Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

Adverse Selection in Insurance Markets with General Risk Distribution

Zuzana Molnárová 2011

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Adverse Selection in Insurance Markets with General Risk Distribution

Master Thesis

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Zvrátený výber na poistnom trhu s všeobecným rozdelením rizikovosti

Diplomová práca

Bc. Zuzana Molnárová

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Cieľ : We study the presence and stability of equilibria of insurance markets under general risk distribution which may depend on a risk aversion parameter. Conditions for the occurrence of adverse selection are focused upon.

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Čestne prehlasujem, že som diplomovú prácu vypracovala samostatne pod dohľadom vedúceho diplomovej práce a s použitím uvedenej literatúry.

I hereby declare that this thesis were written on my own with the only help provided by my supervisor and the referred-to literature.

V Bratislave, dňa 26. 4. 2011 Zuzana Molnárová

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Abstrakt

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V práci sa zaoberáme štúdiom rovnováh na poistných trhoch s asymetrickou informáciou. Zameriavame sa na vplyv zvráteného výberu (adverse selection) na ekvilibriovú cenu a zloženie poistencov. Do úvahy berieme dve rôzne charakteristiky spotrebiteľov, ktoré môžu byť skryté pred ostatnými agentmi na trhu - rizikovosť a mieru rizikoaverzie. Obe popisujeme všeobecným spojitým pravdepodobnostným rozdelením. Ukazujeme, že zvrátený výber môže mať na takomto trhu nepriaznivé dôsledky na efektívnosť rovnovážneho stavu a popisujeme vlastnosti, od ktorých efektívnosť závisí. Venujeme sa ďalej modelu rozšírenému o závislosť medzi spotrebiteľovými skrytými charakteristikami v ktorom je umožnená prítomnosť zvýhodneného výberu (advantageous selection) a uvádzame zdôvodnenie predpokladu závislosti.

Kľúčové slová: asymetrická informácia, zvrátený výber, poistenie, averzia voči riziku, rozdelenie rizikovosti

Abstract

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In this work we study equilibria in insurance markets with asymmetric information. We focus on the effect of adverse selection on the equilibrium price and structure of the insured consumers. Two different characteristics of the consumers that might be hidden to other agents in the market are considered - riskiness and risk aversion, both are described by general continuous probabilistic distributions. We show that adverse selection may have harmful consequences on the market equilibria efficiency and we describe the properties determining this efficiency. We then develop a model extended with dependence between the consumers characteristics which enables the presence of advantageous selection. A reasoning for the assumption of such dependence is also given.

Keywords: asymmetric information, adverse selection, insurance, risk aversion, risk distribution

Contents

Introduction

Asymmetric information is a concept we meet in our everyday lives but rarely think about its consequences. We buy goods and we know only a little about their quality. We vote for politic parties and about their intentions, we know even less. In the microeconomic theory, as in numerous other fields including game theory, contract theory or corporate finance, asymmetric information has been widely studied and discussed. The theoretical research from its very beginning suggests that this concept has harmful effects on market outcomes and tries to find the way to remove these unpleasant effects. Basically whole contract theory field focuses on methods that minimizes the disadvantages caused by the information asymmetry.

Meanwhile, the empirical research of last two or three decades seems much more hesitant in saying whether there are consequences of asymmetric information in the real world economy, at least at the microeconomic level. The credit for the inconsistency is often given to the non-realistic microeconomic assumptions on behavior of individual agents and excessive simplification of the models. A variety of advanced concepts has been developed to describe the behavior of consumers and firms and to fit the reality better.

We believe in the *first things first* approach and we focus on removing the most obvious and most unrealistic assumptions commonly used in the microeconomic models of markets with asymmetric information in our model. Particularly, we consider the continuous distribution of agents' characteristics types and do not focus on just two (bad and good) types which is more realistic and also allows for a better comparison with the empirical results. Secondly, we focus our attention to consumers attitude to risk which we believe vary widely amongst the consumers. The aim of the work is to create a reasonable mathematical model on this basics and extend it with some natural assumptions. As far as we rarely can observe the exact values of parameters of the model we mainly focus on the qualitative results such as occurrence of inefficiencies or determining the type of agents influenced by the information asymmetries rather than focusing on quantitative results. In the work we consider one particular effect of information asymmetry, known as *adverse selection* which has its natural interpretation in insurance markets and we mainly follow this application.

The first chapter introduces the theoretical concepts necessary for understanding of this thesis and summarizes previous works from the field that are important for this work. The second chapter introduces our model and analyses it in detail. In the third chapter we extend the model with some additional assumptions on consumers characteristics which could bring out a possitive effect of so-called *advantageous (propitious) selection*, the effect opposite to adverse selection.

Chapter 1

Theory

We begin with introducing some basic ideas of microeconomic theory that we use in this work. In microeconomic theory, the concept of equilibrium undoubtedly belongs amongst the most important and most interesting ideas. In this work, we examine the equilibria in the markets with some sort of uncertainty and asymmetric information present. Therefore, we devote the first part of this chapter to a short description of the theory of market equilibria in competitive markets and define equilibrium as we use it in the following chapters. We continue the chapter with expected utility theory of markets with uncertainty, attitudes to risk and add some applications in insurance. In the third part of this chapter, we introduce the main ideas of markets with asymmetric information and give a summary of previous works in the field we find most important for our motivation and results. We try to avoid the technical details, so we mention literature sources where the complete explanation can be found for each concept.

1.1 Market Equilibria

In economics, *equilibrium* is understood as a state of world, in which economic forces are in some form of balance, meaning that without an external influence the values of the economic variables are stable. One of the fundamental parts of the microeconomic theory, the general equilibrium theory, states that the (Walrasian) equilibrium at a competitive market is given by the vector of prices and the allocations, such that (a) it is feasible, and (b) each agent is choosing optimally, given that his budget. In a Walrasian equilibrium, if an agent prefers another combination of goods, he can not afford it. According to a well-known result of the theory, First Welfare Theorem, in the competitive markets with symmetric information and no externalities, the equilibrium exists and is always Pareto-efficient. This property makes the equilibrium a socially favourable state of world. However, if the market is not competitive (e.g. there is one agent that has a power to regulate the price), it includes some externalities (negative or positive) or the information is not symmetric, the First Welfare Theorem does not need to hold and the equilibrium may not be efficient or may not exists at all. Full explanation of the general equilibrium theory may be found in Jehle and Reny [10] or Varian [17].

In our work we study the equilibria of markets with asymmetric information. Therefore, we may not take the First Welfare Theorem as granted and the efficiency of the market equilibria becomes a reasonable question to examine. Moreover, in markets with uncertainty a need to reformulate the concept of equilibrium occurs. The equilibrium should involve the specific properties of insurance (or similar) contracts. Rotschild and Stiglitz [16] start formulating of their equilibrium with the assumption that every consumer can only buy one insurance contract. This assumption gives the companies the possibility to specify both prices and quantities of insurance purchased, while in most of the others competitive markets the seller only influence the price and can not control the amounts that the consumers buy. Therefore it is more suitable to define the equilibrium in an insurance market that is specified by prices and quantities, rather than the traditional price equilibrium. The detailed overview of such an equilibrium is given in [16]. The paper states the following definition of equilibrium in a competitive insurance market:

Definition 1.1.1. *An equilibrium in a competitive insurance market is a set of contracts such that, when consumers choose contracts to maximize their expected utility*

- *no contract in the equilibrium makes negative expected profit,*
- *there is no contract outside the equilibrium set such that, if offered, would make a positive profit.*

This equilibrium is of Cournot-Nash type. Each firm assumes that the contracts its competitors offer are independent on it's own actions.

In this work, we follow the assumption that a consumer can buy one insurance contract only. Sometimes we also assume that there is only full insurance offered in the market, therefore the price-quantity equilibrium is not necessary. By the *contract* we then understand a price for which the full insurance is offered to a consumer. Otherwise, we follow the definition given above. The simplification noticeably limit the possibilities of employing a self-selection mechanisms, yet they are not of our interest in the work.

1.2 Market with Uncertainty

Many economic decisions contain some level of uncertainty. The outcomes of an agent's choice can depend on some unknown exogenous factors. For example, the utility of going on a vacation depends on weather, the utility from a new car depends on whether it will be stolen, etc. In expected utility theory we derive the agent's preferences from the average (expected) values of his utility function. In this section we describe the behavior of consumers in the market with a presence of risk based on expected utility theory. We summarize the necessary theory briefly, for details see Jehle and Reny [10], Varian [17] or Brunovský [2] (in Slovak language).

1.2.1 Utility Function

To describe the rational behavior of *consumers*, we characterize them by the utility which the consumption of particular bundles of goods brings to them. By *bundle* we denote vector $x = (x_1, x_2, \dots x_n)$ of the amounts of n goods. We describe the utility that a consumer has from consuming the bundle x by the *utility function* $u(x)$. Consumers do not need to be identical: e. g. while consumer A prefers a bundle containing a piece of sausage, consumer B may prefer a bundle with slim down products instead. We characterize different consumers by different utility functions. For our work, the following properties of the utility function are important:

Properties 1.2.1.

(P1) $u(x)$ *is continuous.*

(P2) $u(x)$ *is strictly quasi concave.*

(P3) $u(x)$ *is strictly increasing.*

Remark 1.2.1. *Since we only require the ability to rank the bundles from the utility function, it is invariant to any positive monotonic transformation of the dependent variable* $u(x)$*. If* h *is a strictly increasing function on the set of values taken on by* u*, then functions* u *and* $h \circ u$ *represent an identical consumer. We say that the function* u *has ordinal character.*

By *rational behavior* of a consumer we understand that he chooses such a bundle feasible for him that maximizes his utility function. *Feasible bundles* of a consumer are given by his *income I* (or his disposable wealth) and the *prices* $p = (p_1, p_2, \dots p_n)$ of goods in the bundle. A rational consumer then chooses the bundle $\hat{x}$ which satisfies the budget constraint

$$\langle p, \hat{x} \rangle \leq I,$$

and maximizes his utility:

$$\hat{x} = \arg\max u(x).$$

Properties 1.2.1 guarantee that the consumer's utility-maximization problem 1.2.1 has a unique solution [10].

In this work, as we usually do in the expected utility theory we want to examine how the consumers appreciate risky outcomes. The consumers utility from uncertain outcomes is given by von Neumann - Morgenstern utility function

$$U(w) = E[u(w)] = \int u(w)\, dF(w),$$

where F represents a distribution of the uncertain outcomes and $u(w)$ is a utility function of only one good - wealth w. In this case, if we want to compare the different utility appreciation of the risky outcomes we need to take into account the exact shape of the utility function of the certain outcomes u (so-called Bernoulli utility function) [14]. Different shapes of the Bernoulli utility functions[1] shows that the agents has different *attitudes towards risk*.

1.2.2 Attitude Towards Risk

If the agent's utility function[2] is concave, he prefers certainty to risk. Conversely, for a convex utility function he prefers uncertain gains/losses to certainty. Hence, we call the agent with a concave utility function *risk-averse* and the agent with a convex utility function *risk-seeking*. Linear utility function express that the agent is *risk-neutral*, indifferent between risky and certain gains and losses with the same expected value. The bigger the curvature of the utility function, the more risk-averse or risk-seeking the agent is. We assume that consumers in insurance markets are always strictly risk-averse and insurance companies are risk-neutral. Still, the consumers may differ in the intensity of their risk aversion and it is therefore useful to be able to measure it.

Definition 1.2.1. *Arrow-Pratt measures of risk aversion.*
The expression

$$A(w) = -\frac{u''(w)}{u'(w)},$$

is called Arrow-Pratt measure (or coefficient) of absolute risk aversion.
The expression

$$r(w) = -\frac{wu''(w)}{u'(w)},$$

is called Arrow-Pratt measure (or coefficient) of relative risk aversion.

The Arrow-Pratt coefficients reflects the curvature of $u(w)$ and stays constant with respect to affine transformations, which we require since expected utility functions of money are not uniquely defined.

Although risk-aversion of a consumer may be expressed by any increasing concave function, there are certain classes of utility functions that are used more commonly. Between the most commonly used are the classes with some special properties, such as CARA and CRRA classes.

[1] Bernoulli utility function is defined up to affine transformation.

[2] from now on, by the utility function we always understand Bernoulli utility function depending only on wealth.

Definition 1.2.2. *CARA and CRRA functions.*
The utility function of the form

$$u(w) = 1 - e^{-\alpha w}, \tag{1.1}$$

or any of its affine transformations exhibits constant absolute risk aversion (CARA)

$$A(w) = \alpha,$$

for all wealth levels w. The utility function of the form

$$u(w) = \frac{w^{1-\rho}}{1-\rho}, \tag{1.2}$$

or any of its affine transformations exhibits constant relative risk aversion (CRRA)

$$r(w) = \rho,$$

for all wealth levels w.

Both CARA and CRRA functions are special cases of a wider class, hyperbolic absolute risk aversion (HARA) functions, that are the most general case of utility functions used in practice. The CARA and CRRA classes brings certain analytical and computational advantages which we fully use later in the work (see e.g. Caballero [3] for some discussion).

In numerical computations included in this work, we assume in consistency with Zavadil [19] values of coefficient of absolute risk aversion $\alpha \in (0, 0.5]$ for risk-averse consumers. For the consumption level at approximately 20 units this corresponds to values of coefficient of relative risk aversion of $\rho \in [2; 10]$. This is roughly the range considered, with some empirical support, by Caballero [3].

1.2.3 Insurance Against Risk

Let $u(w)$ be the utility of owning wealth w of a particular agent, let $X \in M$ be a random variable - the value of agent's wealth ex post (after an event), and let F be the distribution function of the random variable X. Then the agent's *expected utility* is

$$E[u(X)] = \int_M u(x)\, dF(x).$$

Assume that an agent has initial wealth w_0 (for example a car), and π is the probability that he will suffer loss L (e.g. the car radio will be stolen). If the agent is not willing to bear this risk, he may *insure* against it. By insurance we understand that the agent pays an amount of money called *insurance premium* γB and for exchange he receive the amount B, *coverage*, in the case of loss[3]. It has been shown for example in Molnárová [15] that in the competitive market with symmetric information it is optimal for a strictly risk-averse agent to purchase the full insurance contract $B = L$.

1.3 Introduction to Asymmetric Information

A situation in which different agents possess different information is said to be one of *asymmetric information*. As has been showed (e. g. by Akerloff [1]), asymmetric information typically leads to inefficient market outcomes and may cause a total market failure. Under asymmetric information, the First Welfare Theorem no longer holds generally.

There are two possible reasons of this market inefficiency: either there is a subject in the market with hidden *characteristics* (quality of a good, driving abilities etc.) or there is an agent who's *actions* are hidden (e.g. working effort, carefulness). We refer to the possible negative effect of the first situation as to the *adverse selection* effect and of the second as to the *moral hazard* effect. According to Mas-Colell, Winston and Green [14]:

Definition 1.3.1. *Adverse selection is said to occur when the informed agent's decision depends on his unobservable characteristics in a manner that adversely affects the uninformed agents in the market. If the uninformed agent is affected by the informed agent's choice of hidden action, moral hazard is said to occur.*

There is a difference in the time course of adverse selection and moral hazard mechanisms. The characteristics of the market subjects are given *ex ante*, before any contract is signed, and agents at the market has no influence on the hidden

[3] Even though in this work we mostly focus on the loss caused by having an accident or theft, the theory may be easily extended to any form of failure against which we can be theoretically insured like health, success of a business project and so on.

characteristics in the case of adverse selection. On the contrary, moral hazard arises *ex post*, after the contract is signed. The agent who's actions are unobservable decides which action to take and according to the mentioned theory he chooses the action that maximizes his utility. Therefore, by setting the right motivation scheme, the negative effects of moral hazard may be avoided whereas the adverse selection effects may not.

In the case of informational advantage of hidden characteristics on the side of consumers firms can often force individuals to reveal their characteristics by their market behavior. Other things equal, consumers with different characteristics may prefer different contracts. Although this may be an accurate way of finding out consumers characteristics, it is not very proficient. The firms want to know the consumers hidden characteristics ex ante in order to set the appropriate contracts. After the contract is set, the firms can usually gain no benefit from the information.

Sometimes it is also possible to force customers to make choices that both reveal their characteristics and that the firms would have wanted them to make given the characteristics [16]. Such a setting is called a *self-revealing mechanism* and is widely discussed in literature.

In the work we also study the possibilities of a form of *good* selection amongst the agents caused by the asymmetric information, the effect opposite to the adverse selection. This phenomenon is called *advantageous* or *propitious* selection and we define it in consistency with the Definition 1.3.1.

Definition 1.3.2. *Advantageous selection is said to occur when the informed agent's decision depends on his unobservable characteristics in a manner that positively affects outcomes of uninformed agents in the market.*

We examine this concept more deeply in the Chapter 3.

1.3.1 Literature Review

The information asymmetries have been studied in various fields of economics and finance such as game theory, contract theory, corporate finance, but most of the applications have their sources in the microeconomic theory. In this work, we discuss the applications in insurance markets with asymmetric information (some examples may be found in Jehle and Reny [10] or Varian [17]). This section summarize the relevant theoretical and empirical results from literature concerning the insurance markets.

In the core work of insurance markets with asymmetric information from 1976, Rotschild and Stiglitz [16] conclude that the incentive to purchase insurance is greatest for those consumers having private information that they are relatively likely to suffer a loss. Thus, the average risk of the company's clientele may be higher than the average risk amongst all the consumers. Augmenting of this selection effect is moral hazard, the tendency of insurance to dull the incentive to take precautions, thereby intensifying the loss propensity of the insured relative to that of the uninsured. Both of these have negative effect on the market, as there is no longer an equilibrium with fully insured consumers as it is under the prefect competition with full information.

Rotschild and Stiglitz assume that there are two types of consumers in the insurance market: high risk individuals with probability π^H of possessing a loss and low risk individuals with probability $\pi^L < \pi^H$. The insurance company can not recognize consumers' risk types as it is a private knowledge of the consumers. Although a self-revealing mechanism is employed and the company can recognize the consumer's type according to his choice of the contract, high risk (low ability etc. in alternative models) consumers exert a negative externality to the low risk (high ability) consumers. The externality is hidden in the self-revealing mechanism: in order to recognize a type of consumer, insurance companies offers sets of contracts with different prices and different amounts of coverages such that in equilibrium a low-risk consumer always chooses lower coverage than a high-risk consumer. By decreasing the equilibrium quantity of the contract the low risk customers are offered the efficiency of the market equilibrium decreases. Even more interestingly, the equilibrium does not exist under certain quite plausible conditions. Many of the following works in the next decades has developed more or less similar models to Rotschild and Stiglitz's, with similar conclusions.

Rotshild and Stiglitz themselves stated a necessity of a related empirical research that would confirm the presence of the theoretical concept in the real world. The empirical works started to appear from the mid-80's but were not able to give a convincing evidence of presence or non-presence of the negative effects of asymmetric information in the various markets as a significant inconsistency of the results arose. As an example of a recent empirical work we quote the work of Chiappori and Salanié (2000) [5] that finds no evidence of adverse selection or moral hazard among young French drivers. However, since only young drivers are examined, we may have some doubts whether they already have the private information about their own risk level. It is possible that drivers only receive the information about their driving abilities after some time spent by driving[4]. Similar deficiencies often accompany the empirical research in the field and make an interpretation of the results less straightforward and more puzzling.

A recent work of Cohen and Siegelman (2010) [4] summarizes the empirical research in the whole area of insurance markets with asymmetric information. The authors find the differences in the empirical results natural and they try to explain them by the differences of the markets' characteristics - the difference market behavior of consumers in health insurance market and car insurance market is natural as people just tend to value their health much more than their cars and it costs the insurance company much more to cover a liver transplant as to replace a broken bumper.

Hemenway (1990) [9] proposed a theoretical concept of a reverse effect in the insurance market which he called *advantageous selection* or *propitious selection*. Advantageous selection is based on the idea that careful consumers are careful when it comes to both taking precautions and buying insurance. Therefore this *careful* type of consumers have a lower risk of possessing a loss and higher motivation to buy an insurance. The advantageous selection has an opposite effect to the adverse selection and moral hazard and therefore may explain the lack of empirical evidence of negative effects in the empirical research.

Several theoretical models that include different risk-aversion types of consumers were developed to formalize the advantageous selection concept. While de Mezza and Webb (2001) [7] developed a theoretical model in which advantageous selection causes an efficiency gain in the market, with a slightly different assumptions De Donder and Hindriks (2006) [6] shows that the advantageous selection may not exceed the effect of adverse selection. We examine the effect of advantageous selection and the contradiction between the two mentioned results shortly in the Chapter 3.

[4] Of course, it is also possible that drivers never get the information about their own abilities, not even after a long time of driving. The problem of judgment and mis-valuation of risky outcomes has been widely discussed, at least since Kahneman and Tversky [13], 1974.

In the most of the theoretical works authors limit themselves to the case of only two types of consumers' risk types and sometimes also to the two types of consumers risk aversion. In our work, we do not follow this simplification and we show that including a big number of types to the model is not trivial. Considering a continuum of types is more realistic and it also allows for better comparing with the empirical results. We derive the wast majority of the results with assumptions of general distribution of consumers' riskiness and risk aversion as much as with other very general assumptions.

Chapter 2

Adverse Selection in Insurance Market with Continuous Uncertainty in Risk and Risk Aversion

In this chapter we develop and describe a model of insurance market in which consumers have private information about two different characteristics: their risk aversion and their probability of failure (riskiness). Unlike the previous works we do not assume that there are only two types of the characteristics. We make a general assumption that there are continuous distributions of this characteristics amongst the consumers and study the market behavior of the agents. We examine the properties of possible equilibria and focus on the conditions of the presence of adverse selection.

2.1 Model Setup

Consider an insurance market (e. g. car insurance) with many insurance companies and many consumers. Consumers have different probabilities π of accidental loss L (e. g. a car accident), differ in their attitude to risk and are identical otherwise. Let us assume that the attitude to risk may be expressed by a single parameter α of the utility function. An example of such a utility functions are CARA (or CRRA) functions that depends on the coefficient of absolute (relative) risk aversion. We give a more exact definition of the parameter α later in the chapter, after introducing some others necessary concepts. Let the nondegenerate interval $[\underline{\pi}, \overline{\pi}]$ contain the set of all consumers accident probabilities and the nondegerate interval $[\underline{\alpha}, \overline{\alpha}]$ contain the set of all consumers parameters of risk aversion. Consumers are strictly risk-averse, thus they have increasing and strictly concave utility functions. They are rational and make their decisions according to expected utility theory.

Insurance companies are risk-neutral and identical. For simplicity we assume that they only offer full insurance policies, which means that for the price p[1] they promise to pay a consumer the amount L in the case of accident. We suppose that the cost of providing insurance is zero. Hence, if an insurance company sells a policy for p EUR to the consumer whose accident probability is π, the company's expected profit is $(p - \pi L)$ EUR.

Insurance companies are not able to recognize consumers' accident probabilities nor how risk-averse they are. However, they have historical records of consumers and therefore have some information. We suppose that the insurance companies know the distribution of accident probabilities and of risk aversion amongst the consumers and nothing else. We denote $F(\pi, \alpha)$ the joint cumulative distribution function of random variables Π and A on $[\underline{\pi}, \overline{\pi}] \times [\underline{\alpha}, \overline{\alpha}]$ that represents the insurance companies information.

A nontrivial problem that now occurs is the equilibrium price of the insurance policy and which types of the consumers will purchase it. A natural guess might be to set $p^* = E(\pi)L$, where $E(\pi)$ is the expected accident probability of all consumers. Unfortunately, this might be far from the equilibrium value.

Let us suppose $E(\pi)L$ is the price of the insurance. This price might be too high for consumers with low accident probabilities, thus only consumers with higher probabilities actually purchase the insurance. Hence the mean accident probability of the insured consumers is higher than the expected accident probability of all consumers. Consequently, insurance companies would either have negative profits, or they would increase the prices. By doing so, they risk that

[1]Notice that the interpretation of p is the same as γB in the Section 1.2.3 if the coverage B is equal to L.

more low-probability consumers are not willing to pay the premium. The situation where rising the price of the insurance leads to the riskier set of consumers actually purchasing the policy satisfies the Definition 1.3.1, thus is said to be the one of *adverse selection*. Note that according to the definition we say that the adverse selection is present only if the insurance companies are affected by the asymmetric information, e.i. if the average clientele accident probability actually is higher than the expected accident probability amongst all the consumers.

2.1.1 Conditions of Purchasing the Insurance Contract

A consumer with characteristics π, α buys the insurance contract for the price p if his expected utility from purchasing it is greater (or equal) as his expected utility without the insurance:

$$u_\alpha(w-p) \geq \pi \cdot u_\alpha(w-L) + (1-\pi) \cdot u_\alpha(w), \tag{2.1}$$

which can be easily reformulated as

$$\pi \geq \frac{u_\alpha(w) - u_\alpha(w-p)}{u_\alpha(w) - u_\alpha(w-L)} \equiv h(p,\alpha). \tag{2.2}$$

We denote the expression on the right-hand side of the inequality $h(p,\alpha)$. The inequality says that from all the consumers with risk-aversion α only those whose probability of having an accident is higher than $h(p,\alpha)$ are willing to buy the insurance. Note that h is increasing in p. It is plausible to expect that more risk averse consumers are willing to pay more for the insurance than the less risk-averse consumers. In correspondence with this assumption we now define the coefficient of risk aversion:

Definition 2.1.1. *Let us have a utility function of a risk-averse consumer u depending on a parameter $\alpha \in [\underline{\alpha}, \overline{\alpha}]$. We say that α is a* coefficient of risk aversion *if $h(p,\alpha)$ is decreasing in α for any fixed positive value of p.*

Lemma 2.1.1. *Let us have a strictly risk-averse consumers with fixed coefficient of risk aversion α. For each $\tilde{\pi} \in [\underline{\pi}, \overline{\pi}]$, $\tilde{p}$ such that $\tilde{\pi} = h(\tilde{p},\alpha)$ it holds that*

$$\tilde{\pi} L < \tilde{p}. \tag{2.3}$$

Proof. In following we skip α in the notation of the utility function, as it is fixed. Suppose by the way of contradiction that $\tilde{\pi}L \geq \tilde{p}$. Then as u is strictly increasing, we have

$$u(w - \tilde{p}) \geq u(w - \tilde{\pi}L). \tag{2.4}$$

u is strictly concave, therefore

$$\begin{aligned} u(\tilde{\pi}(w - L) + (1 - \tilde{\pi})w) &> \tilde{\pi}u(w - L) + (1 - \tilde{\pi})u(w), \\ u(w - \tilde{\pi}L) &> \tilde{\pi}u(w - L) + (1 - \tilde{\pi})u(w). \end{aligned} \tag{2.5}$$

From (2.4), (2.5) we obtain

$$u(w - \tilde{p}) > \tilde{\pi}u(w - L) + (1 - \tilde{\pi})u(w). \tag{2.6}$$

We can rearange the last inequality and obtain

$$\begin{aligned} \tilde{\pi} &> \frac{u(w) - u(w - \tilde{p})}{u(w) - u(w - L)}, \\ \tilde{\pi} &> h(\tilde{p}, \alpha), \end{aligned}$$

which contradicts the assumptions. Therefore it must hold that $\tilde{\pi}L < \tilde{p}$. $\square$

It is straightforward to show that the weak inequality holds for a weakly risk-averse consumer. The basic intuition behind the Lemma 2.1.1 is that the maximal price that a risk-averse consumer is willing to pay is always higher than his expected loss, e. i. higher than the *fair price*.

2.1.2 Equilibrium Price

Let us recall the definition of the market equilibrium given by Rotschild and Stiglitz (Definition 1.1.1). According to the first condition of the equilibrium, neither the consumers nor the companies can have negative expected profit from buying (selling) the insurance contracts. For the insurance companies to have nonnegative expected profit, the equilibrium price of the insurance must be at least equal to the mean loss of the consumers who actually buy the insurance for the price. Let p^* be the equilibrium price, it must hold that

$$p^* \geq E[\Pi | \Pi \geq h(p^*, A)]L, \tag{2.7}$$

where the expression

$$E[\Pi|\Pi \geq h(p, A)]$$

is the expected accident probability of consumers conditional on $\Pi \geq h(p, A)$. As the market is competitive, none of the insurance companies may set the price higher than the expression on the right-hand side of 2.7 due to market mechanism. Thus, in order to satisfy the first condition of the equilibrium definition the inequality 2.7 must hold as an equation. The candidates for equilibria are given by the equation

$$p^* = E[\Pi|\Pi \geq h(p^*, A)]L. \tag{2.8}$$

The second part of the Definition 1.1.1 requires that there is not any other possible contract that yields higher profit. In the competitive market, companies can never yield a positive profit, their profit is at most zero. Therefore we only need to check the consumers' profits in the prices satisfying 2.8 when analyzing the second condition. Let us have p^* satisfying 2.8. Obviously, an insurance price higher than p^* brings a negative additional profit to those who are insured. A lower price, however, brings a positive profit to those who are insured. Moreover, as h is increasing in p, a price $p < p^*$ may motivate some of the consumers who are not willing to pay the price p^* to buy the insurance. It has no impact on the profits of consumers that are not willing to pay even the new (lowered) price. Therefore, only the lowest price satisfying the first condition satisfies also the second one.

Proposition 2.1.1. *In the insurance market given by our setup, a* competitive equilibrium contract under asymmetric information *is given by the lowest price p such that it satisfies the equation 2.8.*

We look at the expression of the expected loss of consumers conditional on $\Pi \geq h(p, A)$ as a function of price

$$g(p) = E[\Pi|\Pi \geq h(p, A)]L.$$

for $p \in [0, L]$. The conditional expectations are well defined for $h(p, \alpha) \leq \overline{\pi}$. For every $p \in [0, L]$ for which the conditional expectations are not defined we can define the expected value to be $\overline{\pi}$ by continuous extension without influencing the results. Note that u, h and g are continuous. In this notation, searching for the candidates of market equilibria is reduced to searching for the fixed points of the function $g(p)$, that is to solving the equation

$$p = g(p).$$

Since $E[\Pi|\Pi \geq h(p, A)]$ always lies in $[0, \overline{\pi}]$ the continuous function g maps the closed interval $[0, L]$ into itself. Applying the Brouwer fixed point theorem we find that $g(p)$ must have a fixed point $p^* \in [0, L]$. Thus, an equilibrium price exists.

2.2 Special Cases

We begin the analysis of the market equilibria with two special cases in which there is uncertainty in only one of the characteristics. In the first case, there is no uncertainty about riskiness of the consumers, in the second case there is no uncertainty about their risk aversion.

2.2.1 One-Dimensional Uncertainty: Risk Aversion

Assume that there is no uncertainty about riskiness of the consumers. The insurance companies knows the probability π of possessing a loss of each consumer. The insurance company has zero expected profit when setting $p = E(\pi)L$ for each consumer which is also an efficient competitive equilibrium price.

2.2.2 One-Dimensional Uncertainty: Riskiness

Assume that there is no uncertainty about the risk aversion of the consumers. All the consumers have the same coefficient of risk aversion, therefore A is not a random variable, only $\Pi \in [\underline{\pi}, \overline{\pi}]$ is. $F_\Pi(\pi)$ is distribution function of the random variable Π, $f_\Pi(\pi)$ denotes the density function[2]. We wish to examine the fixed

[2]For simplicity we skip the index Π in the notation of distribution function and density function if it is obvious from the context.

points of the function $g(p)$ which now takes the form

$$\begin{aligned} g(p) &= E\left[\Pi|\Pi \geq h(p)\right] L \\ &= \frac{L}{F(\overline{\pi}) - F(h(p))} \int_{h(p)}^{\overline{\pi}} \pi f(\pi)\, d\pi. \end{aligned} \tag{2.9}$$

From the definition of the distribution function we have

$$\begin{aligned} F(\underline{\pi}) &= 0, \\ F(\overline{\pi}) &= 1. \end{aligned}$$

Auxiliary Propositions

At first, we introduce some notation and auxiliary propositions we use later. Let us start with the properties of the function h. Trivially it holds that

$$\begin{aligned} h(0) &= 0, \\ h(L) &= 1, \end{aligned}$$

h is increasing in p, thus there exists an inverse function h^{-1}. Let's denote $\underline{p}$, $\overline{p}$ such that $0 \leq \underline{p} < \overline{p} \leq L$ the prices for which

$$\begin{aligned} \underline{p} &:= h^{-1}(\underline{\pi}), \ \text{ i.e. } \underline{\pi} = h(\underline{p}), \\ \overline{p} &:= h^{-1}(\overline{\pi}), \ \text{ i.e. } \overline{\pi} = h(\overline{p}). \end{aligned} \tag{2.10}$$

The interpretation behind prices $\underline{p}, \overline{p}$ is following: $\underline{p}$ is the maximal price for which a consumer with accident probability $\underline{\pi}$ is willing to buy the insurance. The other way around, $\underline{\pi}$ is a minimal accident probability necessary for the consumer to be willing to buy an insurance for the price $\underline{p}$. The price $\overline{p}$ is defined accordingly for the consumer with accident probability $\overline{\pi}$.

Lemma 2.2.1. *The function $g(p)$ is nondecreasing inside the open interval $(\underline{p}, \overline{p})$, independently of the distribution of the random variable Π. If the probability density function $f(\pi)$ is positive for every π in $(\underline{\pi}, \overline{\pi})$, the function $g(p)$ is increasing in any p in $(\underline{p}, \overline{p})$.*

Proof. Let's examine the derivative of function $g(p)$,

$$g'(p) = \frac{L \cdot f(h(p)) \cdot h'(p) \left[\int_{h(p)}^{\overline{\pi}} \pi f(\pi)\, d\pi - h(p)\big[1 - F(h(p))\big]\right]}{\left[1 - F(h(p))\right]^2}. \tag{2.11}$$

The denominator, the loss L, the density function and the derivative of h with respect to p are always nonnegative. Therefore, the sign of the derivative of the function $g(p)$ depends on the sign of the expression

$$\xi = \int_{h(p)}^{\overline{\pi}} \pi f(\pi)\, d\pi - h(p)[1 - F(h(p))]. \tag{2.12}$$

We can rewrite the expression in the brackets using the Newton-Leibniz formula

$$\begin{aligned} 1 - F(h(p)) &= F(\overline{\pi}) - F(h(p)) \\ &= \int_{h(p)}^{\overline{\pi}} f(\pi)\, d\pi. \end{aligned}$$

The expression 2.12 may now be expressed as

$$\begin{aligned} \xi &= \int_{h(p)}^{\overline{\pi}} \pi f(\pi)\, d\pi - h(p)[1 - F(h(p))] \\ &= \int_{h(p)}^{\overline{\pi}} \pi f(\pi)\, d\pi - h(p) \int_{h(p)}^{\overline{\pi}} f(\pi)\, d\pi \\ &= \int_{h(p)}^{\overline{\pi}} \pi f(\pi)\, d\pi - \int_{h(p)}^{\overline{\pi}} h(p) f(\pi)\, d\pi \\ &= \int_{h(p)}^{\overline{\pi}} (\pi - h(p)) f(\pi)\, d\pi. \end{aligned} \tag{2.13}$$

As we integrate through $\pi \geq h(p)$, ξ is always nonnegative. Therefore, the derivative of function $g(p)$ is nonnegative which means the function is nondecreasing.

It is straightforward to see that if $f(h(p))$ is positive, all terms in the derivative 2.11 are positive for p in $(\underline{p}, \overline{p})$ which proves the second part of the lemma. $\square$

Analysis of Corner Values

Let us have a look at the corner values of the function $g(p)$, $p \in [0, \underline{p}]$ and $p \in [\overline{p}, L]$ with corresponding values of $\pi \in [0, \underline{\pi}]$ and $\pi \in [\overline{\pi}, 1]$. First,

$$\begin{aligned} g(\underline{p}) &= \frac{L}{F(\overline{\pi}) - F(h(\underline{p}))} \int_{h(\underline{p})}^{\overline{\pi}} \pi f(\pi)\, d(\pi) \\ &= \frac{L}{1 - 0} \int_{\underline{\pi}}^{\overline{\pi}} \pi f(\pi)\, d(\pi) \\ &= L \cdot E(\Pi). \end{aligned} \tag{2.14}$$

This result is consistent with basic intuition. The price $\underline{p}$ is defined as the highest price for which *every* consumer is willing to buy the insurance. In such case the expected claim on one insured person for the insurance company is the mean expected loss of all the consumers.

It is useful to stop a little and realize that the expression $L \cdot E(\Pi)$ is limited from both sides

$$L\underline{\pi} < L \cdot E(\Pi) < L\overline{\pi}, \tag{2.15}$$

but in general, we can not decide whether

$$\begin{aligned} &\underline{p} \geq L \cdot E(\Pi), \text{ or} \\ &\underline{p} < L \cdot E(\Pi). \end{aligned} \tag{2.16}$$

Both cases may occur, the illustration is given in the figures 2.2, 2.1. For prices smaller than $\underline{p}$ still all the consumers buy the insurance and the expected claim towards the insurance company stays $L \cdot E(\Pi)$.

We are not able to evaluate the expression $g(\overline{p})$ because it takes the form $0/0$, however, we can evaluate the limit of the function at the point $\overline{p}$. For p going to $\overline{p}$ also $h(p)$ goes to $h(\overline{p})$ and we may state the limit in both ways.

$$\begin{aligned} \lim_{p \to \overline{p}} g(p) &= \lim_{\substack{p \to \overline{p} \\ h(p) \to h(\overline{p})}} \frac{L}{F(\overline{\pi}) - F(h(p))} \int_{h(p)}^{\overline{\pi}} \pi f(\pi)\, d\pi \\ &\overset{L'H}{=} \lim_{\substack{p \to \overline{p} \\ h(p) \to h(\overline{p})}} \frac{-Lh(p) f(h(p))}{-f(h(p))} \\ &= \lim_{\substack{p \to \overline{p} \\ h(p) \to h(\overline{p})}} Lh(p) \\ &= L\overline{\pi}, \end{aligned}$$

where we used the L'Hospital's rule with respect to $h(p)$ in the second equation. The value of the limit is again intuitive: as the price goes closer to $\overline{p}$ only the consumers with accident probability close to $\overline{\pi}$ are willing to purchase the insurance. Therefore the expected loss of the insured consumer goes to $L\overline{\pi}$. The insurance companies in the competitive market never set the prices higher than $L\overline{\pi}$, therefore we do not need to examine the case, however, in order to satisfy the assumptions of the fixed point theorems used in the analysis, we define by the continuous extension

$$g(p) := L\overline{\pi},$$

for $p \in [\overline{p}, L]$. Note that from the Lemma 2.1.1

$$L\overline{\pi} \leq \overline{p}.$$

Thus, the price that the consumer with the highest risk is willing to pay is always higher than the highest price $L\overline{\pi}$ the companies may be willing to set. However, the price that the consumer with the lowest risk is willing to pay may be both higher or lower (2.16) than the minimal price that the companies may be willing to set. As the following analysis shows, the inequality sign of 2.16 is crucial for the properties of the market equilibrium.

Properties of Equilibrium

Let us first suppose that the distribution of Π is such that the right form of 2.16 is

$$\underline{p} < L \cdot E(\Pi) = g(\underline{p}). \tag{2.17}$$

The price that the consumer with the lowest risk is willing to pay is lower than the price that covers the losses of the insurance company, when all the consumers buy the insurance. Therefore we may expect problems conected to adverse selection to occur. The situation is graphically illustrated in the Figure 2.1. The function $g(p)$ is increasing (nondecreasing) on $(\underline{p}, \overline{p})$, starting above the 45 degree line and ending below it. According to the Tarski fixed point theorem[3] there must be a fixed point (not necessarily unique) on $[0, L]$. There is no fixed point for $p < \underline{p}$, therefore the lowest fixed point of the map, i.e. the equilibrium price p^* is higher than $\underline{p}$. Hence, there are consumers who are not willing to buy the insurance for the equilibrium price. It is important for further analysis of dynamics that as $g(\underline{p})$ is greater than $\underline{p}$, in the lowest fixed point p^* the function $g(p)$ has a slope lower than one (it is crossing the 45 degree line from above to below).

[3] We previously showed that as $g(p)$ is continuous contractive map also the Brouver's fixed point theorem may be applied, however, we use the Tarski fixed point theorem here as it gives a better picture of the situation.

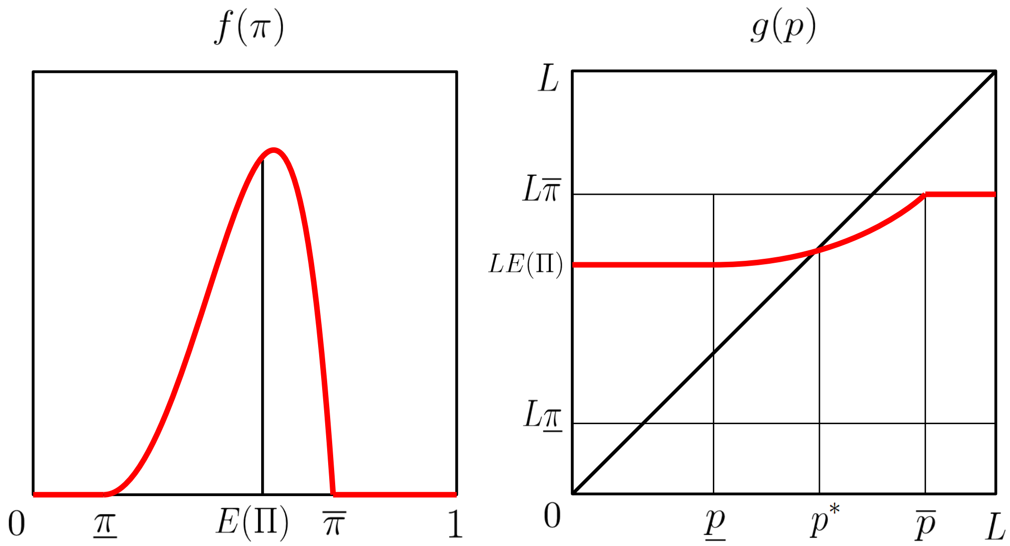

Figure 2.1: The distribution of accident probabilities with high expected value relatively to the lowest accident probability causing $\underline{p} > LE(\Pi)$. The equilibrium price lies between $\underline{p}$ and $\overline{p}$.

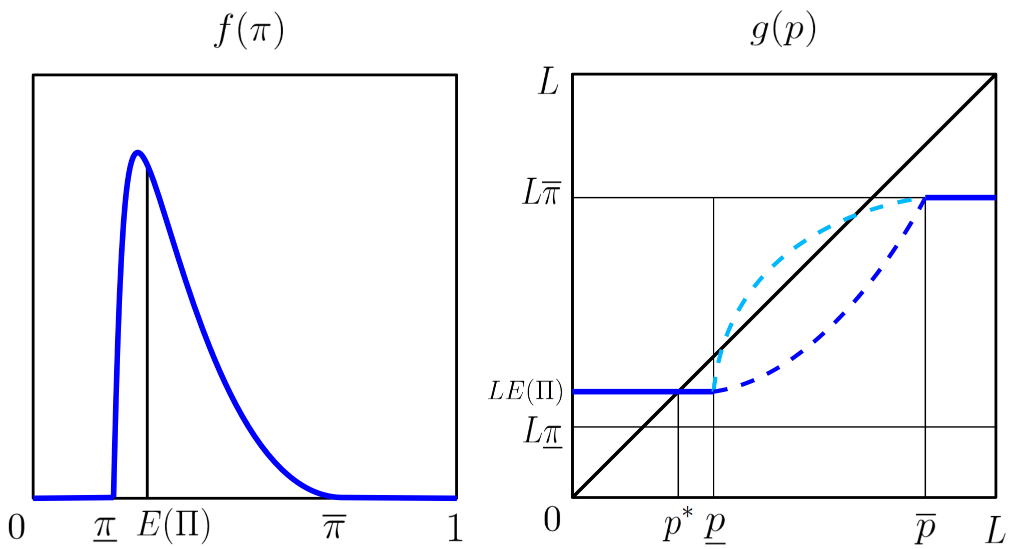

Figure 2.2: The distribution of accident probabilities with low expected value, $\underline{p} \leq LE(\Pi)$. The equilibrium price is lower than $\underline{p}$. The function $g(p)$ may but does not have to have other fixed points.

Let us now suppose that the distribution of Π is such that

$$\underline{p} \geq L \cdot E(\Pi) = g(\underline{p}). \tag{2.18}$$

The price that the consumer with the lowest risk is willing to pay is higher than the price that covers the losses of the insurance company, when all the insurers buy the insurance. Everyone is willing to buy the insurance for the price that makes a nonnegative profit for the insurance company. We expect the adverse selection not to occur. The situation is graphically illustrated in the Figure 2.2. The function $g(p)$ is nondecreasing on $(\underline{p}, \overline{p})$, starting below the 45 degree line and ending below it. There may or might not be some other fixed points of the function $g(p)$ at $(\underline{p}, \overline{p}]$. As $g(p)$ is constant greater than zero on $[0, \underline{p}]$, there exists a single fixed point p^* on $[0, \underline{p}]$, which is an efficient competitive equilibrium price as every risk-averse consumer purchase the insurance policy for this price.

Sensitivity on the Value of Possible Loss

So far, we did not consider that the amount of possible loss L may vary on $(0, w]$. However, it is natural to ask how sensitive is the market equilibrium on the changing value of loss. Although it may appear straightforward that with bigger possible loss the consumers are more motivated to buy the insurance, it does not have to be so. With higher L also the companies expected costs increases and therefore they set higher prices.

Let's examine how the sign of inequality 2.16 changes with L. The derivative with respect to L of the right-hand side is given by $E(\Pi)$. To determine the derivative of the left-hand side it is necessary to reformulate the condition 2.1 such that it shows the maximal price that a consumer is willing to pay conditional on L, π, w and the utility function[4].

$$p = w - u^{-1}[\pi u(w - L) + (1 - \pi)u(w)]. \tag{2.19}$$

[4]In this part we skip the parameter α in the notation of the utility function for simplicity reasons.

The slope of the left-hand side of 2.16 is then given by

$$\begin{aligned}\frac{dp}{dL} &= \frac{du^{-1}(u)}{du}\Big|_{u=\pi u(w-L)+(1-\pi)u(w)} \cdot \frac{du(w_0)}{dw_0}\Big|_{w_0=w-L} \\ &\geq \frac{du^{-1}(u)}{du}\Big|_{u=u(w-L)} \cdot \frac{du(w_0)}{dw_0}\Big|_{w_0=w-L} \\ &= 1,\end{aligned}$$

where we used the convexity of the inverse function and the fact that

$$\pi u(w-L) + (1-\pi)u(w) \geq u(w-L)$$

in the first inequality and the reciprocity of gradients of function and it's inverse function in the second equation. As in the market with uncertainty $E(\Pi)$ is less than one, the left-hand side of the equation rises with L more rapidly than the right-hand side. For the value of possible loss going to infinity, the price that the least risky consumer is willing to pay is always lower than the expected loss from all the consumers, therefore the adverse selection does not occur. However, as the value of loss is limited by the consumers' wealth w, it is not always possible to find such a value of L that guarantees that all the consumers are willing to pay a given price.

Dynamic Properties of Equilibrium

To complete the analysis of the equilibrium we introduce a very simple discrete dynamics: consider the price of the insurance policy at the time $t = 0, 1, \ldots$

- At the time $t_0 = 0$, the insurance company sets the price of insurance p_0. Every consumer decides whether he buys the insurance or not. A consumer buys the insurance if his accident probability π satisfies the condition

$$\pi \geq h(p_0).$$

- Given p_i, the insurance company sets a new price p_{i+1} at the time $i+1$, $i \geq 0$. As the company wishes to have nonnegative profit, it uses the information from the previous time i and sets the price p_{i+1} as

$$p_{i+1} = E(\Pi | \Pi \geq h(p_i))L.$$

  Then, the consumers decide whether they purchase the insurance again.

This model leads us to the simple difference equation

$$p_{i+1} = g(p_i). \tag{2.20}$$

If $|g'(p)| < 1$ in the fixed point p^*, the fixed point is asymptotically stable. Recall that we already showed that in equilibrium, the slope of $g(p)$ is nonnegative and less than one.

Adverse Selection Effect on the Market

The occurrence of adverse selection in the market depends on the relation between the expected value of loss amongst all the consumers and the consumers with the lowest riskiness in the following way:

Proposition 2.2.1. *If the probabilistic distribution of the random variable Π is such that*

- $\underline{p} < L \cdot E(\Pi)$*, then the equilibrium price lies inside the interval* $(\underline{p}, \overline{p})$*. Therefore, in the contrary to the competitive market with full information, there are consumers excluded from purchasing the insurance. The certain efficiency loss occurs due to the adverse selection.*

- $\underline{p} \geq L \cdot E(\Pi)$*, then the equilibrium price lies inside the interval* $[0, \underline{p}]$*. All the risk-averse consumers buy the insurance contract, the equilibrium is thus fully efficient. The decisions of the consumers never affect the insurance companies outcomes negatively, therefore adverse selection is not present.*

In both cases, the equilibria are asymptotically stable from the dynamic point of view. In our simple setting, the equilibrium price may be approximately reached after an iterating process in which market subjects maximizes their utility. It is worth noticing that this iterating process leads to the equilibrium price also if the insurance companies have zero information in the beginning - they do not need to know the distribution function F, they only use the information about their costs at each time period instead. An approximate knowledge of the function F may help to set the starting value of the iterating process close to the equilibrium price and thus to minimize the initial costs of the companies.

2.3 General Model with Two characteristics

Let us now turn back to the general model with two hidden consumers characteristics introduced in the Section 2.1. We wish to examine the fixed points of the function

$$g(p) = E(\Pi|\Pi \geq h(p, A))L,$$

where we can express the conditional expected value of consumers accident probabilities as

$$g(p) = \frac{L}{\int_{\underline{\alpha}}^{\overline{\alpha}} \int_{h(p,\alpha)}^{\overline{\pi}} f(\pi, \alpha)\, d\pi d\alpha} \int_{\underline{\alpha}}^{\overline{\alpha}} \int_{h(p,\alpha)}^{\overline{\pi}} \pi f(\pi, \alpha)\, d\pi d\alpha, \tag{2.21}$$

$f(\pi, \alpha)$ being the joint probability density of random variables Π, A. The function h is increasing in p and decreasing in α. Let us denote $\underline{p}$, $\overline{p}$ the prices, such that $0 \leq \underline{p} < \overline{p} \leq L$, for which

$$\begin{aligned} \underline{\pi} &= h(\underline{p}, \underline{\alpha}), \\ \overline{\pi} &= h(\overline{p}, \overline{\alpha}). \end{aligned} \tag{2.22}$$

The interpretation behind $\underline{p}$, $\overline{p}$ is now as follows: $\underline{p}$ is a maximal price for which a consumer with the accident probability $\underline{\pi}$ and the coefficient of risk aversion $\underline{\alpha}$ buys the insurance. If the consumer with minimal risk aversion and minimal riskiness is willing to buy the insurance, than every consumer is. On the other hand, $\overline{p}$ is the price such that only a consumer with the highest accident probability and the highest coefficient of risk aversion is willing to pay and no one else is.

Properties of the Function g

A remarkable difference from the special case with only uncertainty in riskiness of the consumers is that the function $g(p)$ need not to be increasing here. We show an example of a decrease in the value of g graphically in the Figure 2.3 which shows the set $[\underline{\alpha}, \overline{\alpha}] \times [\underline{\pi}, \overline{\pi}]$. The line $h(p_0, \alpha)$ borders the area of the consumers that buys the insurance for the price p_0, line $h(p_1, \alpha)$ accordingly for p_1 higher than p_0. $E_0 = E(\Pi|\Pi \geq h(p_0, A))$ denotes the expected loss probability of a consumer that buys the insurance for the price p_0. Let us have a probability distribution with zero density in the hatched area bordered by $h(p_0, \alpha)$, E_0 dashed line and $h(p_1, \alpha)$ and a positive density all over the area above the E_0 dashed line. When we move

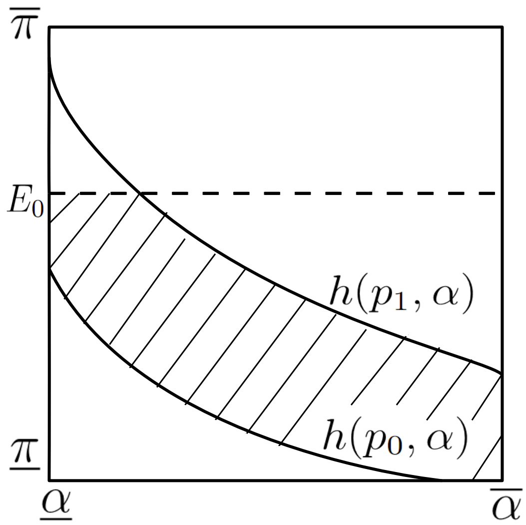

Figure 2.3: The function g does not have to be nondecreasing (example).

the price from p_0 to p_1, all the consumers who stop wanting to buy the insurance (we say that they *drop* the insurance policy) have a risk probability higher than the average probability conditional on p_0. Thus, the expected loss probability decreases and so do $g(p)$.

Because the function g is continuous, the fact that it might be decreasing does not affect the question of the equilibrium existence. However, it might have unpleasant consequences on the stability of the equilibrium if the function g would be decreasing in its fixed point. Luckily, we are able to show that this can never happen.

Proposition 2.3.1. *Let have a function g defined as in 2.21 and it's fixed point . The function g is nondecreasing in p_f.*

Proof. Let p_d be a point in which g is decreasing. There must be a consumer with risk probability $\pi > E(\Pi|\Pi \geq h(p_d, A))$ that drops the insurance at any price higher than p_d. The price p_d must be higher than $E(\Pi|\Pi \geq h(p_d, A))L$, otherwise the consumer would not drop it, as he is risk-averse. Straightforwardly we obtain

$$p_d > E(\Pi|\Pi \geq h(p_d, A))L = g(p_d).$$

The function g can only be decreasing in p such that $p > g(p)$, therefore it may never be decreasing in its fixed point. □

Analysis of Corner Values

In consistency with the one-dimensional case we proceed with the analysis of the corner values of insurance policy prices defined in 2.22.

$$\begin{aligned} g(\underline{p}) &= \frac{L}{\int_{\underline{\alpha}}^{\overline{\alpha}} \int_{h(\underline{p},\alpha)}^{\overline{\pi}} f(\pi,\alpha)\, d\pi d\alpha} \int_{\underline{\alpha}}^{\overline{\alpha}} \int_{h(\underline{p},\alpha)}^{\overline{\pi}} \pi f(\pi,\alpha)\, d\pi d\alpha \\ &= L \cdot E(\Pi), \end{aligned} \tag{2.23}$$

as every consumer wants to buy the insurance for the price $\underline{p}$ by definition, independently on his risk aversion. Correspondingly, for prices p lower than $\underline{p}$ all the consumers buy the insurance and the expected costs of the insurance company on one insured person stays $L \cdot E(\Pi)$.

Again, we may not evaluate the expression $g(\overline{p})$ because it takes the form $0/0$, however, we can evaluate the limit of the function in the point $\overline{p}$. It is clear after a short thought that for p going to $\overline{p}$ also $h(p,\alpha)$ goes to $h(\overline{p},\overline{\alpha})$ and α goes to $\overline{\alpha}$ as only the most risky and risk averse consumers are willing to pay the price.

$$\begin{aligned} \lim_{p\to\overline{p}} g(p) &= \lim_{\substack{p\to\overline{p} \\ h(p,\alpha)\to h(\overline{p},\overline{\alpha})}} \frac{L}{\int_{\underline{\alpha}}^{\overline{\alpha}} \int_{h(p,\alpha)}^{\overline{\pi}} f(\pi,\alpha)\, d\pi d\alpha} \int_{\underline{\alpha}}^{\overline{\alpha}} \int_{h(p,\alpha)}^{\overline{\pi}} \pi f(\pi,\alpha)\, d\pi d\alpha \\ &\overset{L'H}{=} \lim_{\substack{p\to\overline{p} \\ h(p,\alpha)\to h(\overline{p},\overline{\alpha})}} \frac{-L\int_{\underline{\alpha}}^{\overline{\alpha}} h(p,\alpha) f(h(p,\alpha),\alpha) \cdot \frac{\partial h(p,\alpha)}{\partial p}\, d\alpha}{-\int_{\underline{\alpha}}^{\overline{\alpha}} f(h(p,\alpha)) \cdot \frac{\partial h(p,\alpha)}{\partial p}\, d\alpha} \\ &= \frac{-L\int_{\underline{\alpha}}^{\overline{\alpha}} h(\overline{p},\overline{\alpha}) f(h(\overline{p},\overline{\alpha}),\overline{\alpha}) \cdot \frac{\partial h(p,\overline{\alpha})}{\partial p}\Big|_{p=\overline{p}}\, d\alpha}{-\int_{\underline{\alpha}}^{\overline{\alpha}} f(h(\overline{p},\overline{\alpha}),\overline{\alpha}) \cdot \frac{\partial h(p,\overline{\alpha})}{\partial p}\Big|_{p=\overline{p}}\, d\alpha} \\ &= \frac{-L h(\overline{p},\overline{\alpha}) f(h(\overline{p},\overline{\alpha}),\overline{\alpha}) \cdot \frac{\partial h(p,\overline{\alpha})}{\partial p}\Big|_{p=\overline{p}} \int_{\underline{\alpha}}^{\overline{\alpha}} d\alpha}{-f(h(\overline{p},\overline{\alpha}),\overline{\alpha}) \cdot \frac{\partial h(p,\overline{\alpha})}{\partial p}\Big|_{p=\overline{p}} \int_{\underline{\alpha}}^{\overline{\alpha}} d\alpha} \\ &= L \cdot h(\overline{p},\overline{\alpha}), \end{aligned}$$

where we used the L'Hospital's rule with respect to $h(p)$ in the second equation.

We may conclude that

$$\lim_{p \to \overline{p}} g(p) = L \cdot h(\overline{p}, \overline{\alpha}) = L\overline{\pi}. \tag{2.24}$$

The insurance companies in the competitive market never set the prices higher than $L\overline{\pi}$, therefore we do not need to examine the case. Note that it follows from the Lemma 2.1.1

$$L\overline{\pi} \leq \overline{p}.$$

2.3.1 Results

The first important thing to notice is that we get exactly the same values of function $g(p)$ in the corner prices as we get in the case with only uncertainty about riskiness. This is caused by the fact that the risk aversion is not an important characteristic from the supply side of the market: the insurance companies have the same expected costs on two consumers with the same riskiness and different risk aversions. Hence, in the case where all the consumers are insured (similarly when only the worst types are insured), the company's expected costs depend on the distribution of riskiness solely.

Despite the generalization of the model the Proposition 2.2.1 remains in force: although the function $g(p)$ does not have to be increasing, it is continuous, thus the existence of the fixed point is guaranteed. The equilibrium properties and the condition on the adverse selection occurrence do not differ from the Proposition 2.2.1. Moreover, also the asymptotic stability of the equilibrium is still present, although it only holds locally as g might be decreasing outside its fixed points. In order to reach the equilibrium price the companies have to set the initial prices of the iterating process close to the equilibrium value. To be able to make such a good guess it is necessary for the companies to know at least approximate distribution of the consumers characteristics.

The occurrence of the adverse selection depends on the distribution of the riskiness amongst the consumers (right-hand side of the conditioning inequality in 2.2.1 representing the supply side of the market) and the *lower tail* of consumers characteristics (the characteristics closest to $\underline{\pi}$, $\underline{\alpha}$) as those consumers have the highest motivation for not buying the insurance and thus start the adverse selection mechanism. The willingness to pay for the insurance of the lower tail consumers, driven by both of the characteristics, is represented by the left-hand side of the inequality. The occurrence of the adverse selection does not depend on the exact distribution of the risk-aversion except of the least occurring risk aversion.

However, in the case of presence of adverse selection in the market its severity (the exact equilibrium price and the volume of consumers not buying the insurance) depends on the distribution of both hidden characteristics of the consumers.

We examine some possible forms of the characteristics distribution in the next chapter.

Chapter 3

Models with Dependence between Risk Aversion and Accident Probability

In this chapter a concept we did not consider so far, so-called advantageous selection occurs. The basic idea behind the concept is that there exists a connection between the two characteristics examined in the previous chapter and that the connection can positively affect the properties of the equilibrium. This idea, originally coming from Hemenway [9], may appear in the model in different forms. The dependence between risk aversion and riskiness may be included in the model as an exogenous relationship or may be induced by the model endogenously. In the first part of this chapter we study the case of exogenous relationship and provide numerical examples, in the second part we propose an alternative model in which the relationship is a consequence of its setting.

3.1 Exogenous Dependence between Risk Aversion and Accident Probability

In this part we assume that there is a relationship connecting risk aversion with riskiness of consumers and that this relationship is given by a higher force (nature). For example, we might assume that people who are less risk averse are in general worse drivers and that the drivers can not affect their characteristics. So, let us suppose there exists a continuous function $\phi : [\underline{\pi}, \overline{\pi}] \longrightarrow [\underline{\alpha}, \overline{\alpha}]$ connecting risk aversion with accident probabilities of the consumers,

$$\alpha = \phi(\pi),$$

where π is a realization of the random variable Π defined in the Section 2.1, α is a coefficient of risk aversion. The average clientele loss function $g(p)$ takes the form

$$g(p) = E[\Pi | h(p, \phi(\Pi)) \leq \Pi)]L.$$

The finding of the equilibrium price is now complicated by the fact that the condition on buying the insurance is dependent on the consumer's risk in two different ways. It affects not only his expected loss but also his utility function. The question that we are trying to solve now is how the accident probability affects the price that a consumer is willing to pay for the insurance contract. Obviously, what matters is the function ϕ as it describes the relationship between accident probability and risk aversion.

3.1.1 Positive Correlation between Risk Aversion and Accident Probability

Theoretically, we may distinguish two basic qualitative behaviors: a positive or negative correlation of the characteristics. Let us first assume, contrary to Hemenway's [9] ideas that there is a positive correlation between risk and risk aversion. The consumers with higher accident probability tends to be more risk-averse,

$$\phi'(\pi) \geq 0.$$

In such case the coefficient of risk aversion α increases with π. By definition, the function h decreases with α. It follows that h is decreasing with π increasing, thus for higher prices the condition on buying the insurance is satisfied for more risky consumers (Figure 3.1). The information asymmetry has a negative effect on the selection of the consumers from the companies' point of view. This result is consistent with the common sense, as the insurance has higher value for the more risky and more risk-averse consumers.

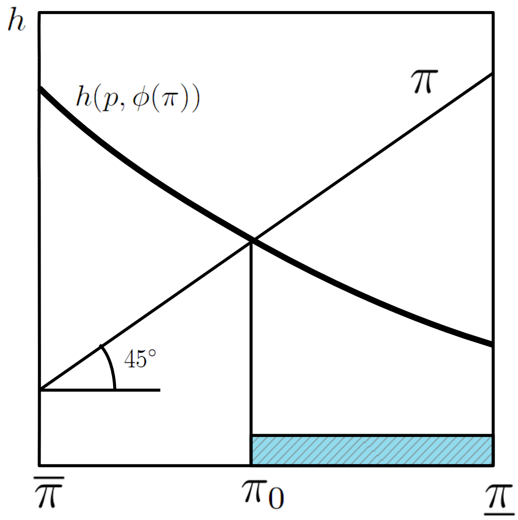

Figure 3.1: The function h is decreasing in π. Only the more risky consumers (hatched) are willing to pay the given price p as the condition $\pi \geq h(p, \phi(\pi))$ is only satisfied for π greater than π_0.

3.1.2 Negative Correlation between Risk Aversion and Accident Probability

This case may be considered more intuitive as the natural guess would be that the more risk averse consumers have less accidents. It is also consistent with Hemenway's proposals. However, in this case it is much more complicated to guess the result by heart without a proper analysis. The behavior of the consumers consists of two opposing effects. With higher π the consumer's expected loss increases, but his coefficient of risk aversion decreases. The resulting effect is determined by the intensity with which the effects influence the consumers behavior, the Figure 3.2 depicts various possibilities.

In the previous chapter we used the condition 2.1 in the form

$$\pi \geq h(p, \alpha), \tag{3.1}$$

as this suited the needs of our analysis. This form is not that useful nor elegant when considering α depending on π because the parameter π appears on the both sides of the inequality. Thus, we reformulate the condition into the form

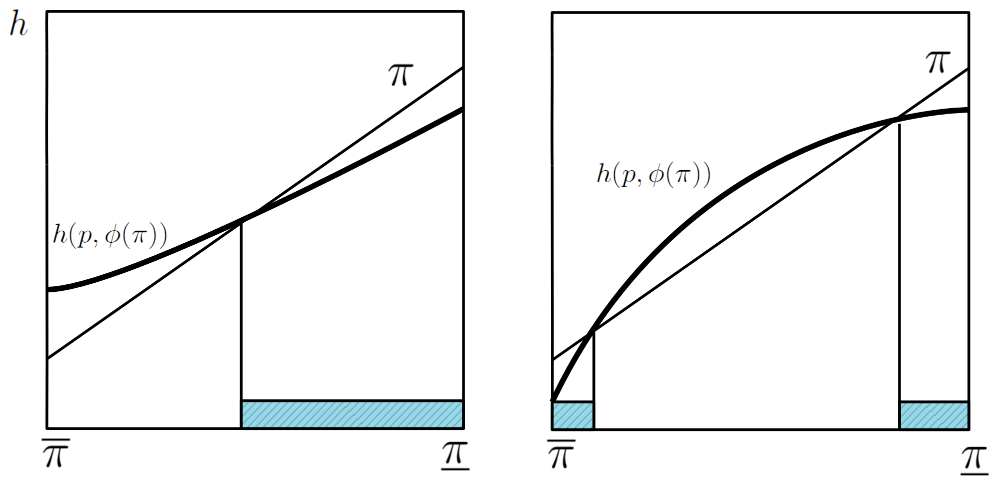

Figure 3.2: The function h is increasing in π. There are various possibilities for the consumers' decision-making, two of them are illustrated.

$$p \leq w - u^{-1}_{\phi(\pi)}[\pi u_{\phi(\pi)}(w - L) + (1 - \pi)u_{\phi(\pi)}(w)], \tag{3.2}$$

similarly to (2.19). It is also helpful to improve the notation of the function u and its inverse in the following way as the functions of two variables:

$$\begin{aligned} u(\alpha, c) &:= u_\alpha(c), \\ v(\alpha, u) &:= u_\alpha^{-1}(u). \end{aligned} \tag{3.3}$$

Now, we can rewrite the condition 3.2 as

$$\begin{aligned} p &\leq w - v[\phi(\pi), \pi u(\phi(\pi), w - L) + (1 - \pi)u(\phi(\pi), w)], \\ p &\leq w - v\big[\phi(\pi), E(u, \pi)\big] =: k(\pi), \end{aligned} \tag{3.4}$$

where $E(u, \pi)$ denotes the expression

$$\pi u(\phi(\pi), w - L) + (1 - \pi)u(\phi(\pi), w).$$

The expression on the right-hand side of 3.4 states the maximal price that a consumer with accident probability π and risk aversion coefficient $\phi(\pi)$ is willing to pay and we denote it $k(\pi)$. The function g may be stated as

$$g(p) = E[\Pi | p \leq k(\Pi)].$$

The mechanism of the adverse selection works exactly the same way as in the previous chapters, therefore it is crucial for determining the properties of the equilibrium to understand which types of consumers are the ones willing to pay the highest prices. We employ the total derivative[1] of the function k with respect to π.

$$\frac{dk}{d\pi} = \frac{\partial v}{\partial \alpha}\Big|_{\alpha=\phi(\pi)} \frac{d\phi(\pi)}{d\pi} + \frac{\partial v}{\partial u}\Big|_{u=E(u,\pi)} \frac{dE(u,\pi)}{d\pi},$$

$$\frac{dk}{d\pi} = \frac{\partial v}{\partial \alpha}\Big|_{\alpha=\phi(\pi)} \frac{d\phi(\pi)}{d\pi} + \frac{\partial v}{\partial u}\Big|_{u=E(u,\pi)} \Big[u(\phi(\pi), w-L) - u(\phi(\pi), w) +$$
$$+ \pi \frac{\partial u}{\partial \alpha}\Big|_{\alpha=\phi(\pi)} \frac{d\phi(\pi)}{d\pi} + (1-\pi)\frac{\partial u}{\partial \alpha}\Big|_{\alpha=\phi(\pi)} \frac{d\phi(\pi)}{d\pi}\Big].$$

Although we may determine the sign of some terms, it is not possible to generally decide whether the expression is positive or negative. We turn to a more concrete form of the utility function to analyze the properties of the market equilibrium. In the rest of this section we will therefore focus on a standard form of constant absolute risk aversion (CARA) utility function

$$u(\alpha, c) = 1 - e^{-\alpha c},$$

c representing consumer's disposable wealth. The CARA class brings analytical and computational simplifications that we believe outweigh, for the purpose of this section, its disadvantages (some discussion is given e.g. in Caballero [3]).

CARA Utility Function

With CARA utility the function k is given as

$$k(\pi) = \frac{1}{\phi(\pi)} \ln[\pi e^{\phi(\pi)L} + (1-\pi)],$$

and the derivative of k with respect to π

$$\frac{dk}{d\pi} = \frac{-\ln\left[(1-\pi) + \pi e^{\phi(\pi)L}\right]\phi'(\pi)}{\phi^2(\pi)} + \frac{e^{\phi(\pi)L} - 1 + L\pi e^{\phi(\pi)L}\phi'(\pi)}{\left[(1-\pi) + \pi e^{\phi(\pi)L}\right]\phi(\pi)}.$$

[1] all the terms in the derivative are expressed in certain points, notation of some of which we skipped to keep the formula understandable. The complete form of the derivative is showed in the Appendix.

$(1-\pi)+\pi e^{\phi(\pi)L}$	positive, at least 1
$\ln[(1-\pi)+\pi e^{\phi(\pi)L}]$	positive
$\phi'(\pi)$	negative
$\phi(\pi)$	positive
$e^{L\phi(\pi)}-1$	positive
$L\pi e^{\phi(\pi)L}\phi'(\pi)\phi(\pi)$	negative

Table 3.1: Signs of the expressions in the numerator.

$-\ln[(1-\pi)+\pi e^{\phi(\pi)L}][(1-\pi)+\pi e^{\phi(\pi)L}]\phi'(\pi)$	positive
$[e^{\phi(\pi)L}-1]\phi(\pi)$	positive
$L\pi e^{\phi(\pi)L}\phi'(\pi)\phi(\pi)$	negative

Table 3.2: Signs of the expressions in the numerator.

The derivative can be equivalently stated as

$$\frac{dk}{d\pi}=-\frac{\ln[(1-\pi)+\pi e^{\phi(\pi)L}][(1-\pi)+\pi e^{\phi(\pi)L}]\phi'(\pi)}{[(1-\pi)+\pi e^{\phi(\pi)L}]\phi^2(\pi)}+$$
$$+\frac{[e^{\phi(\pi)L}-1]\phi(\pi)+L\pi e^{\phi(\pi)L}\phi'(\pi)\phi(\pi)}{[(1-\pi)+\pi e^{\phi(\pi)L}]\phi^2(\pi)}.$$

The common denominator is a positive number, therefore the slope of the function k with respect to π only depends on the numerator N.

$$\begin{aligned}N=&-\ln\left[(1-\pi)+\pi e^{\phi(\pi)L}\right]\left[(1-\pi)+\pi e^{\phi(\pi)L}\right]\phi'(\pi)+\\&+\left[e^{\phi(\pi)L}-1\right]\phi(\pi)+L\pi e^{\phi(\pi)L}\phi'(\pi)\phi(\pi).\end{aligned}\tag{3.5}$$

The Table 3.1 shows the signs of the individual terms of the numerator. To summarize, the first term of the numerator is always positive, the second is always positive and the third is always negative, as we see in the Table 3.2. Let us now try to determine the sign of the slope of the function k.

For the numerator to be negative, the condition

$$\phi'(\pi)\Big[L\pi e^{\phi(\pi)L}\phi(\pi)-\ln\left[(1-\pi)+\pi e^{\phi(\pi)L}\right]\left[(1-\pi)+\pi e^{\phi(\pi)L}\right]\Big]<$$
$$<-\left[e^{\phi(\pi)L}-1\right]\phi(\pi),$$

following from 3.5 has to be satisfied. As right-hand side of the condition is negative, also the left-hand side has to be. Hence, the condition is satisfied if the both of the following inequalities are satisfied

$$\begin{aligned} \phi'(\pi) &< \frac{-\left[e^{\phi(\pi)L}-1\right]\phi(\pi)}{L\pi e^{\phi(\pi)L}\phi(\pi)-\ln\left[(1-\pi)+\pi e^{\phi(\pi)L}\right]\left[(1-\pi)+\pi e^{\phi(\pi)L}\right]}, \\ 0 &< L\pi e^{\phi(\pi)L}\phi(\pi)-\ln\left[(1-\pi)+\pi e^{\phi(\pi)L}\right]\left[(1-\pi)+\pi e^{\phi(\pi)L}\right]. \end{aligned} \tag{3.6}$$

The numerator is negative if and only if the values of the parameters and $\phi'(\pi)$ are such that the inequality 3.6 is satisfied. The function $k(\pi)$ is then decreasing with π, otherwise it is not. Since the condition is stated in a quite complicated form we also use an alternative approach. We derive a satisfying condition for k to be increasing by finding some limiting values of the expressions in it's derivative.

Look at the first expression in the numerator. We may limit the expression

$$(1-\pi)+\pi e^{\phi(\pi)L} \tag{3.7}$$

from the bottom to be at least

$$e^{\pi\phi(\pi)L},$$

as the function $e^{\phi(\pi)c}$ is convex in c. Therefore we may limit the numerator from the bottom as follows:

$$\begin{aligned} N \geq &-\pi L\phi(\pi)[(1-\pi)+\pi e^{\phi(\pi)L}]\phi'(\pi)+ \\ &+[e^{\phi(\pi)L}-1]\phi(\pi)+L\pi e^{\phi(\pi)L}\phi'(\pi)\phi(\pi). \end{aligned}$$

which we can simplify to the form

$$N \geq (1-\pi)\pi L\phi'(\pi)+1.$$

The expression on the right-hand side is positive for

$$\phi'(\pi) > \frac{-1}{(1-\pi)\pi L}, \tag{3.8}$$

thus the numerator is always positive for such values of the parameters and $\phi'(\pi)$. If the inequality 3.8 is satisfied, the function $k(\pi)$ is increasing with π. This condition may be especially useful if we do not have any information about a proper form of the function ϕ.

Proposition 3.1.1. *The maximal price that a consumer with CARA utility function is willing to pay for the insurance contract is decreasing with his accident probability if both of the following inequalities are satisfied*

$$\phi'(\pi) < \frac{-\left[e^{\phi(\pi)L} - 1\right]\phi(\pi)}{L\pi e^{\phi(\pi)L}\phi(\pi) - \ln\left[(1-\pi) + \pi e^{\phi(\pi)L}\right]\left[(1-\pi) + \pi e^{\phi(\pi)L}\right]} =: C_D,$$
$$0 < L\pi e^{\phi(\pi)L}\phi(\pi) - \ln\left[(1-\pi) + \pi e^{\phi(\pi)L}\right]\left[(1-\pi) + \pi e^{\phi(\pi)L}\right] =: C_H$$

and nondecreasing otherwise. If

$$\phi'(\pi) > \frac{-1}{(1-\pi)\pi L} =: C_I,$$

then the function k is increasing.

If the maximal price a consumer is willing to pay is increasing with π, adverse selection may occur because rising the price may lead to a clientele with worse expected accident probability. If it is decreasing, advantageous selection may occur as the less risky consumers are more motivated to buy the insurance as the more risky ones.

Recall that according to the Definition 1.3.2 by advantageous selection we understand a situation where consumers' (informed) choice of insurance contracts positively affects the companies outcomes, in this case by lowering the expected costs of insurance company.

It is worth noticing that the Proposition 3.1.1 is consistent in the way that both the bounding values of the slope of the function ϕ are negative and that

$$C_D \leq C_I$$

holds, meaning that k is not increasing and decreasing at the same time.

Indeed, the conditions on the slope of the function $\phi(\pi)$ seem to be reasonable. If the function ϕ decreases steeply with π, consumers with just a little bit higher accident probabilities are much less risk-averse, therefore the effect of the low risk aversion overweights the effect of higher risk. The insurance tends to be less valuable for the consumer with higher accident probability. Oppositely, if the function ϕ decreases slowly with π, consumers with higher accident probabilities are just a little bit less risk-averse, therefore the effect of the low risk aversion does not overweight the effect of higher risk. The insurance tends to be more valuable for the consumers with higher accident probabilities. As we are considering CARA utility function, the conditions in the proposition 3.1.1 do not depend on the wealth level w of the consumers.

The last note to make to the proposition 3.1.1 is that we consider the given satisfying condition for increase of the function k, the boundary C_I, to be reasonably good as its value can not be generally decreased. For $\phi(\pi)$ going to zero the the slope of the function k is not positive any more if $\phi'(\pi) \leq C_I$.

Properties of Equilibrium and Occurrence of Advantageous Selection

We described the markets with adverse selection in detail in the previous chapter, therefore we only focus on the case of decreasing function k that may lead to the advantageous selection here. As with higher price only the consumers with lower probabilities keep being insured, the function g is nonincreasing. In such market the properties of the equilibrium are different from the previous cases. First of all, the values of the function g in its corner values $\underline{p}, \overline{p}$[2] given by

$$\underline{p} = h(\overline{\pi}, \phi(\overline{\pi})),$$
$$\overline{p} = h(\underline{\pi}, \phi(\underline{\pi})),$$

are as follows:

$$g(\underline{p}) = E[\Pi]L,$$
$$g(\overline{p}) = \underline{\pi}L,$$

as $\underline{p}$ is a price for which all the consumers buy the insurance and $\overline{p}$ is the price for which only the consumer with probability $\underline{\pi}$ (and therefore the highest risk aversion) buys the insurance. It is important to notice, that the price $\underline{p}$ now depends on the consumer with the highest risk. The properties of the equilibrium again depends on the relationship between the price that a consumer with the lowest willingness to pay (highest accident probability) is willing to pay and the expected loss of all the consumers. Recall that according to the Lemma 2.1.1

$$\tilde{p} \geq \tilde{\pi}L, \tag{3.9}$$

for every

$$\tilde{\pi} = h(\tilde{p}, \alpha). \tag{3.10}$$

[2]the definition may seems *opposite* to the definition 2.10 in the Chapter 2.

Substituting $\underline{p}$ for $\tilde{p}$ we get that

$$\underline{p} \geq \overline{\pi} L > E[\Pi] L = g(\underline{p}). \tag{3.11}$$

Proposition 3.1.2. *In the case that k is decreasing in π, the equilibrium price always lies inside the interval $[0, \underline{p})$. All the risk-averse consumers buy the insurance contract, the equilibrium is thus fully efficient.*

The proposition 3.1.2 states that in the case where advantageous selection might occurred it is guaranteed that the market outcomes are fully efficient. The advantageous selection therefore never occurs, although the negative correlation between riskiness and risk aversion has positive effects on the market efficiency, like Hemenway proposed. While the adverse selection sometimes punishes the consumers with good characteristic - low accident probabilities, the advantageous selection does not punishes anyone.

In the next part we illustrate our results with numerical examples.

3.1.3 Numerical Example

The aim of this example is to give a rough idea of how the real world may be reflected in the model. The first issue is to identify the true ranges of the unobservable characteristics π, α. According to empirical data used in Zavadil [19], the accident probabilities of causing a car accident[3] vary roughly from 0.05 to 0.25. We use four-parameter beta distribution[4] of the consumers accident probabilities. Its expected value is given by

$$E[\Pi, \underline{\pi}, \overline{\pi}, a, b] = \underline{\pi} + \frac{a}{a+b}(\overline{\pi} - \underline{\pi}). \tag{3.12}$$

We consider the values of the coefficient of risk aversion $\alpha \in (0, 0.5]$ at approximate consumption (wealth) level scaled to 20, which is roughly the the range considered with some empirical support by Caballero [3] and by Zavadil [19]. Just for this illustration we assume that ϕ is linear function, to fit the empirical values of α and π it takes the form

$$\alpha = \phi(\pi) = -2.5 \cdot \pi + 0.625. \tag{3.13}$$

[3]The data consider the Dutch drivers statistics of bonus-malus insurance system.

[4]The basic characteristics of four-parameter beta distribution may be found in the Appendix.

$\phi(\pi)$	0.50	0.375	0.25	0.125	0.0001
π	0.05	0.10	0.15	0.20	0.25
L			C_D		
0.1	-417.93	-221.12	-156.41	-124.84	-106.67
1	-39.26	-21.21	-15.26	-12.35	-10.67
5	-6.73	-3.83	-2.85	-2.38	-2.13
10	-3.82	-2.08	-1.45	-1.17	-1.07
15	-3.40	-1.73	-1.08	-0.78	-0.71
20	-3.35	-1.65	-0.95	-0.61	-0.53

Table 3.3: Limiting values for slope of $\phi(\pi)$. If the slope is greater then the given value, the function k increases, otherwise it decreases.

In such setting, the expression C_H is always positive. The Table 3.3 shows the bounding values C_D for different values of loss and accident probability.

We see that with growing loss the slope of ϕ bordering an increase and decrease of the function k rises. In our setting, with slope equal to -2.5, k is increasing if the loss does not exceed approximately one quarter of the consumer's wealth level, adverse selection may occur in that case. The condition for decreasing k is never satisfied for low values of π. However, k can still be decreasing at some interval, especially when we skip the assumption that ϕ is linear. In such case the market outcomes are fully efficient.

In the case that k is increasing we need to examine the relationship between the willingness to pay of the consumer with lowest accident probability ($\underline{\pi} = 0.05$) and the expected value of accident probability distribution in order to find the properties of the equilibrium. Let $a = 1$, $b = 10$ be the parameters of beta distribution roughly following the empirical data from Zavadil. For such parameters setup there are lots of consumers with good probabilities and only a few with bad probabilities.

This setting gives the efficient market equilibrium if the possible loss is higher than approximately 0.075 share of the consumers wealth level. For example, saying that the consumer's wealth is 50 000 EUR, the harmful effects of adverse selection shows if considering insurance of losses less than 3750 EUR.

3.2 Endogenous Dependence between Risk Aversion and Accident Probability

In the previous section we assumed that the risk aversion and riskiness of consumers are dependent on each other. Although the single dependence of the two variables seems natural, the reason for such property is at least unclear. In this section we give a possible reasoning following from the consumers' utility maximization. Contrary to the previous cases, this model gives the consumers the ability to influence their accident probabilities and therefore it includes the problems concerning moral hazard which goes beyond the interest of this work. We therefore introduce the model setting explaining the dependence between risk-aversion and riskiness but leave the examination of the market equilibria for the further work.

Let us now assume that there exists a basic probability $\overline{\pi}$ of having an accident that is constant amongst the consumers. The consumers only differ in their coefficient of risk aversion. They may decrease their accident probabilities by exerting some precaution effort $e \in [0, \overline{e}]$, for example by driving carefully, purchasing winter tires, etc. The individual's accident probability is a decreasing function of effort. The effort itself is costly, the cost function $c(e)$ is increasing. The consumers with different risk aversion choose different levels of effort and therefore they have different accident probabilities $\pi(e)$. The values of risk aversion and accident probabilities are unobservable for the insurance companies, therefore the market has asymmetric information.

Clearly, if fully insured, it is optimal for the consumer to take zero precaution effort. However, this changes when the consumer participates on his losses by some deductible amount δL. If the consumer chooses to insure himself, he chooses the effort e that maximizes his utility

$$U_\alpha(e, w, p, \delta) = \pi(e) \cdot u_\alpha(w - \delta L - p - c(e)) + (1 - \pi(e))u_\alpha(w - p - c(e)),$$

where α is the coefficient of risk aversion, w the consumer's wealth, p the price of the insurance contract. We assume that the consumers has the same technology of decreasing the accident probabilities, thus the marginal rate of substitution between wealth and prevention should not depend on the shape of the utility function (Jullien, Salanié and Salanié [12], 2001). The authors show that to fulfill this assumption the costs of precautions has to be expressed as financial costs and included into the utility function in the form $u(R - c(e))$, as we did.

This fact is ignored by both the papers of De Donder and Hindricks [6] and de Meza and Webb [7], and both state the expected utility functions in their own way which may be one of the reasons of the inconsistency in their results. Moreover, while [6] argue for the single-crossing property of preferences, [7] assume double-crossing as a necessary condition. The main difference, however, is that in the De Donder and Hindricks model, moral hazard takes part and increases the accident probabilities of the consumers with high risk aversion (and therefore high willingness to pay). Since in our work we do not include the moral hazard effects, the results of our model are naturally more in the line with [7] as we allow the positive correlation between insurance coverage and risk aversion to occur.

To guarantee that a consumer with higher coefficient of risk aversion chooses a higher precautions we need additional assumptions. The intuition behind says that effort reduces the income in the case of accident, so that a more risk-averse agent facing a high probability of accident may opt for increasing his worst-case income instead of reducing the probability of accident [11]. To summarize, the dependence between the risk aversion and riskiness may take both positive or negative form discussed in the previous section as an exogenous relationship. The literature, however, gives various reasonable but not trivial additional assumptions under which the more risk-averse consumers take more precautions (Jullien, Salanié and Salanié [11], 1999, Dionne et al. [8], 1998).

As the further discussion deals with moral hazard effects in the market we stop here with the argument for the various forms of the dependence between the consumers' characteristics and leave the examination of the market equilibria in this setting for the future.

Conclusions

In this thesis, we develop a model of insurance market with asymmetric information. We consider the continuous distribution of hidden consumers characteristics - risk aversion and riskiness. We create the mathematic model and interpret the results to our economic application. Although there are numerous possible interpretations for the model, we mainly discuss insurance markets of non-life insurance, like car insurance. The focus of our interest is the mechanism and consequences of adverse selection. Oppositely to many works in the field that mostly consider two qualitative types of consumers and employ a game theory in the analysis, we use a little less sophisticated tools in the describing of the equilibria and in exchange we are able to describe more general distributions of the characteristics.

In the first step we develop the model and examine it with only one hidden characteristic. We show the conditions under which the market agents suffer from the inefficiencies caused by adverse selection. Loosely interpreted, we may say that the market tends to search for the equilibrium by itself. We show that the equilibrium prices are asymptotically stable and may be reached by the market with very low requirements on the starting point. We generalize the model by adding the second hidden characteristic. The occurrence of the adverse selection depends on the distribution of the riskiness amongst the consumers (the supply side of the market) and the *lower tail* of consumers characteristics as those consumers have the highest motivation for not buying the insurance and start the adverse selection mechanism. The occurrence of the adverse selection does not depend on the exact distribution of the risk aversion except of the least occurring risk aversion. However, in the case of presence of adverse selection in the market, its severity (the exact equilibrium price and the volume of consumers not buying the insurance) depends on the distribution of both hidden riskiness and risk aversion.

In the last chapter we present the idea of dependence between the two characteristics and a reasoning behind it. We focus on the case of negative correlation of the characteristics which is consistent with some ideas in literature supporting the concept of advantageous selection. However, we find out that in the case where advantageous selection might occurred it is guaranteed that the market outcomes are efficient. Therefore, the advantageous selection as we define it never occurs, although the negative correlation between riskiness and risk aversion have positive effects on the market efficiency. While the adverse selection sometimes causes a negative externality for the consumers with good characteristic - low accident probabilities, the advantageous selection does not causes such externality in our model. Finally, a numerical example and some reasoning for the riskiness to risk aversion dependence are given.

An extension of the model arises naturally when we consider moral hazard issues. We hope we developed an appropriate working environment for such an extension and leave it as a possibility for a further research.

Appendix

Four-Parameter Beta Distribution

According to [20], in probability theory and statistics, the beta distribution is a family of continuous probability distributions defined on the interval $[0, 1]$ parameterized by two positive shape parameters, in our case denoted by a and b. The domain of the beta distribution can be viewed as a probability, and in fact the beta distribution is often used to describe the distribution of an unknown probability value. The probability density function of beta distribution is given by

$$\begin{aligned} f(x, a, b) &= \frac{x^{a-1}(1-x)^{b-1}}{\int_0^1 u^{a-1}(1-u)^{b-1} du} \\ &= \frac{1}{B(a, b)} x^{a-1}(1-x)^{b-1}, \end{aligned}$$

where $B(a, b)$ denotes the beta function with the parameters a, b. The expected value of beta distribution is given by

$$E(X, a, b) = \frac{a}{a+b}.$$

In this work we use a general interval of accident probabilities $[\underline{\pi}, \overline{\pi}]$. To change the support of the distribution we need to employ four-parameter beta distribution which has two additional parameters - the interval limits:

$$f(x, a, b, \underline{\pi}, \overline{\pi}) = \frac{1}{B(a, b)} \frac{(x-\underline{\pi})^{a-1}(\overline{\pi}-x)^{b-1}}{(\overline{\pi}-\underline{\pi})^{a+b-1}},$$

the standard form may be obtained for $y = \frac{x-\pi}{\overline{\pi}-\underline{\pi}}$. The expected value of the four-parameter beta distribution is given by

$$E(X, a, b, \underline{\pi}, \overline{\pi}) = \underline{\pi} + \frac{a}{a+b}(\overline{\pi} - \underline{\pi}).$$

Complete Form of the Derivative of Function k

Here we state the complete form of the derivative used in the Section 3.1.2.

$$\frac{dk}{d\pi} = \frac{\partial v(\phi(\pi), E(u,\pi))}{\partial \alpha}\Big|_{\alpha=\phi(\pi)} \frac{d\phi(\pi)}{d\pi} +$$
$$+ \frac{\partial v(\phi(\pi), E(u,\pi))}{\partial u}\Big|_{u=E(u,\pi)} \Big[u(\phi(\pi), w-L) - u(\phi(\pi), w) +$$
$$+ \pi \frac{\partial u(\phi(\pi), w-L)}{\partial \alpha}\Big|_{\alpha=\phi(\pi)} \frac{d\phi(\pi)}{d\pi} + (1-\pi)\frac{\partial u(\phi(\pi), w)}{\partial \alpha}\Big|_{\alpha=\phi(\pi)} \frac{d\phi(\pi)}{d\pi} \Big],$$

where where $E(u, \pi)$ denotes the expression

$$\pi u(\phi(\pi), w-L) + (1-\pi)u(\phi(\pi), w).$$

Resumé

Diplomová práca sa zaoberá modelovaním poistného trhu s asymetrickou informáciou. Uvažujeme, že spotrebitelia majú informačnú výhodu, keďže lepšie poznajú svoju rizikovosť aj averziu voči riziku. O týchto dvoch vlastnostiach predpokladáme, že ich výskyt je vyjadrený spojitým pravdepodobnostným rozdelením. Vytvárame matematický model a výsledky interpretujeme v danej ekonomickej aplikácii. Aj keď model sa dá interpretovať rôznymi spôsobmi, zameriavame sa hlavne na trhy s neživotným poistením.

V prvej kapitole uvádzame potrebné teóretické základy a sumarizujeme doterajšie výsledky v oblasti. Definujeme pojem ekvilibriového kontraktu, popisujeme správanie agentov na trhu s neistotou a ich vzťah k riziku, zavádzame termín poistenia a venujeme sa základom teórie asymetrickej informácie. Predstavujeme pojmy zvráteného a zvýhodneného výberu a morálneho hazardu. Prehľad najdôležítejších prác začíname prácou Rotschilda a Stiglitza [16](1976), ktorá iniciovala pomerne intenzívny teoretický výskum v danej oblasti. Uvádzame aj empirické výsledky (Chiappori a Salanié, 2000, Cohen a Siegelman, 2010). Upozorňujeme na početné rozpory medzi teoretickými a empirickými prácami.

Samotný model začíname budovať v druhej kapitole. Stanovujeme podmienku, pri ktorej je spotrebiteľ ochotný si kúpiť poistenie ako

$$u_\alpha(w-p) \geq \pi \cdot u_\alpha(w-L) + (1-\pi) \cdot u_\alpha(w), \tag{3.14}$$

kde u_α predstavuje funkciu užitočnosti spotrebiteľa s koeficientom rizikoaverzie α, p je výška poistného, L je strata, ktorú spotrebiteľ utrpí v prípade nehody, π pravdepodobnosť nehody daného spotrebiteľa a w jeho počiatočné bohatstvo. Z tejto podmienky odvodíme funkciu priemernej straty poisťovne pri danej cene p

ako

$$g(p) = E[\Pi|\Pi \geq h(p, A)]L, \tag{3.15}$$

kde Π a A sú náhodné premenné predstavujúce rizikovosť poistenca a jeho koeficient rizikoaverzie, teda jeho skryté charakteristiky a $E[\Pi|\Pi \geq h(p, A)]$ predstavuje podmienenú strednú hodnotu, podmienenú splnením 3.14. Hľadanie rovnovážnej ceny potom redukujeme na hľadanie pevného bodu funkcie $g(p)$.

Pre rovnovážny stav daného poistného trhu odvodzujeme nasledovné vlastnosti:

Tvrdenie 3.2.1. *Ak je rozdelenie náhodnej premennej Π také, že*

- $\underline{p} < L \cdot E(\Pi)$*, potom rovnovážna cena leží vo vnútri inetervalu $(\underline{p}, \overline{p})$. Preto na rozdiel od konkurenčného trhu s úplnou informáciou, sú niektorí spotrebitelia vylúčení s poistného trhu. Efektivita rovnovážeho výstupu je nižšia.*
- $\underline{p} \geq L \cdot E(\Pi)$*, potom rovnovážna cena leží vo vnútri inetervalu $[0, \underline{p}]$. Všetci rizikoaverzní spotrebitelia sa poistia, rovnováha je dokonale efektívna.*

Rovnovážna cena je navyše lokálne asymptoticky stabilná z pohľadu nášho jednoduchého modelu dynamiky. Z primerane nepresnou počiatočnou informáciou teda dynamický systém konverguje k rovnovážnemu stavu. Prítomnosť zvráteného výberu teda závisí od rozdelenia rizikovosti medzi spotrebiteľmi a spodným koncom rozdelenia spotrebiteľov čo sa týka oboch charakteristík, keďže títo spotrebitelia majú najväčšiu motiváciu nekúpiť si poistenie a tým spustiť mechanizmus zvráteného výberu. Výskyt zvráteného výberu teda nezávisí od konkrétneho rozdelenia koeficientu rizikoaverzie. V prípade výskytu zvráteného výberu, samorejme, konkrétna podoba rovnovážneho stavu a výška neefektivity závisia od konkrétnej podoby rozdelení oboch skrytých charakteristík.

V tretej kapitole pridávame do modelu reštrikcie zväzujúce rizikovosť s rizikoaverziou. Ako motiváciu na takéto správanie uvádzame model, v ktorom spotrebitelia vedia svoje vlastnosti ovplyvňovať. Spotrebitelia v takomto modeli diferencujú svoju rizikovosť tak, aby maximalizovali svoju užitočnosť, na základe ich rizokoavreznosti. Závislosť medzi charakteristikami predstavuje funkcia ϕ ktorá pravdepodobnostiam priraďuje hodnoty rizikoaverzie

$$\alpha = \phi(\pi). \tag{3.16}$$

Z takéhoto modelu môže, podľa vlastností funkcie ϕ, vyplývať pozitívna aj negatívna selekcia na trhu (zvýhodnený aj zvrátený výber). My si vyberáme konkrétnejšiu triedu funkcií užitočnosti CARA, a ďalej skúmame vlastnosti rovnovážneho stavu. Platia nasledovné tvrdenia:

Tvrdenie 3.2.2. *Ak je ϕ rastúca funkcia, na trhu sa môže objaviť zvrátený výber, alebo sa neprejaví žiaden efekt asymetrickej informácie. Ak je ϕ klesajúca, potom sa zvýhodnený výber môže prejaviť len ak sú splnené nasledovné dve podmienky:*

$$\phi'(\pi) < \frac{-\left[e^{\phi(\pi)L} - 1\right]\phi(\pi)}{L\pi e^{\phi(\pi)L}\phi(\pi) - \ln\left[(1-\pi) + \pi e^{\phi(\pi)L}\right]\left[(1-\pi) + \pi e^{\phi(\pi)L}\right]},$$
$$0 < L\pi e^{\phi(\pi)L}\phi(\pi) - \ln\left[(1-\pi) + \pi e^{\phi(\pi)L}\right]\left[(1-\pi) + \pi e^{\phi(\pi)L}\right],$$

a zvrátený výber len ak niektorá z podmienok nie je splnená.

Ukazujeme, že zvýhodnený výber sa v našom modeli s asymetrickou informáciou nikdy nemôže prejaviť, teda nikdy nemôže spôsobiť neefektivitu trhu. Negatívna korelácia medzi rizikovosťou a rizikoaverziou ale má pozitívny vplyv na efektivitu rovnovážnych výstupov. Pri splnení vyššie uvedenej podmienky z Tvrdenia 3.2.2 je teda trh vždy plne efektívny. Na záver ukazujeme numerický príklad na dátach z holandského trhu s poistením motorových vozidiel.

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