COMENIUS UNIVERSITY IN BRATISLAVA
FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS
Department of Applied Mathematics and Statistics

ANALYSIS OF THE DEPENDENCE BETWEEN ECONOMIC GROWTH AND ENVIRONMENTAL QUALITY

Diploma Thesis

Bratislava 2011

Pavol Majher

UNIVERZITA KOMENSKÉHO V BRATISLAVE
FAKULTA MATEMATIKY, FYZIKY A INFORMATIKY

ANALÝZA VZŤAHU MEDZI HOSPODÁRSKYM RASTOM A KVALITOU ŽIVOTNÉHO PROSTREDIA

Diplomová práca

Študijný program:	Ekonomická a finančná matematika
Študijný odbor:	9.1.9 Aplikovaná matematika
Školiace pracovisko:	Katedra aplikovanej matematiky a štatistiky
Školitel:	Univ.Prof. Dipl.-Ing. Dr. Mikuláš Luptáčik

Bratislava 2011 Pavol Majher

Acknowledgement

I´m truly indebted to my supervisor, prof. Mikuláš Luptáčik, for his guidance and extensive support during my work on this thesis.

I´m grateful to prof. Gerhard Sorger and prof. Margaréta Halická for their helpful comments on the problems of optimal control theory.

Many thanks and appreciation also goes to Peťka, my family and friends for their love and support.

Most of all, I would like to thank God for all the inspiration and strength that He gave me to complete this work.

Abstract

This thesis aims to contribute to the debate over dependence of economic growth and environmental quality using tools of optimal control. In the first part, we introduce the issue by presenting an empirical evidence along with an overview of literature dealing with this topic. Then we introduce environmental growth model developed by Luptáčik and Schubert [22] and analyze it using advanced theory of optimal control. In the final part, we modify this model to include the endogenous technological change of emission coefficient of the production process. For both models, we derive sufficient conditions for the existence, saddle-point property and local stability of the steady state in the economy. At the same time, we will try to economically interpret these conditions.

Keywords: economic growth model, environment, optimal control theory, steady state, technical change, emission rate of production process

Abstrakt

Cieľom tejto práce je prispieť k diskusii o vzťahu hospodárskeho rastu a kvality životného prostredia pomocou nástrojov teórie optimálneho riadenia. V prvej časti uvedieme túto tému predstavním empirických údajov spolu s prehľadom relevantnej literatúry. Ďalej predstavíme model hospodárskeho rastu a životného prostredia navrhnutý autormi Luptáčikom a Schubertom [22] a analyzujeme ho použitím toórie optimálneho riadenia. V záverečnej časti upravíme tento model zahrnutím endogénnej technologickej zmeny emisných koeficientov produkčného procesu. Pre obidva modely odvodíme postačujúce podmienky pre existenciu, vlastnosť sedlového bodu a lokálnu stabilitu rovnovážneho stavu ekonomiky. Zároveň sa pokúsime ekonomicky interpretovať tieto podmienky.

Kľúčové slová: model hospodárskeho rastu, životné prostredie, teória optimálneho riadenia, rovnovážny stav, technologická zmena, miera emisií produkčného procesu

Obsah

Introduction **9**

1 Economic Growth and Environment **11**

1.1 Motivation . . . 11
1.2 Empirical Evidence . . . 13
  1.2.1 Production and Pollution . . . 13
  1.2.2 Environmental Protection Expenditures . . . 15
  1.2.3 Empirical Comparison of End-of-Pipe and R&D Abatement . . 17
1.3 Different Perspectives . . . 18
1.4 Development of Environmental Policy . . . 20

2 Overview of Literature **23**

2.1 Foundations and Earlier Works . . . 23
  2.1.1 Simple Model of Consumption-Pollution Trade-Off . . . 25
2.2 From Models with Capital Accumulation to Endogenous Growth . . . 27
2.3 Models of Directed Technical Change . . . 29

3 Analysis of Environmental Growth Model **31**

3.1 Formulation of Model . . . 31
3.2 Results Presented in the Original Work . . . 35
  3.2.1 Optimal Solution and Its Analysis . . . 35
  3.2.2 Steady State and Its Stability . . . 37
  3.2.3 Comparative Statics . . . 39
3.3 Further Analysis of Stationary State . . . 40
  3.3.1 Existence of the Steady State . . . 40
  3.3.2 Steady State without Abatement Costs . . . 42

3.3.3 Saddle Point Property of the Steady State 43
3.3.4 Stability Analysis of the Steady State 45

4 Model with the Differentiation of Abatement 48

4.1 Specification of the Model 48
4.2 Optimal Behavior Path 50
4.2.1 Optimal Solution of Our Problem 50
4.2.2 Long-Term Trends in Consumption and Abatement Costs 52
4.2.3 Comparative Statics 53
4.3 Steady State and Stability Analysis 54
4.3.1 Sufficient Conditions for Existence 55
4.3.2 Qualitative Analysis of the Steady State 57
4.3.3 Stability Analysis 58

Conclusion 60

Resumé 62

Literature 65

A Additional Data 70

B Remarks on the Steady State 72

B.1 Sufficient Conditions for Saddle-Point Properties 72
B.2 Global Stability of the System 73

Zoznam tabuliek

1.1 Growth rates of GDP, CO2 Emissions and Waste in the United States . 14
1.2 CO2 emissions and Waste to GDP ratio in the United States 14
1.3 Environmental Protection Expenditures and CO2 Emissions 16

A.1 Average Growth of GDP 70
A.2 Average Growth of CO2 Emissions 71

Introduction

> *Ten years ago, many people thought that there was no climate problem.*
> *They were - unfortunately - wrong.*
> *Five years ago, many people thought that we didn't have the ideas*
> *and technologies to lower emissions dramatically.*
> *Wrong again.*
> Erik Rasmussen
> Founder of the Copenhagen Climate Council

The impact of economic processes on the environmental quality has become obvious since the beginning of industrial revolution. However, the governing authorities (and majority of economists) have been mostly ignoring this relation, while doing economic decisions, for long years. It wasn´t until the 1950s when first independent institution aiming to develop and apply the tools of economic theory was founded, thus laying the foundations of environmental economics (see Pearce in [25]). Since then, the problems related to this issue have gained on importance and drawn a sufficient public attention. Several international meetings (see e.g. [41] and [42] on the UN Conferences) have been held to address the environmental problems caused by human action and to initiate national as well as global measures to deal with these problems (a review concerned with the Kyoto protocol can be found e.g. in [7]).

The main question, which has been inquired by environmental economists, is whether the economic growth and the quality of environment are only in the trade-off relation or there is some complementarity among them, i.e. economic expansion provides for an improvement of natural conditions. It is generally agreed that rapid economic growth since end of 18th century caused a serious deterioration of environmental conditions. On the other hand, the empirical evidence from the recent years has shown that rate of production growth not only significantly surpasses correlated increase of pollution, but also provides additional sources, which can be devoted to the various abatement measures. This "eco-friendly"ability of economic activities has been even more enhanced by the recent research in new technologies.

Our thesis is set within this framework. Its main objective of this thesis is to analyze the relationship between economic growth and quality of environment using optimal control theory, thus aiming to contribute to the discussion of the question whether these indicators are complementary or substitutive goals of economic policy.

The work is divided into the four chapters. First one contains the introduction of the topic. After the motivation, we present an empirical evidence concerned with the development of economic growth and environmental deterioration, along with the findings regarding the abatement measures. Consequently, a different economic perspectives on the environmental issues are mutually confronted (as stated in [43]). The next chapter offers an overview of relevant literature, starting with the simple models of pollution accumulation, through the models incorporating a capital accumulation and endogenous growth, up to the models of the directed technical change, which represent the state of the art in this economic field.

Third chapter is devoted to the environmental growth model developed by Luptáčik and Schubert [22]. After the introduction and preview of results derived by authors, we pursue analysis further by the another methodology. Our special focus is to derive new sufficient conditions for the local stability of the optimal steady state. The authors derived some stability results in their paper, our aim, however, is to apply the methodology by Sorger [37] and possibly propose more conclusive conditions. In the final chapter, we modify the model from the previous chapter to include an endogenous change of the emission coefficient of production process, which can be by devoted capital investments. The thesis is concluded by the analysis of the optimal solution and steady state of this new model.

Kapitola 1

Economic Growth and Environment

In this introductory chapter we will provide the first insight into a problematics of economic decisions dealing with environmental issues, which will be set in the framework of economic growth theory. Firstly we will talk about the motivation that leads to this topic. Besides stating the general reasons, why it should be examined, we will be concerned with an empirical evidence. In order to get better picture how the growth and environmental quality influence each other, we will look at the development of both economic and environmental indicators. The next section will be discussing the different perspectives on this topic, which have developed among the environmental economists. A final part of this chapter will include description of the role that environmental policy had during economic development together with the statement of important historical benchmarks associated with the global environmental policy.

Overall, the purpose of this chapter is to introduce and provide an empirical background for the issue of environmental economics, which will be modelled and analyzed in the later chapters.

1.1 Motivation

Thinking of environment, we can simply describe it as everything natural around us. More generally, it can be understood as a natural world we live in. Either way, the environmental quality is something that affects everyone. Therefore it seems reasonable to take this indicator into account while making decisions aimed at improvement of life. Economic activities, which among others include mining of natural resources or emission of waste during production process, have a crucial impact on nature. Unfortunately, we must say that goals of environmental quality aren´t always on top of the list for the makers of economic decisions at the micro level (firms, consumers) nor at the macro level (government). The decision makers would often use an argument about incompatibility of these goals with welfare measured by amount of production or consumption. However, one must ask whether (or to which extent) are these reasons legitimate. Is it fair to exclude quality of environment from the welfare indicators? And is this incompatibility always inevitable? These questions can be considered as some of the incentives for the analysis realized in this work.

The primary question, which I´m inquiring, is whether the economic growth and environmental quality are the substitutional or complementary targets of economic policy. In order to do so, I will work within the framework of the economic growth theory and optimal control, mainly focusing on dynamic continuous-time models extended by the concept of environment.

Before moving on to the next section, a statement should be made about the approach we will take towards the modelling of environmental indicators. There are several ways to look at environmental issues: we can deal with problems of pollution, exhaustible or renewable resources. All of these topics provide rich and inspirational field for research. Nevertheless, we will focus only on issues of emissions and pollution control in this thesis and the topics of resource economics won´t be addressed here.

While talking about relation between production and environment, we usually differentiate two basic types of abatement measures (adopted according to work by Frondel et al. [18]): cleaner production, where firms lower harmful impact of their activities directly in the production process; and use of end-of-pipe abatement, which means that they utilize an add-on measures to cut down pollution. A utilization of environmentally friendly materials or modification of product design are examples of clean production (also called R&D abatement) activities, while exhaust-gas cleaning equipment or waste water treatment plants are typical end-of-pipe technologies. This concept is covered more in depth in sections 1.2 (empirical evidence) and 4 (extension of model with the consequent analysis).

1.2 Empirical Evidence

In order to make our analysis more relevant, we will look at the empirical data, which will help us to get a better picture of the inquired issue. If this work should bring contribution to the topic, an empirical evidence is truly a necessary part. Knowledge of data should affect our preferences during the creation and modification of model. Moreover, it will help us to evaluate the results of analysis and decide, whether they are reasonable.

1.2.1 Production and Pollution

After the Industrial revolution, and most notably in the recent decades, many economies around the world have experienced economic growth at rate that by far exceeded those during the former economic history. Unfortunately, this economic growth has been accompanied by the growth of natural degradation, which can be represented e.g. by CO2 emissions or municipal waste. Now we will look at how these variables (gpd growth, amount of CO2 emissions and municipal waste) evolved in the recent years, while using data from the OECD Database available online at [24]. At this place, only data for the United States as a representative country will be presented. Overview of data for several other countries and OECD can be found in Appendix A.

Let's start with statistical comparison of growth in real GDP, CO2 emissions and municipal waste. Table 1.1 shows percentage changes of this indicators from the 1970s up to recent years. This will help us to decide about damages, which relate to the production process. Consequently, we may conclude about trends in mutual relationship of economic growth and environmental quality.

As we can see, all of the indicators have grown in majority of observed time periods. This means that in past decades the United States experienced years of economic progress, which was accompanied with increase in produced amount of CO2 emissions as well as municipal waste. According to these data, we can agree with the statement about strong link between economic growth and environmental degradation.

However, on this basis we cannot simply conclude that onward expansion of production will lead to a natural catastrophe, which is a proposition often used by environmental pessimists (see in section 1.3). An important fact here is that although all

Tabuľka 1.1: Growth rates of GDP, CO2 Emissions and Waste in the United States

Time period	1975-80	1980-85	1985-90	1990-95	1995-2000	2000-05
Average annual GDP growth	3.68	3.26	3.24	2.52	4.36	2.42
Average annual change of CO2 emitted amounts	1.23	-0.50	1.36	1.09	2.09	0.32
Average annual change of waste production	NA	1.64	4.53	0.81	2.27	1.23

growth rates are positive, it is apparent that they are mutually quite different. Overall, the highest increase can be observed in GDP, while growth rates of waste production and CO2 emissions are significantly lower. This (positive) disproportion can signify decrease of "pollution requirements"by production, whereby the economy gets closer to the concept of sustainable development. To support this trend, we will introduce another variables describing ratio of GDP to pollution indicators, which will be presented in table 1.2. To make this analysis reasonable, in calculations we have used GDP values in the constant price levels of year 2005.

Tabuľka 1.2: CO2 emissions and Waste to GDP ratio in the United States

Year	1975	1980	1985	1990	1995	2000	2005
GDP to CO2 emissions ratio USD / kg	1.19	1.25	1.51	1.65	1.77	1.97	2.19
GDP to waste ratio USD / kg	NA	42.44	45.91	43.15	46.91	51.76	54.82

Changes of these indicators agree with the ideas presented in the previous paragraph. Both ratios, GDP to CO2 emissions and GDP to waste, grew significantly over the past $25 - 30$ years: GDP to waste ratio increased by about 29% and GDP to CO2 ratio by as much as 84% of their initial values at the beginning of the observed period. Although we have to remark that this statistics describes only part of pollution problems (emission of other gases or waste are neglected), yet it covers one of the most essential problem of production-environment relation, therefore the results should be taken into accout. To sum up, we can conclude following empirical proposition: In recent years, the economic production (of the United States) has grown at a rather significant rate, which, along other effects, was accompanied with the rise in generated pollution amounts. However, the weight of this pay-off between production and environmental quality has been lowered as the value of GDP to emissions ratio has been dropping.

1.2.2 Environmental Protection Expenditures

Our next task is to examine how is the fact of nature deterioration by economy taken into account by the national budget. As we are investigating the empirics of production-environment trade-off, we will shortly overview the development of costs devoted to the environmental protection in several European countries. This indicator will be consequently confronted with the evolvement of pollution capacities in the corresponding economies. We may be able to observe the impact, which have the recent decisions of the environmental policy had on the actual evolvement of the pollution stock.

We will present two indicators: firstly, an average annual expenditures spent on protection of the environment and measured as a percentage of GDP will be overviewed. The source of data was the Eurostat online database [16]. On the other hand, we will show the average annual growth of CO2 emissions, taken from the OECD Database [24] similarly to the previous section. Both this variables are presented in table 1.3 for several EU countries in the three different time periods ('*' means that due to the missing data, only an average for years 1997-2000 has been calculated).

While observing these data, one could notice that we may divide countries into two groups. First group, which includes e.g. Germany or Austria, can be labelled as the abatement advanced countries, meaning that at the time, when the observations have begun (in the 1995), they had a quite high level of pollution control expenditures (about 0.5% of total GDP). Many of these countries gradually lowered their cleaning costs and in some cases, situation with the production of emissions has worsened (see e.g. Finland). On the other hand, there are countries, which have started rather low on environmental protection costs (Portugal or Spain) and high on pollution growth.

Tabuľka 1.3: Environmental Protection Expenditures and CO2 Emissions

	Average Environmental Protection Expenditures (as percentage of GDP)			Average Growth of CO2 Emissions (in percents)		
Year	1995-2000	2000-2005	2005-2007	1995-2000	2000-2005	2005-2007
Germany	0.55	0.49	0.37	-0.95	-0.37	-0.78
Austria	0.65	0.43	0.33	0.73	3.98	-2.70
Portugal	0.21	0.30	0.29	4.38	1.46	-6.45
Finland	0.49	0.41	0.39	-0.57	1.12	8.67
Spain	0.15*	0.24	0.27	4.06	3.69	0.78
UK	0.47*	0.32	0.30	0.30	0.32	-1.03

However, over the observed time period they have begun to address the environmental issues to greater extent in terms of higher part of GDP devoted. At the same time, growth of their pollution capacities started to decrease (or even turned into decline in case of Portugal).

In conclusion, we have seen that situation with the amount of costs devoted to the environmental protection varies with each specific country following its own pattern. As to evolvement of emission process, for the countries lowering their green budget different behaviors were observed. On the other hand, both countries enhancing their environmental protection expenditures have experienced improvement of pollution conditions. Therefore we may propose that devoting additional sources to the improvement of a deteriorating situation with the emissions is quite likely to bring the desired effect. However, if the country follows the opposite direction, which means reduction of the ‶green″expenditures, the outcome on pollution and environmental quality can follow different (even opposite) patterns based on situation regarding to the other indicators in economy.

1.2.3 Empirical Comparison of End-of-Pipe and R&D Abatement

Now it's time to inquire and compare different abatement techniques. In order to do this we will look over an empirical study by Frondel et al. [18]. To examine this diferentiation of abatement, the authors investigated data in several OECD countries (Germany, Japan, United States, and four others). The data are from the OECD survey taken on the facility level, which makes this study even more complex and comprehensive.

The measures taken to reduce the environmental damage caused by production can be divided into two fundamental categories: end-of-pipe abatement, which uses the add-on actions to regulate the emissions level, and abatement with R&D, which can be considered as an investment into the clean technology. Authors further remark that end-of-pipe cleaning activities mostly aim solely at reduction of the harmful by-products of production. On the other hand, R&D abatement can provide for realization of potential benefits such as reduced costs or creation of new markets with goods or processes, which are environmentally friendly. Therefore, this type of abating activity is viewed as more advantageous by the firms. However, it is widely agreed that firms' decisions are still dominated by the end-of-pipe techniques. There are several reasons for this statement, including the fact that for the firms it's not always simple or applicable to meet environmental standards by a sole investment in cleaner technologies. Another reason, stated by the authors, is that there had been only few empirical analyses of this topic because of the scarcity of the available data. The paper inquired a state of the art in this issue together with the factors, which may stimulate firms to use R&D instead of the end-of-pipe.

As for findings of the study, an average over 75% of sample facilities utilized mainly the R&D abatement. Looking at the particular countries, highest proportion was detected in Japan with the contribution 86.5%. On the other hand, Germany displayed lowest share at about 57.5%. This fact can be, according to the authors, explained by the previous strong governmental support of the end-of-pipe abatement in Germany. Overall, authors found out an surprising prevalence of investment in cleaner technologies against the add-on environmental measures, which exists in the surveyed OECD countries.

To be precise on content of the study, we must mention also findings concerning determinants, which influence firms´decision about the type of abatement. To inquire this, an estimation multinomial logit models were used. The results suggested that cost savings, as a decision factor, along with market forces, support choice of R&D abatement, while regulatory weight and extent of the environmental policy enhance utilization of the end-of-pipe techniques. The paper further exploited differentiation of environmental innovations, however we won´t address this issue in the thesis.

In the conclusion, the empirical evidence has shown that amounts of emissions and waste have grown to a tremendous size. This increase was significantly inflicted by boost of the economic production. Although this unpleasant finding, the positive fact is that the rate of the economic growth has exceeded increase of the pollution stock. Moreover, regarding to the type of abatement used by firms, cleaner pollution, which is econmically and environmentally more desirable, has started to gain advantage against the end-of-pipe techniques.

1.3 Different Perspectives

As we have already outlined in motivation, there is much controversy about the economic growth and environmental quality and their mutual relationship. This implies an uprise of diverse scientific views on this issue. An inspirational overview is provided by Van den Bergh and De Mooij in [43]. They describe five different perspectives on growth and environment, particularly focusing on how they view a potential of mutual conformity of growth and environmental preservation. What makes this comparison even more interesting is the fact that the authors came out from the general framework of basic factors in economy, structures of economics activities and their changes. Let´s take a look at a short overview of these perspectives.

The first perspective can be called immaterialistic. A philosophy of these economists can be shortly described by a statement that growth is undesirable. Truly one of the main issues, which they are pursuing, is whether the economic growth is really something that we want, or whether this is an indicator, which should be pursued by the economic policy [14]. Authors usually try to point out the fact that fast economic growth in general doesn´t coincide with a rise in welfare or well-being. Of course, there is some relationship, but according to this group, it´s not really significant. Furthermore, many of them hold an opinion that because economic growth has proven to be often associated with degradation of the natural environment, it should be evaded. Examples can be found in the works of Mishan, Schumacher or Daly [13].

The economists representing next viewpoint are known as the pessimists. According to their opinion, the sustainable economic growth is impossible in the long run. The growth, which remains at levels observed in the past decades, will inevitably lead to the irreversible exhaustion of natural resources. This process may be as well associated with degradation or even destruction of natural and environmental components. Furthermore, these scientists are really pessimistic about the technological potential to prevent further environmental damage. This perspective was worked out in studies by the Club of Rome overviewed by Nordhaus in [23]. Other mentioned authors are Duchin and Lange or Georgescu and Roegen.

Following perspective is advocated by the scientists, who can be described as the technocrats. In their opinion a compatibility exists between growth and environmental quality. Moreover, the economic growth, represented by increasing valuation of man-made goods and services, and expansion of production and consumption in the sense of use of material and energy aren´t related in any particular way. The question, which they are asking, is whether economy can realize an ever-growing value added on the basis of a finite amount of natural resources and environmental capacity. Note that this concept strongly recalls the one of the sustainable growth. They point out several channels that can be used to relax the constraints on growth, including a replacement of non-renewable resources by a renewable capital, or stressing out the importance of investments and technological progress while lowering use of resources. Overall there is a strong emphasis of the significance of ˵environmental technology″in order to harmonize growth and environment. More can be found e.g. in the work of Goeller and Weinberg [19].

Next group is called opportunists. Their perspective was outlined by the statement that growth and environmental degradation are inevitable. According to these economists, we can hardly influence a path of economic development. Reason for this rather confident statement is the fact that both the rate and the direction of growth are result of economic decisions at the micro-level, most notably by households and firms, and as such are not infuenceable by the economic policy maker. Besides that, any effort by government to change economic or social stability is nowadays restricted by number of international agreements and institutions. As a representative for this group, a work by Aalbers was selected.

The last view on growth and environment, as it is stated by authors, is represented by the optimists. These economists have a positive attitude to the economic growth, stating that it may be beneficial for environmental conservation. According to them, positive economic growth can promote such changes in preferences and institutions, which can be substantial for environment as well as for the economy. Another argument by these economists states that global situation concerning environment isn´t as alarming, as it is often presented. They use even empirical evidence to show that economic growth hasn´t damaged nature too severely. Instead it has provided for rise in overall welfare [6]. One of the best known representatives of this ideas is Beckerman.

To conclude, different and often opposing perspectives are undoubtedly an consequence of complexity as well as difficulty of given issue. Obviously, none of them is completely right and each one has its pros and cons from theoretical as well as from empirical point of view. Moreover, we could notice that these viewpoints aren´t strictly disjunct and have lot in common. Nevertheless, each perspective contains something inspiring, which could be included into our analysis. Thus the overview literature and becoming familiar with different opinions turns into an important and inspiring part of this work.

1.4 Development of Environmental Policy

In the final section of this introductory chapter, we will present which actions have been taken to solve the issues of economic growth and environmental quality and how they developed in history. According to Smulders [35], strong connection of environment and economic growth has existed during the entire economic history of the world. This mutual relation has become especially considerable after 18^{th} century and the industrial revolution. A significant rise in the exploitation of natural resources caused by stream of innovations and enormous growth of industrial production began to cause serious deteriorative changes in the environment. For a long time, this has been largely ignored by the producing companies as well as the governing authorities. It wasn't until the 20^{th} century, when the first considerable decisions to improve this unpleasant situation were made.

In the middle of last century, several governments have realized a necessity of action in area of environmental protection, as the negative effects of pollution couldn´t be ignored any longer. In the 1950s, the Resources for the Future, an independent research organization addressing various environmental issues while developing and applying relevant economic theory, was founded in the United States. According to Pearce [25], this laid a foundation for the environmental economics. Later, the National Environmental Policy Act was signed, thus becoming first major environmental law in the United States (see e.g. [11]). Consequently, environmental issues started to be addressed in many countries through the laws, measures or agencies striving to promote and improve environmental quality.

First major international action on the global scale was The United Nations Conference on the Human Environment held in Stockholm in 1972 [41]. One of the most important results was the statement of the Action Plan for the Human Environment, which included specific recommendations for the governments regarding treatment of the environment. This led to an adoption of the first Environmental Action Programme by the European Community (later European Union) [20]. After this first environmental conference, several other meetings took place, beginning with the UN Conference on Environment and Development, held in Rio de Janeiro in 1992 (see [42]) and ending with the Climate Council in Copenhagen in 2009. These meetings, especially the most recent ones, aimed at creation of binding global agreement to lower environmental degradation, mostly by the reduction of greenhouse gas emissions.

One of the most important measures regarding international environmental policy is the Kyoto Protocol. Adopted in 1997, it was motivated by growing public interest in the climate change, which is assumed to be caused by emission of greenhouse gases. This document introduces obligatory emission targets, which should be reached by industrialized countries during the period 2008-2012. As significant as it is, the Protocol brings about opposing viewpoints (see [7]). For the supporters, it is a major accomplishment on the field of internetional climate policy, while opponents argue about defectiveness of its concept, namely regarding set up of the goals and schedules of the emissions´ reduction. Moreover, although it has been accepted by majority of countries, the United States, country producing the biggest amount of emisssions, didn´t ratify it, thus causing serious blow to the expected effect of these measures. Despite this flaws, according to Böhringer [7] is an adoption of this agreement an important step toward future decisions. Because of its flexible international mechanism, the Kyoto Protocol provides valuable foundation for the future creation of efficient environmental policies.

To sum up, topics dealing with the environmental quality and impacting measures have developed into complex and controversial issues that require significant efforts of policy makers world wide. They realize, however, that in order to accept decisions optimal fo social welfare, a sufficient preparation with thorough theoretical background has to be excercised. Thus a place for environmental economists to realize research and apply their findings is created. An overview of approaches within the framework of economic growth theory follows in the next chapter.

Kapitola 2

Overview of Literature

Before moving to the analysis of particular models, we will devote a part of this thesis to the overview of works concerned with question of dependence between economic production and quality of environment. This relation has already providen various inspirations for the academic research. As we have already seen, there are many perspectives, developed by the economists, which address problems outlined by this issue (see section 1.3). Now we will look at the analytical tools used in their analyses, particularly ones concerned with the models of economic growth theory. At the beginning, we will look at the foundations of environmental growth models, which will be illustrated by one of the earlier works devoted to a pure consumption-pollution trade-off. Consequently, a development of theory starting with models that incorporate capital accumulation up to the models with endogenous growth will be overviewed. In the final part, several other growth models using an approach of directed technical change will be mentioned, representing the state of the art of this field.

2.1 Foundations and Earlier Works

It shouldn´t be surprising that issue of environmental quality has been part of economic theory since its very beginning. First influential work, which addressed this topic, was written by Malthus as early as in 18^{th} century. In the course of early development of economic theory, several works addressed various particular environmental problems faced by the growing economies. According to Pearce (see [25]), foundations of the environmental economics can be laid in the 1950s, when Resources for the Future, an independent organization that developed and applied economics to the environmental problems, was founded in the United States. However, it wasn´t until early 1970s, when this economic field gained on importance. During this time period the energy crisis associated with world-wide recession drew considerable attention to the issues regarding scarcity of natural resources or undesirable production of pollution. From this point on, more economists have started to address these problems in their works.

One of the economic fields contributing greatly to issues outlined above is the neoclassical theory of economic growth. Extensive textbooks on this field were written e.g. by D. Romer [31] or D. Acemoglu [2]. As it is stated in paper by Huang and Cai [21], the growth theories were developed in 1950s and 1960s. Some of basic works, which laid foundations for this field of research, were written by Solow [36], Swan [38] or Cass [10]. Providing new opportunities for economic analysis, these tools became employed in wide range of issues, one of them being the topics concerned with the dependence of economic growth and quality of environment. A modelling of the environmental factors, such as pollution or natural resources, has been introduced by many authors in their economic growth models. However, as we have already stated in our motivational section 1.1, in order to be more precise and in-depth we will focus on works concerned with the issues of pollution and emissions, thus omitting topics of exhaustible or non-exhaustible resources. Some of the resource models were proposed by e.g. Anderson [4], Dasgupta and Stiglitz [15] or Tahvonen and Kuuluvainen [40].

At the beginning, the pollution (or waste) control was included in simple models with only one state variable. These works didn't include capital accumulation and the level of production output was given exogenously, as they wanted to keep formulations simple in order to use graphic analytical tools. Examples are papers written by Plourde [26], Smith [34] or Forster [17], who formulated their models as the optimal control problems. Forster´s model will be overviewed more closely later in the section, but we will start with a look at first two models.

In his article, Plourde considers a waste as a state variable, which evolves over time. Its stock is generated by the fixed proportion of production output and abated by waste disposal services together with biodecomposition. Specifically, author differentiates between fixed input capital, which can be alternatively used in production process or abatement activity, and consumption goods generated by production. As to the control variables, the planning authority sets levels of production and abatement capital as well as the consumption and "disservices"proportional to the pollution stock (which is subject to the dynamic equation). The utility function, which is concave and additive, is influenced by level of this consumption good as well as by these "disservicesöf pollution.

Let´s remark on some of the results presented in this work. Author derived and analyzed optimal solution, specifically proposing about the mutual relations between the variables. Interestingly, despite assuming several non-negativity restrictions on the variables, he considered only optimal solution in the interior of the state space, while neglecting the boundary solutions (some of the consumption or capital variable is equal to zero) as they are, according to the author, unlikely to occur. As particularly important part, the steady state solution and its stability were examined. The graphical representation of the phase space provided for the analysis of the stationary point and development of the system for given initial position. Finally, the paper contains remarks about achieving the optimal state by the taxation as well as other possible extensions of the model.

Another similar model was formulated by Smith in [34]. He again considered waste as a state variable, which develops over time. The agents in economy can perform three productive activities: produce new goods, recycle waste and produce commodities for recycling. Each of these activities require part of total amount of input resource and produced capital has to equal investment into abatement, whether through recyclacion or "preparation"process. A different approach was also employed while modelling the utility function, which is, along levels of production and waste, influenced also by the recycling process that causes the utility losses.

2.1.1 Simple Model of Consumption-Pollution Trade-Off

At this place, we will look more closely on the model presented by Forster in [17]. Similarly to the many other works, the article is outlined as a planning problem for governing authority, which aims to maximize a future utility by setting optimal values of certain controlable variable. A contrast with the previous works as well as to works we will be presented later is the focus on a pure consumption-pollution trade-off. Despite this difference, we will present this model because it's a suitable example concerning how the earlier works coped with this issue.

Formally it is a simple model of optimal control theory, containing only one control variable - consumption C, and one state variable - pollution P. The author mathematically formalized given problem in the following way:

$$\max_{C} \int_{0}^{\infty} \mathrm{e}^{-\rho t}\, U(C,P)\, \mathrm{d}t \tag{2.1.1}$$

$$\dot{P} = Z(C) - \alpha P, \quad P(0) = P_0, \quad P(\infty) = free, \tag{2.1.2}$$

$$\Phi^0 - C \geq 0, \quad P \geq 0. \tag{2.1.3}$$

As can be see, the model consists of three important parts. Firstly, there is maximized functional (2.1.1), which represents present value of future utility over infinite time horizont. Notion $U(C,P)$ depicts standard utility function, which will be described in more detail in part 3.1. Dynamic equation with initial (and terminal) conditions (2.1.2) describes evolvement of the pollution over time. Function $Z(C)$ represents an impact that the choice of consumption level has on the change in pollution stock. According to the author, it can be also viewed as a pollution control function, since the selection of consumption level determines the mass of pollution. Term α, on the other hand, relates to the decay of the pollution stock.

The last equation (2.1.3) of the model´s formulation provides constraints on the variables, where Φ^0 is exogenously set level of a total output. This simplification can be considered as the major drawback of the model, because as we are talking about economic growth based on the real data, the level of output can be regarded as anything but fixed. The reason for such assumption, as the paper states, was an aim at focus solely on the consumption-pollution trade-off. In addition, keeping the model simple in terms of modelling only one state variable allowed for a graphical analysis of optimal solutions, which won't be the case in more complex models.

After the analysis of the optimal patterns in the system, paper further inquires an impact of variations in the parameters on the optimal levels of consumption and pollution. Particularly interesting is effect of change in output level Φ^0. Its growth causes increase in consumption level as the productive capacity of the system is higher. However, there can be different impacts on the pollution level, which can be explained through the various extent of indirect effect through the consumption. Based on this findings, author concluded that economic growth doesn't necessarily cause a higher optimal level of the pollution.

The paper is concluded by the phase portraits´ analysis. In the initial model, the equilibrium is shown to be a unique saddle point, with the optimal solution of the model as the trajectory along the stable branch. However, if economy starts in the initial point not belonging to the stable branch, the system will diverge from the optimal trajectory. Author later sharpened analysis by assuming the parametrization of pollution marginal disutility near the clean environment. Depending on the value of this parameter, the solution can be similar to the one from the initial model or the equilibrium is achieved in a clean environment. In certain cases of boundary stationary point, the optimal trajectory reaches a state of the clean environment in a finite time and stays there, as the marginal utility of consumption is lower than its marginal psychic costs.

To sum up, presented earlier models dealing with the environmental issues provided interesting application of the optimal control theory within the theory of economic growth. However, the omission of the capital accumulation (and the economic growth) significantly reduced empirical relevance of these models. As we will see in the following section, the development in theory of optimal control as well as the rise of new approaches in the growth theory provided for more complex and realistic models.

2.2 From Models with Capital Accumulation to Endogenous Growth

The major drawback of models presented so far was their simplicity in terms of using only one state variable. This was caused mainly by the theoretic limitations, as the authors didn't have tools to analyze more complicated systems. With the development of more advanced theory in 1980s, economists began to incorporate more complex features into their models. An example of two dimensional optimal control model was presented in paper by Luptáčik and Schubert [22], which will be introduced in detail in chapter 3.

Later, the authors Tahvonen and Kuulvainen developed another two dimensional model in [40]. Their problem also includes capital acumulation along with the development of pollution stock. The difference is that they consider impact of the policy on environmental quality through the set up of the emission level instead of abatement expenditures. Moreover, these emissions are modelled as a necessary input in the production process, thus making it difficult to possibly incorporate "green"technical change in the model. Later in the work, two extended models were formulated and analyzed: one with the added sector of renewable resource and other considering externality taxes on emissions and harvesting.

Even more interesting than model's formulation are analytical tools used to solve problems and examine stability of stationary state. Authors used knowledge about dynamic systems, most notably the particularly inspirational findings regarding global stability derived by Sorger ([37], see also in appendix B.2). Applying this results, they were able to propose quite simple and straightforward conditions for the problems of existence and uniqueness of steady state along with its stability. We will try to use this approach later in this thesis (chapters 3 and 4) during the analysis of our models.

In the framework of the economic growth theory, a new view was brought up by the introduction of endogenous growth models in late 1980s (see e.g. works by P. Romer [30] or Barro [5]). This new approach started to incorporate the endogenous technological change instead of the exogenous one. Thus, per capita growth in stationary state could grow without restrictions observed in older works. This has led to the more complex models, which were better in correspondence with the empirical evidence. As an example we can mention the works written by Bovenberg and Smulders [8] or Huang and Cai [21], which will be overviewed in more detail.

Similarly to the earlier works, these authors (Huang and Cai) outlined their model as an optimal control problem with maximization of utility over the infinite time horizont. To represent a development of the economy as well as the environment, two dynamic equation are set up, which model capital per capita, generated by production function with technological progress, and pollution. The emission flow is produced by stock of capital and at the same time diminished by two types of variables: an exogenous government spending on the abatement activities and cost of pollution control per capita, which is part of the national budget. Moreover, part of the produced value is subtracted to the taxes, in order to preserve concept of ‶no free lunch″.

This rather complex model provides several interesting propositions. Given by the endogenous technological development, constant growth of capital implies an equal growth of production and consumption. Also, if consumption rises constantly, at rate determined endogenously, the abatement expenditures grow at the same and pollution at slightly lower growth rate. While dealing with the suitability of environmental policy, authors proposed that in economy, where the government considers the abatement expenditures to rise a pollution control´s efficiency, the consumption growth rate is higher than the one in competitive economy. This proposition shows that the environmental measures, which are chosen effectivelly, can actually cause an improvement of welfare in terms of greater consumption. Consequently, a complementarity of these goals from the perspective of the optimal economic policy can be remarked.

2.3 Models of Directed Technical Change

The framework of endogenous growth has laid foundations for the concept of directed technical change, which can be considered as state of art in this field. It proposes that not only the determinants of technical change, but also its direction, or way in which it is biased towards particular factors, is important. This concept was initiated by works of Acemoglu (see e.g. [1]) and it has soon become employed in several models of environmental economics. In the following section, we will present some of them. The main question, which they are asking, is how do the restrictions of theenvironmental policy affect economic growth. While looking through each model, we will introduce its formulation, outline some specifics and also mention some of the proposed results.

One of the earlier works was written by Ricci [29]. In his analysis, he uses the Schumpeterian model of endogenous growth. The model is extended by emissions, which are emitted by capital and they can be at the same time considered as an implicit production input. The contribution of this paper is that it offers an alternative channel of impact of the environmental policy on the economic growth, which lowers the trade-off between environmental quality and growth. This is achieved through the modelling of innovations: not only they improve a capital productivity, but also its pollution intensity. On the other hand, model doesn´t contain the end-of-pipe abatement. As a conclusion of analysis, it is shown that stricter environmental policy influences the economic growth in two opposite ways: it powers the innovations, but the marginal impact of the innovations on productivity growth decreases. The effect is positive only if the firms have little possibility to reduce pollution intensity of innovations.

Very inspirational analysis was put together by Rauscher [28]. To inquire an impact of environmental regulations on innovations and economic growth, a simple model with two types of capital is put together. Unlike other authors, he simplifies this model by assuming only one agent in economy, who employs in all the activities: consumption, saving, production and R&D. However, as author states, only under assumption that markets are perfectly competitive is this model equivalent to the ones with multiple agents. In the matter of outcomes, the work shows that when the environmental standards are tightened, the steady-state rates of investments are decreased in both types of capital. Consequently, a steady-state of growth rate declines as well. One positive impact of the stricter standards is the replacement of the conventional (polluting) capital by the green one (environmentally friendly).

Another similar model can by found in the working paper by Cunha-e-Sá et al. [12] It´s again a two sector model with endogenous growth, where clean or dirty innovations are created by the technical change. Authors show that neither the optimal growth nor the optimal emission rate can be achieved in the decentralized equilibrium. The optimal growth rate decreases with the growing number of consumers, who take the environment into consideration. Furthermore, both R&D subsidies and pollution tax should be used in order to move the research more towards cleaner technology as well as to increase the rate of innovation.

To summarize, this overview presents different approaches to analyze an impact of environmental policy on the economic growth. Authors used different modelling techniques as they are trying to construct a model supported by an empirical evidence. Overall, these works provide a great inspiration for the building process of an growth model, whose purpose is to contribute to the debate on issues of environmental quality and economic growth.

Kapitola 3

Analysis of Environmental Growth Model

After the introduction of topic and overview of literature, we now finally get to the main point of this thesis - an analysis of environmental growth model. As we will try to inquire different issues dealing with the dependence between economic growth and environment, we will focus on the model formulated by authors Luptáčik and Schubert in [22]. We will divide our analysis into several parts. First, we will introduce this model and present its building block. We will be particularly interested in the way of modelling of environmental parts - pollution and abatement activities. Consequently, a results presented in authors´ work will be overviewed, along with notion about used mathematical techniques. In the third part, we will inquire existence and stability of optimal stationary state, while applying different analytical tools.

All in all, our focus isn´t the sole mathematical analysis of the model, but also to draw economic interpretations from these results. In order to make this work reasonable, we will seek an empirical rationale in our theoretical findings.

3.1 Formulation of Model

At the beginning, we should overview a formulation of model, particularly focusing on individual building parts it consists of. As authors outlined in the paper´s introduction, their main aim is to contribute to the discussion about some controversial environmental issues concerning the economic growth. These include questions of trade-off relation between economic growth and environmental quality or time effects of expenditure spent on pollution abatement, while taking into account a nature of chosen strategy. As the analysis and investigation of optimal policies is pursued, the paper takes into account various situations regarding e.g. level of abatement technology or economic development.

In the formulation of their model, authors are applying theory of economic growth, particularly that of optimal control. They set up a model with two control and two state variables, all of them depending on time. A central governing authority can decide about levels of consumption C and abatement expenditures A. Environmental quality (or its inverse) is included in the model through the stock of pollution P. Finally, we should mention economic growth, which is generated by the capital K.

Goals of policy makers are increase in the consumption level (thus making a "consumingëconomy better off) while cutting down on level of pollution. This is expressed via ordinary, strictly concave utility function $U(C,P)$. The assumptions regarding this function, which define relation between two welfare variables, are following:

$$\begin{aligned} &U_C > 0, \qquad U_{CC} < 0, \qquad C > 0, \\ &U_P < 0, \qquad U_{PP} < 0, \qquad P > 0, \\ &U_{CP} \leq 0, \qquad U_{CC}U_{PP} - (U_{CP})^2 \geq 0. \end{aligned} \tag{3.1.1}$$

We can see that marginal utility is positive for consumption and negative for pollution, while both are decreasing. Moreover, last condition along with decrease in marginal consumption utility provides for strict concavity of the utility function.

Particularly interesting is condition laid on the mixed derivative U_{CP}. Assuming $U_{CP} \leq 0$, we state that increase in either consumption or environmental degradation causes diminishment of marginal utility of other varible, for example higher levels of air pollution decreases consumer's utility from additional grilled sausage. However, not every work adapts this approach, as some of them presume conditions $U_{CP} = 0$ or even $U_{CP} \geq 0$ (see e.g. Tahvonen and Kuuluvainen [40]). These authors use an argument that the rise in consumption is likely to cause a reduction of pollution's marginal disutility. Nevertheless, we will use condition as it is stated in (3.1.1).

As we try to evade optimal policy of zero consumption, we will further impose one of Inanda conditions on the utility function:

$$\forall P > 0 \quad \lim_{C \to 0} U_C(C,P) \to +\infty. \tag{3.1.2}$$

Note that we will use sign f_x to represent partial derivative of arbitrary function f with respect to variable x (i.e. $f_x = \partial f / \partial x$).

This utility function expresses only instantaneous utility, while we would like to consider also future welfare levels in our decisions. Thus we calculate present value of all future utilities depending on the levels of consumption and pollution. Using discount rate ρ to describe time preferences and considering infinite time horizont, we can formulate a following optimization problem:

$$\max_{C,\ A} \int_0^\infty \mathrm{e}^{-\rho t}\, U(C, P)\, \mathrm{d}t. \tag{3.1.3}$$

As we have already mention, level of economic growth is influenced by amount of capital K. It is a single production factor of a production function $F(K)$, which holds all the usual assumptions:

$$F'(K) > 0, \qquad F''(K) < 0, \qquad F(0) = 0. \tag{3.1.4}$$

The production Y generated by the capital K (and expressed via $F(K)$) can be exploited in several ways: it can be either invested to increase stock of capital K or spent on consumption C. Third and last option is to use producted value on abatement expenditures A. This can be labeled as national income constraint

$$Y = I + C + A. \tag{3.1.5}$$

As the stock of capital K is generated by investment, this differentiation of total expenditures allows us to define dynamic equation describing capital development over time:

$$\dot{K} = F(K) - C - A - \beta K. \tag{3.1.6}$$

As the last item in this relation, we included also the rate of capital depreciation β.

Now it´s time to address a development of pollution stock. Just to note here, that emissions are considered as a flow variable, while pollution is corresponding stock variable. There are two opposite ways, how do the economic activities affect environmental quality. First, different sectors produce emissions and thus increase mass of pollution. Authors include three such sectors in their model: production, consumption and capital depreciation. Emissions generated by production sector are expressed as $E_Y = \epsilon_1 Y$. Coefficient ϵ_1 (with positive value) represents part of a technology adding up to the environment deterioration and it can change with application of new technologies. This consideration was neglected in the work, however we will address it in chapter 4.

Emissions amanating from consumption activities can be approached in manner similar to the production process. An amount can be described as $E_C = \epsilon_2 C$, where $\epsilon_2 > 0$ displays environmental inefficiency by consumption. Finally, taking into account capital depreciation with rate β and constant emission ratio $\bar{\epsilon}_3$, we can represent environmentally negative effects of amortization by expression $E_K = \bar{\epsilon}_3 \beta K = \epsilon_3 K$.

On the other hand, there are economic activities that can lead to the improvement of environmental conditions. In the paper, they consider an abatement activity of transformation process, where in exchange for current expenditures the level of pollution is lowered. To represent this activities, author use a total abatement function $G(A)$, which affects pollution stock. Regarding assumptions on this function, it meets similar conditions like production function $F(K)$:

$$G'(A) > 0, \qquad G''(A) < 0, \qquad G(0) = 0. \tag{3.1.7}$$

Note that this abatement method by its nature (literally "cleaning"pollution) resembles concept of end-of-pipe techniques (as introduced in 1.2, particularly in 1.2.3). This will become more obvious as we will differentiate abatement techniques in later chapter.

To sum up, after joining this abatement function with channels that produce emissions, we obtain dynamic equation of pollution evolvement:

$$\dot{P} = \epsilon_1 F(K) + \epsilon_2 C + \epsilon_3 K - G(A) - \alpha P. \tag{3.1.8}$$

Since the nature itself has ability to regenerate (transformation of undesirable waste into neutral substances), authors included a rate of pollution decay α in this expression.

Finally, after introduction of all these parts, we are ready to present a mathematical formulation of the model. A subject of optimization (3.1.3) along with dynamic equations (3.1.6) and (3.1.8), which describe development of system, form together with initial and terminal conditions and sign restriction on some of the variables a following optimal control problem:

$$\left.\begin{array}{l}\max\limits_{C,A}\displaystyle\int_0^{\infty} \mathrm{e}^{-\rho t}\, U(C,P)\,\mathrm{d}t\\[1ex] \dot{K} = F(K) - C - A - \beta K\\[1ex] \dot{P} = \epsilon_1 F(K) + \epsilon_2 C + \epsilon_3 K - G(A) - \alpha P\\[1ex] K(0) = K_0, \qquad K(\infty) = free\\[1ex] P(0) = P_0, \qquad P(\infty) = free\\[1ex] A \geq 0, \qquad P \geq 0.\end{array}\right\} \quad \text{Planning Problem 1}$$

3.2 Results Presented in the Original Work

Now, as we have outlined formulation of model, we are able to present analysis realized in the paper. The authors focused on several problems: firstly they derived and analyzed optimal allocation patterns, then and finally inquired about asymptotic convergence of optimal solution towards the steady state. We will follow the structure of the work and address each topic in individual subsection.

3.2.1 Optimal Solution and Its Analysis

Before start of any analysis, authors formulated the Lagrangian function based on the Planning Problem 1 in order to solve the outlined problem:

$$\begin{aligned} L \;=\; & U(C,P) \;+\; \psi_1\left(F(K) - C - A - \beta K\right) \;+\\ +\; & \psi_2\left(\epsilon_1 F(K) + \epsilon_2 C + \epsilon_3 K - G(A) - \alpha P\right) \;+\\ +\; & sA \;+\; r\left(\epsilon_1 F(K) + \epsilon_2 C + \epsilon_3 K - G(A) - \alpha P\right). \end{aligned} \tag{3.2.1}$$

On the basis of the optimal control theory (for overview of topic see e.g. [32] or [33]), variables ψ_1 and ψ_2 are co-state variables, which have an interpretation of shadow prices of state variables K and P, while s and r are the Lagrange multipliers associated with restrictions laid upon A and P $(A \geq 0, P \geq 0)$.

After this formulation, authors applied Pontryagin´s Maximum Principle to derive optimal solutions. The necessary conditions in this case are:

$$L_C = U_C - \psi_1 + \psi_2\epsilon_2 + r\epsilon_2 = 0 \tag{3.2.2}$$

$$L_A = -\psi_1 - \psi_2 G'(A) + s - rG'(A) = 0 \tag{3.2.3}$$

$$s \geq 0, \quad sA = 0 \tag{3.2.4}$$

$$r \geq 0, \quad rP = 0 \text{ and } r\dot{P} = 0. \tag{3.2.5}$$

Based on these expressions, authors formulate several propositions, which interpret analytical results economically. They address mostly marginal utility of consumption and pollution on the optimal path, considering also several particular modifications in parameters.

The paper further examines signs of shadow prices ψ_1 and ψ_2, thus inquiring whether capital (and indirecly investment) and pollution always have same preferences regarding social welfare. Using equations (3.2.2) and (3.2.3), it´s easy to show that $\psi_1 > 0$, meaning that capital investment is always beneficial to the social welfare, as it can be utilized in several useful ways. On the other hand, we obtain $\psi_2 \gtrless 0$ for the shadow price of pollution. This may be surprising, since we would expect pollution stock to be undesirable good (and therefore its shadow price to be negative). However, authors argue that pollution can be valued positively (in terms of welfare) in case, when there is no abatement activity ($A = 0$), and social price of starting it is too high (coefficient s is large).

As you have already noticed, the optimal solution conditions are constructed using current time Hamilton (and Lagrange) function, meaning that expression $\mathrm{e}^{-\rho t}$ is not included in (3.2.1). In accordance with the optimal control theory, authors include discount rate ρ in the costate equations:

$$\dot{\psi}_1 = (\rho - F'(K) + \beta)\,\psi_1 - (\epsilon_1 F'(K) + \epsilon_3)\,(\psi_2 + r) \tag{3.2.6}$$

$$\dot{\psi}_2 = -U_P + (\rho + \alpha)\,\psi_2 + r\alpha. \tag{3.2.7}$$

These expressions are in the paper used to derive particular results concerning growth rates of shadow prices.

An interesting analysis is performed considering compatibility of growth in abatement expenditures and consumption. Althought there is a trade-off relation between these indicators in every period (given by constraint (3.1.5)), authors derived analytical results, which show that a mutual positive growth of cleaning activities and consumption is possible. The proposed sufficient condition for $\dot{C} > 0$ (under assumption of A growing) is

$$\rho + \left(\beta + \frac{\epsilon_3}{G'(A)}\right) + \left(\frac{\sigma(A)}{A} + \frac{G''(A)}{G'(A) + \epsilon_2}\right)\dot{A} \quad < \quad \left(1 - \frac{\epsilon_1}{G'(A)}\right)F'(K),$$

where $\sigma(A) = -A\big[G''(A)/G'(A)\big] > 0$ denotes the elasticity of abatement´s marginal efficiency. The condition proposes that consumption can grow (together with the cleaning costs) on the optimal path, if the net marginal product (marginal product decreased by portion needed to clean additional emissions) is greater than the marginal environmental consequences of this consumption increase for a given abatement growth.

As a final analysis of optimal solution, the paper inquires question of long-term growth in abatement expenditures. To find a suitable and interpretable condition, authors used results already derived in the work. Finally, they formulated following condition:

$$\frac{\dot{\psi}_2}{\psi_2} > \frac{\dot{\psi}_1}{\psi_1} \quad \Rightarrow \quad \dot{A} > 0.$$

As we can see, the long-term growth in abatement costs depends on mutual relation of growth rates of shadow prices. Cleaning expenditures will grow along the optimal path provided that growth rate of marginal social value of environmental quality is greater than that of capital accumulation.

3.2.2 Steady State and Its Stability

Now we will present results of the paper considering the stationary state. It is a situation, where the development of state variables approaches 0, thus halting growth in the system. This means that $\dot{K} = \dot{P} = 0$ as well as $\dot{\psi}_1 = \dot{\psi}_2 = 0$. The steady state values of variables are denoted by upper index ∞, e.g. K^∞. A several propositions, which deal with this state, were derived in the work. Firstly, emission rate due the production process in the steady state must be exceeded by marginal abatementnet, i.e. $G'(A^\infty) > \epsilon_1$. This is an implication of derived result that net marginal product has to be equal to the interest rate and cost of capital depreciations:

$$F'(K^\infty)\left(1 - \frac{\epsilon_1}{G'(A^\infty)}\right) \;=\; \rho \;+\; \beta \;+\; \frac{\epsilon_3}{G'(A^\infty)}\;.$$

Furthermore, it's simple to show that marginal utility of consumption and environmental quality has to be equal, with the second one being discounted by augmented interest rate:

$$U_C(C^\infty, P^\infty) = -\frac{U_P(C^\infty, P^\infty)\big(G'(A^\infty) + \epsilon_2\big)}{\rho + \alpha}.$$

As the last stationary state proposition, the "no change"condition implies investment at the level just replacing depreciated capital

$$I^\infty = F(K^\infty) - C^\infty - A^\infty = \beta K^\infty, \tag{3.2.8}$$

and abatement equal to the total emission rate

$$\epsilon_1 F(K^\infty) + \epsilon_2 C^\infty + \epsilon_3 K^\infty = G(A^\infty) + \alpha P^\infty. \tag{3.2.9}$$

In the final part of this work, authors analyze question of optimal solution convergence to the stationary state. From the analytical point of view, this is the most interesting as well as challenging part of the paper, since convergence question for problems with more than one state variable (and discounted future utility) is non-trivial. In order to contribute to this issue, authors tried to apply results from the work by Brock and Scheinkman (see discussion paper [9]).

After application of theoretical findings to their planning problem and derivation of mutual effects between variables (see section 3.2.3), the authors were able to conclude stability conditions. They proposed, under stricter assumptions $A > 0$, $P > 0$ and $U_{CP} < 0$, that the steady state is locally asymptotically stable (with few exceptions) in case that expression

$$G'(A^\infty)\rho + \epsilon_3 + \beta\epsilon_1 - \left(\epsilon_2 \frac{U_{CP}}{U_{CC}} + \alpha\right)\big(G'(A^\infty)/\epsilon_1\big)$$

is negative and there exist a closed curve Γ, which contains this steady state and the vector field described by dynamic system

$$\dot{\underline{k}} \;=\; H_{\underline{p}}\big(\underline{p}(\underline{k}), \underline{k}\big)$$

is "pointing inward", in other words k_i near zero implies $\dot{k}_i > 0$ and k_i large enough provides for $\dot{k}_i > 0$. Note that we have used notation of vectors of state variables by $\underline{k} = (k_1, k_2) = (K, P)$, control variables by $\underline{x} = (C, A)$, and shadow prices by $\underline{p} = (\psi_1, \psi_2)$. As it mentioned in the paper [9], factual verification of inward pointing hypothesis is rather nontrivial task. In this thesis, we won´t inquire this topic any further, as the alternative approach will be applied. However, readers interested in this methodology should see cited literature for more details.

3.2.3 Comparative Statics

To conclude this section, we will mention derived comparative statics. During the stability analysis realized in previous part, the authors had to calculate what effects do the state variables and shadow prices have on the control variables along the optimal path. Since we will be using them later in our analysis, it is therefore useful to mention these conditions. Based on (3.2.2) and (3.2.3), which characterized the optimal solution, the following relations concerning consumption C can be derived:

$$\frac{\partial C}{\partial K} = 0, \qquad \frac{\partial C}{\partial P} < 0, \qquad \frac{\partial C}{\partial \psi_1} < 0, \qquad \frac{\partial C}{\partial \psi_2} > 0. \tag{3.2.10}$$

As we can see, level of consumption along the optimal path isn´t affected by changes in capital level, only by changes in its social value. If this shadow price increases (meaning capital is perceived as more valuable), it is optimal to raise capital investment, thus dropping level of consumption. Regarding changes of environmental quality, its deterioration (implied by growth of pollution stock) cause consumers to lower their consumption as to prevent continuation of this effect. Similarly, if subjective value of pollution decreases (meaning people are more sensitive about environmental problems), an expenditures are again switched away from consumption.

Using analogical approach, we can receive similar expressions for the effects on abatement expenditures A:

$$\frac{\partial A}{\partial K} = 0, \qquad \frac{\partial A}{\partial P} = 0, \qquad \frac{\partial A}{\partial \psi_1} < 0, \qquad \frac{\partial A}{\partial \psi_2} < 0. \tag{3.2.11}$$

This time, while following the optimal patterns, abatement expenditures A shouldn´t be affected by change in neither of the state variables K or P. In shadow price of capital rises, a costs devoted to the environmental cleaning should be lowered, because of the similar reasons like in case of consumption effect. On the other hand, decrease in social value of pollution causes more capital resources to be devoted to the abatement.

3.3 Further Analysis of Stationary State

In the previous section we have presented analysis of Planning Problem 1, particularly concerned with the stationary state of the system, which was performed in the surveyed paper. Now it´s time to solve and analyze this problem using different theoretical background. In our inquiry, we will be using approach from the work by authors Tahvonen and Kuuluvainen [40].

3.3.1 Existence of the Steady State

Firstly, we will again derive expressions representing the steady state and try to formulate conditions for its existence. Let K^{∞}, P^{∞}, C^{∞}, A^{∞}, ψ_1^{∞} and ψ_2^{∞} denote steady state values of state, control and costate variables (to simplify notation, we will suppress an upper index of ∞ for these variables in this section). We assume that these values follow the optimal path, thus meeting the necessary conditions (3.2.2) - (3.2.5). Moreover, because they are in the stationary state, the relations $\dot{K} = \dot{P} = 0$ as well as $\dot{\psi}_1 = \dot{\psi}_2 = 0$ must be satisfied (as already stated in section 3.2.2). To sharpen our analysis, let's assume that the steady state belongs to the interior of the control and state space, i.e. $A > 0$ and $P > 0$. This presumption has several important implications: firstly, the values of Lagrangian multipliers hold $r = 0$ and $s = 0$. Moreover, according to statement concerning signs of shadow prices in section 3.2.1, the costate variable of pollution ψ_2 should be negative.

The stationary point is a solution of following system of equations:

$$0 = F(K) - C - A - \beta K \tag{3.3.1}$$

$$0 = \epsilon_1 F(K) + \epsilon_2 C + \epsilon_3 K - G(A) - \alpha P \tag{3.3.2}$$

$$0 = U_C(C,P) - \psi_1 + \psi_2 \epsilon_2 \tag{3.3.3}$$

$$0 = -\psi_1 - \psi_2 G'(A) \tag{3.3.4}$$

$$0 = \big(\rho - F'(K) + \beta\big)\psi_1 - \big(\epsilon_1 F'(K) + \epsilon_3\big)\psi_2 \tag{3.3.5}$$

$$0 = -U_P(C,P) + (\rho + \alpha)\,\psi_2 \tag{3.3.6}$$

Firstly, let´s use equation (3.3.1) to express A as a function of K and C:

$$A(K,C) = F(K) - C - \beta K. \tag{3.3.7}$$

Moreover, equation (3.3.2) along with previous condition (3.3.7) defines following dependence of P on values of K and C:

$$P(K,C) \;=\; \frac{1}{\alpha}\big(\epsilon_1 F(K) + \epsilon_2 C + \epsilon_3 K - G\left[A(K,C)\right]\big). \tag{3.3.8}$$

As can be seen, we can express ψ_2 and consequently ψ_1 as functions of C and K using equations (3.3.6), (3.3.4). Thus we receive

$$\psi_2(K,C) \;=\; \frac{U_P\left[C, P(K,C)\right]}{(\rho+\alpha)} \tag{3.3.9}$$

and

$$\psi_1(K,C) \;=\; -\psi_2\, G'(A) \;=\; -\frac{U_P\left[C, P(K,C)\right]}{(\rho+\alpha)}\, G'\left[A(K,C)\right]. \tag{3.3.10}$$

After this calculations, we can substitute derived conditions (3.3.7) - (3.3.10) to the expressions in remaining steady state equations (3.3.3) and (3.3.5) and formulate two functions $\mathcal{F}$ and $\mathcal{G}$:

$$\mathcal{F}(K,C) \;:=\; U_C\big[C, P(K,C)\big] + \psi_2(K,C)\,\big(G'[A(K,C)] + \epsilon_2\big), \tag{3.3.11}$$

$$\mathcal{G}(K,C) \;:=\; \big(\rho - F'(K) + \beta\big)\,\psi_1(K,C) - \big(\epsilon_1 F'(K) + \epsilon_3\big)\,\psi_2(K,C). \tag{3.3.12}$$

We can see that task of proving an existence of the stationary state is now changed into equivalent problem of finding common root to these two functions.

Firstly, denote capital level K_1 that holds $F'(K_1) = \beta$, and K_2 such that $F(K_2) = \beta K_3$. Obviously $K_1 < K_2$ because of the assumptions (3.1.4) laid on the production function F. Moreover, we can see that any level of capital K in the steady-state must be lower than K_2 because of positivity of A (and as well C) and condition (3.3.7).

Let´s look at function $\mathcal{F}$, more specifically on problem $\mathcal{F} = 0$. Using the assumptions laid on functions U and G, along with effects calculated from the conditions (3.3.7) - (3.3.10), we receive $\partial\mathcal{F}/\partial C < 0$. Consequently, for $K \in (K_1, K_2)$ we have also $\partial\mathcal{F}/\partial K < 0$. Denote K_3 a capital level such that for $K > K_3$ a condition $\partial\mathcal{F}/\partial K < 0$ holds (note that from the continuity of derivative that $K_3 < K_1$). Thus using implicit function theorem, for capital level $K \in (K_3, K_2)$ consumption C is strictly decreasing (and therefore injective) function of K ($C = C(K), C'(K) < 0$) implicitly given by the expression $\mathcal{F} = 0$.

Now we will focus on the function $\mathcal{G}$. After substituting variable C by the functional dependence $C(K)$, which has been derived above, $\mathcal{G}$ will be dependent only on the level of capital K, given:

$$\mathcal{G}(K) \;=\; \big(\rho - F'(K) + \beta\big)\,\psi_1(K, C(K)) - \big(\epsilon_1 F'(K) + \epsilon_3\big)\,\psi_2(K, C(K)).$$

Proof that this function reaches zero value will be equivalent to the existence of the stationary state. One can notice that $\psi_1(K, C(K)) > 0$ for any level of K. Similarly, $(\epsilon_1 F'(K) + \epsilon_3) > 0$ and $\psi_2(K, C(K)) < 0$. Thus the equality $\mathcal{G}(K) = 0$ can be only achieved, if the condition $F'(K) > \rho+\beta$ is satisfied and $\psi_1(K, C(K)) > 0$ is high enough for some levels of capital K. This provide for existence of a level of capital K^* (and implied level of consumption C^*) such that $\mathcal{G}(K^*) = 0$. Since F is concave function, K^* must be less than K_1. Moreover, in order to meet the constraint on implicitly given function $C(K)$, K^* has to be greater than K_3.

After this inquiry, we can finally propose a sufficient condition for existence of the optimal steady state, where the variables follow their pattern according the conditions of Maximum Principle. This stable point exists, if there is an interval of capital levels K, where the marginal product of capital F' is greater than sum of discount rate ρ and depreciation rate β and at the same time the social value of capital is sufficiently positive. This proposition holds if rates of discount and depreciation are small enough, marginal productivity is high enough, or additively social price of capital is high and social value of environmental quality is relatively low. Moreover, we can see that in case that $\lim\limits_{K\to 0^+} F'(K) < \rho + \beta$, no steady state with $A > 0$ and $P > 0$ exists.

3.3.2 Steady State without Abatement Costs

Concerning the existence of the steady state at the boundary, let´s consider the case when no abatement expenditures are spent, i.e. $A = A_0 = 0$ and thus $s \geq 0$. In this case, we have to make several changes into accoust during formulation of conditions: $G(A_0) = 0$ and $G'(A_0) = G'_0 > 0$. Thus, again using necessary conditions 3.2.2 and 3.2.3 together with the dynamic equations for state and costate variables in stationary state, we receive similar stationary state system of equations to the previous part (see expressions (3.3.1) - (3.3.6)), with (3.3.1), (3.3.2) and (3.3.4) changed as follows:

$$\begin{aligned} 0 &= F(K) - C - \beta K \\ 0 &= \epsilon_1 F(K) + \epsilon_2 C + \epsilon_3 K - \alpha P \\ 0 &= -\psi_1 - \psi_2 G'_0 + s. \end{aligned}$$

Applying similar approach like in section 3.3.1, we can express C, P, ψ_1, ψ_2 and s as the functions dependent on level of capital K:

$$
\begin{aligned}
C(K) &= F(K) - \beta K && (3.3.13)\\
P(K) &= \frac{1}{\alpha}\big(\epsilon_1 F(K) + \epsilon_2 C(K) + \epsilon_3 K\big) && (3.3.14)\\
\psi_2(K) &= \frac{U_P\left[C(K), P(K)\right]}{(\rho+\alpha)} && (3.3.15)\\
\psi_1(K) &= U_C(C(K), P(K)) + \psi_2(K)\epsilon_2 && (3.3.16)\\
s(K) &= \psi_1(K) + \psi_2(K)G_0'. && (3.3.17)
\end{aligned}
$$

Thus our task to proove an existence of the stationary state is reduced to the question of finding root of following expression, derived from (3.3.5):

$$0 \;=\; \big(\rho - F'(K) + \beta\big)\psi_1(K) - \big(\epsilon_1 F'(K) + \epsilon_3\big)\psi_2(K) \;=:\; \mathcal{H}(K). \qquad (3.3.18)$$

Let´s denote K_1 a capital level such that $F'(K_1) = \beta + \rho$. Using similar reasoning, one can see that expressions $\psi_1(K) > 0$ and $\epsilon_1 F'(K) + \epsilon_3 > 0$ as well as $-\psi_2(K)$ are all positive for any level of K, thus $\mathcal{H}(K) > 0$ for any $K > K_1$. However, if $K \to 0^+$, we obtain $C \to 0$ and $P \to 0$, with U_C growing over any bound and U_P bounded (see assumptions 3.1.1), thus providing $\mathcal{H}(K) < 0$.

Suming up, an optimal stationary state in the stage of system without abatement can exist given that marginal productivity for low levels of capital is sufficienly high, specifically $\lim\limits_{K\to 0^+} F'(K) > \rho + \beta$. We could go on and analyze boundary steady state in the case of clean environment with abatement in process ($P = 0$ and $A > 0$) using similar approach. There is no need to address stage without pollution and abatement, as this implies zero values of capital and consumption, which is non-optimal (and in fact unattainable) based on assumption laid on utility function (3.1.2).

3.3.3 Saddle Point Property of the Steady State

One important question concerning the steady state is whether it has local properties of saddle point. To inquire this task, we will apply an approach used by Tahvonen and Kuuluvainen [40], which is presented in appendix, section B.1. The modified Hamiltonian dynamic system for our Planning Problem 1 is expressed by the dynamic equations (3.1.6) and (3.1.8) for the state variables and (3.2.6) and (3.2.6) for the costate variables. We construct Jacobi matrix (using comparative statics conditions (3.2.10) and (3.2.11)) as follows:

$$J = \begin{pmatrix} F'(K)-\beta & -C_P & -C_{\psi_1}-A_{\psi_1} & -C_{\psi_2}-A_{\psi_2} \\ \epsilon_1 F'(K)+\epsilon_3 & \epsilon_2 C_P-\alpha & \epsilon_2 C_{\psi_1}-G'(A)A_{\psi_1} & \epsilon_2 C_{\psi_2}-G'(A)A_{\psi_2} \\ -(\psi_1+\epsilon_1\psi_2)F''(K) & 0 & \rho+\beta-F'(K) & -\epsilon_1 F'(K)-\epsilon_3 \\ 0 & -U_{CP}C_P-U_{PP} & -U_{CP}C_{\psi_1} & -U_{CP}C_{\psi_2}+\rho+\alpha \end{pmatrix}. \tag{3.3.19}$$

Consequently, we can calculate Ω as it was defined in (B.1.2):

$$\Omega = \begin{vmatrix} F'(K)-\beta & -C_{\psi_1}-A_{\psi_1} \\ -(\psi_1+\epsilon_1\psi_2)F''(K) & \rho+\beta-F'(K) \end{vmatrix} + \begin{vmatrix} \epsilon_2 C_P-\alpha & \epsilon_2 C_{\psi_2}-G'(A)A_{\psi_2} \\ -U_{CP}C_P-U_{PP} & -U_{CP}C_{\psi_2}+\rho+\alpha \end{vmatrix} + \\ + 2\begin{vmatrix} -C_P & -C_{\psi_2}-A_{\psi_2} \\ 0 & -\epsilon_1 F'(K)-\epsilon_3 \end{vmatrix} \tag{3.3.20}$$

In order to prove that our steady state has a local saddle properties, we have to find conditions under which $\Omega < 0$ and $\Delta = det(J) > 0$.

Let´s start with the sign of Ω. The signs of particular elements are given by the assumptions and the comparative statics. Therefore it´s easy to see that last matrix in expression (3.3.20) is always positive. In the first two matrices, the sign of only one unit in each of them is ambiguous. Therefore, we can conclude that Ω is positive given if following inequalities hold:

$$\psi_1+\epsilon_1\psi_2 > 0 \tag{3.3.21}$$

$$U_{CP}C_P+U_{PP} < 0. \tag{3.3.22}$$

The examination of the sign of determinant Δ is more difficult because of the density and dimension of the Jacobi matrix (3.3.19). To derive the expression of this determinant, we use computer algebra system Maxima, which can execute calculations with symbolic values. Keeping in mind conditions derived for the sign of Ω along with the assumptions, we can reduce the question of the sign of Δ to the inquiry of the sign of two expressions:

$$C_{\psi_2}+A_{\psi_2}, \tag{3.3.23}$$

which describes reaction of optimal consumption and abatement costs level with regard to the change in social value of pollution, and expression

$$\epsilon_2 C_{\psi_1} - G'(A) A_{\psi_1}, \tag{3.3.24}$$

that describes the effect that change in the price of capital has on the pollution development. Finally, after careful examination, we are able to find the sufficient conditions. We derive that our stationary point has the local properties of the saddle, if following statement holds true: expression (3.3.23) is equal to zero and (3.3.24) is negative. Furthermore, we require $F''(K)$ to be close enough to zero or $F'(K)$ to be sufficiently high. Let´s note that other sufficient conditions may be formulated as well.

To sum up, by the application of methodology used in article by Tahvonen and Kuuluvainen [40], we were able to derive some of the characteristics, which may sufficiently provide for the saddle point behavior of our dynamic system locally near the steady state. If we interpret these conditions economically, we receive that the social price of capital must be in absolute values higher than the price of pollution proportional to the production process (3.3.21) ; a change of marginal pollution disutility due to the pollution increase must be negative (3.3.22); the changes of optimal levels of consumption and cleaning expenditures caused by the social valuation of environmental quality must be equal (equation (3.3.23)); a rise in the social price of capital has to decrease a growth rate of pollution (equation (3.3.24)); and finally marginal productivity must be high.

3.3.4 Stability Analysis of the Steady State

After we have inquired the question of existence and characteristics of the stationary state, let's now focus on stability conditions of this point. The stability in the dynamic systems is understood in sense of optimal solution´s convergence to this steady state. To derive analytical results, we will use theoretical propositions proposed in article [37] by Sorger. The applied findings are outlined in appendix part B.2. Following an assumption in the Luptáčik and Schubert´s Paper (see p. 466 in [22]), we will again consider interior point of the state space, i.e. $A > 0$ and $P > 0$.

First, we will define current value Hamiltonian for our Planning Problem 1 following the expression (B.2.1):

$$\begin{aligned} H(K,P,\psi_1,\psi_2) &= \max_{C,A}\Big\{U(C,P)+\psi_1\big(F(K)-C-A-\beta K\big)+ \\ &+\ \psi_2\big(\epsilon_1 F(K)+\epsilon_2 C+\epsilon_3 K-G(A)-\alpha P\big)\Big\} = \\ &= U(C(P,\psi_1,\psi_2),P)+\psi_1\left(F(K)-C(P,\psi_1,\psi_2)-A(\psi_1,\psi_2)-\beta K\right)+ \\ &+\ \psi_2\left(\epsilon_1 F(K)+\epsilon_2 C(P,\psi_1,\psi_2)+\epsilon_3 K-G(A(\psi_1,\psi_2))-\alpha P,\right) \end{aligned} \tag{3.3.25}$$

where $C(P,\psi_1,\psi_2)$ and $A(\psi_1,\psi_2)$ denote optimal responses of consumption and abatement expenditures on the levels of the state and costate variables, which are derived from the necessary conditions (3.2.2) and (3.2.3). The effects, or signs of derivatives, have been presented in the part concerned with the comparative statics 3.2.3.

Now we apply results by Sorger. The sufficient condition of global stability of the steady state is negative definiteness of curvature matrix $\mathbb{C}$. Now we will derive this matrix from our system following the equation (B.2.4) while setting coefficient $\gamma = 0$ (just to note that $x = (K,P)$ and $p = (\psi_1,\psi_2)$). The matrix H_{xx} for our problem is

$$H_{xx} = \begin{pmatrix} (\psi_1+\epsilon_1\psi_2)F'' & 0 \\ 0 & U_{PP}+U_{CP}C_P \end{pmatrix}. \tag{3.3.26}$$

Concerning the matrix $-H_{pp}$, we use again an implicit function theorem to derive following expression

$$-H_{pp} = \begin{pmatrix} C_{\psi_1}+A_{\psi_1} & G'A_{\psi_1}-\epsilon_2 C_{\psi_1} \\ G'A_{\psi_1}-\epsilon_2 C_{\psi_1} & G'A_{\psi_2}-\epsilon_2 C_{\psi_2} \end{pmatrix}, \tag{3.3.27}$$

After composing the matrix $\mathbb{C}$, our task is to inquire the conditions for its negative definiteness. In order to do so, we will use the Sylvester´s criterion, which is necessary and sufficient condition for this matrix characteristic. Since all of the diagonal elements of $\mathbb{C}$ are negative under assumptions (3.3.21) and (3.3.22), the following two conditions will provide desired result:

$$-H_{KK}\,H_{\psi_1\psi_1} - \frac{\rho^2}{4} > 0, \tag{3.3.28}$$

$$H_{KK}\left[H_{PP}\left(H_{\psi_1\psi_1}H_{\psi_2\psi_2}-H^2_{\psi_1\psi_2}\right)+\frac{\rho^2 H_{\psi_1\psi_1}}{4}\right] + \frac{\rho^2}{4}\left(H_{PP}H_{\psi_2\psi_2}+\frac{\rho^2}{4}\right) > 0. \tag{3.3.29}$$

Finally, let´s formulate derived sufficient conditions for the local stability of the stationary point. Assume that this steady state of our system exists in the interior of the state space. We propose that this rest point is locally asymptotically stable, if the conditions (3.3.28) and (3.3.29) as well as (3.3.21) and (3.3.22) hold. This restrictions can be satisfied if e.g. the discount rate ρ is sufficiently small.

At this place, we have to note that because of these restrictions, the stationary state does´t meet the requirements for global stability from the Sorger´s corollary, as we can't ensure the negative definiteness of the matrix $\mathbb{C}$ for all $(x, p) \in \mathbb{R}^n \times \mathbb{R}^n$ (see conditions (3.3.21) and (3.3.22)). Nevertheless, we will pursue our analysis while inquiring only the local stability of our steady state (i.e. condition (B.2.3) holds only locally around this state) under assumption that the derived conditions are satisfied in some open area around this point.

To conclude, this chapter presented an application of the optimal control theory to the economic issues of the dependence between economic growth and environmental quality. Although the analyzed model provided quite thorough insight in the topic, we may have missed possibility to influence the emissions´ rates, particularly the one describing environmental burden of production process. Analysis of this additional abatement technique could bring a deeper insight into the economic progress´ impact on the state of environment.

Kapitola 4

Model with the Differentiation of Abatement

As we´ve seen in empirical study by Frondel (presented in section 1.2.3), the type of cleaning technique does matter in the process of abatement activities. The differentiation between R&D abatement in sense of investments into cleaner production process and end-of-pipe technique, which cleans harmful by-products already emitted by production, was introduced. Therefore, in order to make our analysis more realistic, we will now modify Planning Problem 1 analyzed in the chapter 3 with this concept of differentiated abatement activities, specifically using endogenously changed rate of pollution generated by the production process. At the beginning, we will introduce this model and explain the modifications compared to the previous one. In following part, we will analyze it and solve the relevant planning problem. Moreover, we will examine qualitative characteristics of shadow prices, analyze the long-term trends in optimal levels of variables and derive conditions of the comparative statics. Consequently, a steady state of the system and its stability will be inquired. During this calculations, there will be an overview of differences between this and original model. Finally, we will derive results for some special cases, where one of the abatement activities is neglected.

4.1 Specification of the Model

In the previous chapter, we have analysed model which included environmental policy measures in terms of general abatement expenditures A. These costs were part of total spendings in economy and their impact on the level of pollution was modelled through the general function $G(A)$. Within the frame of abatement differentiation, this approach can resemble the end-of-pipe activities, where pollution from the environment is cleaned up. On the other hand, flow of emissions caused by the production process was determined by coefficient ϵ_1, whose value was given exogenously.

In our new model, we divide the abatement expenditures A into two distinct variables. Firstly, A_1 represents R&D expenditures as an investment devoted to lower environmental impact of production. This effect is achieved by making the emission coefficient ϵ_1 function dependant on A_1 with following conditions:

$$\forall A_1 \geq 0 : \ \epsilon_1(A_1) \ > \ 0, \qquad \frac{\partial \epsilon_1}{\partial A_1} \ < \ 0, \qquad \frac{\partial^2 \epsilon_1}{\partial A_1^2} \ > \ 0. \tag{4.1.1}$$

To intepret this, production cannot be made completely emission-free only by investment into the cleaner technologies, as there is always some pollution generated. The next point is that the higher investment is spent on R&D abatement, the lower polluting by-products are emitted in the production process. However, an absolute value marginal improvement is decreasing, as it is always harder to find new ways to improve environmental efficiency with the many means already being implemented.

On the other hand, A_2 will stand for techniques, which tackle pollution present in environment. In mathematical representation of our model, this variable will take place of previous general abatement A, i.e. it will be input in the function $G(A_2)$, which decreases the rate of emissions with properties defined by expression (3.1.7). Both these indicators will be considered as the control variables in our optimal control problem.

To sum up, we have again an optimal control problem with two state (P and K) and three control variables (consumption C along with A_1 and A_2). In order to focus on analysis of abatement techniques and their comparison, we will omit some parts from previous model, particularly the depreciation and the additional sources of pollution other than production. The mathematical formulation of problem is as follows:

$$\left.\begin{aligned}
&\max_{C,A_1,A_2} \int_0^\infty \mathrm{e}^{-\rho t}\, U(C,P)\, \mathrm{d}t \\
&\dot{K} = F(K) - C - A_1 - A_2 \\
&\dot{P} = \epsilon_1(A_1)F(K) - G(A_2) - \alpha P \\
&K(0) = K_0, \quad K(\infty) = free \\
&P(0) = P_0, \quad P(\infty) = free \\
&A_1 \geq 0, \qquad A_2 \geq 0, \qquad P \geq 0.
\end{aligned}\right\} \quad \text{Planning Problem 2}$$

4.2 Optimal Behavior Path

4.2.1 Optimal Solution of Our Problem

Solution of the model again starts with formulation of current value Lagrange function

$$\begin{aligned}
L = {} & U(C,P) \ + \ \psi_1\left(F(K) - C - A_1 - A_2\right) \ + \\
& + \ \psi_2\left(\epsilon_1(A_1)F(K) - G(A_2) - \alpha P\right) \ + \\
& + \ s_1 A_1 \ + s_2 A_2 \ + \ r\left(\epsilon_1(A_1)F(K) - G(A_2) - \alpha P\right),
\end{aligned} \tag{4.2.1}$$

where ψ_1 and ψ_2 denote (in correspondence to the Planning Problem 1) the shadow prices of captial and pollution, s_1 and s_2 stand for Lagrangian multipliers belonging to the restrictions of different abatement expenditures and r is multiplier of capital restriction.

Now we apply Pontryagin´s Maximum Principle to receive following necessary conditions:

$$L_C = U_C(C,P) - \psi_1 = 0 \tag{4.2.2}$$

$$L_{A_1} = -\psi_1 + \psi_2 F(K)\epsilon_1'(A_1) + s_1 + rF(K)\epsilon_1'(A_1) = 0 \tag{4.2.3}$$

$$L_{A_2} = -\psi_1 - \psi_2 G'(A_2) + s_2 - rG'(A_2) = 0 \tag{4.2.4}$$

$$s_1 \geq 0, \quad s_2 \geq 0, \qquad s_1 A_1 = 0, \quad s_2 A_2 = 0 \tag{4.2.5}$$

$$r \geq 0, \qquad rP = 0 \quad \text{and} \quad r\dot{P} = 0. \tag{4.2.6}$$

Moreover, variables ψ_1 and ψ_2 must follow the costate equations:

$$\dot{\psi}_1 = \rho\psi_1 - \psi_1 F'(K) - \psi_2\epsilon_1(A_1)F'(K) - r\epsilon_1(A_1)F'(K) \tag{4.2.7}$$

$$\dot{\psi}_2 = \rho\psi_2 - U_P(C,P) + \psi_2\alpha + r\alpha. \tag{4.2.8}$$

Expressions (4.2.2) - (4.2.4) provide several interesting interpretations as the patterns for optimal planning. As we compare them to the propositions stated in the original paper [22] (see section 3.2.1), several differences can be noticed. First of all, marginal utility of consumption has to be equal a production cost of consumed capital. Since we don´t consider pollution effect caused by consumption, this equality isn´t affected by any environmental indicator. Concerning differentiation of abatement we can see that marginal contribution of end-of-pipe activities $G'(A_2)$ has to equal that of cleaner production process $(-F(K)\epsilon_1'(A_1))$, when both types are used (i.e. $A_1 > 0$, $A_2 > 0$ and $s_1 = s_2 = 0$).

Several remarks should be made regarding signs of shadow variables ψ_1 and ψ_2. From the condition (4.2.2) we receive immediately:

$$\psi_1 \ = \ U_C(C,P) \ > \ 0 \tag{4.2.9}$$

Similarly to the results in original work, the marginal social value of capital is always positive as the additional units of this variable can be always spent in beneficial way. On the other hand, condition of ψ_2, which can be derived from (4.2.3) or (4.2.4), is slightly more complex:

$$\psi_2 \ = \ \frac{\psi_1}{F(K)\epsilon_1'(A_1)} - r - \frac{s_2}{F(K)\epsilon_1'(A_1)} \ \gtrless \ 0, \tag{4.2.10}$$

or alternatively

$$\psi_2 \ = \ \frac{-\psi_1}{G'(A_2)} - r + \frac{s_2}{G'(A_2)} \ \gtrless \ 0. \tag{4.2.11}$$

Inspired by the propositions in the original paper, we can interpret this as follows: in case that there is positive level of (either type of) abatement in the system, the marginal social value of pollution holds negative. The only possibility for pollution to be valued positively by the society is the state without any abatement and with high cost of starting it (s_1 or s_2 is large).

4.2.2 Long-Term Trends in Consumption and Abatement Costs

In order to sharpen our analysis, let´s consider an interior point of the state spate ($A_1 > 0$, $A_2 > 0$ and $P > 0$). As we are following optimal patterns of resource allocation, a question arises whether we are able to enhance our consumption behavior ($\dot{C} > 0$) and at the same time maintain (or even rise) the investments into the protection of environmental quality ($\dot{A}_1 > 0$ or $\dot{A}_2 > 0$). In the short time horizont, this task is unattainable because of the instantaneous constraint on the national income $F(K) = C + A_1 + A_2 + I$, where I stands for the capital investment. However, holds this statement true also in the long term development of these variables? We will try to inquire this question in the following part.

Firstly, we take the expression (4.2.10) and differentiate it with respect to time. Thus we receive

$$\frac{\dot{\psi}_2}{\psi_2} \;=\; \frac{U_{CC}}{\epsilon_1'(A_1)F(K)}\,\dot{C} \;-\; \frac{\epsilon_1''(A_1)U_C}{\epsilon_1'(A_1)}\,\dot{A}_1. \tag{4.2.12}$$

Consequently, we combine (4.2.8) and (4.2.10) to obtain

$$\frac{\dot{\psi}_2}{\psi_2} \;=\; \rho + \alpha - \frac{U_P}{U_C}F(K)\epsilon_1'(A_1). \tag{4.2.13}$$

Let´s denote, with accordance to the original paper, the elasticity of marginal consumption utility $\sigma(C)$ and the elasticity of marginal efficiency of clean production investments $\sigma(A_1)$, and define them as follows:

$$\sigma(C) \;=\; -C\,\frac{U_{CC}}{U_C} \;>\; 0, \quad \text{and} \quad \sigma(A_1) \;=\; -A_1\,\frac{\epsilon_1''(A_1)}{\epsilon_1'(A_1)} \tag{4.2.14}$$

At last, we merge the equations (4.2.12) and (4.2.12) with the substitution of defined variables (4.2.14) to receive

$$\frac{\dot{C}}{C} \;=\; -\frac{1}{\sigma(C)}\left(\rho + \alpha - \sigma(A_1)\frac{\dot{A}_1}{A_1} - \frac{U_P}{U_C}F(K)\epsilon_1'(A_1)\right).$$

Assume that the expenditures devoted to the "green"technological progress are growing, i.e. $\dot{A}_1 > 0$. Then it’s easy to derive that

$$\dot{C} \;>\; 0 \quad \text{if} \quad \rho \;+\; \alpha \;<\; \frac{U_P}{U_C}\,F(K)\,\epsilon_1(A_1) \;+\; \sigma(A_1)\,\frac{\dot{A}_1}{A_1}.$$

To interpret this condition, we can propose following statement: if the elasticity of the marginal efficiency of costs devoted to the cleaner production, which is proportional to the growth of these expenditures ($\sigma(A_1)\,(\dot{A}_1/A_1)$), exceeds impact of natural self-cleaning ability (α), and the utility change resulting from the marginal change in these costs ($(U_P/U_C)\,F(K)\,\epsilon_1'(A_1)$) is higher than rate of discount (ρ) than there is possible consumption growth along the path of optimal solution ($\dot{C} > 0$).

Using similar approach, we can derive the propositions concerning the dependence between a growth in the consumption and increase of the end-of-pipe abatement costs. This time we start with the expression (4.2.11), which is again differentiated and than combined with (4.2.8) and (4.2.11). We also define elasticity of the marginal efficiency of end-of-pipe abatement $\sigma(A_2)$ as follows

$$\sigma(A_2) \;=\; -A_2\,\frac{G''(A_2)}{G'(A_2)}. \tag{4.2.15}$$

Finally we receive

$$\frac{\dot{C}}{C} \;=\; -\frac{1}{\sigma(C)}\left(\sigma(A_2)\,\frac{\dot{A}_2}{A_2} - \rho - \alpha - \frac{U_P}{U_C}G'(A_2)\right).$$

and thus, under assumption of growing abatement expenditures $\dot{A}_2 > 0$, we obtain following requirement for the increase of consumption:

$$\dot{C} \;>\; 0 \quad \text{if} \quad \rho \,+\, \alpha \;<\; \sigma(A_2)\,\frac{\dot{A}_2}{A_2} \;-\; \frac{U_P}{U_C}\,G'(A_2). \tag{4.2.16}$$

This condition can be interpreted analogously to the case of cleaner production investments A_1, i.e. the level of consumption grows in the long term horizont, if proportional elasticity of abatement activity surpasses rate of pollution decay in the nature and utility change is greater than the rate of value decrease (aka discount rate).

4.2.3 Comparative Statics

After proposing conditions for optimal solution and long-term trends, we can mention comparative statics of the variables. In order to make these computations more straightforward (and with respect to assumptions in further analysis), we assume that system is in the state with positive level of pollution ($P > 0$, i.e. $r = 0$) and both types of abatement are utilized (i.e. $A_1 > 0$ and $A_2 > 0$). Based on the necessary conditions (4.2.2) - (4.2.4), the effects of state and costate variables on optimal level of C, A_1 and A_2 will be derived using implicit function theorem. Let´s first focus on consumption, where we obtain following results:

$$\frac{\partial C}{\partial K} \;=\; 0, \qquad \frac{\partial C}{\partial P} \;<\; 0, \qquad \frac{\partial C}{\partial \psi_1} \;<\; 0, \qquad \frac{\partial C}{\partial \psi_2} \;=\; 0. \tag{4.2.17}$$

Compared to the results concerning original model presented in part 3.2.3, the only difference is that optimal level of consumption isn´t affected by the social value of pollution. The reason can be found in the omission of term representing the emissions caused by consumption.

The other effects, which have to be examined, are that concerned with the abatement expenditures. In contrast to the Planning Problem 1, this time we have to calculate and interpret changes in two different types of cleaning activities. Firstly, expressions related to the expenditures A_1, which are devoted to the cleaner production, will be calculated:

$$\frac{\partial A_1}{\partial K} > 0, \qquad \frac{\partial A_1}{\partial P} = 0, \qquad \frac{\partial A_1}{\partial \psi_1} < 0, \qquad \frac{\partial A_1}{\partial \psi_2} < 0. \tag{4.2.18}$$

An increase in capital level should enhance optimal level of cleaner production costs. Since the greater production brings additional emission burden, more resources should be devoted to preserve environmental quality. These resources, however, are made available through the corresponding capital growth. On the other hand, rise in pollution doesn´t affect these expenditures, as sources used to fight pollution disutility are taken away from consumption, thus causing another utility drop. Concerning social values of capital and pollution, both of them have negative effect on A_1 level. A higher price of capital leads to greater investments, which pulls resources off the abatement. Also higher (or less negative) valuation of pollution takes abatement away from social concern.

Finally, the effects of variables on end-of-pipe abatement costs will be derived:

$$\frac{\partial A_2}{\partial K} = 0, \qquad \frac{\partial A_2}{\partial P} = 0, \qquad \frac{\partial A}{\partial \psi_1} < 0, \qquad \frac{\partial A}{\partial \psi_2} < 0. \tag{4.2.19}$$

These effects are identical with the ones calculated for cleaning expenditures A in section 3.2.3, thus supporting our proposition that this type of the abatement in the original model represents concept of end-of-pipe abatement activities.

4.3 Steady State and Stability Analysis

In order to pursue our analysis of Planning Problem 2, we will now devote some space to the inquiry of the stationary point and its stability. As we have already mentioned, this concept of steady state is particuralry important, because it holds information about long-term sustainable position of the system. Consequently, a question, whether our optimal solution converge to this point, is essential if any relevant propositions about this optimal pattern should be made. Before starting, let´s note that throughout this section we will be considering only interior point steady state in sense that inequalities $P > 0$, $A_1 > 0$ and $A_2 > 0$ hold.

4.3.1 Sufficient Conditions for Existence

Similarly to the previous chapter, at the beginning the sufficient conditions for the existence of stationary state will be presented. According to the part 3.2.2, in this point the system doesn't change in any variable, which means $\dot{K} = \dot{P} = 0$ as well as $\dot{\psi}_1 = \dot{\psi}_2 = 0$. It is a position of the system, where the total production it completely spent on consumption and cleaning (without any further capital investments), and the abatement measures are at the level, where amount of pollution in environment doesn't change. To express this mathematically (while using same notation convention like in previous chapter), we recieve:

$$0 = F(K) - C - A_1 - A_2 \tag{4.3.1}$$

$$0 = \epsilon_1(A_1)F(K) - G(A_2) - \alpha P \tag{4.3.2}$$

$$0 = \big(\rho - F'(K)\big)\psi_1 - \epsilon_1(A_1)F'(K)\psi_2 \tag{4.3.3}$$

$$0 = -U_P(C, P) + \big(\rho + \alpha\big)\psi_2 \tag{4.3.4}$$

Moreover, we must consider also the necessary conditions for the optimal solution (4.2.2), (4.2.3) and (4.2.4), which must hold in the steady state:

$$0 = U_C(C, P) - \psi_1 \tag{4.3.5}$$

$$0 = -\psi_1 + \psi_2 F(K)\epsilon_1'(A_1) \tag{4.3.6}$$

$$0 = -\psi_1 - \psi_2 G'(A_2) \tag{4.3.7}$$

Thus a problem of stationary state existence is reduced to equivalent task of finding solution to system of equations (4.3.1) - (4.3.7).

As we have already stated, the optimal values of control variables C, A_1 and A_2 can be expressed in the dependence of state and costate variables by conditions (4.3.5) - (4.3.7). The following functions are received

$$C = C(P, \psi_1), \qquad A_1 = A_1(K, \psi_1, \psi_2), \qquad A_2 = A_2(\psi_1, \psi_2), \tag{4.3.8}$$

with the signs of derivatives already expressed in part 4.2.3. Consequently, we derive a relation for P using (4.3.2) along with (4.3.8), obtaining

$$P \;=\; P(K,\psi_1,\psi_2) \;=\; \frac{1}{\alpha}\Bigg(\epsilon_1\big[A_1(K,\psi_1,\psi_2)\big]F(K) \;-\; G\big[A_2(\psi_1,\psi_2)\big]\Bigg), \qquad (4.3.9)$$

and substitute these relations into the conditions (4.3.1) and (4.3.4). From the implicit function theorem we are able to propose that the unique functions $\psi_1(K)$ and $\psi_2(K)$ are implicitly given by the system of equations

$$\begin{aligned} 0 \;&=\; F(K) \;-\; C\big[P(K,\psi_1,\psi_2),\psi_1\big] \;-\; A_1(K,\psi_1,\psi_2) \;-\; A_2(\psi_1,\psi_2), \\ 0 \;&=\; -U_P\Bigg(C\big[P(K,\psi_1,\psi_2),\psi_1\big], P(K,\psi_1,\psi_2)\Bigg) \;+\; (\rho+\alpha)\psi_2, \end{aligned}$$

if the following condition holds true

$$\Bigg(-U_{CP}\big(C_P\,P_{\psi_1} \;+\; C_{\psi_1}\big) \;-\; U_{PP}\,P_{\psi_1}\Bigg)\Bigg(-U_{CP}\,C_P\,P_{\psi_2} \;-\; U_{PP}\,P_{\psi_2} \;+\; \rho \;+\; \alpha\Bigg) \;<\; 0. \qquad (4.3.10)$$

Thus we have reduced our problem of finding the solution of the system (4.3.1) - (4.3.7) to the existence of root of equation (4.3.3), which has following form with regard to the corresponding substitutions:

$$0 \;=\; \big(\rho - F'(K)\big)\,\psi_1(K) \;-\; \epsilon_1\Bigg(A_1\big[K,\psi_1(K),\psi_2(K)\big]\Bigg)\,F'(K)\,\psi_2(K). \qquad (4.3.11)$$

Now we apply a similar reasoning like in the previous chapter (see part 3.3.1). It´s obvious that factors ψ_1, $-\psi_2$, ϵ_1 and F' are always positive, thus only possibility of the root existence is $\rho - F'(K) < 0$. If the marginal product of capital is sufficiently high (and discount rate is small enough), than the equation 4.3.11 has solution in some capital level, which consequently implies the values of the remaining variables.

In conclusion, a stationary point of economy with the two types of measures dealing with an environmental deteriorations, cleaner production´s investments and end-of-pipe abatement, can exists, if the production generated by an additional unit of capital is high enough and the discount rate is rather small (condition (4.3.11)). Moreover, we require that a marginal pollution disutility reacts negatively to the change of the social value of capital and its potential decrease as a result of the pollution social value´s growth doesn´t exceed the discount rate and the rate of natural self-abatement (condition (4.3.10)). Note that the last proposal may also hold vice versa.

4.3.2 Qualitative Analysis of the Steady State

After finding some of the sufficient conditions for the stationary point´s existence, it's again useful to inquire about its qualitative properties. For the Planning Problem 1, we have derived several conditions, which, in case they are satisfied in the steady state, imply the local saddle point behaviour. Now we will inquire whether it is possible to draw a similar conclusion with regard to the Planning Problem 2. As we will analyze the steady state, whose existence has been examined, we again assume location in the interior of the state space.

During our analysis, we will again apply an approach used by Tahvonen and Kuuluvainen in their work [40], in the same manner like in part 3.3.3. Firstly, we derive a Jacobi matrix J for the dynamic system of state and costate equations (following pattern (B.1.1)) along with the variable Ω (see (B.1.2)). To recap (according to the propositions included in appendix B.1), the saddle point property of the rest point requires $\Delta > 0$ and $\Omega < 0$, where $\Delta := \det J$.

The derivation of Δ and Ω as well as the calculation of their sign using conditions of the comparative statics (4.2.17) - (4.2.19) is more difficult than the one realized in the section 3.3.3. Because of their complexity, we won't present an explicit form of J and Ω and limit ourself to the statement of results. After the thorough analysis, using the effects from part 4.2.3, we obtain following list of sufficient conditions:

$$\psi_1 \;+\; \epsilon_1 \psi_2 \;>\; 0, \qquad \left|\frac{\epsilon_1''}{\epsilon_1'}\right| \;>\; \left|\frac{\epsilon_1'}{\epsilon_1}\right|, \qquad \sigma(A_1)\,F(K) \;>\; A_1. \tag{4.3.12}$$

where is defined in (4.2.14). Moreover, we require F' to be sufficiently high

$$U_{CP}\,C_P + U_{PP} \;<\; 0, \qquad \left|U_{CP}\,C_P + U_{PP}\right| \;<<\; 1. \tag{4.3.13}$$

Note that some of the conditions are identical with the ones in the analysis realized in previous chapter.

To sum up, the steady state of the economy, which is subject to the Planning Problem 2, has the properties of the saddle point if:

- the social price of capital is greater than the valuation of the environmental quality proportional to the emission rate of production process;
- the rate of change in green R&D investments must be higher than the corresponding rate of change in marginal efficiency of these investments;

- the expenditures devoted to the cleaner production are lower than the production proportional to the elasticity of the marginal efficiency of these costs;
- an additional unit of capital causes sufficiently high production increase;
- the loss of marginal pollution disutility resulting from the pollution increase must exceed, though only slightly, its potential increase due to the consumption drop caused by this pollution growth.

4.3.3 Stability Analysis

Our analysis of the stationary point for the system Planning Problem 2 will be concluded by remarks on stability. We will again apply the proposals proved by Sorger in [37], which are overviewed in appendix B.2. Let´s begin with the construction of the curvature matrix $\mathbb{C}$, which follows the expression (B.2.4). By the differentiation of the current value Hamiltonian for this problem

$$\begin{aligned} H(K,P,\psi_1,\psi_2) &= U\big[C(P,\psi_1),P\big]+ \\ &+ \psi_1\Big(F(K)-C(P,\psi_1)-A_1(K,\psi_1,\psi_2)-A_2(\psi_1,\psi_2)\Big)+ \\ &+ \psi_2\Big(\epsilon_1\big[A_1(K,\psi_1,\psi_2)\big]F(K)-G\big[A_2(\psi_1,\psi_2)\big]-\alpha P\Big), \end{aligned}$$

as well as an application of the effects (4.2.17) - (4.2.19), we receive the matrices H_{xx}

$$H_{xx} = \begin{pmatrix} (\psi_1+\epsilon_1\psi_2)F'' & 0 \\ 0 & U_{PP}+U_{CP}C_P \end{pmatrix}$$

and $-H_{pp}$

$$-H_{pp} = \begin{pmatrix} C_{\psi_1}+(A_1)_{\psi_1}+(A_2)_{\psi_1} & G'(A_2)_{\psi_1}-\epsilon_1'F(A_1)_{\psi_1} \\ G'(A_2)_{\psi_1}-\epsilon_1'F(A_1)_{\psi_1} & G'(A_2)_{\psi_2} \end{pmatrix}.$$

Note that H_{xx} is exactly the same to the matrix (3.3.26) derived in the previous chapter.

The question whether $\mathbb{C}$ is negative definite, being a sufficient condition for the global asymptotic stability of the rest point, will be again inquired by the Sylvester´s criterion. Under assumption of first condition from (4.3.12) and condition (4.3.13) we obtain negative sign of all diagonal elements of the matrix $\mathbb{C}$. Thus the property of negative definiteness again depends on the following conditions:

$$-H_{KK}\,H_{\psi_1\psi_1}\;-\;\frac{\rho^2}{4}\;>\;0, \tag{4.3.14}$$

$$H_{KK}\left[H_{PP}\left(H_{\psi_1\psi_1}H_{\psi_2\psi_2}-H^2_{\psi_1\psi_2}\right)+\frac{\rho^2}{4}-H_{\psi_1\psi_1}\right]\;+\;\frac{\rho^2}{4}\left(H_{PP}H_{\psi_2\psi_2}+\frac{\rho^2}{4}\right)\;>\;0. \tag{4.3.15}$$

If these inequalities are satisfied, than the steady state of our system is locally asymptotically stable. This is more likely to hold in case that the discount rate ρ is small enough. Because of the restrictions (4.3.12) and (4.3.13), we are concerned only with the local, not the global asymptotic stability of our rest point (see reasoning in 3.3.4).

To conclude this chapter, we have modified the model developed by Luptáčik and Schubert in [22] by the inclusion of an additional channel of abatement through the technological progress. An analysis of this model and its stationary state, much resembling the approach from the previous chapter, has brought forward several interesting implications about the long-term trends in a development of different indicators along with the sufficient conditions for the existence, saddle-point behavior and the local stability of the rest point.

Conclusion

To sum up, our thesis addresses the environmental issues within the framework of the economic growth theory. Its main goal is to contribute to the question whether economic growth and quality of environment are complementary or substitutive goals of economic policy using the optimal control theory.

In the first chapter, we introduce the topic, stating why it is important to inquire it. Moreover, a several kinds of empirical evidence are presented. We overview and compare the trends in GDP growth and emission rate development in the United States, than the different policies regarding the total environment protection costs are presented. A study by Frondel [18], concerned with the comparison of R&D and end-of-pipe cleaning activities, concludes the empirical evidence. The chapter continues by mention about different scientific perspectives on this issue (based on paper [43]), which have evolved over time. Finally, we shortly look over the development of environmental policies, starting with the first UN conference dealing with the environmental issues up to the mention of the Kyoto Protocol.

The next chapter offers an overview of relevant literature. As our thesis is set within the framework of the models of economic growth and the optimal control theory, the similiar works are presented. We start with the simple models of pollution accumulation, where the level of output is set exogenously. Than, the examples of the environmental models incorporating a capital accumulation are presented. The overview is concluded by the examples of the environmental models with endogenous growth and finally the models of the directed technical change, which represent the state of the art in this economic field.

In the third chapter, we use the environmental growth model developed by Luptáčik and Schubert [22] to derive particular results and findings on this topic. The chapter starts with an overview of the planning problem and the results introduced by the authors. Our focus is the inquiry about the optimal steady state of the system, which is a state that the economy remains at. Firstly we derive sufficient conditions for its existence, obtaining that this point can exist if the marginal product of the capital is higher than the sum of discount and depreciation rate for some levels of capital and at the same time the social value of capital is sufficiently high.

Consequently we inquired the qualitative properties of the the rest point using the methodology from the paper by Tahvonen and Kuuluvainen [40]. We find out that it has local behavior of the saddle point if several conditions hold, most notably the social price of capital must exceeds the value of the environmental quality proportional to the emission rate of the production process (condition (3.3.21)) and the change of marginal pollution disutility caused by the increase of pollution stock must be negative (condition (3.3.22)).

The pivotal analysis of this thesis is concerned with the stability of the steady state is terms of optimal solutions´ convergence to this rest point. We apply theoretical findings by Sorger [37], thus receiving the sufficient conditions (3.3.28) and (3.3.29) for the local stability of the stationary point of our system. In comparison to the results from the original paper, we haven´t obtained more simple conditions, however our approach proved to be more straightforward.

Our thesis is concluded by the chapter devoted to the model with differentiated abatement. We modify an original model to include an endogenous change of emission coefficient of the production process through the capital invesments. A planning problem is outlined and analyzed in the similar manner to the previous chapter. Concerning the local stability, a several sufficient conditions are proposed, which hold if e.g. a discount rate is sufficiently small. Overall we can state that the differentiation of abatement, introduced in this chapter, makes our model somehow more plausible.

There are several possibilities to pursue our analysis even further. We may receive interesting results by using the production function of ‟endogenous growth″type. Moreover, as the more powerful tools of optimal control theory have been developed, our model could be extended to include the renewable or non-renewable resources. Overall, this field provides truly inspirational field for the economic research, which can find application in the decision-making process of the environmental policy.

Resumé

V našej práci sa zaoberéme problematikou znečistenia životného prostredia zasadenou v rámci teórie hospodárskeho rastu a optimálneho riadenia. Našim hlavným cieľom je prispieť k debate o vzájomnou vzťahu týchto dvoch faktorov a to v otázke či ide o substitučné alebo komplementárne ciele hospodárskej politiky. Dôležitým teoretickým výsledkom je analýza rovnovážneho stavu a odvodenie postačujúcich podmienok pre jeho lokálnu stabilitu aplikovaním pokročilých výsledkov teórie optimálneho riadenia pri dvoch spojitých modeloch hospodárskeho rastu.

Prvá kapitola tvorí úvod do tejto problematiky. Dôležitou časťou je poukázanie na vzťahy vychádzajúce z empirických pozorovaní. Na základe údajov z databázy OECD [24] ukážeme, že v ekonomike Spojených štátov rástol v posledných desaťročiach ukazovateľ HDP ako aj produkcie emisií CO2 a tvorba komunálneho odpadu. Na dátach môžeme pozorovať, že hospodárska produkcia rastie značne vyššou mierou než ukazovatele znečisťovania, čo môžeme pripísať vysokému technologickému pokroku a následnému znižovaniu produkcie emisií. Ďalej porovnáme podiely výdavkov ochrany životného prostredia na celkovej produkcii vo viacerých krajinách. Záver empirických pozorovaní tvorí štúdia prezentovaná Frondelom et al. [18], ktorá sa zaoberá porovnaním dvoch typov čistiacich aktivít: investícií do čistejšej produkcie a čistenia vyprodukovaného znečistenia.

Úvodná kapitola ďalej obsahuje pasáž týkajúcu sa rozličných vedeckých pohľadov na túto problematiku. Vychádzajúc z článku Van den Bergha a De Mooija [43] uvedieme päť rozdielnych názorov týkajúcich sa vzájomného vzťahu rastu a životného prostredia, od "nematerialistov" zavrhujúcich hospodársky rast až po optimistov, ktorí sú presvedčení o jeho potrebe pre zlepšenie stavu životného prostredia. Napokon sa stručne pozrieme na historický vývoj politík venujúcich sa tejto téme, začínajúc od prvej konferencie OSN týkajúcej sa tejto témy a končiac Kjótskym protokolom.

V nasledujúcej kapitole prezentujeme prehľad literatúry, ktorá sa zaoberá vzťahom hospodárskeho rastu a životného prostredia z pohľadu teórie hospodárskeho rastu. Na začiatku uvedieme príklady modelov akumulácie znečistenia, kde je úroveň produkcie daná exogénne. Sú to jednoduché modely, kde môže byť stabilita rovnovážneho stavu analyzovaná graficky. Nasledujú modely, kde je zahrnutá aj akumulácia kapitálu a tým pádom hospodársky rast. Pokračujeme príkladmi modelov s endogénnym rastom a kapitola je ukončená modelmi s usmernenou technickou zmenou, ktoré predstavujú aktuálny stav tejto problematiky.

Tretia kapitola ukazuje využitie rastového modelu sformulovaného Luptáčikom a Schubertom [22] na odvodenie konkrétnych výsledkov. Kapitola začína prehľadom modelu formulovaného ako úloha optimálneho riadenia a výsledkov odvodených v pôvodnej práci. Našim hlavným cieľom je skúmanie optimálneho rovnovážneho stavu tohto systému. Pomocou teórie optimálneho riadenia odvodíme postačujúce podmienky pre existenciu takéhoto stavu. Vyžadujeme, aby hraničný produkt $F'(K)$ bol vyšší ako súčet diskontnej miery ρ a miery amortizácie β, zároveň musí byť spoločenská hodnota kapitálu ψ_1 dostatočne vysoká. Podobné podmienky odvodíme aj pre prípad rovnovážneho stavu, kde uvažujeme nulové výdavky na čistenie.

Následne vyšetrujeme kvalitatívne vlastnosti stacionárneho bodu použitím postupu z článku Tahvonena a Kuuluvainena [40]. Ukážeme, že tento bod má lokálne správanie typu sedlového bodu, ak je splnených niekoľko podmienok. Spoločenská hodnota kapitálu musí prevýšiť spoločenskú hodnotu pre kvalitu životného prostredia, pričom táto je proporcionálna miere emisií generovných produkciou (podmienka (3.3.21)). Taktiež uvádzame, že zmena hraničnej užitočnosti zo znečistenia spôsobená rastom množstva znečistenia musí byť záporná (podmienka (3.3.22)).

Ťažisková analýza tejto práce sa zaoberá lokálnou stabilitou rovnovážneho stavu. Na odvodenie postačujúcich podmienok použijeme teoretické výsledky z článku Sorgera [37], pričom dostávame nerovnosti (3.3.28) a (3.3.29) spolu s potrebou platnosti postačujúcich podmienok na sedlovosť. Dané nerovnosti môžu byť splnené napr. ak diskontný faktor ρ je dostatočne malý. V porovnaní s výsledkami v pôvodenj práci sme nezískali kvalitatívne výrazne odlišné podmienky, ale náš postup bol pomerne jednoduchší. Je dôležité poznamenať, že pôvodná Sorgerova teória sa zaoberá globálnou stabilitou, avšak naše ohraničenia zúžili platnosť tejto vlastnosti na lokálnu.

Naša práca je zakončená kapitolou venovanou modelu s rozlíšenými metódami čistenia. Model z predchádzajúcej kapitoly obsahoval iba spôsoby odbúravajúce už uvoľnené znečistenie. Rozšírime tento koncept tak, že môžeme endogénne vstupovať do emisného koeficientu prislúchajúceho produkčnému procesu. Znečisťujúca náročnosť produkcie môže byť znižovaná cez kapitálové investície. Toto opäť sformulujeme ako úlohu optimálneho riadenia, získame optimálne riešenie použitím štandarných analytických nástrojov a prezentujeme ho spolu s jeho interpretáciou.

Následne analyzujeme rovnovážny stav podobným spôsobom ako v predchádzajúcej kapitole, pričom dostávame postačujúce podmienky pre jeho existenciu, sedlovosť a stabilitu. Výsledky sú podobné tým z predchádzajúceho modelu, opäť požadujeme reštrikcie týkajúce sa spoločenskej hodnoty kapitálu a kvality životného prostredia ako aj vývoja hraničnej neužitočnosti znečistenia.

Ako napokon uvádzame v závere, táto práca ponúka viaceré možnosti na rozšírenie, či už použitím iného typu produkčnej funcie alebo modelovaním vyčerpateľných nerastných zdrojov. Každopádne je to oblasť, ktorej skúmanie sa ukazuje ako veľmi podnetné a zaujímavé.

Literatúra

[1] ACEMOGLU, D.: *Directed Technical Change.* The Review of Economic Studies **69(4)**, 781-809. 2002.

[2] ACEMOGLU, D.: *Introduction to Modern Economic Growth.* Princeton: Princeton University Press. 2009. ISBN 9780691132921.

[3] ACEMOGLU, D., et al.: *The Environment and Directed Technical Change.* National Bureau of Economic Research. Working Paper No. 15451. 2009.

[4] ANDERSON, K.P.: *Optimal Growth when the Stock of Resources is Finite and Depletable.* Journal of Economic Theory **4(2)**, 256-267. 1972.

[5] BARRO, R.J.: *Government Spending in a Simple Model of Endogenous Growth.* The Journal of Political Economy **98(5)**, 103-125. 1990.

[6] BECKERMAN, W.: *A Pro-growth Perspective.* Handbook of Environmental and Resource Economics, 622-634. 1999.

[7] BÖHRINGER, CH.: *The Kyoto Protocol: A Review and Perspectives.* Centre for European Economic Research. Discussion Paper No. 03-61. 2003.

[8] BOVENBERG, A.L., SMULDERS, S.: *Environmental Quality and Pollution-Augmenting Technological Change in a Two-Sector Endogenous Growth Model.* Journal of Public Economics **57**, 369-391. 1995.

[9] BROCK, W.A., SCHEINKMAN, J.A.: *The Global Asymptotic Stability of Optimal Control with Applications to Dynamic Economic Theory.* Discussion Paper No. 151. 1975.
Available at http://www.kellogg.northwestern.edu/research/math/papers/151.pdf
(Date when accessed: 22.3.2011)

[10] CASS, D.: *Optimal Growth in an Aggregative Model of Capital Accumulation.* Review of Economic Studies **32(3)**, 233-240. 1965.

[11] COUNCIL ON ENVIRONMENTAL QUALITY: *A Citizen's Guide to the NEPA. Having Your Voice Heard.* 2007.
Available at http://ceq.hss.doe.gov/nepa/Citizens_Guide_Dec07.pdf (Date when accessed: 14.3.2011)

[12] CUNHA-E-SÁ, M.A., LEITÃO, A., REIS, A.B.: *Growth, Innovation and Environmental Policy: Clean vs. Dirty Technical Change.* 2008.

[13] DALY, H.E.: *From Concept and Theory to Operational Principles.* Population and Development Review **16**, 25-43. 1990.

[14] DALY, H.E.: *Growth and Development: Critique of a Credo.* Population and Development Review **34(3)**, 511-518. 2008.

[15] DASGUPTA, P., STIGLITZ, J.E.: *Resource Depletion under Technological Uncertainty.* Econometrica **49(1)**, 85-104. 1981.

[16] EUROPEAN COMMISSION: *Eurostat Statistics Database.*
Available at http://epp.eurostat.ec.europa.eu/portal/page/portal/statistics/search_database (Date when accessed: 2.4.2011)

[17] FORSTER, B.A.: *On a One State Variable Optimal Control Problem. Consumption-pollution Trade-offs.* Application of Control Theory to Economic Analysis, 35-56. 1977.

[18] FRONDEL, M., HORBACH, J., RENNINGS, K.: *End-of-Pipe or Cleaner Production? An Empirical Comparison of Environmental Innovation Decisions Across OECD Countries.* Centre for European Economic Research. Discussion Paper No. 04-82. 2007.
Available at ftp://ftp.zew.de/pub/zew-docs/dp/dp0482.pdf (Date when accessed: 14.3.2011)

[19] GOELLER, H.E., WEINBERG, A.M.: *The Age of Substitutability.* The American Economic Review **68 (6)**, 1-11. 1978.

[20] HEY, CH.: *EU Environmental Policies: A short history of the policy strategies.* EU Environmental Policy Handbook: A Critical Analysis of EU Environmental Legislation, 17-30. 2006.
Available at http://www.eeb.org/publication/chapter-3.pdf (Date when accessed: 14.3.2011)

[21] HUANG, CH.-H., CAI, D.: *Constant-Returns Endogenous Growth with Pollution Control.* Environmental and Resource Economics 4, 383-400. 1994.

[22] LUPTÁČIK, M., SCHUBERT. U.: *Optimal Economic Growth and the Environment.* Economic Theory of Natural Resources. Wurzburg-Wien. 1982.

[23] NORDHAUS, W.D.: *Lethal Model 2: The Limits to Growth Revisited.* Brooking Papers on Economic Activity 2, Cowles Foundation Paper 831. 1992. Available at http://cowles.econ.yale.edu/P/cp/p08a/p0831.pdf (Date when accessed: 14.3.2011)

[24] OECD: *OECD Statistics.*
Available at http://stats.oecd.org/ (Date when accessed: 14.3.2011)

[25] PEARCE, D.: *An Intellectual History of Environmental Economics.* Annual Review Energy Environment **27**, 57–81. 2002.

[26] PLOURDE, C.G.: *A Model of Waste Accumulation and Disposal.* Canadian Journal of Economics **5(1)**, 119-125. 1972.

[27] PORTER, M.E., VAN DER LINDE, C.: *Toward a New Conception of the Environment-Competitiveness Relationship.* The Journal of Economic Perspectives **9(4)**, 97-118. 1995.

[28] RAUSCHER, M.: *Green R & D versus End-of-Pipe Emission Abatement: A Model of Directed Technical Change.* Thünen-Series of Applied Economic Theory. Working Paper No. 106. 2009.

[29] RICCI, F.: *Environmental Policy and Growth when Inputs are Differentiated in Pollution Intensity.* Fondazione Eni Enrico Mattei.Working Paper No. 16/2002. 2002.

[30] ROMER, P.M.: *Increasing Returns and Long-Run Growth.* The Journal of Political Economy **94(5)**, 1002-1037. 1986.

[31] ROMER, D.: *Advanced Macroeconomics.* New York: McGraw-Hill. 1996. ISBN 0-07-053667-8.

[32] SEIERSTAD, A., SYDSAETER, K.: *Optimal Control Theory with Economic Applications.* Amsterdam: Elsevier. 1987. ISBN 978-0-444-87923-3.

[33] SETHI, S.P., THOMPSON, G.L.: *Optimal Control Theory: Applications to Management Science and Economics.* New York: Springer. 2nd ed. 2000. ISBN 0-387-28092-8.

[34] SMITH, V.L.: *Dynamics of Waste Accumulation: Disposal versus Recycling.* Quarterly Journal of Economics **86**, 600-616. 1972.

[35] SMULDERS, S.: *Endogenous Growth Theory and the Environment.* Handbook of Environmental and Resource Economics, 610-621. 1999.

[36] SOLOW, R.M.: *A Contribution to the Theory of Economic Growth.* The Quarterly Journal of Economics **70(1)**, 65-94. 1956.

[37] SORGER, G.: *On the Optimality and Stability of Competitive Paths in Continuous Time Growth Models.* Journal of Economic Theory **48**, 526-547. 1989.

[38] SWAN, T.W. *Economic Growth and Capital Accumulation.* Economic Record **32(3)**, 334-361. 1956.

[39] TAHVONEN, O.: *On the Dynamics of Renewable Resource Harvesting and Pollution Control.* Environmental and Resource Economics **1**, 97-117. 1991.

[40] TAHVONEN, O., KUULUVAINEN, J.: *Economic Growth, Pollution, and Renewable Resources.* Journal of Environmental Economics and Management **24**, 101-118. 1993.

[41] THE UNITED NATIONS: *Report of the United Nations Conference on the Human Environmen*t. The United Nations Conference on the Human Environment. Stockholm. 1972.

Available at http://www.unep.org/Documents.Multilingual/Default.asp?documentid=97 (Date when accessed: 14.3.2011)

[42] THE UNITED NATIONS: *UN Conference on Environment and Development (1992)*. 1997.
Available at http://www.un.org/geninfo/bp/enviro.html (Date when accessed: 14.3.2011)

[43] VAN DEN BERGH, J.C.J.M., DE MOOIJ, R.A.: *An Assessment of the Growth Debate*. Handbook of Environmental and Resource Economics, 643-655. 1999.

Dodatok A

Additional Data

Our first part of appendix will contain additional data concerned with comparison of economic growth and production of pollution. We have already seen these indicators in section 1.2 for the United States. At this place, more countries at different levels of economic development will be presented.

In our first table A.1, we will show the average rates of annual GDP growth for different time periods. Similarly to the chapter 1, the data are taken from the OECD Database [24] and are expressed as percentage rates.

Tabuľka A.1: Average Growth of GDP

Time period	1975-80	1980-85	1985-90	1990-95	1995-2000	2000-05
China	NA	10.78	7.94	12.28	8.62	9.58
France	3.44	1.54	3.28	1.16	2.80	1.68
Germany	3.36	1.36	3.32	2.22	2.00	0.75
India	NA	5.24	5.96	5.00	6.16	6.50
Italy	4.46	1.68	3.16	1.28	1.92	0.90
Japan	4.40	3.10	4.82	1.54	1.02	1.30
United Kingdom	1.76	2.14	3.34	1.66	3.44	2.52
OECD Countries	3.64	3.25	3.54	2.16	3.42	2.18

As we can see, different growth patterns can be observed over the past decades in these sample countries. Particularly high growth of GDP can by noted for China and India, which can be nowadays considered as top world producers, although regarded as less developed countries before. On the other hand, highly developed European countries haven't shown such overwhelming growth. Nevertheless, their production has increased. A specific case is Japan, which has experienced a stagnation starting in the 1990s after previous long years of expanssion. Finally, the production in developed countries associated in OECD has grown at solid rate between 2% and 4% on average. Overally, these values represent a progress of economic production in all of the included countries, which, however, can be associated with strong environmental degradation.

To pursue an issue of environmental damage caused by production in our sample countries, we will present one of its indicators in our next table A.2. The observed variable is an average annual change in production of CO2 emissions again evaluated as percentage rate.

Tabuľka A.2: Average Growth of CO2 Emissions

Time period	1975-80	1980-85	1985-90	1990-95	1995-2000	2000-05
China	6.13	3.98	5.37	6.21	0.40	10.90
France	1.47	-4.78	-0.41	0.22	1.32	0.59
Germany	1.66	-0.76	-1.31	-1.76	-0.95	-0.37
India	4.13	7.43	7.05	5.87	4.53	3.43
Italy	2.46	-0.72	2.80	0.63	0.68	1.38
Japan	0.63	-0.05	4.05	1.49	0.62	0.63
United Kingdom	-0.16	-0.88	0.30	-1.25	0.30	0.32
OECD Countries	1.80	-0.48	1.29	0.89	1.54	0.68

Comparing pollution development with the growth rates of economic production brings forth several interesting correlations. The generation of emissions in countries developing at the high rates (China and India) has grown rather significantly. As the main goal of these countries is high economic growth, they haven't given environmental issues much credit. On the other hand, countries with already high levels of production display little increase or even decrease in production of polluting residuals, as the more resources are devoted to the abatement activities and research in cleaner production. Note that decrease of pollution is most likely to happen in countries, where the nature suffered much from economic development beforehand (Germany or United Kingdom).

Dodatok B

Remarks on the Steady State

B.1 Sufficient Conditions for Saddle-Point Properties

In our analysis of trade-off between economic growth and environmental quality using the optimal control theory, we pay significant attention to the existence of stationary point and its qualitative properties. Over the analysis realized in our work, particularly in chapters 3 and 4, we have used the approaches realized in work by Tahvonen and Kuuluvainen [40]. In what follows, the findings included in Appendix 1 (p.114) are applied.

Let's consider the modified Hamiltonian system for the general optimal control problem with the two state variables x and y, and two corresponding costate variables ϕ and ψ:

$$\begin{aligned}
\dot{x} &= \mathcal{H}_\phi(x, y, \phi, \psi), \\
\dot{y} &= \mathcal{H}_\psi(x, y, \phi, \psi), \\
\dot{\phi} &= \rho\phi - \mathcal{H}_x(x, y, \phi, \psi), \\
\dot{\psi} &= \rho\psi - \mathcal{H}_y(x, y, \phi, \psi),
\end{aligned}$$

where ρ stands for discount rate and $\mathcal{H}$ denotes maximized Hamiltonian. The conditions for the stationary point are $\dot{x} = \dot{y} = \dot{\phi} = \dot{\psi} = 0$. Furthermore, we consider the Jacobian of this system:

$$J = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y & \partial\dot{x}/\partial\phi & \partial\dot{x}/\partial\psi \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y & \partial\dot{y}/\partial\phi & \partial\dot{y}/\partial\psi \\ \partial\dot{\phi}/\partial x & \partial\dot{\phi}/\partial y & \partial\dot{\phi}/\partial\phi & \partial\dot{\phi}/\partial\psi \\ \partial\dot{\psi}/\partial x & \partial\dot{\psi}/\partial y & \partial\dot{\psi}/\partial\phi & \partial\dot{\psi}/\partial\psi \end{pmatrix} \tag{B.1.1}$$

which is evaluated at the steady state. To analyze local properties of this point, we have to analyze properties of the eigenvalues of this matrix, or in other words the roots of the characteristic polynomial. Tahvonen and Kuuluvainen further denoted Δ as the determinant of J matrix and defined Ω as follows:

$$\Omega = \begin{vmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial\phi \\ \partial\dot{\phi}/\partial x & \partial\dot{\phi}/\partial\phi \end{vmatrix} + \begin{vmatrix} \partial\dot{y}/\partial y & \partial\dot{y}/\partial\psi \\ \partial\dot{\psi}/\partial y & \partial\dot{\psi}/\partial\psi \end{vmatrix} + 2 \begin{vmatrix} \partial\dot{x}/\partial y & \partial\dot{x}/\partial\psi \\ \partial\dot{\phi}/\partial y & \partial\dot{\phi}/\partial\psi \end{vmatrix}. \tag{B.1.2}$$

After these introductory statements, the authors finally presented a theoretical conclusion from the article by Tahvonen [39]. The proposition holds that if the conditions $\Delta > 0$ and $\Omega < 0$ are satisfied, the steady state has saddle point properties. This finding is also important for our work. As we inquire about qualitative properties of the stationary point of our system, it can significantly contribute to our analysis of economy's behavior with regard to the trade-off between economic growth and environmental quality.

B.2 Global Stability of the System

Some of the inspirational results concerning optimal solutions and stability of steady states in growth models were proposed by Sorger in his article [37]. In our thesis, we apply these finding in analysis of stationary state stability in terms of optimal solutions' convergence. It's therefore suitable to overview used theoretical results.

Author considered following optimal control problem, where $x \in \mathbb{R}^n$ represents state and $v \in \mathbb{R}^m$ control variables. Function $f : \mathbb{R}^n \times \mathbb{R}^m \times [0, \infty) \to \mathbb{R}^n$ describes the dynamics of the system and $U : \mathbb{R}^n \times \mathbb{R}^m \times [0, \infty) \to \mathbb{R}$ denotes standard utility function. Coefficient ρ is a discount factor.

$$\left.\begin{aligned}
&\max \int_0^\infty \mathrm{e}^{-\rho t}\, U(x(t),v(t),t)\ \mathrm{d}t\\
&\dot{x} = f(x(t),v(t),t) \quad \text{for almost all } t \in [0,\infty),\\
&x(0) = x_0,\\
&v(t) \in V(t) \quad \text{for almost all } t \in [0,\infty).
\end{aligned}\right\}\quad \text{Planning Problem 3}$$

Author further defines Hamiltonian od this problem, using $\langle .,. \rangle$ to denote usual scalar product of $\mathbb{R}^n$ vectors:

$$H(x,p,t) = \sup_{v \in V(t)} \big\{ U(x,v,t) + \langle p, f(x,v,t) \rangle \big\}. \tag{B.2.1}$$

As we are looking for the optimal solution, the necessary conditions can be formulated through the Hamiltonian dynamic system:

$$\begin{aligned}
\dot{p}(t) &= \rho p(t) - H_x(x(t),p(t))\\
\dot{x}(t) &= H_p(x(t),p(t)).
\end{aligned} \tag{B.2.2}$$

It´s defined that the steady state of this dynamic system is globally asymptotically stable if the condition

$$\lim_{t\to\infty} (x(t),p(t)) \ = \ (\overline{x},\overline{p}) \tag{B.2.3}$$

holds for all bounded solutions $(x(t),p(t))$ of B.2.2.

Author further pursues a theory of canonical transformations of this system and its implications regarding stationary state and its stability properties. This topic won't be addressed in depth in our thesis.

As the result valuable in our analysis, we will mention one important finding. Formulated as a Corollary 2 ([37], p.540), this proposition states that the steady state is globally asymptotically stable for bounded sollutions of dynamic system (B.2.2), if there exists a scalar $\gamma \in \mathbb{R}$ such that the curvature matrix

$$\mathbb{C} \ = \ \begin{pmatrix} H_{xx} + \gamma\,[H_{xp} + H_{px}] & -\frac{\rho}{2} I + \gamma H_{pp} \\ -\frac{\rho}{2} I + \gamma H_{pp} & -H_{pp} \end{pmatrix} \tag{B.2.4}$$

is negative definite for all $(x,p) \in \mathbb{R}^n \times \mathbb{R}^n$. This hold under assumption that a stationary state exists.