COMENIUS UNIVERSITY IN BRATISLAVA
Faculty of Mathematics, Physics and Informatics

Optimal Portfolio Management in Incomplete Markets

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Mathematics of Economics and Finance

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Declaration

I declare that all parts of this thesis have been written by myself and that I have only used references explicitly referred to in the text.

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Abstract

Mean-Variance hedging presents a way to evaluate total hedging error in incomplete markets. We use this framework to define and derive an approximate formula for covariance between european options in a matrix form. Using this result, we compare existing hedging strategies used by market-makers (gamma strategy) with a strategy based on mean-variance preferences. The covariance matrix derived in previous step is used as a way to express the volatility of portfolio. Total performance of both trading strategies is compared using monte carlo simulation that provides approximate distributions of their respective profits.

Keywords: mean-variance hedging, covariance matrix, monte carlo simulation, quadratic programming, expected utility maximization

Abstrakt

Kvadratické zaisťovanie predstavuje možný prístup na určenie zaisťovacej chyby na nekompletnom trhu. Tento rámev používame na definíciu a odvodenie aproximácie kovariancie európskych opcií v maticovej forme. Použitím tohto výsledku porovnávame bežne používané zaisťovacie stratégie so stratégiou založenou na kvadratických preferenciách. Kovariančnú maticu odvodenú v predošlom kroku používame ako prostriedok na vyjadrenie volatility portfólia. Celkovú výkonnosť oboch stratégii porovnávame pomocou monte carlo simulácie, ktorá poskytuje aproximatívnu distribúciu ziskov z nich.

Kľúčové slová: kvadratické zaisťovanie, kovariančná matica, monte carlo simulácia, kvadratické programovanie, maximalizácia očakávanej užitočnosti

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Introduction

Derivative securities form an important aspect of financial markets. While previously used mainly as a form of insurance, we can see an increase in their trade by speculators who seek opportunities like arbitrage or so called good deals. Since their establishment as financial instruments, derivative securities have come a long way to the stage in which we know them today. While many aspects have pushed the theory behind them forward, the most crucial and famous article in this field was probably the paper written by Fischer Black and Myron Scholes in 1973 [3].

Although their model explains some characteristics that a given no-arbitrage priced derivative security should posess, it is today still considered mainly as a benchmark for other more complicated methods, mostly due to its robustness. The main reason why the result of Black and Scoles cannot be taken too seriously in the real world is the fact that its underlying assumptions do not necessarily hold at all times. For example, the underlying asset does not have to follow the Brownian motion but tends to jump now and then instead, or there might be transaction costs when trading in the derivative security.

In order to look into the problem more deeply, one can lose some of the Black-Scholes assumptions and watch what happens afterwards. For example, we may assume that the market is in fact not complete, e.g. that not every derivative can be hedged perfectly. In this case, we always have to expect some possible variance in the hedging portfolio that may occure when trading in the underlying asset.

Different methods have been introduced in order to evaluate risk that stems from the market incompleteness. The mean-variance hedging belongs to common approaches in this area. The concept was introduced for the first time by Hans Föllmer and Martin Schweizer in 1988 [8]. The whole idea is a minimization of the squared hedging error which is the difference between the terminal payoff of the derivative and the value of self-financing hedging portfolio.

Aim of this thesis is twofold. Firstly, we want find a way to define a covariance matrix for a given portfolio of hedged positions in terms of the mean-variance hedging theory. Since the computation of such matrix is too extensive using the classical theory, we intend to derive an approximation that reduces the computational time to a minimum. We derive and create this matrix for a portfolio of hedged european options where the hedging is done in the mean-variance sense.

And secondly, after this structure is created, we use it to define a particular option trading strategy that assumes mean-variance preferences of investor. More precisely, we present an alternative to commonly used techniques such as delta hedging or gamma hedging. We also simulate the underlying price process as a geometric Brownian motion in order to explore the efectiveness of our strategy in comparison with delta and gama hedging using monte carlo simulation.

The organization of thesis looks as follows. First chapter introduces the concept of mean-variance hedging and explains the derivation of a particular hedging strategy together with its hedging error. The second chapter defines a covariance between two hedged positions and uses this definition to create an approximative covariance matrix for a given portfolio of hedged european options. And finally, the third chapter uses this concept to present a particular trading strategy and evaluate its efectiveness with respect to the gamma hedging strategy. Fourth chapter summarizes the results and some proofs and lemmas are mentioned in the last chapter.

Chapter 1

Mean-Variance Hedging Strategies

This chapter serves as an introduction to the theory of mean-variance hedging. We present notation and assumptions used throughout the thesis and explain the difference between two different hedging strategies. This introduction is almost completely based on the paper from Černý and Kallsen [7]. Under the term mean-variance hedging we understand a problem of minimizing the squared difference between the derivative's payoff and the value of self-financing replicating portfolio.

Our goal is to solve the following problem:

$$\inf_{\vartheta} \mathrm{E}[(\nu + \vartheta \cdot S_T - H)^2], \tag{1.1}$$

where ν is an admissible initial endowment, ϑ is an admissible trading strategy, S is the price of underlying asset, H is a derivative we are going to hedge and $\vartheta \cdot S_T$ stands for our gains from trading in the time interval $[0, T]$. This approach contrasts with the local risk minimization in a sense that it minimizes the total hedging error.

1.1 Notation and Assumptions

Consider trading on a time horizon $T \in \mathbb{N}$ and the set of trading dates $\tau := 0, 1, \ldots, T$. We fix a probability space $(\Omega, P, \mathcal{F})$ a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \in \tau}$, $\mathcal{F}_T = \mathcal{F}$ and an $\mathcal{F}_T$ - measurable contingent claim $H \in L^2(P)$. We use the following notation for conditional expectations for a given random variable $X : \Omega \to \mathbb{R}$:

$$\mathrm{E}_t[X] := \mathrm{E}[X|\mathcal{F}_t], \tag{1.2}$$

$$\mathrm{Var}_t[X] := \mathrm{E}_t[X^2] - (\mathrm{E}_t[X])^2 \tag{1.3}$$

The discounted stock price process $\{S_t\}_{t \in \mathcal{T}}$ is adapted to $\mathbb{F}$ and we assume that $\forall t$: S_t is locally square-integrable meaning that for $\Delta S_{t+1} := S_{t+1} - S_t$ we obtain

$$\mathrm{E}_t[(\Delta S_{t+1})^2] < \infty \qquad \forall t < T \tag{1.4}$$

Definition 1. *We say that process S admits no arbitrage if $\forall t \in \mathcal{T} \backslash \{0\}$ and all $\mathcal{F}_{t-1}$ measurable portfolios ϑ_t we have that $\vartheta \Delta S_t \geq 0$ almost surely implies $\vartheta \Delta S_t = 0$ almost surely.*

Definition 2. *We say that (ν, ϑ) is an admissible endowment - strategy pair if and only if ν is $\mathcal{F}_0$-measurable, $\vartheta = \{\vartheta_t\}_{t \in \mathcal{T} \backslash \{0\}}$ is predictable and*

$$\nu + \vartheta \cdot S_T := \nu + \sum_{t=1}^{T} \vartheta_t \Delta S_t \in L^2(P). \tag{1.5}$$

The set of admissible strategies is denoted by $\overline{\Theta(\nu)}$ (depending on the initial endowment). We use $\overline{\Theta}$ as a shorthand for $\overline{\Theta(0)}$.

1.2 Hedging strategies

There are two hedging strategies presented in [6], the locally optimal and the globally optimal one. The globally optimal strategy solves (1.1). It computes the hedging error we obtain when hedging dynamically, while the local hedging error is obtained when using the locally optimal strategy. The whole idea behind the local risk minimization approach is about pretending that we are able to choose the value of self-financing portfolio arbitrarily and that it does not depend on previous trading.

In other words, unlike the locally optimal strategy, the dynamically optimal strategy is path-dependent, meaning that in every time we need to know not only the asset price and the time remaining to maturity, but the value of replicating portfolio as well. And according to fact that this value depends on the previous trading, it may be computationally too demanding to use the globally optimal strategy. The locally optimal strategy does not take into account the actual value of self-financing portfolio. It behaves as if we have so far traded in such a way that its value equals the value of derivative we are trying to hedge. However, this property is not necessarily satisfied in incomplete markets. On the other hand, the globally optimal strategy takes into account the value of self-financing portfolio and adjusts the amount of stock we are buying by its difference from the optimal value (which is the price of derivative we are hedging).

The locally optimal strategy was for the first time introduced in an influential paper from Föllmer and Schweizer [8]. They explain that the result of Black and Scholes is simply a special case of sequential regressions that minimizes the local one-period hedging error. More specifically, they introduced a way to handle the option hedging in the case of incomplete markets and that their result actually reduces to the Black-Scholes formula if the market becomes complete and if we allow for continuous hedging.

1.2.1 Derivation of hedging strategies

In the following section, we are going to introduce explicit formulae for both locally and globally optimal strategy with restriction to a case when returns are IID. It is possible to find a solution for a more general non-IID case as well using so-called opportunity neutral measure, but for the sake of keeping things simple we are not going to mention it here. Further details about this extended approach can be found in [7].

Suppose that $S_0 > 0$ and that $\{R_t\}_{t \in \mathcal{T}}$ are IID random variables with finite second moment such that $R_t > 0$ almost surely. Define

$$S_t := S_0 \prod_{j=1}^{t} R_j \quad \forall t \geq 1. \tag{1.6}$$

- **Locally Optimal Strategy**

  In the case of local risk minimization, the problem can be formulated as follows:

$$\{V_{t-1}, \xi_t\} := \arg \min_{\nu_{t-1}, \vartheta_t} \{\mathrm{E}_{t-1}[(\nu_{t-1} + \vartheta_t \Delta S_t - V_t)^2]\} \tag{1.7}$$

$$V_T := H, \tag{1.8}$$

  where ν_{t-1} and ϑ_t are $\mathcal{F}_{t-1}$-measurable. If we look at the problem as if it were a least squares regression, we can see that the solution looks as follows:

$$V_{t-1} = \mathrm{E}^Q_{t-1}[V_t] \tag{1.9}$$

$$\xi_t = \frac{\mathrm{Cov}_{t-1}(V_t, \Delta S_t)}{\mathrm{Var}_{t-1}(\Delta S_t)} = \frac{\mathrm{E}_{t-1}\left[(V_t - V_{t-1})\Delta S_t\right]}{\mathrm{E}_{t-1}\left[(\Delta S_t)^2\right]}, \tag{1.10}$$

  where Q stands for the minimal martingale measure which is defined in a following way with respect to the objective real-world measure P:

$$\frac{\mathrm{d}Q}{\mathrm{d}P} := \prod_{t=1}^{T} \frac{1 - \tilde{\lambda}_t \Delta S_t}{1 - \Delta \tilde{K}_t}. \tag{1.11}$$

  The quantities $\tilde{\lambda}_t$ and $\Delta \tilde{K}_t$ are in fact the coefficients and the sum of explained squares from an auxiliary regression of the constant onto the explanatory variable ΔS_t,

$$\tilde{\lambda}_t := \arg \min_{\vartheta_t \in \mathbb{R}} \mathrm{E}\left[(\vartheta_t \Delta S_t - 1)^2\right] = \frac{\mathrm{E}_{t-1}\left[\Delta S_t\right]}{\mathrm{E}_{t-1}\left[(\Delta S_t)^2\right]} \tag{1.12}$$

$$\Delta \tilde{K}_t := \frac{\left(\mathrm{E}_{t-1}\left[\Delta S_t\right]\right)^2}{\mathrm{E}_{t-1}[(\Delta S_t)^2]} = 1 - \mathrm{E}_{t-1}\left[(1 - \tilde{\lambda}_t \Delta S_t)^2\right] \tag{1.13}$$

- **Globally Optimal Strategy**

  In the case of dynamically optimal strategy, we are trying to minimize the expected squared replication error at maturity where the expectation is taken at time zero. Denote the dynamically optimal delta by $\phi(\nu) = \{\phi_t(\nu)\}_{t\in\mathcal{T}}$. The problem can therefore be written in a following way:

$$\min_{\vartheta} \mathrm{E}\left[(G_T^{\nu,\vartheta} - V_T)^2\right], \tag{1.14}$$

  where $V_T := H$ and the self-financing condition $G_{t+1}^{\nu,\phi(\nu)} = G_t^{\nu,\phi(\nu)} + \vartheta_{t+1}\Delta S_{t+1}$ is required to hold throughout the whole process. The actual hedging strategy depends on initial wealth ν. We also suppress its dependence on H because we treat the derivative's terminal payoff as if it were constant.

  After plugging the self financing condition into (1.14), we obtain that the globally optimal delta should solve

$$\phi_t(\nu) = \arg\min_{\vartheta} \mathrm{E}_{t-1}\left[(G_{t-1}^{\nu,\phi(\nu)} + \vartheta_t\Delta S_t - V_t)^2\right]. \tag{1.15}$$

  Similarly to the locally optimal case, this can be viewed as a least squares regression. The only difference is that this time we do not include intercept because of the self-financing condition which already implies a particular value for $G_{t-1}^{\nu,\phi(\nu)}$ (i.e. it is already given by our previous trading in the stock). Therefore now we have a regression with dependent variable $V_t - G_{t-1}^{\nu,\phi(\nu)}$, explanatory variable ΔS and with no intercept. This implies the hedging strategy has to take on the following form:

$$\phi_t(\nu) = \frac{\mathrm{E}_{t-1}\left[(V_t - G_{t-1}^{\nu,\phi(\nu)})\Delta S_t\right]}{\mathrm{E}_{t-1}\left[(\Delta S_t)^2\right]} = \tag{1.16}$$

$$= \frac{\mathrm{E}_{t-1}\left[(V_t - V_{t-1})\Delta S_t\right]}{\mathrm{E}_{t-1}\left[(\Delta S_t)^2\right]} + \tilde{\lambda}_t(V_{t-1} - G_{t-1}^{\nu,\phi(\nu)}) = \tag{1.17}$$

$$= \xi_t + \tilde{\lambda}_t(V_{t-1} - G_{t-1}^{\nu,\phi(\nu)}). \tag{1.18}$$

  We have used (1.10) in the last equality.

As we can see, the globally optimal strategy is always more precise in finding a solution to the dynamical problem, because what we are actually trying to find is a strategy minimizing total hedging error, not only one-period errors. So the question remains: Why should we even mention the locally optimal strategy, if the global strategy behaves always better?

The reason is simple: If we compare the two strategies according to the expected hedging error they imply, the locally optimal strategy behaves only slightly worse in comparison to the dynamically optimal strategy, as showed in [6]. According to fact that the locally optimal strategy is easier to implement computationally, it is obvious why we mentioned it here. The next section will compare the hedging errors incurred from both strategies more thoroughly.

1.2.2 Hedging Errors

This section provides formulae for both strategies.

- **Locally Optimal Strategy**
  In order to simplify the derivation, denote by $e_t := V_{t-1} + \xi_t \Delta S_t - V_t$ the one-period locally optimal hedging error.

  Denote by ψ_t the following expression:

  $$\psi_t := \mathrm{E}_{t-1}[e_t^2] = \mathrm{Var}_{t-1}[V_t] - \xi_t \mathrm{Cov}_{t-1}[\Delta S_t, V_t]. \tag{1.19}$$

  Set $V_T := H$ and suppose that the value of our portfolio at time t is defined as a sum of initial wealth and the profit (loss) incurred from trading up to the time t:

  $$G_t^{\nu,\xi} := \nu + \xi \cdot S_t. \tag{1.20}$$

  If we apply the self-financing condition $G_{t+1}^{\nu,\xi} = G_t^{\nu,\xi} + \xi_{t+1}\Delta S_{t+1}$, we get the expected hedging error in a following form:

$$\mathrm{E}\left[(G_T^{\nu,\xi} - V_T)^2\right] = \mathrm{E}\left[\mathrm{E}_{T-1}\left[(G_T^{\nu,\xi} - V_T)^2\right]\right] = \tag{1.21}$$

$$= \mathrm{E}\left[\mathrm{E}_{T-1}\left[(G_{T-1}^{\nu,\xi} - V_{T-1} + V_{T-1} + \xi_T \Delta S_T - V_T)^2\right]\right] = \tag{1.22}$$

$$= \mathrm{E}\left[\mathrm{E}_{T-1}\left[\underbrace{(G_{T-1}^{\nu,\xi} - V_{T-1})^2}_{T-1 \text{ measurable}} + \right.\right. \tag{1.23}$$

$$+ 2\underbrace{(G_{T-1}^{\nu,\xi} - V_{T-1})}_{T-1 \text{ measurable}}\underbrace{(V_{T-1} + \xi_T \Delta S_T - V_T)}_{\mathrm{E}(.)\text{equals zero}} + \tag{1.24}$$

$$\left.\left. + \underbrace{(V_{T-1} + \xi_T \Delta S_T - V_T)^2}_{e_T^2}\right]\right] = \tag{1.25}$$

$$= \mathrm{E}\left[(G_{T-1}^{\nu,\xi} - V_{T-1})^2 + \psi_t\right]. \tag{1.26}$$

Now it is obvious that after repeating this T times, we get the following result:

$$\mathrm{E}\left[(G_T^{\nu,\xi} - V_T)^2\right] = (\nu - V_0)^2 + \sum_{t=1}^{T} \mathrm{E}[\psi_t]. \tag{1.27}$$

- **Globally Optimal Strategy**

  Analogically to the previous case, we use the property of zero expected value of one-period hedging error. Consequently we can express the expected value of one-period squared hedging error as follows:

$$\mathrm{E}_{t-1}\left[(G_{t-1}^{\nu,\phi(\nu)} + \phi_t(\nu)\Delta S_t - V_t)^2\right] = (1 - \Delta\tilde{K}_t)(G_{t-1}^{nu,\phi(\nu)} - V_{t-1})^2 + \psi_t \tag{1.28}$$

At this point, we can define process L_t in a following way:

$$L_t = \prod_{j=t+1}^{T} (1 - \Delta\tilde{K}_j), \qquad L_T = 1 \tag{1.29}$$

Due to fact that L is deterministic, we can express the total hedging error in a global strategy case similarly to the previous local strategy case:

$$\mathrm{E}\left[(G_T^{\nu,\phi(\nu)} - V_T)^2\right] = L_0(\nu - V_0)^2 + \sum_{t=1}^{T} \mathrm{E}[L_t \psi_t]. \tag{1.30}$$

Now we can see why it was convenient to work with returns that are IID, so that L remains deterministic. In case we would like to explore model with non-IID one period returns, we could do it by changing the measure. Černý and Kallsen define this measure as a so-called "opportunity neutral measure". Further details can be found in [7].

Chapter 2

Covariance matrix

In this chapter, I would like to derive an approximate formula for the covariance between european options where under the term "covariance between options", we actually mean the covariance between their hedged positions. Aim of this whole chapter is to find a formula that expresses the covariance between two european options (call or put, it does not matter as we will see later) in the mean-variance hedging framework. After that, we are able to create a covariance matrix for a given portfolio of their hedged positions.

So far, we have spoken about two different types of hedging error. One of them was a result of dynamical hedging strategy while the other one was a result of locally optimal hedging strategy. We will continue to work with the local hedging error as it performs only slightly worse in comparison with its dynamical counterpart.

2.1 Definition

First, we need to define the covariance between two options properly (or between their hedged positions to be more precise). Following is a well known identity between two random variables X and Y.

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y). \tag{2.1}$$

From this, we can see that a covariance should generally satisfy:

$$\mathrm{Cov}(X,Y) = \frac{\mathrm{Var}(X+Y) - \mathrm{Var}(X) - \mathrm{Var}(Y)}{2}. \tag{2.2}$$

If we assume that the variance of a given hedged position is in fact its squared hedging error, then the covariance between two positions of european options should satisfy the following definition:

Definition 3 (Covariance)**.** *Suppose we have two positions, X_t and Y_t at a given time $t \in [0,T]$, and that both of them consist from holding one european option and ξ_{X_t} (ξ_{Y_t}) underlying assets, where ξ_X and ξ_Y are their respective mean-variance hedging strategies. We define the covariance between these two positions as follows:*

$$\mathrm{Cov}(X_t,Y_t) = \frac{\epsilon^2_{0X+Y} - \epsilon^2_{0X} - \epsilon^2_{0Y}}{2} \tag{2.3}$$

where under ϵ we understand the mean-variance hedging error and the subscripts X, Y and $X+Y$ mean that we are working with terminal payoffs of options from positions X, Y and the sum of their payoffs $X+Y$ respectively. We may use a term "covariance of options" in the the rest of the thesis. In that case, we always mean a covariance of their hedged position where the hedging is done according to the mean-variance theory from chapter one.

This way, we can define covariance using the theory behind mean-variance hedging. In other words, all we need to do is to take the formulas derived in previous section and plug them into (2.3) to obtain the desired covariance. However, it would be computationaly too extensive which is the reason why we intend to derive an approximation.

2.2 Approximation I

Following lines can be found in [6] for a case of a single derivative. We extend this analysis further. Černý shows in his book that the hedging error of a certain derivative can be approximated in a following way using the Black-Scholes model:

$$\epsilon_0^2 \approx \left(\frac{\text{Kurt}(R) - 1}{2}\right) \frac{1}{2}(\text{Var}(R))^2 \sum_{t=0}^{T-1} R_f^{2(T-t-1)} \text{E}[(\gamma_t S_t^2)^2] \qquad (2.4)$$

where γ_t is a gamma of the derivative, R is a random one period return and R_f is a risk free rate. In order to evaluate this expression, one needs to find the value of $\text{E}[(\gamma_t S_t^2)^2]$. For european options this problem can be solved analytically as we will see in the following lines. Toft[13] has already done so in a case of a single derivative. However, we will have to derive a more general formula. It is now also clear that in case of european options there is no need to distinguish between put and call options simply because their gamma is always the same, no matter what type of option we are talking about (see (5.0.3)). Since there is nothing else except gamma in the expression that could distinguish different types of options from each other, it remains the same for both put and call options.

After plugging (2.4) into (2.3), the covariance equals:

$$
\begin{aligned}
\mathrm{Cov}(X,Y) &= \frac{\mathrm{Var}(X+Y)-\mathrm{Var}(X)-\mathrm{Var}(Y)}{2} = \\
&= \frac{\epsilon^2_{0X+Y}-\epsilon^2_{0X}-\epsilon^2_{0Y}}{2} = \\
&= \frac{1}{2}\Bigg[\left(\frac{\mathrm{Kurt}(R)-1}{2}\right)\frac{1}{2}(\mathrm{Var}(R))^2\sum_{t=0}^{T-1}R_f^{2(T-t-1)}\mathrm{E}[(\gamma_{X+Y\,t}S_t^2)^2] - \\
&\quad - \left(\frac{\mathrm{Kurt}(R)-1}{2}\right)\frac{1}{2}(\mathrm{Var}(R))^2\sum_{t=0}^{T-1}R_f^{2(T-t-1)}\mathrm{E}[(\gamma_{X\,t}S_t^2)^2] - \\
&\quad - \left(\frac{\mathrm{Kurt}(R)-1}{2}\right)\frac{1}{2}(\mathrm{Var}(R))^2\sum_{t=0}^{T-1}R_f^{2(T-t-1)}\mathrm{E}[(\gamma_{Y\,t}S_t^2)^2]\Bigg] = \\
&= \frac{1}{2}\Bigg[\left(\frac{\mathrm{Kurt}(R)-1}{2}\right)\frac{1}{2}(\mathrm{Var}(R))^2\,\cdot \\
&\quad \cdot\sum_{t=0}^{T-1}R_f^{2(T-t-1)}\left(\mathrm{E}[(\gamma_{X+Y\,t}S_t^2)^2]-\mathrm{E}[(\gamma_{X\,t}S_t^2)^2]-\mathrm{E}[(\gamma_{Y\,t}S_t^2)^2]\right)\Bigg] = \\
&= \frac{1}{2}\Bigg[\left(\frac{\mathrm{Kurt}(R)-1}{2}\right)\frac{1}{2}(\mathrm{Var}(R))^2\,\cdot \\
&\quad \cdot\sum_{t=0}^{T-1}R_f^{2(T-t-1)}\left(\mathrm{E}[((\gamma_{X\,t}+\gamma_{Y\,t})S_t^2)^2]-\mathrm{E}[(\gamma_{X\,t}S_t^2)^2]-\mathrm{E}[(\gamma_{Y\,t}S_t^2)^2]\right)\Bigg] = \\
&= \left(\frac{\mathrm{Kurt}(R)-1}{2}\right)\frac{1}{2}(\mathrm{Var}(R))^2\sum_{t=0}^{T-1}R_f^{2(T-t-1)}\left(\mathrm{E}[\gamma_{X\,t}\gamma_{Y\,t}S_t^4]\right)
\end{aligned}
$$

In the process of derivation we have used lemma (5.0.1). As we can see, the problem reduces to a simple issue of evaluating expression

$$\mathrm{E}[\gamma_{X\,t}\gamma_{Y\,t}S_t^4] \tag{2.5}$$

since distribution of one period return as well as the risk free rate are both known to us.

2.3 Evaluation of $\mathrm{E}[\gamma_{Xt}\gamma_{Yt}S_t^4]$

This particular issue has already been solved in [13] by Klaus Bjerre Toft, however only for a case where $\gamma_X = \gamma_Y$. As our case is more complicated and the solution more complex, we will derive it in the following lines. Lemmata (5.0.2) and (5.0.3) will be used in the process, see appendix for more details. Density function of normalized normal distribution is denoted by $\phi(.)$.

$$\mathrm{E}[\gamma_{Xt}\gamma_{Yt}S_t^4] = \frac{1}{\sigma^2(T-t)}\mathrm{E}\left[\phi(d_{1X})\phi(d_{1Y})S^2\right] =$$

$$= \frac{S_0^2}{\sigma^2(T-t)}\mathrm{E}\left[\frac{S^2}{S_0^2}\phi\left(\frac{\overbrace{\ln\frac{S}{S_0}}^{=:\mathrm{X}} + \overbrace{\ln\frac{S_0}{K_X} + (r+\frac{\sigma^2}{2})(T-t)}^{=:a_X}}{\sigma\sqrt{T-t}}\right)\cdot\right.$$

$$\left.\cdot\,\phi\left(\frac{\overbrace{\ln\frac{S}{S_0}}^{=:\mathrm{X}} + \overbrace{\ln\frac{S_0}{K_Y} + (r+\frac{\sigma^2}{2})(T-t)}^{=:a_Y}}{\sigma\sqrt{T-t}}\right)\right] =$$

$$= \frac{S_0^2}{\sigma^2(T-t)}\mathrm{E}\left[e^{2x}\phi\left(\frac{x+a_1}{\sigma\sqrt{T-t}}\right)\phi\left(\frac{x+a_2}{\sigma\sqrt{T-t}}\right)\right] =$$

$$= \frac{S_0^2}{\sigma^2(T-t)}\mathrm{E}\left[e^{2x}\frac{1}{2\pi}e^{-\frac{1}{2}\left(\frac{(x+a_1)^2+(x+a_2)^2}{\sigma^2(T-t)}\right)}\right] =$$

$$= \frac{S_0^2}{2\pi\sigma^2(T-t)}\mathrm{E}\left[\exp\left(-\frac{1}{2}\left(\frac{(x+a_1)^2+(x+a_2)^2}{\sigma^2(T-t)}\right)+2x\right)\right] =$$

$$= (*)$$

So far, we have only simplified the expression using explicit formulae for gamma of european call and put option. We have defined a new variable $X = \ln\frac{S}{S_0}$. Since our approximation is based on the Black-Scholes model, it holds that $X_t \sim \mathcal{N}(\mu t - \frac{\sigma^2}{2}t, \sigma^2 t)$ due to the character of S (which behaves as a geometric Brownian motion and therefore has a lognormal distribution at maturity). Using the fact that X has normal distribution, we can continue in the derivation:

$$(*) = \frac{S_0^2}{2\pi\sigma^2(T-t)}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2 t}}\cdot$$

$$\cdot\exp\left[-\frac{1}{2}\left(\frac{(x+a_1)^2+(x+a_2)^2}{\sigma^2(T-t)}\right)+2x-\frac{(x-\overbrace{(\mu t-\frac{\sigma^2}{2}t)}^{=:\,\tilde{\mu}t})^2}{2\sigma^2 t}\right]\mathrm{d}x=$$

$$=\frac{S_0^2}{(2\pi)^{\frac{3}{2}}\sigma^3\sqrt{t}(T-t)}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\left(\frac{2x^2+2(a_1+a_2)x+a_1^2+a_2^2}{\sigma^2(T-t)}\right)+\right.$$

$$\left.+2x-\frac{x^2-2\tilde{\mu}tx+\tilde{\mu}^2t^2}{2\sigma^2 t}\right]\mathrm{d}x=$$

$$=\frac{S_0^2}{(2\pi)^{\frac{3}{2}}\sigma^3\sqrt{t}(T-t)}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\left(\frac{1}{\sigma^4 t(T-t)}\right)\cdot\right.$$

$$\cdot\left(2\sigma^2tx^2+2(a_1+a_2)\sigma^2tx+(a_1^2+a_2^2)\sigma^2t-4\sigma^4t(T-t)x+\right.$$

$$\left.\left.+\sigma^2(T-t)x^2-2\sigma^2t(T-t)\tilde{\mu}x+\sigma^2t^2(T-t)\tilde{\mu}^2\right)\right]\mathrm{d}x=$$

$$=\frac{S_0^2}{(2\pi)^{\frac{3}{2}}\sigma^3\sqrt{t}(T-t)}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\left(\underbrace{\frac{1}{\sigma^4 t(T-t)}}_{=:k}\right)\cdot\right.$$

$$\cdot\left([\underbrace{2\sigma^2t+\sigma^2(T-t)}_{=:A}]x^2+\right.$$

$$+[\underbrace{2(a_1+a_2)\sigma^2t-4\sigma^4t(T-t)-2\sigma^2t(T-t)\tilde{\mu}}_{=:B}]x+$$

$$\left.\left.+\ [\underbrace{(a_1^2+a_2^2)\sigma^2t+\sigma^2t^2(T-t)\tilde{\mu}^2}_{=:C}]\right)\right]\mathrm{d}x=$$

$$=\frac{S_0^2}{(2\pi)^{\frac{3}{2}}\sigma^3\sqrt{t}(T-t)}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2k}\left(Ax^2+Bx+C\right)\right]=$$

$$=(**)$$

At this point, we have reduced the former expression to a more suitable form. Using (5.0.2), we are able to simplify this integral form even further. However, the derivation is too extensive and it would span about a dozen of pages, using only basic algebraic operations from now on that are not interesting in any way. Therefore, we omit it here and state the final form directly.

$$\begin{aligned}\mathrm{E}[\gamma_{X_t}\gamma_{Y_t}S^4] = (**) = &\frac{S_0^2 e^{2\mu t}}{2\pi\sigma^2\sqrt{T^2-t^2}} \exp\left[-\frac{1}{4\sigma^2(T^2-t^2)}\cdot\right. \\ &\cdot \big(2T(\lambda_X^2+\lambda_Y^2) + 2\sigma^2(\lambda_X+\lambda_Y)(T-t)(T+2t) - \\ &\left. -4t\lambda_X\lambda_Y + \sigma^4 T^2(T-t)\big)\right]\end{aligned} \tag{2.6}$$

$$\lambda_X = \ln\frac{S_0}{K_X} + r(T-t) + \mu t \tag{2.7}$$

$$\lambda_Y = \ln\frac{S_0}{K_Y} + r(T-t) + \mu t. \tag{2.8}$$

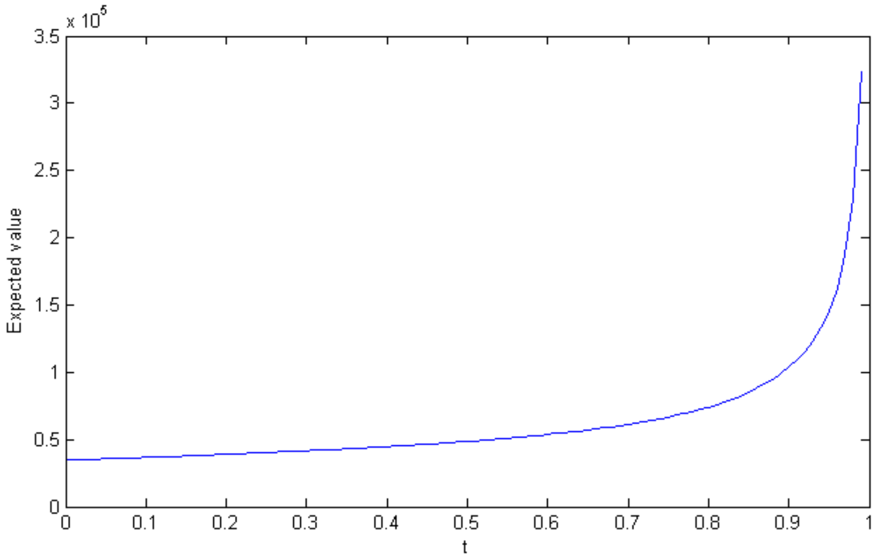
where r is the risk-free rate per unit of time, μ is the expected rate of return per unit of time, and σ^2 is the variance of log return per unit of time.

We have successfully derived explicit formula for $\mathrm{E}[\gamma_X\gamma_Y S^4]$. We have also tested it using monte carlo simulations to ensure that no errors were made during the long derivation. Two graphs are presented in figures (2.1) and (2.2), showing the character of $\mathrm{E}[\gamma_{X_t}\gamma_{Y_t}S^4]$ in two different scenarios (equal and different strike prices). We can see both from the figures and from the shape of the solution, that it is quantitatively different for equal and for different strike prices.

2.4 Approximation II

Now we are able to plug the result back to the original formula. However, few modifications will take place. Due to fact, that we are already approximating, there is no need for exactness in evaluating the sum that our expression

Figure 2.1: $\mathrm{E}[\gamma_{Xt}\gamma_{Yt}S^4]$ *computed for* $t \in [0, T]$ ($S = 100$, $T = 1$, $K_1 = 110$, $K_2 = 110$, $\mu = 0.05$, $\sigma = 0.2$, $r = 0$)



contains. Therefore, we modify the approximation in a following way:

$$\mathrm{Cov}(X,Y) \approx \left(\frac{\mathrm{Kurt}(R)-1}{2}\right)\frac{1}{2}(\mathrm{Var}(R))^2\sum_{t=0}^{T-1}R_f^{2(T-t-1)}\mathrm{E}[\gamma_{Xt}\gamma_{Yt}S_t^4] \approx$$

$$\approx \left(\frac{\mathrm{Kurt}(R)-3+2}{2}\right)\frac{1}{2}(\sigma^2\Delta t)^2\sum_{t=0}^{T-1}R_f^{2(T-t-1)}\mathrm{E}[\gamma_{Xt}\gamma_{Yt}S_t^4] =$$

$$= \left(\frac{\mathrm{Kurt}(R)-3}{2}\right)\frac{1}{2}\sigma^4(\Delta t)^2\sum_{t=0}^{T-1}R_f^{2(T-t-1)}\mathrm{E}[\gamma_{Xt}\gamma_{Yt}S_t^4] +$$

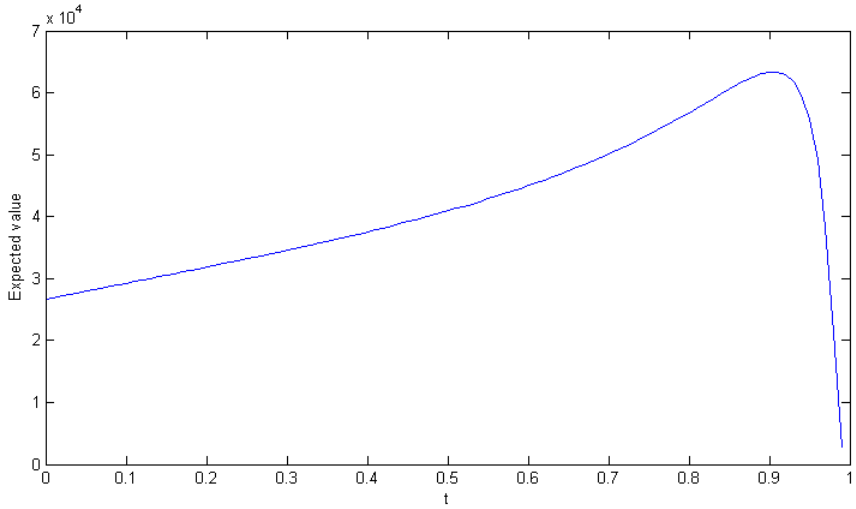
$$+ \frac{1}{2}\sigma^4(\Delta t)^2\sum_{t=0}^{T-1}R_f^{2(T-t-1)}\mathrm{E}[\gamma_{Xt}\gamma_{Yt}S_t^4] =$$

$$= \overbrace{(\mathrm{Kurt}(R)-3)(\Delta t)}^{\text{const. for } \Delta t \to 0}\frac{1}{4}\sigma^4\overbrace{\sum_{t=0}^{T-1}R_f^{2(T-t-1)}\mathrm{E}[\gamma_{Xt}\gamma_{Yt}S_t^4]\Delta t}^{\text{Definite integral for } \Delta t \to 0} +$$

$$\underbrace{+ \frac{1}{2}\sigma^4(\Delta t)^2\sum_{t=0}^{T-1}R_f^{2(T-t-1)}\mathrm{E}[\gamma_{Xt}\gamma_{Yt}S_t^4]}_{\text{This expression goes to zero for } \Delta t \to 0} =$$

$$= (\mathrm{ExKurt}(R))\frac{1}{4}\sigma^4\int_0^T R_f^{2(T-t-1)}\mathrm{E}[\gamma_{Xt}\gamma_{Yt}S_t^4]\mathrm{d}t$$

Figure 2.2: $\mathrm{E}[\gamma_{X_t}\gamma_{Y_t}S^4]$ *computed for* $t \in [0,T]$ ($S = 100$, $T = 1$, $K_1 = 100$, $K_2 = 110$, $\mu = 0.05$, $\sigma = 0.2$, $r = 0$)



Under the expression ExKurt(R), we understand the excess kurtosis:

$$\mathrm{ExKurt}(R) := \mathrm{Kurt}(R) - 3 \tag{2.9}$$

It can be shown that in Lévy model this expression goes to ∞ as $\Delta t \to 0$ by order $\frac{1}{\Delta t}$. Therefore if multiplied by Δt, it remains constant for sufficiently small Δt. We need it to be close to zero to lower the probability of price jumps. For further references see [6]. Analogically, we have approximated the sum by a definite integral after multiplying it by Δt and sending it to zero. For more details regarding this concept, refer to [2].

At this moment, we can use the result from (2.6) and insert it into our expression. After we do so, the task reduces to a problem of definite integral evaluation.

2.5 Numerical evaluation of definite integrals

Last key step that has to be completed in order to find the desired approximation is a numerical evaluation of the definite integral we have derived in previous sections. More precisely, we are going to find a value of

$$\int_0^T R_f^{2(T-t-1)} \mathrm{E}[\gamma_{X_t} \gamma_{Y_t} S_t^4] \mathrm{d}t \qquad (2.10)$$

with $\mathrm{E}[\gamma_{X_t} \gamma_{Y_t} S_t^4]$ defined previously in (2.6). After examining the convergence, one will obtain that in case of equal strike prices ($K_1 = K_2$), the value of function we are trying to integrate is going to ∞ with $t \to T$. This property can be easily seen also from figures (2.1) and (2.2). According to fact that we are going to approximate this integral numerically using adaptive Simpson quadrature [10] in MATLAB, it is neccessary to somehow solve this little inconvenience. Note that the formula for $K_1{=}K_2$ has already been derived in [13] and that this is the only case in which our expected value goes to infinity with time close to expiration date.

We are going to express (2.6) differently using the per partes integration such that the expression no longer contains a definite integral that goes to ∞. Obviously, the part in the exponential can be expressed as

$$\exp\left[-\frac{1}{4\sigma^2(T^2 - t^2)}\left(At^2 + Bt + C\right)\right], \qquad (2.11)$$

where A,B and C are some constants. Therefore after assuming continuous compound interest the whole expression (2.10) can be seen in a following way:

$$\int_0^T \mathrm{E}[\gamma_{X\,t}\gamma_{Y\,t}S_t^4]\mathrm{d}t = \tag{2.12}$$

$$= \int_0^T \frac{S_0^2 e^{\mu t}}{2\pi\sigma^2\sqrt{T^2-t^2}} e^{2r(T-t)}\cdot \tag{2.13}$$

$$\underbrace{\cdot\exp\left[-\frac{1}{4\sigma^2(T^2-t^2)}\left(At^2+Bt+C\right)\right]}_{=:f(t)}\mathrm{d}t = \tag{2.14}$$

$$= \frac{S_0^2 e^{2rT}}{2\pi\sigma^2}\int_0^T \frac{1}{\sqrt{T^2-t^2}} e^{2t(\mu-r)} f(t)\mathrm{d}t = \tag{2.15}$$

$$= \frac{S_0^2 e^{2rT}}{2\pi\sigma^2}\left[\arcsin\frac{t}{T} e^{2t(\mu-r)} f(t)\Big|_0^T - \right. \tag{2.16}$$

$$\left. - \int_0^T \arcsin\frac{t}{T} e^{2t(\mu-r)}(f'(t)+2f(t)(\mu-r))\mathrm{d}t\right] \tag{2.17}$$

Since $f(T)$ and $f(0)$ is finite for $K_1 = K_2$ as we can see in [13] or [6], the per partes integration changes the former problem to a problem of finite definite integral evaluation which we can be solved by using the same method we have used in case $K_1 \neq K_2$. The new function we need to integrate is depicted in figure 2.3 and as we can see it does not go to infinity like its previous counterpart.

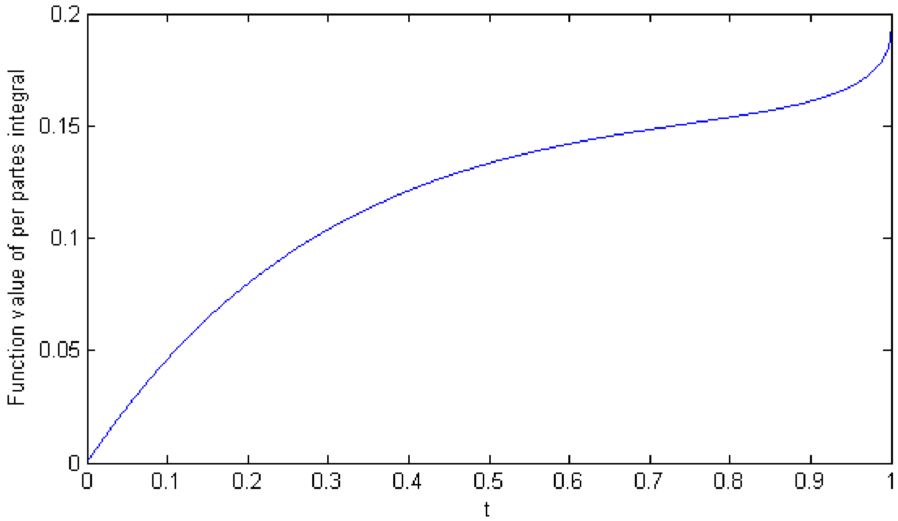
2.6 Final shape of covariance matrix

At this moment, we are finally able to create the covariance matrix for a given portfolio of european options. Previous sections were necessary both for obtaining the explicit formula as well as for explaining its further numerical approximation.

The final step we are going to do is expressing the volatility of a given portfolio in terms of covariance matrix.

Definition 4 (Covariance matrix)**.** *Suppose we have a portfolio of k call options and l put options. Denote the covariance of any two options X_i, X_j*

Figure 2.3: *Per partes new definite integral computed for* $t \in [0, T]$ $(S = 100, T = 1, K_1 = 110, K_2 = 110, \mu = 0.05, \sigma = 0.2, r = 0)$



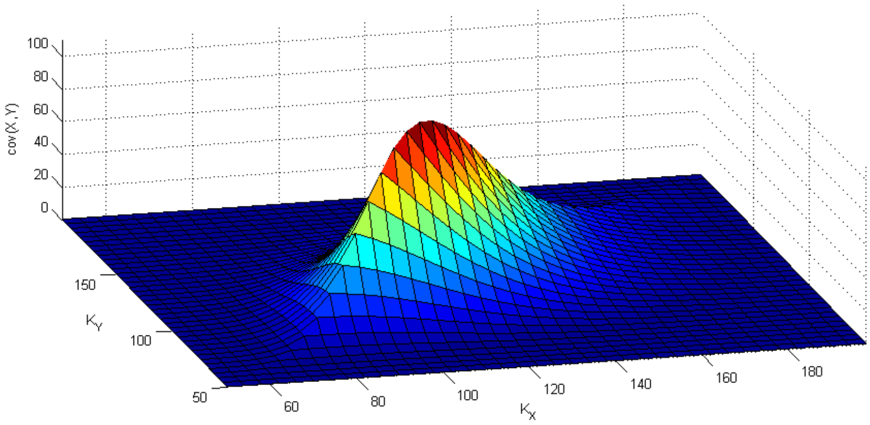
by: $\sigma_{ij}^2 := \text{Cov}[X_i, X_j]$ *where the covariance is meant in the sense described in previous sections. Consequently, the covariance matrix of our portfolio* Σ *is defined as follows:*

$$\Sigma := \begin{bmatrix} \sigma_{11}^2 & \sigma_{12}^2 & \dots & \sigma_{1k}^2 & \sigma_{1,k+1}^2 & \sigma_{1,k+2}^2 & \dots & \sigma_{1n}^2 \\ \sigma_{21}^2 & \sigma_{22}^2 & \dots & \sigma_{2k}^2 & \sigma_{2,k+1}^2 & \sigma_{2,k+2}^2 & \dots & \sigma_{2n}^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_{k1}^2 & \sigma_{k2}^2 & \dots & \sigma_{kk}^2 & \sigma_{k,k+1}^2 & \sigma_{k,k+2}^2 & \dots & \sigma_{kn}^2 \\ \sigma_{k+1,1}^2 & \sigma_{k+1,2}^2 & \dots & \sigma_{k+1,k}^2 & \sigma_{k+1,k+1}^2 & \sigma_{k+1,k+2}^2 & \dots & \sigma_{k+1,n}^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_{n1}^2 & \sigma_{n2}^2 & \dots & \sigma_{nk}^2 & \sigma_{n,k+1}^2 & \sigma_{n,k+2}^2 & \dots & \sigma_{nn}^2 \end{bmatrix} \quad (2.18)$$

where the first k *derivatives are call options and the rest are put options and for any* $i, j \in \{1, \dots, k\}$*:* $i < j \Rightarrow K_i < K_j$ *(analogically for any* $i, j \in \{k+1, \dots, n\}$*:* $i < j \Rightarrow K_i < K_j$*).*

We have defined the covariance matrix and explained its approximation.

Figure 2.4: *One of four identical blocks of covariance matrix computed for* $t = 0$, $S = 100$, $T = 1$, $\mu = 0.05$, $\sigma = 0.2$, $r = 0$. *X and Y axes represent the strike prices and Z axis represents the actual value of covariance between the respective options.*



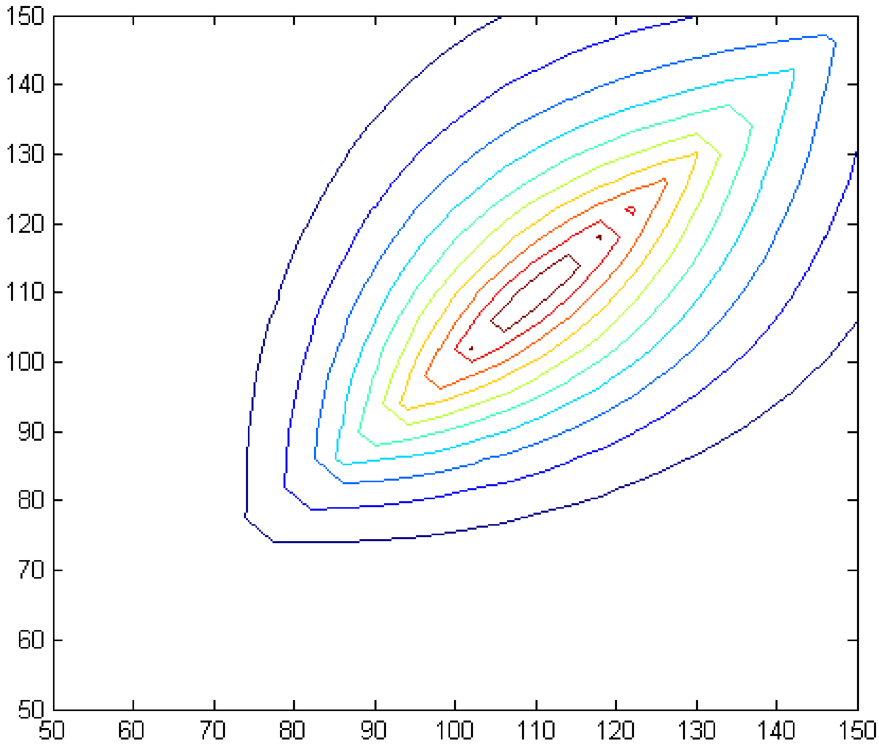
Note however, that the computational time used to evaluate it can be decreased dramatically if we assume the same strike prices for both call and put options (which is the case of real market data). At this moment, the matrix consists of four blocks, each of which corresponds to different pairs of options. In other words, we can rewrite it in the following block form :

$$\Sigma := \begin{bmatrix} \Sigma_{CC} & \Sigma_{CP} \\ \Sigma_{PC} & \Sigma_{PP} \end{bmatrix} \tag{2.19}$$

where Σ_{CC} and Σ_{PP} correspond to covariance matrices for call and put options respectively, while Σ_{CP} and Σ_{PC} stand for their combinations.

The restriction of equal strike prices for both types of options will help us significantly, since our matrix will attain a particular special form. More precisely, the four blocks that the matrix is currently formed from are identical in this special case. This follows directly from the way we have defined the matrix in the first place. Since there is the same number of calls and puts and their gammas do not differ, it is obvious that it does not matter what block we choose in case of equal strike prices, they are simply identical.

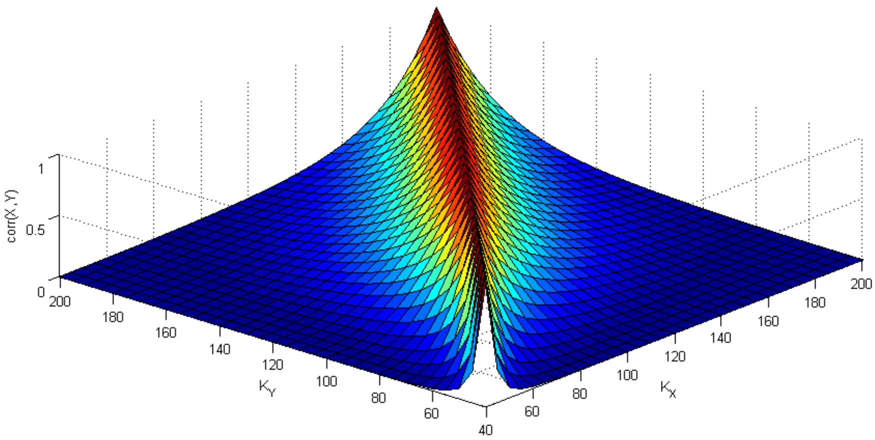
Figure 2.5: *One of four identical blocks of covariance matrix computed for $t = 0$, $S = 100$, $T = 1$, $\mu = 0.05$, $\sigma = 0.2$, $r = 0$. X and Y axes represent the strike prices. The colour of figure represents the level of covariance between respective options.*



However, this is not the only improvement we are able to achieve when trying to optimize our task. Obviously, thanks to the symmetry of covariance, all of these four equal blocks have to be symmetric as well. Therefore, it is not necessary to evaluate the whole block, the upper or lower half will do the same trick in almost a half of the former computation time.

In summary, we can reduce the computation time by exploiting the specific shape of covariance matrix. This way, instead of computing n^2 integrals numerically, we only need to compute $\frac{n^2}{8} + \frac{n}{4}$ of them. One of four identical

Figure 2.6: *One of four identical blocks of correlation matrix computed for $t = 0$, $S = 100$, $T = 1$, $\mu = 0.05$, $\sigma = 0.2$, $r = 0$. X and Y axes represent the strike prices and Z axis represents the actual value of correlation between the respective options.*

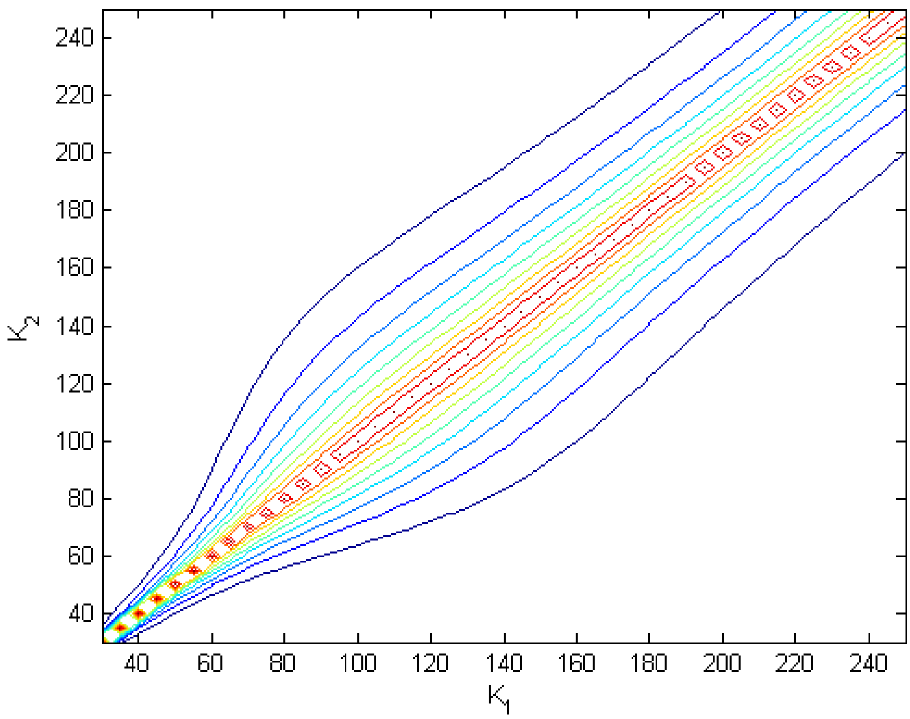


blocks that the covariance matrix is composed from can be seen in figures (2.4) and (2.5).

Also, for a better overview, we are including figures of correlation matrix, which is made analogically to the covariance matrix, with the only difference that every element is divided by the product of respective option variances (their squared hedging errors in our case). Graphs of this stucture are depicted in figures (2.6) and (2.7).

This whole chapter was supposed to bring some light in the definition and derivation of covariance matrix for european options. We have first defined it properly, using results from mean-variance hedging theory. Afterwards, we have used some known approximations and derived a formula that expresses a key part of one of them. At last, we have depicted the final shape in a figure and for a better understanding, we have also included a closely related correlation matrix. This whole concept will be used in the next chapter to create a trading strategy that can be compared with some already used approaches.

Figure 2.7: *One of four identical blocks of correlation matrix computed for* $t = 0$, $S = 100$, $T = 1$, $\mu = 0.05$, $\sigma = 0.2$, $r = 0$. *X and Y axes represent the strike prices. The colour of figure represents the level of correlation between respective options.*



Chapter 3

Portfolio Management

This chapter introduces a stragety for portfolio management in incomplete markets. It is based on the results from previous chapters, most importantly on the covariance matrix concept. We compare existing hedging strategies used commonly by market makers with a mean-variance hedging strategy using simulated stock prices in order to evaluate their effectiveness. The actual comparison is provided by a monte carlo simulation of stock price process which allows us to trade options in discrete time. We assume the risk free rate to be zero throughout the whole chapter in order to keep things simple. There are T trading periods: $t \in \{0, \ldots, T-1\}$.

3.1 Problem Formulation

Suppose we have a bank (or any other market maker) which decides to issue certain number of call or put options. All of these options are issued on the same underlying asset, but they differ in their strike prices. We assume that this market maker was initially able to sell them with a certain profit, i.e. for a price higher than some fair price (for example Black-Scholes price [3]) and is now trying to hedge the position with the underlying asset in some way. We allow for changes in the option portfolio in the course of time and we are looking for a best way to do so.

Usually, the first approach that a company would choose in this scenario is a delta or gamma hedging. However, if the issuer wants to weight the risk more precisely, she may want to choose a mean-variance hedging strategy that we propose instead. Its advantage over the commonly used gamma hedging is its variability. In other words, we can adjust it according to the level of risk aversion of the issuer. But more importantly, as we will show in the simulations, it behaves better than the standard gamma hedging in terms of expected profit.

Definition 5. *We define processes π^δ, π^γ and π^{MV} as follows:*

$$
\begin{aligned}
{\pi_t}^\delta &:= \pi_{t-1}^\delta + {\beta_{t-1}^\delta}^T (C_{BSt}\delta_{t-1}(S_t - S_{t-1}) - C_{BSt-1}) && (3.1)\\
{\pi_t}^\gamma &:= \pi_{t-1}^\gamma + {\beta_{t-1}^\gamma}^T (C_{BSt}\delta_{t-1}(S_t - S_{t-1}) - C_{BSt-1}) + (\beta_t^\gamma - \beta_{t-1}^\gamma)^T C_{BSt} - \\
&\quad - \left[(\beta_t^\gamma - \beta_{t-1}^\gamma)^{+T} c_{ask} - (\beta_t^\gamma - \beta_{t-1}^\gamma)^{-T} c_{bid} \right] && (3.2)\\
{\pi_t}^{MV} &:= \pi_{t-1}^{MV} + {\beta_{t-1}^\gamma}^T (V_t - \xi_{t-1}(S_t - S_{t-1}) - V_{t-1}) + (\beta_t^\gamma - \beta_{t-1}^\gamma)^T V_t - \\
&\quad - \left[(\beta_t^\gamma - \beta_{t-1}^\gamma)^{+T} c_{ask} - (\beta_t^\gamma - \beta_{t-1}^\gamma)^{-T} c_{bid} \right] && (3.3)\\
\pi_0^\delta &:= W_0 + {\beta^\delta}^T C_{BS0} && (3.4)\\
\pi_0^\gamma &:= W_0 + {\beta^\gamma}^T C_{BS0} && (3.5)\\
\pi_0^{MV} &:= W_0 + {\beta^{MV}}^T V_0. && (3.6)
\end{aligned}
$$

All variables mentioned in this definition are time-dependent (this is represented by the time subscript) and they have the following meaning:

- π^δ *(scalar) - profit from following delta strategy. Under this term, we understand the joint value of all options plus the amount of money on a risk-free bank account plus the value we have in underlying asset.*

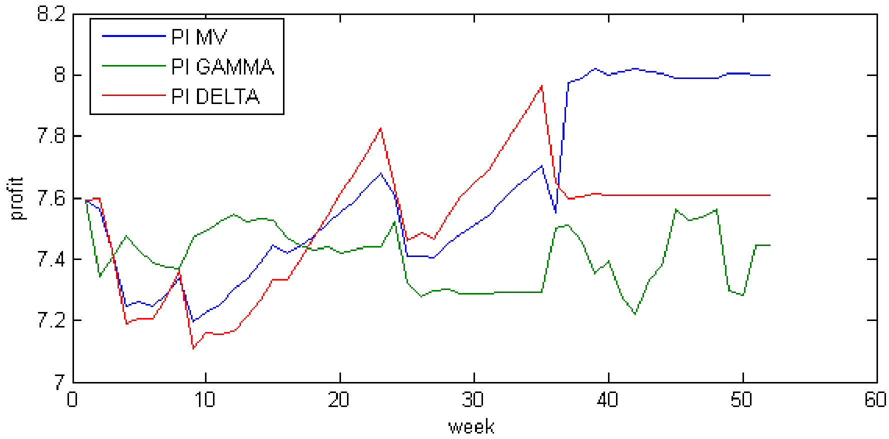
- π^γ *(scalar) - profit from following gamma strategy.*

- π^{MV} *(scalar) - profit from following mean-variance strategy.*
- β^{δ} *(vector) - amount of options in delta portfolio.*
- β^{γ} *(vector) - amount of options in gamma portfolio.*
- β^{MV} *(vector) - amount of options in mean-variance portfolio.*
- C_{BS} *(vector) - Black - Scholes price of options in portfolio [3].*
- C_{bid} *(vector) - Bid prices of options (prices somebody is willing to buy the options for)*
- C_{ask} *(vector) - Ask prices of options (prices that I can buy the options for in the market)*
- W_0 *(scalar) - Initial wealth (the initial profit from selling the options for a higher than Black - Scholes price).*
- V *(vector) - mean value process, for more details see [6]. We approximate it by a Black-Scholes option price. For a justification, see again [6].*
- δ *(vector) - Black - Scholes delta.*
- ξ *(vector) - locally optimal hedging strategy. In our case, it is again approximated by the Black - Scholes delta.*
- S *(scalar) - underlying asset price. In the simulation, we use a geometric Brownian motion.*

Processes π^{δ}, π^{γ} and π^{MV} serve as a comparison of our three strategies and they are very important characteristics. We compare their distribution in time T using monte carlo simulation.

Following sections define the delta trading strategy, the delta-gamma trading strategy and the mean-variance trading strategy which we are going to compare afterwards more thoroughly.

Figure 3.1: *Realization of processes π^{δ}, π^{γ} and π^{MV} using monte carlo simulation*



3.2 Delta Trading Strategy

This is the simplest of the three approaches we are working with. It uses the standard delta hedging, e.g. it keeps the delta of portfolio equal to zero. When using this strategy, the portfolio of options does not change in time, we only trade in the underlying asset. This is by the way the major difference between delta hedging and the other two strategies, because both delta-gamma and mean-variance approaches dynamically change number of options in portfolio in time. Nevertheless, we include the variable β in (3.1) in order to remain consistent with the other two strategies, even though $\beta_0^{\delta} = \beta_1^{\delta} = \ldots = \beta_T^{\delta}$ is always satisfied.

3.3 Delta - Gamma Trading Strategy

The delta - gamma hedging strategy is a very simple method to manage a given portfolio of derivatives. Its main idea is keeping the total gamma of the portfolio close to zero such that the fluctuations of underlying asset do not affect its value too much. This is done using gammas of options included in portfolio. Furthermore, the total delta is put equal to zero by some amount of the underlying asset. But firstly, we want to achieve a zero gamma.

In case of european options, the task is very simple since the total gamma of the portfolio Γ equals the weighted sum of gammas of all the options in this portfolio and we are able to analytically find a formula that expresses them in the Black Scholes model (see lemma (5.0.3)). Strictly speaking, if we had n options denoted by $i \in \{1, 2, \ldots, n\}$ in every time period then the gamma hedging constraint would take on the following form:

$$\Gamma = \sum_{i=1}^{n} \beta_i \gamma_i = 0. \tag{3.7}$$

Since there are n derivatives in our portfolio, there is obiously not a unique solution to (3.7), but a set of solutions instead. Therefore, we have to find a unique gamma-hedged portfolio in each time period. Since there are multiple ways to do so, we have to pick one particular. Therefore we are going to solve the following problem in every period $t \in \{0, 1, \ldots, T-1\}$:

$$\{\Delta\beta_t^+, \Delta\beta_t^-\} := \arg \max_{x \geq 0, y \geq 0} \left[c_{BS}^T(\beta_{t-1} + x - y) - (x^T c_{ask} - y^T c_{bid}) \right] \tag{3.8}$$

$$s.t. \qquad (\beta_{t-1} + x - y)^T \gamma_t = 0 \tag{3.9}$$

What it actually states is that we want to maximize the portfolio Black-Scholes value with the least possible cost while keeping the Γ equal to zero. The new β is consequently defined as follows:

$$\beta_t := \beta_{t-1} + \Delta\beta_t^+ - \Delta\beta_t^- \tag{3.10}$$

It is understandable that this approach is rather locally optimal, because we are not considering the character of S in a long time horizon. Instead, we only focus on a one period optimization.

The actual optimization is done using MATLAB, more precisely using its built-in function called linprog. linprog uses a projection method to solve the medium scale linear programming problems which is exactly our case [1].

Now the only thing that needs to be done is keeping the portfolio delta equal to zero. This is done simply by delta hedging each option that we are trading in and can be seen from the way process π_γ is defined in the beginning of chapter (3.2).

In the simulation, we stop trading in options some time before the expiration date in order to maintain lower standard deviation. The reason is very well described by the quotation from [12]: "The instability comes about because gamma goes to zero extremely quickly—exponentially squared fast—when the strike moves away from the spot. The closer we are to expiry, the worse things are."

3.4 Mean-Variance Trading Strategy

The Mean-Variance trading strategy does not differ from the gamma strategy that much. Just like in the previous strategy, we are working both with options and stock again. The difference is in the way how portfolio is created. As we will see, the task cannot be reduced to a linear programming problem which was the case of gamma strategy, but to a quadratic programming problem instead.

Since the correlation matrix derived in previous chapter was defined for a given hedged positions, it is understandable that we have to work with them if we want to use this matrix in our approach. Therefore, we are not trading only in options, but in the underlying asset as well.

[1] It is an active set method which means it is derived from a well known simplex method proposed by Dantzig, more details in [5].

The actual Mean-Variance trading strategy rebalances the portfolio vector β_t in every period $t \in \{1, 2, \ldots, T-1\}$ such that the following utility is maximized:

$$U(\pi_T) = \mathrm{E}[\pi_T] - \frac{\alpha}{2}\mathrm{Var}[\pi_T] \tag{3.11}$$

$$\pi_T := \pi_t + (\beta_t^T V_T - \beta_t^T \xi \cdot S_t - \beta_t^T V_t), \tag{3.12}$$

where again, $\beta_t^T \xi \cdot S_t$ stands for any trading in stock that was made between t and T, with ξ being a vector now.

In other words, we are trying to find a beta as of time t that would follow our mean-variance preferences if not changed again in due course. This is the reason why no transaction costs were included in the definition of U(.). We are simply looking for an optimal β, given that we are not going to change it again. Obviously:

$$\begin{aligned}
\mathrm{E}_t[\pi_T] &= \mathrm{E}_t[\pi_t + (\beta_t^T V_T - \beta_t^T \xi \cdot S_t - \beta_t^T V_t)] = \\
&= \mathrm{E}_t[\pi_t] + \mathrm{E}_t[(\beta_t^T V_T - \beta_t^T \xi \cdot S_t - \beta_t^T V_t)] = \\
&= \mathrm{E}_t[\pi_t] = \\
&= \pi_t \\
\mathrm{Var}_t[\pi_T] &= \mathrm{Var}_t[\pi_t + (\beta_t^T V_T - \beta_t^T \xi \cdot S_t - \beta_t^T V_t)] = \\
&= \mathrm{Var}_t[\beta_t^T V_T - \beta_t^T \xi \cdot S_t - \beta_t^T V_t] = \\
&= \mathrm{Var}_t[\beta_t^T (V_T - \xi \cdot S_t - V_t)] = \\
&= \beta_t^T \mathrm{Var}_t[V_T - \xi \cdot S_t - V_t]\beta_t = \\
&= \beta_t^T \Sigma_t \beta_t,
\end{aligned}$$

where Σ is our well known covariance matrix of hedged positions.

Therefore, we are able to express the utility at time t in a following way:

$$U(\pi_T) = \pi_t - \frac{\alpha}{2}\beta_t^T \Sigma_t \beta_t \tag{3.13}$$

The question we are going to solve looks as follows. Suppose we have just arrived at time t, which means we have a current portfolio value of π_{t-1} plus some changes in the option and stock price that have occurred since the last period. This portfolio value consists from a set of β_{t-1} options and $-\beta_{t-1}^T\xi_{t-1}$ stocks, the rest is money on a risk-free bank account. The question we are asking ourselves is how can we change the option portfolio such that the utility $U(\pi_T)$ from (3.11) is maximized?

In summary, the problem formulated at time t takes on the following form if we replace β_t with $\beta_{t-1}+x-y$ which is how we construct the new portfolio vector:

$$\begin{aligned}\{\Delta\beta_t^+, \Delta\beta_t^-\} := \arg\max_{x\geq 0, y\geq 0} \Big[&\pi_{t-1} + (x-y)^T V_t + \\ &+ \beta_{t-1}^T(V_t - \xi_t(S_t - S_{t-1}) - V_{t-1}) - \\ &-(x^T c_{ask} - y^T c_{bid}) - \frac{\alpha}{2}(\beta_{t-1}+x-y)^T\Sigma(\beta_{t-1}+x-y)\Big].\end{aligned} \tag{3.14}$$

As we have already mentioned, the new vector β_t is constructed in the same fashion as it was in the previous strategy. That means:

$$\beta_t := \beta_{t-1} + \Delta\beta_t^+ - \Delta\beta_t^- \tag{3.15}$$

This, however, is where the similarity ends. Unlike the problem in gamma hedging, this optimization is not linear and therefore cannot be solved using linear programming techniques. In order to solve it, we are going to transform it to a quadratic programming problem of a specific shape which can be solved numerically.

Define vector $\Delta\beta_t^*$ and matrix Σ_t^*:

$$\Delta\beta_t^* := \begin{bmatrix}\Delta\beta_t^+ \\ \Delta\beta_t^-\end{bmatrix} \tag{3.16}$$

$$\Sigma_t^* := \begin{bmatrix}\Sigma_t & -\Sigma_t \\ -\Sigma_t & \Sigma_t\end{bmatrix} \tag{3.17}$$

$$\tag{3.18}$$

Then (3.14) can be rewritten as:

$$\begin{aligned}\Delta\beta_t^* := \arg\max_{z\geq 0} \Bigg[& \pi_{t-1} + z^T \begin{bmatrix} V_t \\ -V_t \end{bmatrix} + \\ & + \beta_{t-1}^T (V_t - \xi_t(S_t - S_{t-1}) - V_{t-1}) - \\ & - z^T \begin{bmatrix} c_{ask} \\ -c_{bid} \end{bmatrix} + z^T \begin{bmatrix} -\alpha\Sigma\beta_{t-1} \\ \alpha\Sigma\beta_{t-1} \end{bmatrix} - \frac{\alpha}{2} z^T \Sigma_t^* z \Bigg] = \\ = \arg\max_{z\geq 0} \Bigg[& z^T \begin{bmatrix} V_t - c_{ask} - \alpha\Sigma\beta_{t-1} \\ c_{bid} - V_t + \alpha\Sigma\beta_{t-1} \end{bmatrix} - \frac{\alpha}{2} z^T \Sigma_t^* z \Bigg]. \end{aligned} \tag{3.19}$$

Clearly, (3.19) is a problem where the optimized function takes on form $f^T x - \frac{1}{2} x^T H x$ which is already a shape that can be solved using MATLAB's function quadprog [2].

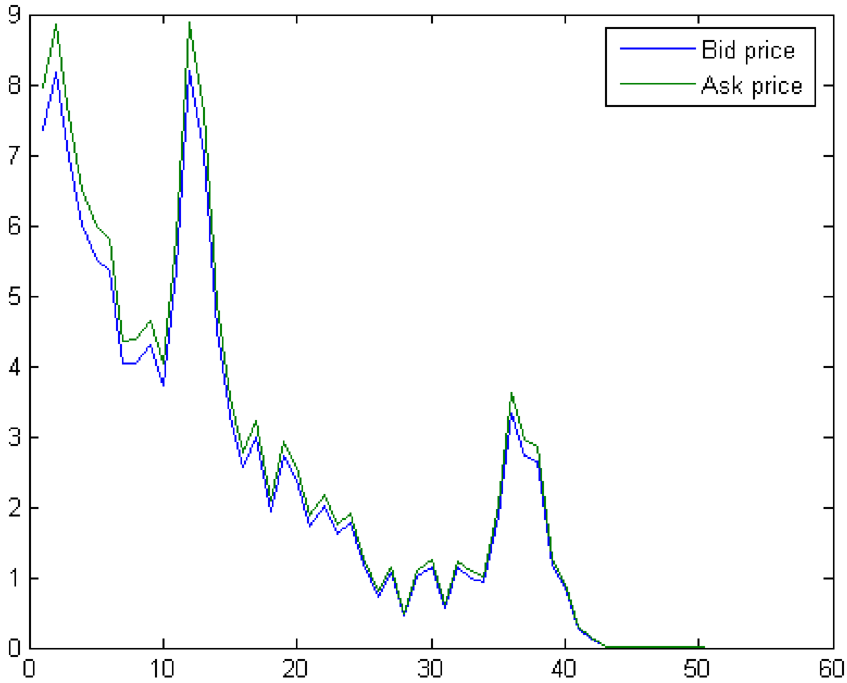
3.5 Monte Carlo Simulation

We have defined all three strategies in the previous section. Now we are going to compare them using a monte carlo simulation. We assume to have a stock with initial value S_0 at time zero which follows a geometric Brownian motion without price jumps. There is a spread in the option price just like in the real market with the Black-Scholes value being the benchmark from which it is created. A particular relization of the spread can be seen in figure (3.2). We create the spread according to the value of the option and to the change in the underlying stock.

We assume the Black-Scholes price to be lower than the bid price of the option. This corresponds with the real market data. The actual comparison of strategies is based on the processes π^δ, π^γ and π^{MV} from (3.1),(3.2) and (3.3), since they represent the total profit from following a particular strategy, including the value obtained from trading in the underlying asset and the total value of all options.

[2] This function is based on the interior-reflective Newton method described in [4].

Figure 3.2: *Realization of a bid-ask spread with* $S_0 = 100, K = 100, \mu = 0.05, \sigma = 0.2, T = 1$



Simulation of Mean-Variance strategy for different levels of risk aversion is depicted in figure (3.3).

Both gamma and delta approaches tend to have higher standard deviation and lower expected profit than the mean-variance strategy we propose. We have simulated their evolution using monte carlo technique. The distribution of profit in this case depends on the risk aversion coefficient as well, because delta and gamma strategies both start from the portfolio which is optimal in a mean-variance sense. In order to create such a portfolio, coefficient α is required. Therefore, the distribution of profit depends on it. Consequently, the characteristics of the two strategies look as shown in figures (3.4) and (3.5).

In order to clarify the approach, we present the step-by-step algorithm

Figure 3.3: *Monte Carlo simulation of profit from following the Mean-Variance strategy for different values of risk aversion coefficient* α *with parameters* $S_0 = 100, \mu = 0.05, \sigma = 0.2, T = 1$ *in case* $C_{BS} < C_{bid} < C_{ask}$

α	Avg	CI of Avg	Std	CI of Std
0.05	10.0783	[9.7972,10.3608]	3.1600	[2.9521,3.4048]
0.1	9.9430	[9.7085,10.1644]	2.5765	[2.4177,2.7548]
0.5	9.5200	[9.4423,9.5950]	0.8787	[0.8278,0.9313]
1	7.9354	[7.8871,7.9817]	0.5525	[0.5192,0.5882]
5	6.6575	[6.6111,6.7024]	0.5289	[0.4982,0.5615]
100	6.2986	[6.2498,6.3494]	0.5791	[0.5374,0.6257]

for generating the monte carlo simulation:

1. **Initialization of market**
   We set the basic parameters of the market. These include the time to expiry, the number of trading days per year, the parameters of underlying price process or for example the investor's risk aversion.

2. **Creation of starting portfolio**
   This portfolio maximizes the mean-variance preferences for a given risk aversion coefficient α.

3. **Simulation**
   We generate random stock prices such that the underlying price process follows geometric Brownian motion. Strategies trade in underlying and/or options according to their definitions during the whole simulation.

4. **Collecting Results**
   Sample distribution of profit from following a particular strategy is created for all of them and some characteristics are evaluated (mean, standard deviation and confidence intervals for both parameters).

Figure 3.4: *Monte Carlo simulation of profit from following the Delta strategy for different values of risk aversion coefficient* α *with parameters* $S_0 = 100, \mu = 0.05, \sigma = 0.2, T = 1$ *in case* $C_{BS} < C_{bid} < C_{ask}$

α	Avg	CI of Avg	Std	CI of Std
0.05	10.0676	[9.7817,10.3447]	3.2906	[3.0582,3.5952]
0.1	9.9745	[9.6920,10.2385]	3.0748	[2.8767,3.2985]
0.5	9.6106	[9.4409,9.7728]	1.8895	[1.7615,2.0562]
1	7.6103	[7.5101,7.7016]	1.0627	[0.9888,1.1600]
5	5.9256	[5.9082,5.9420]	0.1931	[0.1788,0.2097]
100	5.5583	[5.5574,5.5591]	0.0100	[0.0092,0.0107]

In every simulation, we assume to issue a portfolio of options that expires in one year. This portfolio is dynamically rebalanced depending on the strategy we use (except for delta strategy which does not change the amount of options). We are initially assuming to sell options with profit and then we are trying to hold the portfolio steady and obtain some positive profit from the whole process.

As we can see from the figures, the mean-variance strategy behaves much better than the other two strategies do both in the matter of expected profit and in the matter of standard deviation. For an example of approximate distribution for all strategies refer to figure (3.6).

Furthermore, it is worth noting what happens for different levels of risk aversion. Clearly, if we put α closer to zero, we are actually stating that risk does not mean that much to us. And of course as we can see from the figures, the mean-variance characteristics of this case are close to the characteristics of basic delta hedging, because that is exactly how delta hedging is defined.

Therefore, we can understand our strategy as a modification of delta hedging, where we are able to decrease the standard deviation of profit in exchange for a decrease in its expected value. Then, depending on our subjective preferences, we can adjust the value of α such that the resulting

Figure 3.5: *Monte Carlo simulation of profit from following the Delta strategy for different values of risk aversion coefficient* α *with parameters* $S_0 = 100, \mu = 0.05, \sigma = 0.2, T = 1$ *in case* $C_{BS} < C_{bid} < C_{ask}$

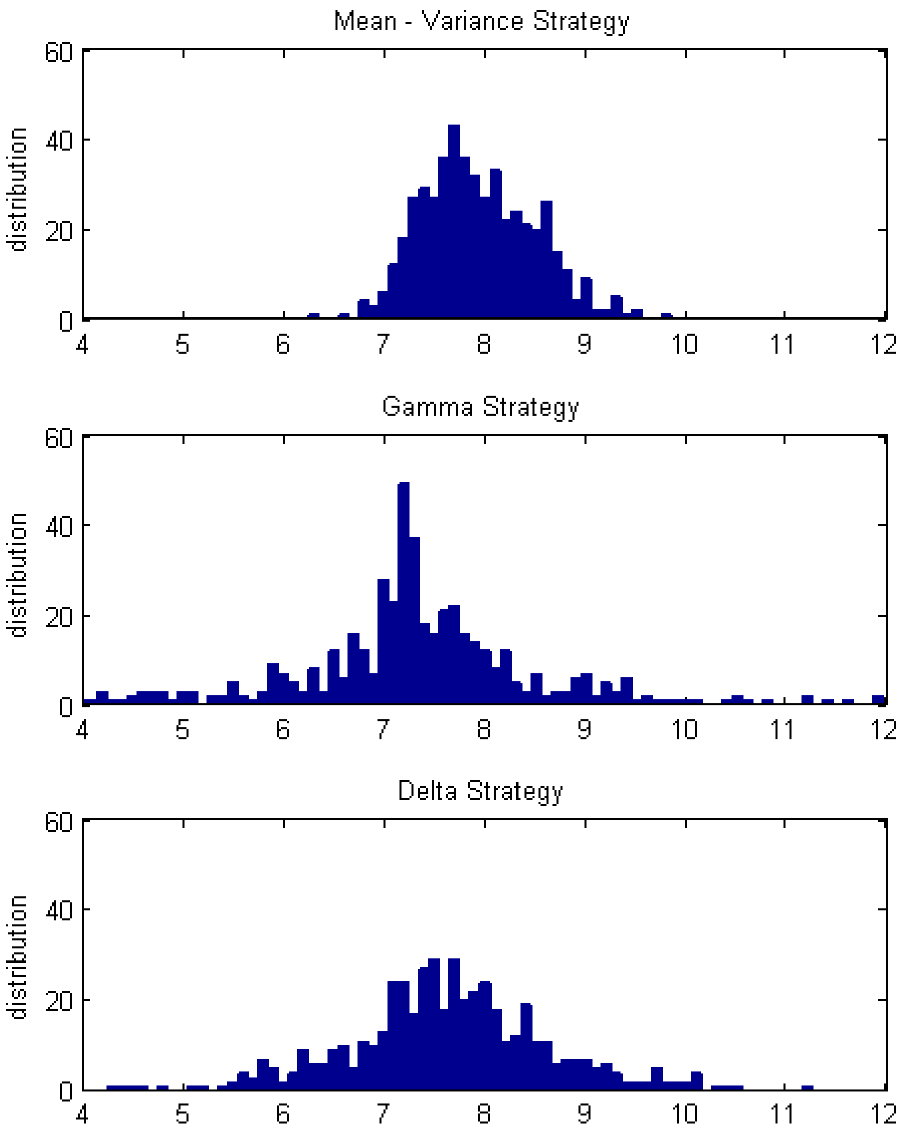
α	Avg	CI of Avg	Std	CI of Std
0.05	8.8758	[8.4714 9.2170]	4.2958	[3.5750 5.8114]
0.1	9.0280	[8.6934 9.3775]	3.9389	[3.1533 5.4118]
0.5	8.9437	[8.6638 9.2717]	3.4948	[2.9006 5.0554]
1	7.1680	[6.9712 7.3531]	2.1465	[1.7671 2.8073]
5	5.9117	[5.8618 5.9696]	0.6216	[0.4737 0.9050]
100	5.5555	[5.5536 5.5574]	0.0218	[0.0177 0.0303]

distribution reflects them better.

3.5.1 Summary

We have tested the mean-variance strategy in two different cases of which one was meant for its theoretical evaluation and the other one tried to mimic real markets by the choice of bid and ask prices above the Black-Scholes price. We have included only one of them in the thesis because they behave almost identically. In both variants, we have observed that our strategy dominates both delta and gamma hedging in terms of expected profit and expected variance. In other words, we are able to reduce the risk incurred from hedging european options in discrete time significantly by following this approach.

Figure 3.6: *Approximate distribution of profit from following a particular strategy. Computed for $\alpha = 1$ and 500 observations from monte carlo*



Chapter 4

Results & Conclusions

Despite the fact, that the primary goal of the Mean-Variance theory is to evaluate risk associated with trading in derivatives, we have shown that it is actually possible to use it for expressing covariance between hedged positions and moreover to create an interesting trading strategy.

Contribution of this thesis is twofold. One major contribution is the derivation of covariance matrix for a given portfolio of options described in chapter 2. Despite really extensive derivation which we did not include completely[1], it can be easily implemented due to its quite simple final shape.

Although this concept by itself is already very interesting, we have decided to go even further and to use it for creating a trading strategy, which is the second contribution of our thesis. As we have seen, this strategy behaves very well in comparison with two basic approaches that are commonly used today. For an investor with Mean-Variance preferences, our strategy is exactly what he is looking for. And since the Mean-Variance preferences are realistic when dealing with uncertainty, we believe that our strategy is a very reasonable alternative to the other ones we have compared it to.

[1]The derivation spans a huge amount of pages and includes only some basic algebraic operations. Therefore it is not interesting to mention it here in its full length

Chapter 5

Appendix

Lemma 5.0.1. *Let X and Y be two european options and let $X+Y$ be their combination (i.e. a position of holding one X and one Y). Denote by γ_X, γ_Y and γ_{X+Y} their respective gammas. Then the following identity holds:*

$$\gamma_{X+Y} = \gamma_X + \gamma_Y \tag{5.1}$$

Proof. From [3] we know the price P_{X+Y} of the position $X+Y$ equals:

$$P_{X+Y} = P_X + P_Y \tag{5.2}$$

Therefore it holds that:

$$\frac{\partial^2 P_{X+Y}}{\partial S^2} = \frac{\partial^2 P_X}{\partial S^2} + \frac{\partial^2 P_Y}{\partial S^2} \tag{5.3}$$

$$\gamma_{X+Y} = \gamma_X + \gamma_Y \tag{5.4}$$

□

Lemma 5.0.2. *Suppose A, B and C are real numbers and moreover that $A > 0$. Then the following equality holds:*

$$\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(Ax^2 + Bx + C\right)\right] \mathrm{d}x = \frac{\sqrt{2\pi}}{\sqrt{A}} \exp\left[-\frac{1}{2}\left(C - \frac{B^2}{4A}\right)\right] \tag{5.5}$$

Proof. Firstly, we will focus on the expression in brackets inside the integral. It can be easily seen that:

$$Ax^2 + Bx + C = Ax^2 + 2\sqrt{A}x\frac{B}{2\sqrt{A}} + \frac{B^2}{4A} - \frac{B^2}{4A} + C = \tag{5.6}$$

$$= \left(\sqrt{A}x + \frac{B}{2\sqrt{A}}\right)^2 + \left(C - \frac{B^2}{4A}\right). \tag{5.7}$$

Now let us focus on the original integral and its evaluation. Using the previous result, we obtain the following formula:

$$\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(Ax^2 + Bx + C\right)\right] \mathrm{d}x = \tag{5.8}$$

$$= \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\left(\sqrt{A}x + (\frac{B^2}{2\sqrt{A}})\right)^2 + \left(C - \frac{B^2}{4A}\right)\right)\right] \mathrm{d}x = \tag{5.9}$$

$$= \exp\left[-\frac{1}{2}\left(C - \frac{B^2}{4A}\right)\right] \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\sqrt{A}x + \frac{B}{2\sqrt{A}}\right)^2\right] \mathrm{d}x = \tag{5.10}$$

$$= \frac{\sqrt{2\pi}}{\sqrt{A}} \exp\left[-\frac{1}{2}\left(C - \frac{B^2}{4A}\right)\right] \cdot \tag{5.11}$$

$$\cdot \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\sqrt{A}x - \left(-\frac{B}{2\sqrt{A}}\right)\right)^2\right]}_{\text{Density function of normal distribution}} \mathrm{d}x = \tag{5.12}$$

$$= \frac{\sqrt{2\pi}}{\sqrt{A}} \exp\left[-\frac{1}{2}\left(C - \frac{B^2}{4A}\right)\right] \tag{5.13}$$

$$\tag{5.14}$$

,where we have used that the density of any statistical distribution including normal distribution must have integral from $-\infty$ to ∞ equal to 1. In our case, the random variable was $\sqrt{A}X$ with mean equal to $-\frac{B}{2\sqrt{A}}$ and variance equal to 1. □

Lemma 5.0.3. *Gammas of european put and call options are the same and equal to*

$$\gamma = \frac{\phi(d_1)}{S\sigma\sqrt{T-t}} \tag{5.15}$$

where $\phi(.)$ is a density of normalized normal distribution and

$$d_1 = \frac{\ln \frac{S}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{5.16}$$

Proof. We will derive the gamma for both types of option separately.

- **Call option**

  Since γ is a second partial derivative of option price with respect to the underlying asset price, we can evaluate it as a partial derivative of option's delta with respect to the same variable. Therefore, we will evaluate the option's delta first. Denote the price of european call option in time t by $C(S,t)$. It follows from [3] that

$$C(S,t) = \Phi(d_1)S - \Phi(d_2)Ke^{-r(T-t)} \tag{5.17}$$

  where $\Phi(.)$ is a cumulative distribution function of standard normal distribution and

$$d_1 = \frac{\ln \frac{S}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{5.18}$$

$$d_2 = \frac{\ln \frac{S}{K} + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{5.19}$$

  Therefore the option delta equals:

$$\Delta = \frac{\partial C(S,t)}{\partial S} = \Phi(d_1) + S\frac{\partial \Phi(d_1)}{\partial S} - Ke^{-r(T-t)}\frac{\partial \Phi(d_2)}{\partial S} = \tag{5.20}$$

$$= \Phi(d_1) + S\frac{\partial \Phi(d_1)}{\partial d_1}\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}\frac{\partial \Phi(d_2)}{\partial d_2}\frac{\partial d_2}{S} = \tag{5.21}$$

$$= \Phi(d_1) + S\frac{1}{\sqrt{2\pi}}e^{\frac{-d_1^2}{2}}\frac{1}{S\sigma\sqrt{T-t}} - \tag{5.22}$$

$$- Ke^{-r(T-t)}\frac{1}{\sqrt{2\pi}}e^{\frac{-d_1^2}{2}}\frac{S}{K}e^{r(T-t)}\frac{1}{S\sigma\sqrt{T-t}} = \tag{5.23}$$

$$= \Phi(d_1) + \frac{S}{S\sigma\sqrt{2\pi(T-t)}}e^{\frac{-d_1^2}{2}} - \frac{S}{S\sigma\sqrt{2\pi(T-t)}}e^{\frac{-d_1^2}{2}} = \tag{5.24}$$

$$= \Phi(d_1). \tag{5.25}$$

This follows from the fact that

$$\frac{\partial \Phi(d_1)}{\partial d_1} = \phi(d_1) = \frac{1}{\sqrt{2\pi}} e^{\frac{-d_1^2}{2}} \tag{5.26}$$

$$\frac{\partial \Phi(d_2)}{\partial d_2} = \phi(d_1) = \frac{1}{\sqrt{2\pi}} e^{\frac{-d_2^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{\frac{-(d_1-\sigma\sqrt{T-t})^2}{2}} = \tag{5.27}$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{-d_1^2}{2}} e^{d_1\sigma\sqrt{T-t}} e^{\frac{-\sigma^2(T-t)}{2}} = \tag{5.28}$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{-d_1^2}{2}} e^{\ln\frac{S}{K}+\left(r+\frac{\sigma^2}{2}\right)(T-t)} e^{\frac{-\sigma^2(T-t)}{2}} = \tag{5.29}$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{-d_1^2}{2}} \frac{S}{K} e^{r(T-t)} \tag{5.30}$$

$$\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}. \tag{5.31}$$

Derivation of gamma for a call option is now straightforward:

$$\gamma = \frac{\partial^2 C(S,t)}{\partial S^2} = \frac{\partial \delta}{\partial S} = \frac{\partial \Phi(d_1)}{\partial S} = \frac{\partial \Phi(d_1)}{\partial d_1}\frac{\partial d_1}{\partial S} = \tag{5.32}$$

$$= \frac{\phi(d_1)}{S\sigma\sqrt{T-t}}. \tag{5.33}$$

- **Put option**

Denote the price of european put option in time t by $P(S,t)$. We could proceed analogically to the previous case of call option, but the derivation is much more simple using the put call parity [11].

$$P(S,t) = Ke^{-r(T-t)} - S + C(S,t) \tag{5.34}$$

It is obvious that this implies the delta of put option equal to:

$$\delta = \Phi(d_1) - 1 \tag{5.35}$$

And therefore since derivative of a constant is zero, it is clear that γ of a put option is the same as the γ of a call option which is

$$\gamma = \frac{\phi(d_1)}{S\sigma\sqrt{T-t}}. \tag{5.36}$$

□

Bibliography

[1] Zuzana Batmendijnová. Mean-variance hedging with uncertain trade execution. Master's thesis, Comenius University Bratislava, June 2009.

[2] Dimitris Bertsimas, Leonid Kogan, and Andrew W. Lo. When is time continuous? *Journal of Financial Economics*, 55:173–204, 2000.

[3] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *The Journal of Political Economy*, 81(3):637–654, May-Jun 1973.

[4] T.F Coleman and Y. Li. A reflective newton method for minimizing a quadratic function subject to bounds on some of the variables. *SIAM Journal on Optimization*, 6(4):1040–1058, 1996.

[5] G.B. Dantzig, A. Orden, and P. Wolfe. Generalized simplex method for minimizing a linear form under linear inequality restraints. *Pacific Journal of Mathematics*, 5:183–195, 1995.

[6] Aleš Černý. *Mathematical Techniques in Finance: Tools for Incomplete Markets*. Princeton University Press, 41 William Street, Princeton, New Jersey 08540, 2 edition, 2009.

[7] Aleš Černý and Jan Kallsen. Hedging by sequetial regressions revisited. *Mathematical Finance*, 19(4):591–617, October 2009.

[8] Hans Föllmer and Martin Schweizer. Hedging by sequential regression: An introduction to the mathematics of option trading. *ASTIN Bull*, 18:147–160, 1988.

[9] Michael Kamal. Quantitative strategies research notes. Goldman Sachs Equity Derivatives Research, 1998.

[10] G.F. Kuncir. Algorithm 103: Simpson's rule integrator. *Communications of the ACM*, 5(6):347, 1962.

[11] Samuel Armstrong Nelson. *The ABC of options and arbitrage*. New York, Nelson, 1904.

[12] Rolf Poulsen. Four things you might not know about the black-scholes formula.

[13] Klaus Bjerre Toft. On the mean-variance tradeoff in option replication with transaction costs. *Journal of Financial and Quantitative Analysis*, 31(2):233–263, Jun 1996.