá podmnožina množiny U. Rovnako ako v predošlom prípade môžeme použiť vetu 2.3.1 na identifikáciu všetkých smerov, v ktorých existuje perturbačná krivka spĺňajúca stavovú rovnicu. Navyše chceme, aby takáto perturbácia patrila aj do množiny U_t pre každé t. Ak pre každý čas $t\in\mathbb{N}_0$ označíme

$$\begin{aligned} I_t(\hat{u}_t) &= \{k\in\{1,\dots,m_t\} : s_t^k(\hat{u}_t)=0\},\\ \tilde{s}_t(\hat{u}_t) &= \{s_t^k(\hat{u}_t)\}_{k\in I_t(\hat{u}_t)},\\ \delta U_t(\hat{u}_t) &= \{\eta\in\mathbb{R}^m : D_u\tilde{s}_t(\hat{u}_t)\eta\le 0\},\\ L_t &= \delta U_t(\hat{u}_t), \end{aligned}$$

tak pre množinu všetkých kriviek, v ktorých existuje perturbácia spĺňajúca ohraničenia na riadenie, platí

$$(\boldsymbol{\xi},\boldsymbol{\eta})\in\mathbf{L}=l_1^n\times(l_1^m\cap(L_0\times L_1\times L_2\times\cdots)).$$

Takže každý smer, v ktorom existuje prípustná perturbácia musí spĺňať

$$(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbf{L} \cap \mathbf{K}, \tag{2.30}$$

kde $\mathbf{K} = \mathcal{N}(\iota_0, (\boldsymbol{\sigma} - \tilde{\mathbf{A}}, -\tilde{\mathbf{B}}))$. Navyše v takomto smere účelová funkcia klesá $DJ(\hat{\mathbf{x}}, \hat{\mathbf{u}})(\boldsymbol{\xi}, \boldsymbol{\eta}) \leq 0$. Tieto rovnice vieme prepísať v zmysle definície polárneho kužeľa ako

$$DJ(\hat{\mathbf{x}}, \hat{\mathbf{u}})(\boldsymbol{\xi}, \boldsymbol{\eta}) \leq 0 \ \ \forall (\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbf{K} \cap \mathbf{L} \ \Rightarrow \ DJ(\hat{\mathbf{x}}, \hat{\mathbf{u}}) \in (\mathbf{K} \cap \mathbf{L})^{\circ}.$$

Ak je splnená niektorá z postačujúcich podmienok (ako je uvedená v príklade 10), môžeme posledný vzťah napísať v tvare

$$DJ(\hat{\mathbf{x}}, \hat{\mathbf{u}}) \in (\mathbf{K} \cap \mathbf{L})^{\circ} = \mathbf{K}^{\circ} + \mathbf{L}^{\circ} = \mathcal{R}((\iota_0, (\sigma - \tilde{\mathbf{A}}, -\tilde{\mathbf{B}}))^*) + \mathbf{L}^{\circ}.$$

Tým pádom znova dostávame nutné podmienky optimality.

$$\begin{aligned} D_x f(\hat{x}_t, \hat{u}_t) &= \psi_{t-1} - A_t^* \psi_t \ \ \forall t \in \mathbb{N}, \\ D_u f(\hat{x}_t, \hat{u}_t) &= -B_t^* \psi_t + q_t \ \ \forall t \in \mathbb{N}_0, \end{aligned}$$

kde $q_t \in [\delta U_t(\hat{u}_t)]^{\circ}$ pre všetky $t \in \mathbb{N}_0$ a $\boldsymbol{\psi} \in (l_1^n)^* = l_\infty^n$.

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