COMENIUS UNIVERSITY IN BRATISLAVA
FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS

APPLICATIONS OF NUMERICAL METHODS FOR THE TRANSFORMED NONLINEAR BLACK-SCHOLES EQUATION

Master's Thesis

Marek Uhliarik Bratislava 2011

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Master Thesis

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Aplikácie numerických metód pre riešenie transformovanej nelineárnej Black-Scholesovej rovnice

Diplomová práca

Marek Uhliarik

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Názov : Applications of numerical methods for the transformed nonlinear Black-Scholes equation

Cieľ : In this thesis we focus on the nonlinear Black-Scholes equation and its solving by numerical methods. Nonlinear Black-Scholes models improve the linear onesin the way that volatility is no longer constant but it takes into consideration some extra variables. It can be, for example, transaction costs, a risk from a portfolio, preferences of a large trader, etc. We shall work with the transformed Black-Scholes equation (Gamma equation) yielding more robust numerical approximation schemes.

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I declare on my honour that this work is based only on my knowledge, references and consultations with my supervisor.

Marek Uhliarik

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Abstract

Uhliarik, Marek: *In this thesis we focus on the nonlinear Black-Scholes equation and its solving by numerical methods. Nonlinear Black-Scholes models improve the linear ones in the way that volatility is no longer constant but it takes into consideration some extra variables. It can be e.g. transaction costs, a risk from a portfolio, preferences of a large trader, etc. We shall work with the transformed Black-Scholes equation (Gamma equation) yielding more robust numerical approximation schemes.*
This master thesis is organised in the following way. In the first chapter we offer some short introduction into the theory of the financial derivatives. The second chapter is devoted to the volatility models which are further used in the thesis. We work with e.g. Jumping volatility model, Leland's model or RAPM model. In the third and fourth chapter we introduce used numerical schemes and derivation of the Gamma equation. In the last chapter, there are numerical results from our experiments. [Master thesis], Comenius University in Bratislava, Faculty of Mathematics, Physics and Informatics, Department of Applied Mathematics and Statistics. Bratislava, 2011, 66 p.
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Keywords: nonlinear Black-Scholes equation, Gamma equation, numerical methods, Khaliq-Liao method, RAPM model, Model with variable transaction costs according to Amster and et al.

Abstrakt

Uhliarik, Marek: *V tejto diplomovej práci sa zameriavame na nelineárnu Black-Scholesovu rovnicu a na možnosti riešenia tejto rovnice pomocou numerických metód. Nelineárne Black-Scholesove modely vylepšujú lineárne tým, že volatilita už nie je konštantou, ale je to funkcia závislá od viacerých premenných a tým pádom dokáže lepšie aproximovat' skutočnost'. K takýmto premenným patria napríklad transakčné náklady, riziko z portfólia alebo preferencie vel'kého investora na trhu. Pracujeme s tzv. Gamma rovnicou, ktorá je odvodená od Black-Scholesovej rovnice a poskytuje lepšie možnosti pre numerické riešenie.*
Usporiadanie diplomovej práce je nasledovné. V prvej kapitole uvedieme krátky prehl'ad z oblasti finančných derivátov. Druhá kapitola sa bude venovat' modelom volatility, ktoré sa v práci d'alej využívajú. Je to napríklad Amsterov model, Lelandov alebo RAPM model. V tretej a štvrtej kapitole predstavíme numerické schémy ako aj odvodenie Gamma rovnice z Black-Scholesovej rovnice. Posledná kapitola bude venovaná numerickým výsledkom. [Diplomová práca], Univerzita Komenského v Bratislave, Fakulta matematiky, fyziky a informatiky, Katedra aplikovanej matematiky a štatistiky. Bratislava, 2011, 66 s.
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Kl'účové slová: nelineárna Black-Scholesova rovnica, Gamma rovnica, numerické metódy, Khaliq-Liao metóda, RAPM model, Model s variabilnými transakčnými nákladmi podl'a Amstera a kol.

Contents

Introduction

In finance, a financial derivative is a instrument whose value depends on some more basic underlying asset. A big part of the financial market is devoted right to the financial derivatives. For some of them it is difficult to find a proper price and therefore there is a big effort to find new models or to improve current ones which would approximate the real price of the derivative as good as possible. Model which is particularly used for this purpose is Black-Scholes model and in this thesis we will work with it. Generally used Black-Scholes model considers volatility to be constant. However, there are extensions of this model which take some extra variables (transaction costs, presence of a big trader on the market, ...) into consideration. Volatility in these extended models is not constant anymore and therefore term nonlinear Black-Scholes models is used for such models. This thesis is focused right on these volatility models.

In the beginning of the thesis we will introduce theory from the area of the financial derivatives. We will focus particularly on the options. Then we will introduce some nonlinear volatility models. Among others there will be Leland's model, RAPM model or Model with variable transaction costs. We shall work with a transformed Black-Scholes equation (Gamma equation) yielding more robust numerical approximation schemes. Also these numerical schemes will be introduced.

Chapter 1

Financial derivatives

A financial derivative (contingent claim) can be defined as a security which value depends on the value of a more basic underlying asset. To the most common variables affecting financial derivatives belong the price of the traded asset, interest rate, time to maturity, exercise price and so on.

The primary purpose to trade financial derivatives is to minimise potential losses, caused by unpredictable movements of the underlying asset. The basic financial derivatives are particularly forwards and options.

A forward contract (also future if traded on an exchange) represents an agreement between two parties, that one party will purchase an asset on a certain time for a predetermined price from a counterparty.

On the other hand, options offer a right but not the obligation (compared to forwards) to sell or buy an asset for a predetermined price at a certain time in the future. Therefore, options can be considered as the way of an "insurance" against unpleasant movements of the price. Except from the right for selling / buying the asset, the option's owner has to pay some fee, called premium, for entering the option contract. On the contrary, by a forward contract no such fee exists. In this thesis we will focus especially on the call and put options.

European call (put) option is a contract which gives the holder the right, but not the obligation to buy (sell) a predescribed asset, known as an underlying asset by a certain date T (expiration date or maturity) for a predetermined price E (called strike or exercise price). If the holder of the option wants to exercise the call option, the writer has the obligation to sell (buy from) him the underlying asset for an agreed strike price (then it is called exercised option). The holder of the European option can exercise this option only at the expiration time T.

At the time T, when the holder has the possibility to exercise the call option, three different scenarios of the asset's current price $S(T)$ can come true.

- $S(T) > E$ - in this case the holder of the option can buy an asset for the strike price which is lower than the current price. Therefore he will exercise the option and he has the possibility to sell immediately the asset for the price $S(T)$. Then his gain $S(T) - E$ will be positive. This option is called *in-the-money option*.

- $S(T) = E$ - the cash flow following immediate exercising of the option has zero value. This option is called *at-the-money option*.

- $S(T) < E$ - exercising of the option would result in negative cash-flow. This last case is called *out-the-money option*.

In the last two cases the holder will not exercise the option because at the time T he has the possibility to buy an asset on the market for a price equal to or lower than E. It would be therefore pointless to exercise the call option and to pay more than the asset's market price. The value of the European call option is represented by the following function (so-called *pay-off function*)

$$V(S,T) = \max(S(T) - E, 0) = (S(T) - E)^+.$$

In Figure (1.1) the payoff function for the European Call option is depicted.

The same three cases may appear for a European Put option. If $S(T) < E$, the option is *in-the-money*. The holder can sell the asset for a price higher than its current price. If $S(T) = E$ it is *at-the-money* and in the last case we refer to the *out-the-money* option. The pay-off function for an European put option (see Figure 1.2) is given by

$$V(S,T) = \max(E - S(T), 0) = (E - S(T))^+.$$

Already mentioned pay-off functions were from the perspective of the holder of the option, i.e. of the *long position*. The holder has the possibility to buy the underlying asset and become the owner of the asset. We can obtain pay-off

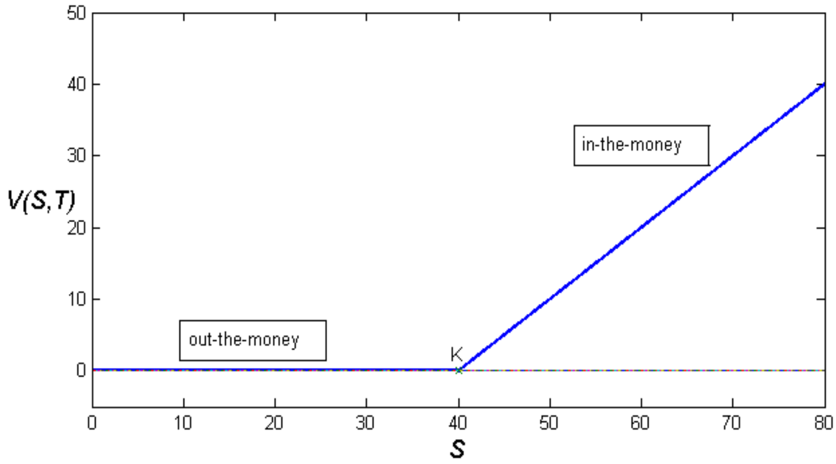

Figure 1.1: Pay-off function for a Call option.

functions for the writer of the option (*short position*) by multiplying the pay-off function for the holder by (-1).

The second type of the options are *American options*. In contrast to the European options, American options can be exercised at any time until the expiration date. This extra right has to be also reflected in the price of the option. Therefore, the value of an American option is never smaller than the value of the European option (the holder of the American option has at least the same rights as the holder of the European option). This extra premium is called the *early exercise premium*.

The price of the stock option is affected by the following factors: the current stock price (S_0), the strike price (E), time to expiration $(T-t)$, the volatility of the stock (σ), the risk-free interest rate (r), dividends expected during the life time of the option. The good way of an explanation of the influence of these variables on the American / European options offers Table 1.1 taken from Hull [15].

The influence of the strike price and the current stock price at the time T is evident from the payoff function of the option. The time effect is obvious. The holder of the American option with $T_2 > T_1$ has all the opportunities for exercising the option as the holder of the American option with the exercise time T_1 and even more. Therefore, with the increasing time the value of the option also increases for the American options. The time effect for the European options is not so clear. As the volatility of the underlying asset increases, the chance that the

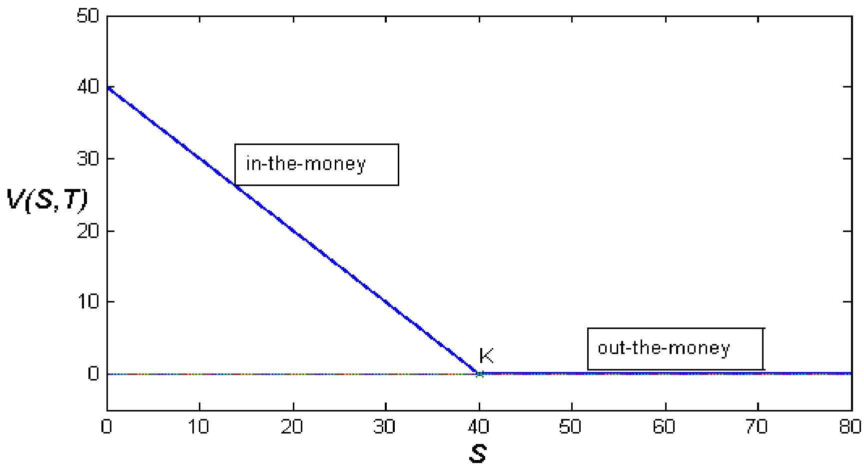

Figure 1.2: A pay-off function for a Put option (K is the strike price).

value of the option will do very well or poorly increases. On the other hand, the holder of the option has only limited losses. So with increasing volatility the value of the option also increases. By increasing interest rate in economy, the expected return for investors has to increase too. The present value of the future cash flow decreases. So the value of the put decreases and the value of the call increases. The last factor which influences the value of options is the presence of dividends. Dividends reduce the stock price on the "exdividend date" (time after dividends are paid out). The value of the call option is therefore decreasing and the value of a put is increasing.

The presence of dividends is described by a *dividend yield* q. If there are no dividends and all other parameters are the same, it is supposed that the value of the American and European Call to be equal. On the other hand, exercising of the American Put option prior to expiry can be often more advantageous than waiting to expiry. So the value of the American Put option is higher than the value of the European Put option.

Variable	European call	European put	American call	American put
Current stock price	+	−	+	−
Strike price	−	+	−	+
Time to expiration	?	?	+	+
Volatility	+	+	+	+
Risk free interest rate	+	−	+	−
Dividends	−	+	−	+

Table 1.1: The effects of increasing one variable on option price, while others are fixed

The American and European options are called *vanilla options*. This means, that their value depends only on the value of the asset at focus on the final time T. Options which value depends on the path of the underlying asset are called *exotic* or *path-dependent options*. To this type of options belong for example *Asian options, Barrier options and others*. However, in this thesis we focus particularly on the vanilla European options.

1.1 The Black-Scholes equation

1.1.1 The linear Black-Scholes equation

In the beginning of the 1970's Fisher Black, Myron Scholes [6] and Robert Merton [21] made a great progress in the option pricing theory. Their classical model had a great influence on how traders priced options and also classical hedge options. The famous *Black-Scholes (linear) partial differential equation* (PDE)

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} - rV = 0 \tag{1.1}$$

was first time introduced in 1973 by Black and Scholes [6]. They derived it by using Itô 's lemma (see Appendix) and used that

$$dS = \mu S dt + \sigma S dW. \tag{1.2}$$

The derivation was based on a synthetized portfolio

$$\Pi = V + \delta S, \tag{1.3}$$

where $\delta = -\frac{\partial V}{\partial S}$ is a trading strategy. In equation (1.1) S stands for $S(t) > 0$ and time $t \in (0, T)$. This equation gives us an option pricing formula for both the American and European option in dependence on the terminal conditions. The rather restrictive assumptions of this equation are following [24]

1. *There are no arbitrage opportunities* - it means there is no possibility to make a risk free profit ("no free lunch").

2. *The price of the asset follows the Geometric Brownian motion (GBM)* - this means, that the price of the asset S fulfills equation (1.2).

3. *The trend (drift) μ, the risk-free interest rate r and the volatility σ are constant.*

4. *The market is frictionless* - market is without any transaction costs (no fees and no taxes), the interest rates for borrowing and lending money are equal, all parties have an immediate access to complete information. All securities, informations and credits are available at any time and at any size. Therefore, all variables are perfectly divisible (can take any real number). It is also assumed that the individual trading will not influence the price.

Kwok [19] mentions even some more assumptions. He assumes also that the trading takes place continuously in time, there are no dividends and no penalties to the short selling and the full use of proceeds is permitted.

Under this assumptions we talk about a *complete market*, i.e. any derivative can be replicated or hedged with a portfolio of other assets in the market. The derivation of the Black-Scholes formula can be found for example in Seydel's book [24].

1.1.2 Nonlinear Black-Scholes equations

As we can see from the assumptions of the linear Black-Scholes equation (1.1) some of its assumptions are rather restrictive. Furthermore, some of them are never fulfilled in reality. The problem is particularly with the presence of transaction fees, incomplete markets or large investors preferences. In recent years some assumptions have been relaxed to solve these restrictions. Here comes the motivation to study the Black-Scholes equation in a nonlinear way. In this thesis we consider different models concerning the volatility to be not constant. This means that it depends on time to maturity $(T-t)$, the asset price (S) or on the second derivative of the option price $\partial_S^2 V$, i.e. we put

$$\tilde{\sigma^2} = \tilde{\sigma^2}\left(T-t, S, \frac{\partial^2 V}{\partial S^2}\right).$$

Consequently, the nonlinear Black-Scholes equation has the following form

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\tilde{\sigma^2}(T-t, S, \frac{\partial^2 V}{\partial S^2})S^2\frac{\partial^2 V}{\partial S^2} - rV = 0, \tag{1.4}$$

with $dS = \mu S dt + \tilde{\sigma} S dW, S > 0$.

It is worthwhile noting, that in the financial world partial derivatives from the Black-Scholes equation are often represented by so-called *Greeks*

$$\frac{\partial V}{\partial t} =: \Theta; \qquad \frac{\partial V}{\partial S} =: \Delta; \qquad \frac{\partial^2 V}{\partial S^2} =: \Gamma.$$

The nonlinear Black-Scholes equation (1.4) has then following form

$$\Theta + rS\Delta + \frac{1}{2}\tilde{\sigma}^2(T-t, S, \Gamma)S^2\Gamma - rV = 0. \tag{1.5}$$

1.2 The terminal and boundary Conditions

In order to find the solution of the Black-Scholes equation (1.4) we have to supply boundary and terminal conditions.

1.2.1 The European Options

The European Call option

The terminal condition for the European Call option was already mentioned (condition at time $t = T$). The boundary conditions are located at $S = 0$ and $S \to \infty$. To summarise it, the solution of (1.4) for the Call option defined on the domain $S \in [0, \infty)$ and $t \in [0, T]$ has the following conditions

$$
\begin{aligned}
V(S,T) = \max(S-E,0), &\quad S \in [0,\infty), \\
V(0,t) = 0, &\quad t \in [0,T], \\
V(S,t) \sim S - Ee^{-r(T-t)}, &\quad S \to \infty.
\end{aligned}
$$

With these boundary and terminal conditions we can determine the value for the European Call option from the equation (1.4) as

$$
C(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2), \tag{1.6}
$$

where

$$
\begin{aligned}
d_1 &= \frac{\ln(\frac{S}{E}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \qquad (1.7) \\
d_2 &= \frac{\ln(\frac{S}{E}) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T}.
\end{aligned}
$$

and $N(x)$ is normal cumulative distribution function.

The European Put option

In the same way we formulate terminal and boundary conditions for the European Put option on the domain $S \in [0,\infty)$ and $t \in [0,T]$

$$
\begin{aligned}
V(S,T) = \max(E-S,0), &\quad S \in [0,\infty), \\
V(0,t) = Ee^{-r(T-t)}, &\quad t \in [0,T], \\
V(S,t) = 0, &\quad S \to \infty.
\end{aligned}
$$

The value of the European Put option with the above mentioned terminal and boundary conditions is

$$
P(S,t) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1), \tag{1.8}
$$

where d_1 and d_2 are given in (1.7).

Chapter 2

Volatility models

As it was already mentioned in the first chapter, the considered nonlinear Black-Scholes equation differs from the linear Black-Scholes equation right in the volatility. In the case of the nonlinear Black-Scholes equation we cannot consider volatility as constant, but it depends on some variables. In this chapter we will mention five volatility models including transaction costs: Leland's model, the Risk adjusted pricing methodology model (RAPM), Model with variable transaction costs according to Amster and et al., Jumping volatility model and Barles-Soner's model which consideres also investor's preferences. Furthermore, we present one extra model using the idea of Amster et al.'s model.

2.1 Leland's model

The first model concerning transaction costs we mention is Leland's model [20]. This model was introduced in 1985 by Leland and further extended by Hoggard, Whalley and Wilmott [12]. We are concerning a portfolio $\Pi = V + \delta S$. Change of the portfolio is equal to the change of a riskless bond (i.e. portfolio with a risk-free interest rate). On the other hand, trading of an asset leads also to some nontrivial transaction costs TC, that should be also added to the change of the portfolio. Therefore we get

$$r\Pi dt = d\Pi = d(V + \delta S) - dTC, \tag{2.1}$$

where

$$dTC = C|k|S/2. \tag{2.2}$$

Here C denotes the round trip transaction cost per unit dollar. Then

$$C = \frac{S_{ask} - S_{bid}}{S}, \tag{2.3}$$

where S_{ask}, S_{bid} stand for Ask and Bid prices of assets, respectively (see Appendix) and S is the mid value of the asset (average of Bid and Ask prices).

As k stands for amount of the traded assets [20], it can be written as $k = d\delta$ (change of the traded amount over a time step dt - if the sign is negative we sell, if positive we buy the asset). As the trading strategy standard delta hedging $\delta = -\frac{\partial V}{\partial S}$ is used. Therefore applying Itô's lemma (see Appendix) on this strategy leads to

$$d\delta = -\frac{\partial^2 V}{\partial S^2}\sigma S dW - \frac{\partial^2 V}{\partial S^2}\mu dt - \frac{1}{2}\frac{\partial^3 V}{\partial S^3}\sigma^2 dt. \tag{2.4}$$

We approximate $d\delta$ in dW and get

$$d\delta = -\sigma S \frac{\partial^2 V}{\partial S^2} dW. \tag{2.5}$$

Following Leland's approach from [20] we approximate $|dW|$ by $E(|dW|)$ and

$$E(|dW|) = E(|\Phi|)\sqrt{dt} = \sqrt{\frac{2}{\pi}}\sqrt{dt}. \tag{2.6}$$

Therefore

$$dTC = S^2 \frac{C\sigma}{\sqrt{2\pi}} \left| \frac{\partial^2 V}{\partial S^2} \right| \sqrt{dt}.$$

Finally, after some adjustments and applying Itô's lemma on the function V, approximating the equation in dt and finally after deviding whole equation by dt we get the governing equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \left(1 + Le \,\mathrm{sign}\left(\frac{\partial^2 V}{\partial S^2}\right)\right)\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{2.7}$$

Therefore the volatility in the case Leland's model is

$$\tilde{\sigma}^2 = \sigma^2 \left(1 + Le \,\mathrm{sign}\left(\frac{\partial^2 V}{\partial S^2}\right)\right) \tag{2.8}$$

and Le is so called Leland's constant defined as

$$Le = \sqrt{\frac{2}{\pi}}\frac{C}{\sigma\sqrt{dt}}. \tag{2.9}$$

2.2 The Risk Adjusted Pricing Methodology

Another model discussing transaction costs is the *Risk Adjusted Pricing Methodology (RAPM) model*. This model was introduced by Kratka [18] and improved by Ševčovič and Jandačka [16] . Improvement of [16] concludes with a model, which is scale invariant and mathematically well-posed, what was missing in Kratka's model. In this paper except from the transaction costs also risk from a volatile portfolio is concerned. Both risk and transaction costs are dependent on the time lag between two consecutive adjustments of the portfolio. With increasing time lag, the risk from volatility of the portfolio is increasing, on the other hand with decreasing time lag the transactions costs are increasing. In [16] authors look for an optimal time difference between consecutive adjustments.

2.2.1 Derivation of the RAPM model

By derivation of the scale-invariant RAPM model we assume that the asset pays no dividends and the asset price follows (1.2). Following the ideas of Black and Scholes a portfolio (1.3) is constructed. Originally, in Black-Scholes theory is assumed, that

$$\Delta\Pi = r\Pi\Delta t. \tag{2.10}$$

However, such a simplified assumption is not satisfied and risk should be also considered. In the RAPM model we consider $\Delta\Pi = r\Pi\Delta t + r_R S\Delta t$ [16]. The total risk (r_R) per unit asset price consits of the transaction risk r_{TC} and risk, which comes from the volatility of the portfolio r_{VP}, so $r_R = r_{TC} + r_{VP}$. The model of [16] considers separately transaction costs and risk of portfolio and then, as already mentioned, minimises the total risk.

Modeling the risk from transaction costs and volatile portfolio

Transaction costs - Similarly as in the case of the Leland's model, the transaction costs are considered to be $CS/2|k|$. The change of portfolio can be again written as $d\Pi = \Delta V + \delta dS - C|d\delta|S/2$, where δ is again the hedging strategy. And finally we arrive at

$$\Delta\Pi = \Delta V + \delta\Delta S - r_{TC}S\Delta t, \tag{2.11}$$

where the coefficient for the risk of transaction costs is of the form [16]

$$r_{TC} = \frac{C\sigma S}{\sqrt{2\Pi}} \left|\partial_S^2 V\right| \frac{1}{\sqrt{\Delta t}}. \tag{2.12}$$

As we can see, by increasing time the lag between two adjustments of portfolio we are decreasing risk from the transaction costs.

Volatile portfolio - when an investor invests into portfolio, which is highly volatile, mostly he awaits some extra compensation. The variance of the portfolio can be measured through the relative increments of the replicating portfolio ($\Pi' = V + \delta S$), i.e. var$((\Delta\Pi')/S)$. We can write [16]

$$r_{VP} = R\frac{\mathrm{var}(\Delta\Pi'/S)}{\Delta t}. \tag{2.13}$$

The constant R in (2.13) denotes the risk premium coefficient. With increasing R the investor seems to be more risk aversive. By using relations for variance and after some more adjustments we arrive at the final formula

$$r_{VP} = \frac{1}{2}R\sigma^4 S^2(\partial_S^2 V)^2\Delta t. \tag{2.14}$$

Again, from the previous relation we can see, that with increasing time interval, there is a higher risk from the volatile portfolio.

Finally, by minimising the total risk [16] through the time lag

$$\Delta t \to r_R = \frac{1}{2}R\sigma^4 S^2(\partial_S^2 V)^2\Delta t + \frac{C\sigma S}{\sqrt{2\Pi}} \left|\partial_S^2 V\right| \frac{1}{\sqrt{\Delta t}} \tag{2.15}$$

we come to value of r_R with this optimal Δt

$$r_R(\Delta t_{opt}) = \frac{3}{2}\left(\frac{C^2 R}{2\pi}\right)^{1/3} \sigma^2 |S\partial_S^2 V|^{4/3}, \tag{2.16}$$

where

$$\Delta t_{opt} = \frac{K^2}{\sigma^2|S\partial_S^2 V|^{2/3}}, \qquad K = \left(\frac{C}{R\sqrt{2\pi}}\right)^{1/3}. \tag{2.17}$$

It is worth to remind the reader, that this relation is valid only in case, when Δt is sufficiently small, i.e. $0 < \Delta t << 1$.

Finally, by taking into consideration both risk from the volatile portfolio and risk from transaction costs we can write the change of portfolio as $\Delta\Pi = \Delta V +$

$\delta\Delta S - r_R S\Delta t$. From this portfolio with value of r_R in optimal time lag Ševčovic and Jandačka arrive at the so called *Risk-adjusted Black-Scholes equation*

$$\partial_t V + \frac{\sigma^2}{2}S^2\left(1 + 3\left(\frac{C^2 R}{2\pi}S\partial_S^2 V\right)^{1/3}\right)\partial_S^2 V = r(V - S\partial_S V) \qquad (2.18)$$

and

$$\tilde{\sigma}^2 = \sigma^2\left(1 + 3\left(\frac{C^2 R}{2\pi}S\partial_S^2 V\right)^{1/3}\right). \qquad (2.19)$$

Further, following the notation from [16], in Matlab code term $3(\frac{C^2R}{2\pi})^{1/3}$ is also denoted as μ.

The risk adjusted Black-Scholes equation (2.18) can be backward parabolic equation if and only if the function

$$\beta(S\Gamma) = \frac{\sigma^2}{2}(1 - \mu(S\Gamma)^{1/3})S\Gamma \qquad (2.20)$$

will be an increasing function in $S\Gamma$. This condition will be fulfilled when

$$S\Gamma < \kappa := \left(\frac{3}{4\mu}\right)^3 \qquad (2.21)$$

what is the point where function (2.20) has its maximum value.

Early exercise

In the RAPM model we consider Leland's model [20] for transaction costs. One of the assumptions in this model is that the time lag between two adjustments is sufficiently small compared to $T - t$. A natural way to satisfy this condition is to disallow adjustments of portfolio near exercise time. One of ways is to divide the interval interval $(0, T)$ into two subintervals $(0, t_\star)$ and $(t_\star, T)$. While on the first subinterval portfolio adjustments are allowed, on the second they are not. The time $t_\star$ is so called *switching time*. Before this switching time the risk-adjusted equation takes place and because of fact that there are no portfolio adjustments after this time allowed, on the second interval we can use for pricing of options just formulas for pricing European options [16].

The next problem is how to find this switching time. The idea of finding it is based on finding the last portfolio adjustment before the expiry. If our hedging strategy follows optimal time lag (2.17), we can approach this problem in the following way [16]

$$T - t_\star = \min_{S>0} \Delta t_{opt}(S, t_\star), \tag{2.22}$$

what is adequate to looking for

$$K^2\sigma^{-2}(\max_{S>0} S\Gamma(S, t_\star))^{-2/3}. \tag{2.23}$$

Finally, we come to the relation

$$T - t_\star = \frac{C}{R\sigma^2} \tag{2.24}$$

and as $t_\star$ must be positive, $(T - t_\star < T)$ following has to be true

$$C < \sigma^2 RT.$$

For the existence of a solution of the Risk adjusted Black-Scholes equation on interval $(0, t_\star)$, it is necessary to fulfill the condition of backward parabolicity. The maximum from (2.23) is

$$\max_{S>0} S\Gamma(S, t_\star) = (2\pi\sigma^2(T - t_\star))^{1/3} \tag{2.25}$$

(for more details see [16]). As relation (2.21) has to be valid for all $S\Gamma$, together with condition for $t_\star$ (2.24), finally we come to

$$CR < \frac{\pi}{8}, \tag{2.26}$$

what assures backward parabolicity of the equation on the interval where risk adjusted Black-Scholes equation is valid.

Scale invariance

The RAPM model was first introduced by Kratka [18]. The improvement of this model made by Ševčovič and Jandačka [16] was in scale invariance property. It means the term $S\partial_S^2 V$ remains unchanged after scaling of V and S by a factor κ. The main difference lies in the definition of risk from volatile portfolio (2.13), which was in Kratka's work defined as

$$r_{VP} = R\frac{\text{var}(\Delta\Pi')}{\Delta t}. \tag{2.27}$$

In [17] Jandačka mentiones an example for this scale invariance. In the model without the scale invariance change of the currency unit from euors to cents causes an increase of the risk premium 10000 times.

2.3 Barles and Soner's model

The second model we will mention, is Barles and Soner's model [5] which is taking to considerations investor's preferences. The investor's preferences are described by the utility function of the investor with a constant investor's risk aversion. We again assume that the stock pays no dividends and follows stochastic differential equation with a nonconstant volatility.

Barles and Soner derived in their paper [5] this model with introducing stochastic processes X_t and Y_t which stand for dollar holding in money market and amount of shares of stocks owned, respectively. They introduced trading strategy on $[t, T]$ as pair of left nondecreasing functions (L_t, M_t) such that $L_t = M_t = 0$. The interpretation of these functions can be as number of shares of stocks transfered from the money market to the stock (L_t) and vice versa (M_t). In this model are also included proportional transaction costs μ ($S_{ask} = (1+\mu)$; $S_{bid} = (1-\mu)S$, where S is average of S_{ask} and S_{bid} - see Appendix). The increments of dollar amount on the market and amount of shares in the portfolio, respectively are then expressed as [5]

$$\begin{aligned} dX &= -S(1+\mu)dL + S(1-\mu)dM, &\quad (2.28)\\ dY &= dL - dM. &\quad (2.29)\end{aligned}$$

As it was already mentioned, this model considers utility function of investor. In general a utility function is increasing and concave. In the derivation of Barles and Soner's model was used exponential utility function with constant absolute risk aversion

$$U^\epsilon(\xi) = U(\xi/\epsilon),$$

where

$$U(r) = 1 - e^{-r}$$

and the parameter

$$\epsilon = \frac{1}{\gamma N} \quad (2.30)$$

is considered to be small, $0 < \epsilon << 1$.

We price European call option through utility maximization which was proposed by Hodges and Neuberger in [11]. They consider two optimization problems. In the first one (v^f), there are no options in investor's wealth in terminal time, while in the second there are N European call options (V) [5]. These stochastic optimization problems can be written as

$$v^f(x,y,s,t) = \sup_{L,M} E(U(X_T + Y_T S_T)), \tag{2.31}$$

$$v(x,y,s,t) = \sup_{L,M} E(U(X_T + Y_T S_T - N(S_T - E)^+)), \tag{2.32}$$

where $x = X_t$, $y = Y_t$ and $s = S_t$. In other words we are maximizing the expected utility from the final wealth with respect to all trading strategies (L_t, M_t). Hodges and Neubringer in [11] postulate that the price of the call option is equal to maximal solution $\Lambda = \Lambda(x,y,s,t,\gamma,N)$ of the equation

$$v(x + N\Lambda, y, s, t) = v^f(x,y,s,t). \tag{2.33}$$

In the option price Λ we can see a linearity argument. Selling N options with risk aversion factor γ is the same as selling one option with risk aversion $N\gamma$. For simplification Barles and Soner introduce two auxiliary functions $z^\epsilon(x,y,s,t)$ and $z^{f,\epsilon}(x,y,s,t)$ in the following way [5]

$$v(x,y,s,t) = U^\epsilon(x + ys - z^\epsilon), \tag{2.34}$$

$$v^f(x,y,s,t) = U^\epsilon(x + ys - z^{f,\epsilon}). \tag{2.35}$$

At time $t = T$ we have $z^\epsilon(x,y,s,T) = (s-E)^+$ and $z^{f,\epsilon}(x,y,s,T) = 0$ and due to linearity and (2.30) we get value of option price as

$$\Lambda(x,y,s,t,1/\epsilon,1) = z^\epsilon(x,y,s,t) - z^{f,\epsilon}(x,y,s,t). \tag{2.36}$$

Using knowledge of stochastic dynamic programming Barles and Soner come to the equation

$$\begin{aligned} 0 \;=\; & \inf_{\dot{L},\dot{M}} -\frac{\partial v}{\partial t} - \rho s\frac{\partial v}{\partial s} - \frac{\hat{\sigma}^2 s^2}{2}\frac{\partial^2 v}{\partial s^2} + \left(s(1+\mu)\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y}\right)\dot{L} \\ + \; & \left(\frac{\partial v}{\partial y} - s(1-\mu)\frac{\partial v}{\partial x}\right)\dot{M}, \end{aligned}$$

which is valid for v and v^f. This equation can be transformed to a minimising problem [5]

$$\min\left(-\frac{\partial v}{\partial t} - \rho s\frac{\partial v}{\partial s} - \frac{\hat{\sigma}^2 s^2}{2}\frac{\partial^2 v}{\partial s^2}; s(1+\mu)\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y}; \frac{\partial v}{\partial y} - s(1-\mu)\frac{\partial v}{\partial x}\right). \tag{2.37}$$

From the previous minimising equation we can get to maximising equation for z (according to assumption that utility function U^ϵ is increasing).

We assume proportional transaction costs μ to be expressed as $\mu = a\sqrt{\epsilon}$, where a is some constant. Finally, as ϵ is approaching 0 we get

$$z^{f,\epsilon}(y,s,t) \to 0, \qquad z^\epsilon(y,s,t) \to V(s,t), \tag{2.38}$$

where $V(s,t)$ is a solution of nonlinear Black-Scholes equation with the volatility function

$$\tilde{\sigma}^2(S^2V_{SS}^2, T-t) = \hat{\sigma}^2(1 + \Psi(a^2e^{r(T-t)}S^2V_{SS}^2)). \tag{2.39}$$

Here $\hat{\sigma}$ stands for constant volatility of the underlying stock price and function Ψ can be counted from nonlinear singular ordinary differential equation

$$\frac{d\Psi}{dA}(A) = \frac{\Psi(A)+1}{2\sqrt{\Psi(A)A} - A}; \qquad \Psi(0) = 0. \tag{2.40}$$

In [7] the volatility function Ψ of Barles and Soner is implicitely defined as

$$A = \left(-\frac{\text{arcsinh}\sqrt{(\Psi)}}{\sqrt{\Psi+1}} + \sqrt{\Psi}\right)^2, \qquad \text{if } \Psi > 0, \tag{2.41}$$

$$A = -\left(\frac{\text{arcsinh}\sqrt{(-\Psi)}}{\sqrt{\Psi+1}} - \sqrt{-\Psi}\right)^2, \qquad \text{if } 0 > \Psi > -1. \tag{2.42}$$

Furthermore, Ψ is a one to one increasing function mapping the real line onto the interval $[-1,\infty]$.

For more details about Barles and Soner's model see [23].

2.4 Jumping volatility model

In this model we again assume that the volatility is not known precisely but we know some boundaries inbetween which the volatility lies. These two extremes (σ_{min} and σ_{max}) can be inferred e.g. from the extreme values of the historical volatility. They can be viewed as defining a confidence interval for the future volatility values and can be modeled as functions of the time to maturity and price of the underlying asset. Throughout this thesis we consider a simple Jumping volatility model and set these functions to be constant over the time and independent of S. In this model the volatility can switch between these two values depending on the sign of the second derivative of the option price

$$\sigma^2(S^2\partial_S^2 V, S, \tau) = \begin{cases} \hat{\sigma_1^2}, & \text{if } \partial_S^2 V < 0 \\ \hat{\sigma_2^2}, & \text{if } \partial_S^2 V > 0. \end{cases} \tag{2.43}$$

As we can see, when both volatilities are the same we have classical Black-Scholes equation.

2.5 Model with variable transaction costs according to Amster and et al.

Another model using Leland's approach in transaction costs was introduced by Amster, Averbuj, Mariani and Rial. In this thesis we will refer to this model as Amster et al.'s model. In [2] transaction costs behave as a nonincreasing linear function $h(S) = a - bS$. The idea behind this model is that the value of the transaction costs is decreasing function of the amount of traded assets, i.e. by the increasing amount of the traded assets there is some kind of discount.

Similarly as in derivation of the Leland's model, we start from the equation (2.1). In contrast to Leland's model, the transaction costs here have form

$$dTC = (a - b|k|)S|k|. \tag{2.44}$$

Again $k = d\delta$ and as a hedging strategy $\delta = -\frac{\partial V}{\partial S}$ is used. Applying the expected value of the Wiener process (**??**) we gain the expected value of the transaction costs in case of Amster et al.'s model as

$$E((a - b|k|)S|k|) = \left|\frac{\partial^2 V}{\partial S^2}\right| \sigma S^2 \sqrt{\frac{2}{\pi}}\sqrt{dt}a - bS^3\left(\frac{\partial^2 V}{\partial S^2}\right)^2\left(\frac{2}{\pi}\right)\sigma^2 dt. \tag{2.45}$$

Finally we obtain the equation

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}-a\left|\frac{\partial^2 V}{\partial S^2}\right|\sigma S^2\sqrt{\frac{2}{\pi dt}}+\left(\frac{\partial^2 V}{\partial S^2}\right)^2 b\left(\frac{2}{\pi}\right)S^3\sigma^2+r\left(\frac{\partial V}{\partial S}S - V\right) = 0. \tag{2.46}$$

The nonlinear volatility for Amster et al.'s model has the form

$$\tilde{\sigma}^2 = 1 - \frac{a}{2}\mathrm{sign}\left(\frac{\partial^2 V}{\partial S^2}\right)\sqrt{\frac{2}{\pi dt}} + 2b\left(\frac{2}{\pi}\right)S\frac{\partial^2 V}{\partial S^2}. \tag{2.47}$$

According to [2] in the Amster et al.'s model there is one extra condition for a

$$\sigma^2\left(1 - \frac{a}{\sigma}\sqrt{\frac{2}{\pi dt}}\right) > 0, \tag{2.48}$$

therefore constant a has to be sufficiently small.

In this model we see one thing which could be improved. When we look in (2.44), the first part of the expression tells us that although until $k = \frac{a}{b}$ the transaction costs are increasing, after this point they decrease. It would mean that when we trade a big amount of the asset there will be only small transaction costs for this amount or even the transaction costs will be negative.

As a solution to his problem we tried to change the transaction costs function and instead of $(a - b|k|)$ we tried to consider decreasing exponencial function

$$dTC = a\exp\left(-\frac{b}{a}|k|\right)|k|. \tag{2.49}$$

This function would assure that there would be also discount for traded amount but the total transaction costs would increase. However while working with this problem we realised that using Taylor expansion of the exponential function

$$a\exp\left(-\frac{b}{a}|k|\right)|k| = a\left(1 - \frac{b}{a}|k| + \frac{|k|^2 b^2}{2a^2} + \dots\right)|k|, \tag{2.50}$$

leads again to Amster model for $\lim |k| \to 0$ as $E(|k|) \approx \sqrt{dt}$ (see 2.5 and 2.6) for $dt \to 0$. Therefore we were working only with transaction costs function (2.44).

Chapter 3

Gamma equation

In this chapter we will introduce Gamma equation [23] and present its derivation from Black-Scholes equation. The aim of this chapter is to transform fully non-linear parabolic equation into a quasilinear equation. For such an equation more effective numerical schemes for approximation can be constructed.

The original nonlinear Black-Scholes equation (1.4) can be written as

$$\frac{\partial V}{\partial S} + S\beta(S\Gamma) + (r-q)S\frac{\partial V}{\partial S} - rV = 0, \tag{3.1}$$

where the Greek Γ (see Section 1.1.2) stands for $\partial_S^2 V$ and the volatility from nonlinear Black-Scholes equation (1.4) is concerned in the function $\beta(S\Gamma)$. In the derivation of the Gamma equation, there are necessary some standard change of independent variables: $x = \ln(S/E)$, $x \in (-\infty, \infty)$ and $\tau = T - t$, $\tau \in (0, T)$. Furthermore, as term $S\Gamma = S\partial V_S^2$ is present in the equation (3.1), the following transformation is introduced

$$H(x,\tau) = S\Gamma = S\partial V_S^2.$$

3.1 Derivation of the Gamma equation

The Gamma equation is derived by taking the second derivative of the nonlinear Black-Scholes equation (1.4) with respect to x. Next we show this Gamma equation's derivation.

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}\left(\frac{\partial V}{\partial t}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial^2 V}{\partial S\partial t}\frac{\partial S}{\partial x}\right) = \frac{\partial^3 V}{\partial t\partial^2 S}S^2 + \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial t}\right),\\
\frac{\partial^2}{\partial x^2}\left(S\beta(H)\right) &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(S\beta(H))\right) = \frac{\partial}{\partial x}\left(S\beta(H) + S\frac{\partial\beta(H)}{\partial x}\right) =\\
&= S\frac{\partial\beta(H)}{\partial x} + S\frac{\partial^2\beta(H)}{\partial x^2} + \frac{\partial}{\partial x}(S\beta(H)),\\
\frac{\partial^2}{\partial x^2}\left((r-q)S\frac{\partial V}{\partial S}\right) &= \frac{\partial}{\partial x}\left((r-q)S\frac{\partial V}{\partial S} + (r-q)S^2\frac{\partial^2 V}{\partial S^2}\right) =\\
&= 2(r-q)S^2\frac{\partial^2 V}{\partial S^2} + (r-q)S^3\frac{\partial^3 V}{\partial S^3} + \frac{\partial}{\partial x}\left((r-q)S\frac{\partial V}{\partial S}\right),\\
\frac{\partial^2 V}{\partial S^2}(-rV) &= \frac{\partial}{\partial x}\left(-rS\frac{\partial V}{\partial S}\right) = -rS^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial}{\partial x}(-rV).
\end{aligned}$$

When we sum all the terms on the right hand side we get

$$\begin{aligned}
0 &= \frac{\partial^3 V}{\partial t\partial^2 S}S^2 + S\frac{\partial\beta(H)}{\partial x} + S\frac{\partial^2\beta(H)}{\partial x^2} + 2(r-q)S^2\frac{\partial^2 V}{\partial S^2} + (r-q)S^3\frac{\partial^3 V}{\partial S^3} +\\
&- rS^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial t} + S\beta(H) + (r-q)S\frac{\partial V}{\partial S} - rV\right).
\end{aligned}$$

As the last bracket is equal to 0 (see 3.1) after deviding the whole equation by S we get

$$\begin{aligned}
-\frac{\partial^3 V}{\partial t\partial^2 S}S &= \frac{\partial\beta(H)}{\partial x} + \frac{\partial^2\beta(H)}{\partial x^2} + 2(r-q)S\frac{\partial^2 V}{\partial S^2} + (r-q)S^2\frac{\partial^3 V}{\partial S^3} + -rS\frac{\partial^2 V}{\partial S^2}\\
&= \frac{\partial\beta(H)}{\partial x} + \frac{\partial^2\beta(H)}{\partial x^2} + (r-q)S\frac{\partial^2 V}{\partial S^2} + (r-q)S^2\frac{\partial^3 V}{\partial S^3} - qS\frac{\partial^2 V}{\partial S^2}.
\end{aligned}$$

As

$$\begin{aligned}
\frac{\partial H}{\partial\tau} &= \frac{\partial}{\partial\tau}\left(S\frac{\partial^2 V}{\partial S^2}\right) = S\frac{\partial^3 V}{\partial t\partial^2 S}\frac{\partial t}{\partial\tau} = -S\frac{\partial^3 V}{\partial^2 S\partial t},\\
(r-q)\frac{\partial H}{\partial x} &= (r-q)\frac{\partial}{\partial x}\left(S\frac{\partial^2 V}{\partial S^2}\right) = (r-q)S\frac{\partial^2 V}{\partial S^2} + (r-q)S^2\frac{\partial^3 V}{\partial S^3},\\
-qH &= -qS\frac{\partial^2 V}{\partial S^2},
\end{aligned}$$

we finally get the *Gamma equation* , i.e.

$$\frac{\partial H}{\partial\tau} = \frac{\partial^2\beta(H)}{\partial x^2} + \frac{\partial\beta(H)}{\partial x} + (r-q)\frac{\partial H}{\partial x} - qH. \tag{3.2}$$

Another derivation of the Gamma equation can be found e.g. in [23].

As an approximation of the initial Dirac delta function we will use $\bar{H}(x) = N'(d)/(\hat{\sigma}\sqrt{\tau^\star})$, where $\tau^\star$ is sufficiently small, $\hat{\sigma}$ is the constant volatility, $N(d)$ is the cumulative distribution function of the standard normal distribution and $d = (x + (r - q - \hat{\sigma}^2/2)\tau^\star)/\hat{\sigma}\sqrt{\tau^\star}$. The form of $N'(d)$ is following

$$N'(d) = \frac{1}{\sqrt{2\pi}} e^{-d^2/2}.$$

The initial condition for (3.2) at $\tau = 0$ are

$$H(x, 0) = \bar{H}(x) \tag{3.3}$$

and $\bar{H}(x)$ is the Dirac δ function, i.e.

$$\begin{aligned} \int_{-\infty}^{\infty} \delta(x - x_0)\phi(x)dx &= \phi(x_0), \\ \int_{-\infty}^{\infty} \delta(x)dx &= 1, \end{aligned}$$

where $\phi(x)$ is a smooth function and on the set where it cannot have zero values it is also bounded. The boundary conditions of the function $H(x, \tau)$ are

$$H(-\infty, \tau) = H(\infty, \tau) = 0. \tag{3.4}$$

The solution of the financial derivative $V(S, t)$ can be finally computed from the Gamma equation as

$$\partial_S V(S, t) = \partial_S V(0, t) + \int_0^S \frac{1}{s} H(\ln(s/E), T - t)ds = \int_{-\infty}^{\ln(S/E)} H(x, T - t)dx$$

and by integration we get formula for the call option

$$V(S, t) = \int_{-\infty}^{\infty} (S - Ee^x)^+ H(x, T - t)dx, \tag{3.5}$$

as $\partial_S V(0, t) = V(0, t) = 0$. Similarly, the value of put option is

$$V(S, t) = \int_{-\infty}^{\infty} (Ee^x - S)^+ H(x, T - t)dx. \tag{3.6}$$

3.2 β functions

In this section we introduce β functions for different nonlinear models. We just remind that nonlinear Black-Scholes equation has form

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\tilde{\sigma}^2(T-t, S, \frac{\partial^2 V}{\partial S^2})S^2\frac{\partial^2 V}{\partial S^2} - rV = 0.$$

- **RAPM model**

  The volatility function in RAPM model is given as

$$\tilde{\sigma}^2 = \sigma^2\left(1 + 3\left(\frac{C^2R}{2\pi}S\partial_S^2 V\right)^{1/3}\right)$$

  we use substitution $\mu = 3\left(\frac{C^2R}{2\pi}\right)^{1/3}$ and we come to the β-function for RAPM model

$$\beta(H) = \frac{\hat{\sigma}^2}{2}(1 + \mu H^{\frac{1}{3}})H. \tag{3.7}$$

- **Barles's and Soner's model**

  The volatility function in this case has form

$$\begin{aligned}\tilde{\sigma}^2(S^2V_{SS}^2, T-t) &= \hat{\sigma}^2(1 + \Psi(a^2e^{r(T-t)}S^2V_{SS}^2)) \\ &= \hat{\sigma}^2(1 + \Psi(a^2e^{r(T-t)}e^{\ln S/E}ESV_{SS}^2))\end{aligned}$$

  and the β-function of Barles's and Soner's model has form

$$\beta(H, x, \tau) = \frac{\hat{\sigma}^2}{2}(1 + \Psi(Ea^2e^{r\tau+x}H))H. \tag{3.8}$$

- **Jumping volatility model**

  The volatility function of Jumping volatility model has form

$$\sigma^2(S^2\partial_S^2V, S, \tau) = \begin{cases}\hat{\sigma_1^2}, & \text{if } \partial_S^2 V < 0 \\ \hat{\sigma_2^2}, & \text{if } \partial_S^2 V > 0.\end{cases}$$

  So the β-function is easy to derive and its form is

$$\beta(H) = \begin{cases}\frac{\hat{\sigma_1^2}}{2}H, & \text{if } H < 0 \\ \frac{\hat{\sigma_2^2}}{2}H, & \text{if } H > 0.\end{cases} \tag{3.9}$$

- **Leland's model** The volatility of Leland's model is of the form

  $$\tilde{\sigma}^2 = \sigma^2 \left(1 + Le \text{ sign}\left(\frac{\partial^2 V}{\partial S^2}\right)\right),$$

  so the derived $\beta-$ function is

  $$\beta(H) = \frac{\sigma^2}{2}\left(1 + Le \text{ sign}(H)\right) H. \tag{3.10}$$

- **Amster et al.'s model**

  The volatility

  $$\tilde{\sigma}^2 = 1 - \frac{a}{\sigma}\text{sign}\left(\frac{\partial^2 V}{\partial S^2}\right)\sqrt{\frac{2}{\pi dt}} + 2b\left(\frac{2}{\pi}\right) S\frac{\partial^2 V}{\partial S^2}$$

  transforms to

  $$\beta(H) = \frac{\sigma^2}{2}\left(1 - \frac{2a}{\sigma}\text{ sign}(H)\sqrt{\frac{2}{\pi dt}}\right) H + 2b\left(\frac{2}{\pi}\right) H^2. \tag{3.11}$$

Chapter 4

Numerical methods

In this section we will introduce the numerical way we will work with in this thesis. Throughout this chapter we will work with grid where $k = \Delta t$ is a time step for the time variable t and we donote a spatial step as $h = \Delta x$.

4.1 Explicit numerical method

This method is defined through approximations of derivative ∂_t by a forward difference, ∂_x by a central difference and ∂_x^2 by a symmetric forward difference, i.e.

$$\begin{aligned}
\partial_t f(t+k,x) &= \frac{f(t+k,x)-f(t,x)}{k} + O(k),\\
\partial_x f(t+k,x) &= \frac{f(t+k,x+h)-f(t+k,x-h)}{2h} + O(h^2),\\
\partial_x^2 f(t+k,x) &= \frac{f(t+k,x+h)-2f(t+k,x)+f(t+k,x-h)}{h^2} + O(h^2).
\end{aligned}$$

The stability condition for explicit method in the case of Black Scholes model requires the so called Courant-Lewy-Fridrichs condition, i.e.

$$\frac{\sigma^2 k}{h^2} \le 1.$$

In the case of explicit method the Gamma equation can be written as

$$\begin{aligned}
\frac{H_i^{j+1}-H_i^j}{k} &= \frac{1}{h^2}\left(\beta'(H_i^j)(H_{i+1}^j - H_i^j) - \beta'(H_{i-1}^j)(H_i^j - H_{i-1}^j)\right) \quad (4.1)\\
&+ \beta'(H_i^j)\frac{H_{i+1}^j - H_{i-1}^j}{2h} + r\frac{H_{i+1}^j - H_{i-1}^j}{2h}.
\end{aligned}$$

This form of the explicit method is also known as FTCS (Forward time central space). It means that in time we move one step forward (left side of the equation) and for the derivative on the right side central derivative is used.

In case of explicit method it is possible to write the equation we are working with in a matrix form $x^{j+1} = Ax^j + b^j$. The matrix A is a tridiagonal matrix with the nonnegative elements on the diagonals and the maximum norm of the elements is at most 1.

4.2 Implicit numerical method

As we can see the relation between k, h and σ is rather restrictive. For example in the case when $h = 0.01$ and $\sigma = 0.4$ the longest time step we can take is $k = 1/1600$. Implicit method overcomes this problem and therefore we can work with shorter time step (there is no more such restriction for the time step). The approximations are defined as following

$$\begin{aligned}
\partial_t f(t,x) &= \frac{f(t+k,x) - f(t,x)}{k} + O(k), \\
\partial_x f(t,x) &= \frac{f(t,x+h) - f(t,x-h)}{2h} + O(h^2), \\
\partial_x^2 f(t,x) &= \frac{f(t,x+h) - 2f(t,x) + f(t,x-h)}{h^2} + O(h^2).
\end{aligned}$$

4.3 Semi-Implicit numerical method

Semi-Implicit method is somehow combination of explicit and implicit method. The nonlinear terms $\beta'_H(H,x,\tau)$ and $\beta'_x(H,x,\tau)$ are evaluated from the previous time step τ_{j-1} and the linear terms are evaluated in the current time level, i.e. the Gamma equation has the following form

$$\begin{aligned}
\frac{H_i^{j+1} - H_i^j}{k} &= \frac{1}{h^2}\left(\beta'(H_i^j)(H_{i+1}^{j+1} - H_i^{j+1}) - \beta'(H_{i-1}^j)(H_i^{j+1} - H_{i-1}^{j+1})\right) \\
&+ \beta'(H_i^j)\frac{H_{i+1}^{j+1} - H_{i-1}^{j+1}}{2h} + r\frac{H_{i+1}^{j+1} - H_{i-1}^{j+1}}{2h}.
\end{aligned}$$

This form of the equation can be again reffered as BTCS (backward time central space) numerical method. Similarly as in case of FTCS method the central derivative is used in the method but we move one time step back in this method. In Chapter 5 we are comparing results from explicit and implicit model in case of Jumping volatility model.

4.4 Khaliq-Liao method

In following we are going to present scheme introduced by Liao and Khaliq in [13]. Khaliq-Liao method is using Padé approximation and Richardson extrapolation and instead of solving a single convection-diffusion equation,

$$u_t = \beta u_{xx} + \lambda u_x$$

a system of two equations is considered. We introduce a new unknown function

$$v(x,t) = u_x(x,t)$$

and the original convection-diffusion equation is converted into the following system

$$\begin{aligned} u_t &= \beta u_{xx} + f(u,v), \\ v_t &= \beta v_{xx} + \lambda u_{xx} + g(u,v). \end{aligned}$$

The boundary and initial conditions for variable u are given as $u(x,0) = u_0(x)$, $u(0,t) = b_0(t)$ and $u(1,t) = b_1(t)$. In case when u is not smooth enough, we have to approximate these boundary conditions. In the derivation of the boundary/initial conditions for the new variable v we will follow the notation from [13]. Let difference operator Δ_x be defined as

$$\Delta_x u_i = u_{i+1} - u_{i-1},$$

so the approximation of variable v is

$$v(h,t) = \frac{u(2h,t) - u(0,t)}{2h} = \frac{\Delta_x}{2h} u(h,t).$$

This second order approximation can be improved to fourth order approximation taking $\Delta_x/(1+1/6\delta_x^2)$, $\delta_x^2 u_i = ui + 1 - 2u_i + u_{i-1}$. Therefore we get from

$$\left(1 + \frac{1}{6}\delta_x^2\right) v(h,t) = \frac{\Delta_x}{2h} u(h,t) \tag{4.2}$$

the boundary conditions

$$\begin{aligned} v(0,t) &= \frac{3}{h}(u(2h,t)-u(0,t)) - 4v(h,t) - v(2h,t), \\ v(1,t) &= \frac{3}{h}(u(1-2h,t)-u(1,t)) - 4v(1-h,t) - v(1-2h,t). \end{aligned}$$

Finally, after Padé approximation Khaliq-Liao method can be written as

$$\begin{aligned} \left(1+\frac{\delta_x^2}{12}-\frac{h^2\beta}{2\Delta t}\delta_x^2\right)u_i^{n+1} &= \left(1+\frac{\delta_x^2}{12}+\frac{h^2\beta}{2\Delta t}\delta_x^2\right)u_i^n \\ &+ \frac{\Delta t}{2}\left(1+\frac{\delta_x^2}{12}\right)(f_i^{n+1}+f_i^n), \\ \left(1+\frac{\delta_x^2}{12}-\frac{h^2\beta}{2\Delta t}\delta_x^2\right)v_i^{n+1} &= \left(1+\frac{\delta_x^2}{12}+\frac{h^2\beta}{2\Delta t}\delta_x^2\right)v_i^n + \lambda\frac{h^2}{2\Delta t}\delta_x^2(u_i^{n+1}+u_i^n) \\ &+ \frac{\Delta t}{2}\left(1+\frac{\delta_x^2}{12}\right)(g_i^{n+1}+g_i^n). \end{aligned}$$

Terms f_i^{n+1} or f_i^n stand for $f(u_i^{n+1}, v_i^{n+1})$ or $f(u_i^n, v_i^n)$ respectively. The same notation is for the function $g(u,v)$. The truncation error of Khaliq-Liao method is in the form of $K_1\Delta t^2 + K_2\Delta t^4 + K_3\Delta h^4$. However, Richardson extrapolation can be used for the solution where we can eliminate the term Δt^2 and consequently we get method which is of the fourth order in both time and spatial dimension.

Chapter 5

Numerical experiments

Throughout this thesis we were working with the function $H(x,t)$. As it was already introduced in Chapter 4 the domain of this function is $(-\infty,\infty)\times(0,T)$. Anyway, for numerical purposes we had to somehow shorten the interval $(-\infty,\infty)$. As the variable x stands for $x=\ln\frac{S}{E}$ an appropriate interval can be $x\in(-1.5,1.5)$ as it was already used in ([23]).

5.1 Jumping volatility model

First of all we will present numerical results considering Jumping volatility model. These results are based on the bull spread strategy. A spread trading strategy takes position in two or more options of the same type. A bull spread strategy is created by buying a call option with exercise price E_1 and selling another call option with another exercise price E_2 both with the same expiration date. This strategy gives us the opportunity to work with the Jumping volatility model as it concerns both positive and negative $\frac{\partial^2 V}{\partial S^2}$.

Strike prices in this model were $E_1=25$ for the call option we are buying and $E_2=30$ for the call option we are selling, $r=0.011$, $\sigma_1=0.2$ (volatility in case $H\leq 0$), $\sigma_1=0.4$ (in case $H>0$) and $T=1$.

When we compare results among a numerical scheme we use CPU time (in seconds) and the difference from the benchmark in the euclidean norm. As the benchmark were used results from the gentlest grid (with the time step $k=1/7000$). The spatial step was proportional to the second square root of the time step ($h \sim \sqrt{k}$).

The initial function in this case when we work with spread trading strategy has form

$$H(x,0) = \exp\left(\frac{-((x+(r-q-\sigma_1^2/2)\tau^*)/(\sigma_1\sqrt{\tau^*}))^2}{2}\right)\frac{1}{(\sigma_1\sqrt{\tau^*}\sqrt{2\pi})}$$
$$-\exp\left(\frac{-((x-\ln(30/25)+(r-q-\sigma_2^2/2)\tau^*)/(\sigma_2\sqrt{\tau^*}))^2}{2}\right)\frac{1}{\sigma_2\sqrt{\tau^*}\sqrt{2\pi}}.$$

In Figure (5.1) we can see the function $H(x,\tau)$ in the points $\tau = 0$ or $\tau = T$. From the values of this function is finally the value of the derivative computed.

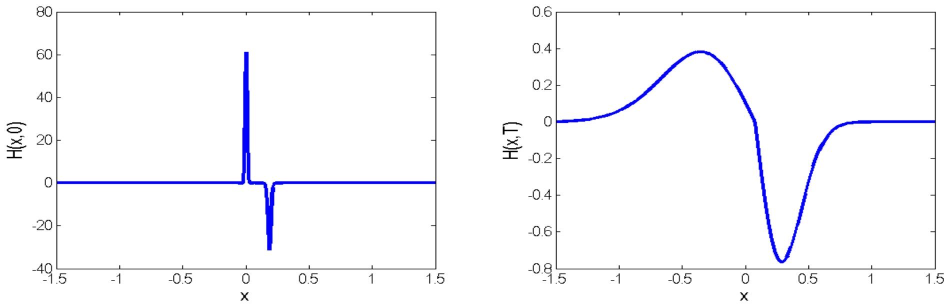

Figure 5.1: Function $H(x,\tau)$ in point $\tau = 0$ (left) and $\tau = T$ (right).

Figure (5.2) shows the price of the derivative with different volatility and pay-off of this derivative. Price of the derivative using jumping volatility is computed by Khaliq-Liao method. In [23] was previous computation also done but authors work with another numerical method.

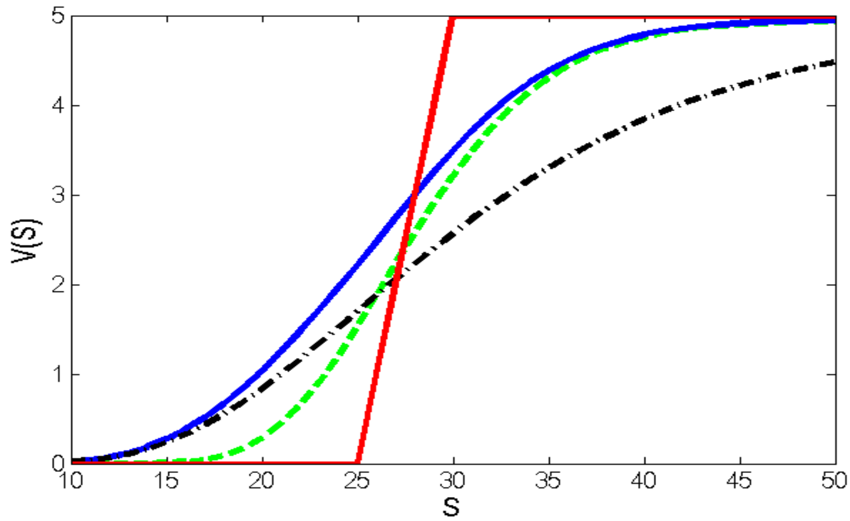

Figure 5.2: Comparison of the price V(S), where the volatility was from jumping volatility model (blue solid line), constant volatility $\sigma = 0.4$ (dashed line) or $\sigma = 0.2$ (dash-dot line), pay-off of the derivative (red solid line)

5.1.1 Numerical results

In this subsection we present results from three numerical methods considering Jumping volatility model. For each method table with CPU time and convergence graph is given.

Khaliq-Liao method

The next graph (Figure 5.3) depicts difference between the benchmark and the result of the function $H(x, \tau)$ at the time 0.

m	k	CPU(s)	difference
250	1/250	2.4376	3.93863
1000	1/1000	32.5185	0.01912
1750	1/1750	101.7569	0.01538
2500	1/2500	212.3195	0.00041
3250	1/3250	360.2153	0.00461
4000	1/4000	561.1594	0.00140
4750	1/4750	790.4845	0

Table 5.1: Comparison of results with different time step (Khaliq-Liao method).

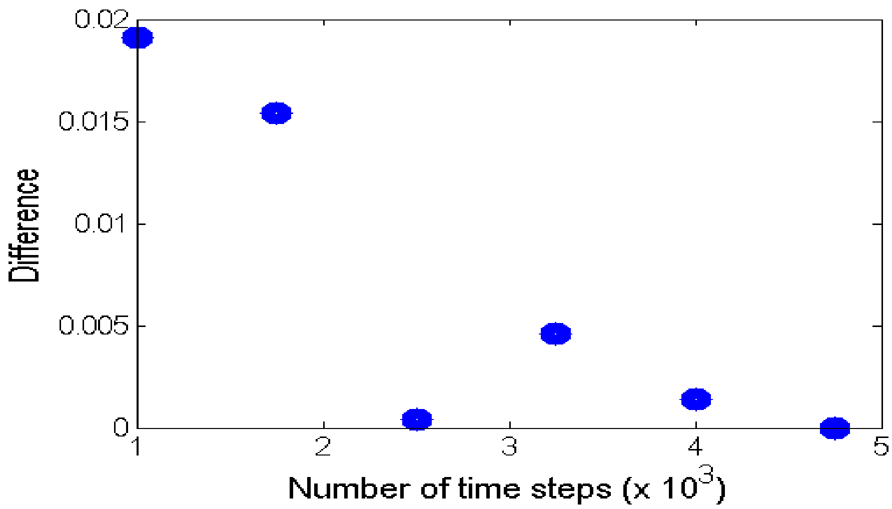

Figure 5.3: Convergence of Khaliq-Liao method

Implicit method

The next graph (see Figure 5.4) depicts difference between the benchmark and the result of the function $H(x, \tau)$ at the time 0 in case of implicit method.

m	k	CPU(s)	difference
250	1/250	0.816	4.08019
1000	1/1000	8.143	0.04811
1750	1/1750	25.565	0.00065
2500	1/2500	50.874	0.01046
3250	1/3250	82.173	0.00083
4000	1/4000	124.629	0.00315
4750	1/4750	169.845	0

Table 5.2: Comparison of results with different time step (Implicit method).

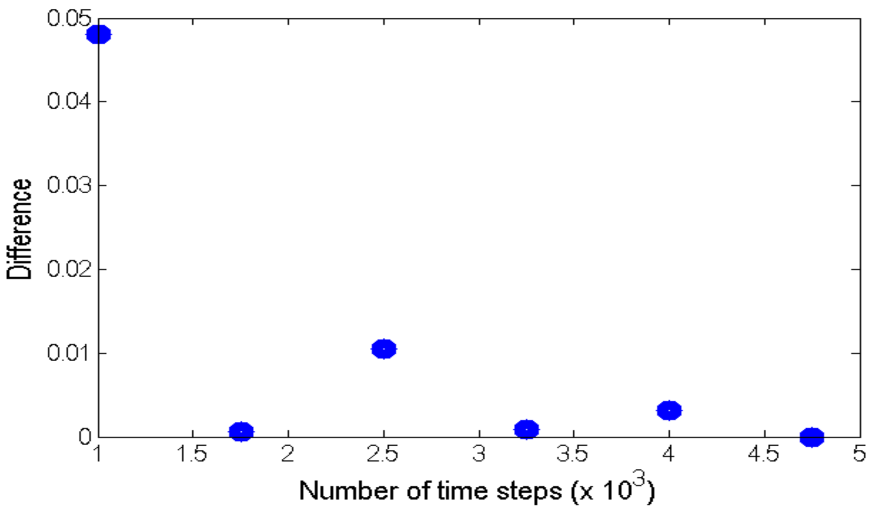

Figure 5.4: Convergence of Implicit method

Explicit method

The next graph (see Figure 5.5) depicts again the difference between the benchmark and the result of the function $H(x, \tau)$ at the time 0 in case of using explicit method.

m	k	CPU(s)	difference
250	1/250	0.029	4.08019
1000	1/1000	1.326	0.04811
1750	1/1750	6.199	0.00065
2500	1/2500	15.162	0.01046
3250	1/3250	29.200	0.00083
4000	1/4000	47.017	0.00315
4750	1/4750	71.188	0

Table 5.3: Comparison of results with different time step (Explicit method).

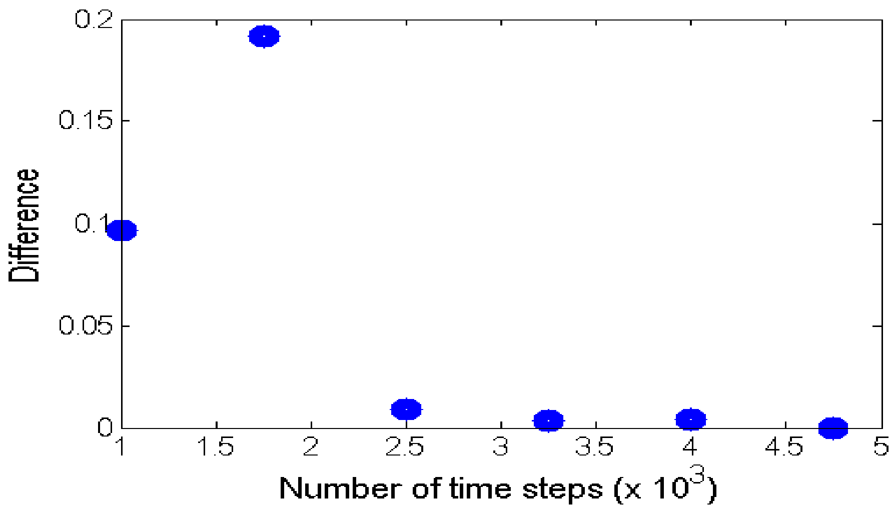

Figure 5.5: Convergence of Explicit method

5.1.2 Comparison of explicit and implicit method

As it was mentioned in Chapter 4.1 the stability condition in the case of Black-Scholes equation requires so called Courant-Lewy-Fridrichs condition ($\frac{\sigma^2 k}{h} \leq \frac{1}{2}$). In the following we are comparing explicit and implicit methods when the Courant-Lewy-Fridrichs condition does not hold. As a benchmark for comparing these two methods we are using number of steps $m = 2000$ and spatial step $h = 0.0089$. The time step k is then computed according to the CLF ratio, where as the parameter σ the higher volatility (σ_2) is used.

CLF-ratio	Explicit	Implicit
1	0	0
1.1	0.098	5.699E-07
1.2	0.714	3.236E-07
1.3	6.443	1.012E-06
1.4	1.104E+02	3.146E-07
1.5	6.343E+03	2.091E-07
1.6	1.031E+117	1.003E-06

Table 5.4: Comparison of Explicit and Implicit method

As it was already mentioned, from comparison of explicit and implicit method we can see that explicit is faster than implicit, on the other hand the stability condition is rather restrictive.

5.2 RAPM model

In this section we present numerical results based on the RAPM model. Throughout this computation we set the paramters as $C = 0.01$ (round trip transaction costs per unit dollar) and $R = 30$ (risk premium coefficient). Therefore the value of the parameter μ is $\mu = 0.2345$. Call option was used for the computation in case of this RAPM model. The form of the initial condition for the function $H(x, \tau)$ is

$$H(x,0) = \exp\left(-\frac{((x + (r - q + \sigma^2/2)\tau^*)/(\sigma * \sqrt{\tau^*}))^2}{2}\right)\frac{1}{(\sigma\sqrt{\tau^*}\sqrt{2\pi})}. \quad (5.1)$$

The following picture shows form of the function $H(x, \tau)$ at the beginning and also the final form of this function.

As it was already introduced in Chapter 3 the Gamma equation (3.2) has form

$$\frac{\partial H}{\partial \tau} = \frac{\partial^2 \beta(H)}{\partial x^2} + \frac{\partial \beta(H)}{\partial x} + (r - q)\frac{\partial H}{\partial x} - qH.$$

There are two possibilities how to proceed with the second derivative $\frac{\partial^2\beta(H)}{\partial x^2}$. The first possibility is

$$\frac{\partial^2 \beta(H)}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \beta}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \beta}{\partial H}\frac{\partial H}{\partial x}\right) = \frac{\partial^2 \beta}{\partial H^2}\left(\frac{\partial H}{\partial x}\right)^2 + \frac{\partial \beta}{\partial H}\frac{\partial^2 H}{\partial x^2}.$$

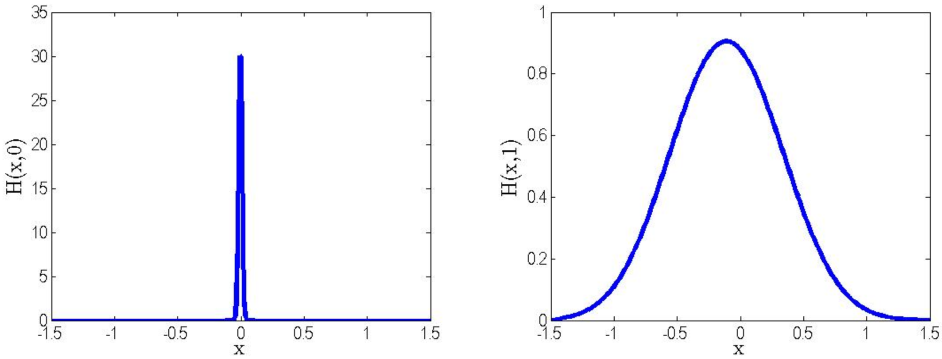

Figure 5.6: Comparison of the function $H(x, \tau)$ in point $\tau = 0$ (left) and $\tau = T$ (right).

As in case of RAPM model $\frac{\partial^2 \beta}{\partial H^2}$ would contain $H^{-\frac{2}{3}}$, the second approximation of the second derivative is more appropriate (through numerical approximation), i.e.

$$\frac{\partial^2 \beta(H)}{\partial x^2} = (\beta'(H(i,j))H(i,j)_x - \beta'(H(i-1,j))H(i-1,j)_x)\frac{1}{h}.$$

5.2.1 CLF condition-fixing time or spatial step

As one of the assumptions of explicit numerical method is that the mesh ratio $\frac{\sigma^2 k}{h^2} \leq 1$, there were two possibilities how to ensure this condition to be valid. One of them was to fix the spatial step h and then compute the time step k and the second possibility was first to fix the time step k and consequently compute the spatial step h according to the CLF condition. While working with RAPM and Amster et al.'s model we were trying both approaches but when the mesh ratio was small enough there was problem with the initial condition $H(x, 0)$ when we fixed the time step k. Mesh for the initial function was not dense enough and the results were not precise. Therefore in case of RAPM and Amster et al.'s model we fixed spatial step at the value $h = 0.008$ and worked further with this value.

5.2.2 Numerical results considering RAPM model

Table 5.5 shows computed values of the financial derivative $V(S,t)$ for different values of parameters R and C at variable time points. The constant τ^* was computed for each values according to the relation (2.24). Explicit numerical method was used for the computation.

	S=E=25			S=20			S=30		
R	0	30	45	0	30	45	0	30	45
C	0	0.01	0.0087	0	0.01	0.0087	0	0.01	0.0087
μ	0	0.234	0.2446	0	0.234	0.2446	0	0.234	0.2446
$V(S,0)$	3.989	3.442	3.423	1.611	1.2	1.183	7.335	6.843	6.836
$V(S,0.5)$	2.787	2.341	2.326	0.762	0.496	0.484	6.167	5.83	5.833
$V(S,\frac{2}{3})$	2.253	1.862	1.847	0.444	0.26	0.251	5.693	5.477	5.458

Table 5.5: Computed values of $V(S,t)$ - RAPM model

When the parameter μ is equal to 0, the derivative's prices $V(S,0), V(S,0.5)$ and $V(S,\frac{2}{3})$ computed by explicit method were on the level of the prices computed by Black-Scholes equation (1.6). In this case nonlinear Black-Scholes equation changes to linear because the nonlinear term disapperas and therefore it is equal to the value computed by formula (1.6). The difference is on the level of discretisation error.

5.3 Amster et al.'s model

Similarly as in case of RAPM model in Amster et al.'s model we also make computation on the call option. As it was already mentioned in Chapter 2.3 transaction costs in this model are considered to be as a nonincreasing linear function $h(S) = a - bS$. According to the note in [2] the constant a should fulfill the following relationship

$$\sigma^2\left(1 - \frac{a}{\sigma}\sqrt{\frac{2}{\pi dt}}\right) > 0.$$

Considering this, we fixed the value of a to be equal $\frac{\sigma}{\sqrt{2/(\pi k)}}0.99$ and the value of the constant b we considered to be equal to 1. Similarly as in case of RAPM model we fixed the value of the spatial step $h = 0.008$ and the time step was computed according to the mesh ratio $\frac{\sigma^2 k}{h^2} = \frac{1}{80}$. It was $k = 5.0 \exp_{10}(-6)$. Following two pictures (see Figure 5.7 and Figure 5.8) show development of the call option with the exercise price $E = 25$ in case of Amster et al.'s model. We were using explicit forward time central space method to compute the value of the derivative.

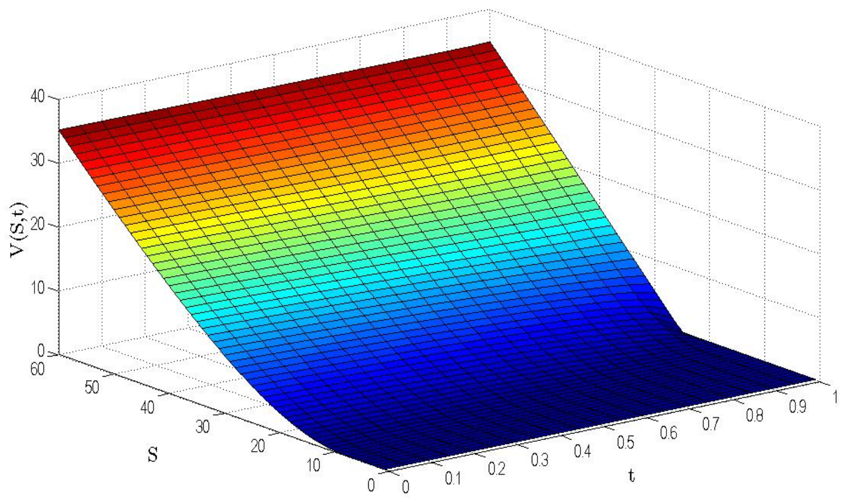

Figure 5.7: Development of the call option in Amster et al.'s model

The following table (Table 5.6) contains values of the derivative's price with some concrete values of parameters a, b and the asset's price. Value a^* stands for the value also used in the Matlab code (see Appendix). Semi-implicit method was used for the computation.

	S=E=25			S=20			S=30		
a	0	a^*	0.001	0	a^*	0.001	0	a^*	0.001
b	0	1	0.7	0	1	0.7	0	1	0.7
$V(S,0)$	3.987	5.583	4.633	1.609	2.239	2.155	7.333	8.829	7.83
$V(S,0.5)$	2.785	4.312	3.592	0.76	1.903	1.341	6.166	7.506	6.759
$V(S,\frac{2}{3})$	2.252	3.705	3.092	0.443	1.427	0.974	5.692	6.882	6.621

Table 5.6: Computed values of $V(S,t)$ - Amster et al.'s model

When the values of the parameters were set to $a = b = 0$, the derivative's prices $V(S,0), V(S,0.5), V(S,\frac{2}{3})$ computed by numerical method were on the level of the prices computed by Black-Scholes equation (1.6). When mentioned

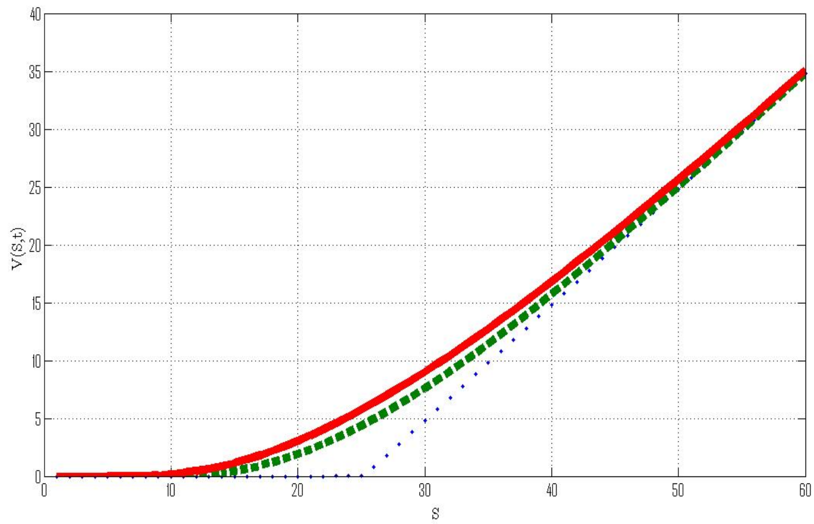

Figure 5.8: Development of the call option in case of Amster et al.' s model: blue dotted line at time t=1, green dashed line at time t=1/2 and the solid red line shows the price of the derivative at the beginning (t=0).

parameters a, b are set to those values, similarly as in case of RAPM model, we have linear Black-Scholes equation.

5.4 Leland's model

The last section is devoted to the presentation of the numerical results in case of Leland's model (see Table 5.7). We present results for different value of the parameter σ which sonsequently changes the value of the Leland's number. Again all results are presented for three different prices S. In the case of Leland's model Khaliq-Liao method was used for the computation. Constant $C = 0.00005$ was used as an input into the Leland's number.

	S=E=25			S=20			S=30		
σ	σ=0.3	σ=0.4	σ=0.5	σ=0.3	σ=0.4	σ=0.5	σ=0.3	σ=0.4	σ=0.5
$V(S,0)$	3.087	4.056	4.903	0.942	1.658	2.339	6.490	7.409	8.229
$V(S,\frac{1}{3})$	2.494	3.299	4.086	0.567	1.102	1.690	5.943	6.662	7.421
$V(S,0.5)$	2.143	2.842	3.537	0.374	0.793	1.28	5.644	6.231	6.878

Table 5.7: Computed values of $V(S,t)$ - Leland's model

Conclusion

The aim of this thesis was to work with the nonlinear Black-Scholes equation. Nonlinear Black-Scholes equation differs from the linear one in the way that the volatility is not constant but it is a function dependent on some extra variables. We focused particularly on the volatility functions concerning transaction costs, i.e. Leland's model, RAPM model, Amster et al.'s model and Jumping volatility model.
We were working with the transformed Black-Scholes equation. As the used variable $H = S\partial_S^2 V$ includes $\partial_S^2 V$, known in financial world as Gamma, this transformed equation was named *Gamma equation*. The Black-Scholes equation yields more robust numerical approximation schemes.

Thesis is organised in the following way. The first chapter offers a short introduction into the field of financial derivatives. It concerns particularly theory about options. Except from mentioned volatility models also Barles's and Soner's model is introduced in the second chapter. The third chapter is devoted to the derivation of the Gamma equation and in the fourth chapter the theory from used numerical schemes (explicit, implicit and Khaliq-Liao method) are presented. The last chapter presents numerical results.

In the numerical computation we used two different initial functions. The first one was bull spread strategy for Jumping volatility model. This strategy behaves differently depending on the sign of the ∂_S where we could use the properties of Jumping volatility model. The second initial function was call option and computation with Leland's, RAPM's, Amster et al.'s model or Model with nonegative transaction costs function was done.

The numerical methods are presented seperately for each model. On the Jumping volatility model convergence and CPU time of all numerical schemes are compared. In case of RAPM, Amster et al.'s and Leland's model we introduce results of semiimplicit, explicit and Khaliq-Liao method, respectively. Parameters as input into the model were changed and further compared. When the parameters were set in the way that nonlinear term dissapeared, the results were in line with the results computed using Black-Scholes formulas (the difference was in line with the discretisation error).

Resume

Cieľom tejto práce bolo numerické spracovanie transformovanej nelineárnej Black-Scholesovej rovnice. Rozdiel medzi lineárnou a nelineárnou Black-Scholesovou rovnicou spočíva vo volatilite. Lineárna Black-Scholesova rovnica totiž predpokladá konštantnú volatilitu. Funkcia volatility, ktorá vstupuje do nelineárnej rovnice dokáže lepšie aproximovať skutočnosť, keďže berie do úvahy faktory, ktoré lineárnu rovnicu neovplyvňujú. Existuje preto viacero modelov, ktoré riešia otázku nelinearity rozdielnym spôsobom. Môžu sa zaoberať transakčnými nákladmi (RAPM model, Lelandov model, Model zohľadňujúci variabilné transakčné náklady), funkciou užitočnosti obchodovateľa alebo jeho preferenciami (Barles-Sonerov model). Prítomnosti veľkého investora na trhu, ktorý svojim konaním dokáže istým spôsobom ovplyvňovať ceny, je zas venovaný Freyov model.

Prvá kapitola tejto diplomovej práce je venovaná teórii finančných derivátov. Sú tam predstavené základné finančné deriváty (forwardy, furturity, európske a americké opcie). Keďže ďalej sa venujeme predovšetkým európskym opciam, opciam ako celku je aj venovaná podstatná časť tejto kapitoly. Opcie môžu byť ovplyvnené viacerými premennými. Patria k nim súčasná cena aktíva, realizačná cena, čas do expirácie, volatilita, bezriziková úroková cena alebo dividendy. Vplyv týchto premenných na jednotlivé opcie (európske a americké put alebo call opcie) je prehľadne znázornený v tabuľke 1.1. Je tu aj teória k Black-Scholesovej rovnici (lineárnej aj nelineárnej).

Modely nelineárnej Black-Scholesovej rovnice sú bližšie rozpracované v druhej kapitole. Prvým modelom, ktorý je tu bližsie predstavený Lelandov model skúmajúci transakčné náklady. Tie sú zahrnuté formou $dTC = C|k|S/2$, kde premenná k predstavuje počet zobchodovaných aktív (podľa znamienka určujeme, či sa jedná o predaj alebo kúpu) a premenná C označuje transakčné náklady na jednu menovú jednotku. Lelandov model je základom pre ďalšie modely, ktorým sa v práci venujeme. RAPM model zohľadňuje okrem transakčných nákladov aj riziko z volatilného portfólia a hľadá optimálny časový krok medzi jeho dvomi úpravami. Ďalší model (Model s variabilnými transakčnými nákladmi podľa Amstera a kol.) sa zaoberá myšlienkou výšky transakčných nákladov podľa množstva zobchodovaných aktív. Čím je toto množstvo vyššie, tým obchodník dostáva vyššiu "zľavu". Posledným modelom založeným na Lelandovom modeli je Jumping volatility model, ktorého volatilita sa pohybuje medzi istými hranicami. Okrem spomenutých modelov, ktorým sa ďalej venujeme v praktickej časti tejto práce uvádzame ešte jeden model, ktorého základom sú preferencie veľkého investora na trhu. Na tento model už ale numerické schémy neaplikujeme.

Ako už bolo spomenuté na začiatku práce, venujeme sa transformovanej Black-Scholesovej rovnici. Táto transformácia je založená na zmene nezávislých premenných času a ceny a transformácii $H(x,\tau) = S\Gamma = S\partial_S^2 V$. Výhodou tejto transformovanej rovnice je, že pre túto rovnicu vieme odvodiť efektívne numerické schémy. Navyše okrajové podmienky tejto rovnice sú $H(-\infty,\tau)$=$H(\infty,\tau) = 0$.

V práci používame explicitnú a semiimplicitnú numerickú metódu, ktorú používame na všetky modely a navyše pracujeme aj s Khaliq-Liaovou metódou, ktorú aplikujeme na Lelandov model a na Jumping volatility model.

V práci porovnávame konvergenciu a dĺžku výpočtového času jednotlivých metód (na Jumping volatility modeli), skúmame CLF podmienku medzi explicitnou a implicitnou numerickou metódou. Táto podmienka je istým nedostatkom explicitnej metódy a hovorí o pomere medzi časovým a priestorovým krokom. Tieto numerické riešenia sú spracované na európskej call opcii, prípadne na bullish spread. Bullish spread je pre Jumping volatility model, keďže hodnota volatility v tomto modeli závisí od znamienka druhej derivácie funkcie $V(S,t)$ podľa ceny.

List of Symbols

Option Variables

$t,\ T$	Time, Expiration time.
τ	Time to maturity, $\tau = T - t$.
E	Exercise price ('strike price') of an option.
$S,\ S(t)$	Price of the underlying asset at time t .
$V,\ V(S,t)$	Price of the financial derivative at time t and asset's price S.
$(S(T) - E)^+$	Payoff function at time T, $(= \max(0, S(T) - E))$.
σ	Constant volatility.
$q,\ r$	Dividend yield rate.
a, b	Parameters from Amster's model.
C, R	Parameters from RAPM model (transaction costs, risk premium).
Le	Parameter from Leland's model, Leland's number.
x	Transformed spatial variable $x = \ln \frac{S}{E}$.
$H(x,\tau)$	Variable from Gamma equation, $H = S\partial V_S^2$.
$\tilde{\sigma}(.)$	Nonconstant volatility function.

Grid

$k,\ h$	Time step, Spatial step.
j	Index for time step.
i	Index for spatial step.
$m,\ n$	Number of time steps, Number of spatial steps.

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Appendix

1. Itô's lemma

Theorem Consider a function $V(S,t)$ and suppose that $S(t)$ follows Itô's process

$$dS = a(S,t)dt + b(S,t)dW,$$

where $W(t)$ is the standard Wiener's process. Then V follows an Itô's process with the same Wiener process $W(t)$:

$$dV = (aV_S + \frac{1}{2}b^2V_{SS}^2 + V_t)dt + bV_S dW,$$

where a:=a(S,t) and b:=b(S,t).

In our case, where $a(S,t) = \mu S$ and $b(S,t) = \sigma S$ (in nonlinear Black-Scholes equation $b(S,t) = \tilde{\sigma}S$) we come to

$$dS = \mu S dt + \sigma S dW.$$

or in the case of nonconstant volatility

$$dS = \mu S dt + \tilde{\sigma} S dW.$$

Using Itô's lemma we come to

$$\begin{aligned} dV &= \left(\mu S V_S + \frac{1}{2}\sigma^2 S^2 V_{SS}^2 + V_t\right) dt + \sigma S V_S dW \\ &= \left(\frac{1}{2}\sigma^2 S^2 V_{SS}^2 + V_t\right) dt + V_S dS. \end{aligned}$$

and again in the case of the nonconstant volatility function we have

$$\begin{aligned} dV &= \left(\mu S V_S + \frac{1}{2}\tilde{\sigma}^2 S^2 V_{SS}^2 + V_t\right) dt + \tilde{\sigma} S V_S dW \\ &= \left(\frac{1}{2}\tilde{\sigma}^2 S^2 V_{SS}^2 + V_t\right) dt + V_S dS. \end{aligned}$$

S_{ask} market price for selling an asset (trader wants to buy),
S_{bid} market price for buying an asset (trader wants to sell).

2. **Bid and Ask Prices**

If the transaction costs are available on the market, there are different prices for buying and selling an asset. We donote as

3. **Matlab codes**

In this Appendix we present Matlab codes we were using. The first code stands for the explicit numerical method used for RAPM model.

```
% Variables used in this code have the same
notation as in the thesis.
taustar=0.0021;
L=1.5; %variable x is defined on <-L,L>
E=25;
T=1;
h=0.008;
sigma_hat=0.4;
k=h^2/4^2/sigma_hat^2/5;
x=-1.5;
m=floor(T/k);
r=0.011;
q=0;
R=30;
C=0.01;
mu=3*(C^2*R/2/pi)^(1/3);
n=floor(2*L/h)+1;
%initial value of the vector H according to the
%approximation of the function N(d)
for i=1:1:n;
Hint(i)=exp(-((x+(r-q+sigma_hat^2/2)*taustar)/...
(sigma_hat*sqrt(taustar)))^2/2)/(sigma_hat*...
sqrt(taustar)*sqrt(2*pi));
```

```
x=x+h;
end
%matrix initializing with appropriate size
H=zeros(n,m+1);
H(:,1)=Hint';
x=1.5;
%following four constants are further
%used in the  cycle
c_1=k/h^2;
c_2=k/2/h;
c_3=sigma_hat^2/2;
c_4=2/3*sigma_hat^2*mu;
prem=0;
for j=1:m
for i=1:n
if(i==1)
H(i,j+1)=c_1*((c_3-c_4*H(i,j)^(1/3))*...
(H(i+1,j)-H(i,j))-(c_3-c_4*prem^(1/3))*...
(H(i,j)-prem))+c_2*(c_3-c_4*H(i,j)^(1/3))*...
(H(i+1,j)-prem)+r*c_2*(H(i+1,j)-prem)+H(i,j);
elseif(i==n)
H(i,j+1)=c_1*((c_3-c_4*H(i,j)^(1/3))*...
(prem-H(i,j))-(c_3-c_4*H(i-1,j)^(1/3))*...
(H(i,j)-H(i-1,j)))-c_2*(c_3+c_4*...
H(i,j)^(1/3))*(prem-H(i-1,j))+r*c_2*(prem-...
H(i-1,j))+H(i,j);
else
H(i,j+1)=c_1*((c_3-c_4*H(i,j)^(1/3))*...
(H(i+1,j)-H(i,j))-(c_3-c_4*H(i-1,j)^(1/3))*...
(H(i,j)-H(i-1,j)))+c_2*(c_3-c_4*H(i,j)^(1/3))*...
(H(i+1,j)-H(i-1,j))+r*c_2*(H(i+1,j)-H(i-1,j))+H(i,j);
end
end
end
```

The second Matlab code is working with Amster's model and semi-implicit numerical method.

```
taustar=0.001;
L=1.5;
E=25;
T=0.999;
x=-1.5;
sigma_hat=0.4;
h=0.008;
k=h^2/4^2/sigma_hat^2/5;
m=floor(T/k)
r=0.011;
q=0;
a=sigma_hat/(sqrt(2/pi/k)*1.9);
b=1;
%initial value of the vector H according to the
%approimation of the function N(d)
for i=1:1:floor(2*L/h)+1;
Hint(i)=exp(-((x+(r-q+sigma_hat^2/2)*taustar)/...
(sigma_hat*sqrt(taustar)))^2/2)/(sigma_hat*...
sqrt(taustar)*sqrt(2*pi));
x=x+h;
end
n=floor(2*L/h)+1;
x=1.5;
c_1=sigma_hat^2/2;
c_2=-c_1*a/sigma_hat*sqrt(2/pi/k);
H=zeros(n,m);
H(:,1)=Hint';
n=size(H(:,1));
n=n(1,1);
```

```
A=zeros(n);
c_1=k/h;
c_2=r*c_1;
c_3=sigma_hat^2/2;
c_4=-a/sigma_hat*c_3*sqrt(2/pi/k);
c_5=2*b*sigma_hat^2;
for j=1:m
for i=1:n
for l=1:n
if(i==1 && l==1) A(i,l)=1+k/h^2*(c_3+c_4+c_5*...
H(i,j)+c_3);
elseif(i==1 && l==2)A(i,l)=-k/h^2*c_3-k*r/2/h-...
k/2/h*(c_3+c_4+c_5*H(i,j));
elseif(i==l) A(i,l)=1+k/h^2*(c_3+c_4+c_5*H(i,j)+...
c_3+c_4+c_5*H(i-1,j));
elseif(i==l+1)A(i,l)=-k/h^2*(c_3+c_4+c_5*...
H(i-1,j))+k*r/2/h+k/2/h*(c_3+c_4+c_5*H(i,j));
elseif(i==l-1) A(i,l)=-k/h^2*(c_3+c_4+c_5*...
H(i,j))-k*r/2/h-k/2/h*(c_3+c_4+c_5*H(i,j));
end
end
end
H(:,j+1)=linsolve(A,H(:,j));
end
for S=10:1:45
for i=2:floor(2*L/h)
V(S,1)=0;
V(S,i)=V(S,i-1)+h*max(S-E*exp(-L+i*h),0)*H(i,m+1);
end
end
```

The last presented Matlab code is concerning Khaliq-Liao method applied on Leland's model.

```
taustar=0.001;
L=1.5;
E_1=25;
T=1-taustar;
h=0.006;
sigma_hat=0.4;
k=h^2/4^2/sigma_hat^2;
m=floor(T/k);
x=-1.5;
Co=0.00005;
Le=sqrt(2/pi)*Co/(sigma_hat*sqrt(k));
r=0.011;
q=0;
n=floor(2*L/h+1)
for i=1:1:n;

Hint(i)=exp(-((x+(r-q-sigma_hat^2/2)*taustar)/...
(sigma_hat*sqrt(taustar)))^2/2)/(sigma_hat*...
sqrt(taustar)*sqrt(2*pi));
x=x+h;
end
x=1.5;
H=zeros(n,m);
H(:,1)=Hint';
c_1=sigma_hat^2/2;
c_2=c_1*Le;
for i=1:1:(floor(2*L/h)+1)
if(Hint(i)>0) v(1,i)=(c_1+c_2);
elseif(Hint(i)<=0)v(1,i)=(c_1-c_2);
end
end
```

```
H(:,1)=Hint';
A_1=zeros((floor(2*L/h)+1));
A_2=zeros((floor(2*L/h)+1));
B_1=zeros((floor(2*L/h)+1));
B_2=zeros((floor(2*L/h)+1));
C_1=zeros((floor(2*L/h)+1));
C_2=zeros((floor(2*L/h)+1));
D_1=zeros((floor(2*L/h)+1));
D_2=zeros((floor(2*L/h)+1));
E_1=zeros((floor(2*L/h)+1));
E_2=zeros((floor(2*L/h)+1));

A_1=diag(ones((floor(2*L/h)+1),1)*...
(5/6+(c_1-c_2)*k/h^2))+...
diag(ones((floor(2*L/h)+1)-1,1)*...
(1/12-(c_1-c_2)/2*k/h^2),1)+...
diag(ones((floor(2*L/h)+1)-1,1)*...
(1/12-(c_1-c_2)/2*k/h^2),-1);
A_2=diag(ones((floor(2*L/h)+1),1)*...
(5/6+(c_1+c_2)*k/h^2))+...
diag(ones((floor(2*L/h)+1)-1,1)*...
(1/12-(c_1+c_2)/2*k/h^2),1)+...
diag(ones((floor(2*L/h)+1)-1,1)*...
(1/12-(c_1+c_2)/2*k/h^2),-1);

B_1=diag(ones((floor(2*L/h)+1),1)*...
(5/6-(c_1-c_2)*k/h^2))+...
diag(ones((floor(2*L/h)+1)-1,1)*...
(1/12+(c_1-c_2)/2*k/h^2),1)+...
diag(ones((floor(2*L/h)+1)-1,1)*...
(1/12+(c_1-c_2)/2*k/h^2),-1);
B_2=diag(ones((floor(2*L/h)+1),1)*...
(5/6-(c_1+c_2)*k/h^2))+...
diag(ones((floor(2*L/h)+1)-1,1)*...
```

```
(1/12+(c_1+c_2)/2*k/h^2),1)+...
diag(ones((floor(2*L/h)+1)-1,1)*...
(1/12+(c_1+c_2)/2*k/h^2),-1);

C_1=diag(ones((floor(2*L/h)+1),1)*...
(5/6+(c_1-c_2)*k/h^2))+...
diag(ones((floor(2*L/h)+1)-1,1)*...
(1/12-(c_1-c_2)/2*k/h^2),1)+...
diag(ones((floor(2*L/h)+1)-1,1)*...
(1/12-(c_1-c_2)/2*k/h^2),-1);
C_2=diag(ones((floor(2*L/h)+1),1)*...
(5/6+(c_1+c_2)*k/h^2))+...
diag(ones((floor(2*L/h)+1)-1,1)*...
(1/12-(c_1+c_2)/2*k/h^2),1)+...
diag(ones((floor(2*L/h)+1)-1,1)*...
(1/12-(c_1+c_2)/2*k/h^2),-1);

D_1=diag(ones((floor(2*L/h)+1),1)*...
(5/6-(c_1-c_2)*k/h^2))+...
diag(ones((floor(2*L/h)+1)-1,1)*...
(1/12+(c_1-c_2)/2*k/h^2),1)+...
diag(ones((floor(2*L/h)+1)-1,1)*...
(1/12+(c_1-c_2)/2*k/h^2),-1);
D_2=diag(ones((floor(2*L/h)+1),1)*...
(5/6-(c_1+c_2)*k/h^2))+...
diag(ones((floor(2*L/h)+1)-1,1)*...
(1/12+(c_1+c_2)/2*k/h^2),1)+...
diag(ones((floor(2*L/h)+1)-1,1)*...
(1/12+(c_1+c_2)/2*k/h^2),-1);

E_1=diag(ones((floor(2*L/h)+1),1)*...
((-2)*(r+(c_1-c_2))*k/2/h^2))+...
diag(ones((floor(2*L/h)+1)-1,1)*...
((r+(c_1-c_2))*k/2/h^2),1)+...
```

```
diag(ones((floor(2*L/h)+1)-1,1)*...
((r+(c_1-c_2))*k/2/h^2),-1);
E_2=diag(ones((floor(2*L/h)+1),1)*...
((-2)*(r+(c_1+c_2))*k/2/h^2))+...
diag(ones((floor(2*L/h)+1)-1,1)*...
((r+(c_1+c_2))*k/2/h^2),1)+...
diag(ones((floor(2*L/h)+1)-1,1)*...
((r+(c_1+c_2))*k/2/h^2),-1);

for j=1:m
for(i=1:(floor(2*L/h)+1))
A(i,:)=A_1(i,:).*(H(:,j)<=0)'+A_2(i,:).*...
(H(:,j)>0)';
B(i,:)=B_1(i,:).*(H(:,j)<=0)'+B_2(i,:).*...
(H(:,j)>0)';
vec=(c_1-c_2).*(H(:,j)<=0)'+(c_1+c_2).*...
(H(:,j)>0)';
end
if(j==1)
for prem=1:(floor(2*L/h)+1)
if prem==1 v(1,prem)=0;
elseif prem==(floor(2*L/h)+1) v(1,prem)=0;
else v(1,prem)=(Hint(prem+1)-Hint(prem-1))/2/h;
end
end
for prem=1:1:(floor(2*L/h)+1)
if prem==1
jo(j,prem)=(vec(prem)+r)*v(j,prem)+...
(-2*v(j,prem)+v(j,prem+1))/12*(vec(prem)+r);
elseif prem==(floor(2*L/h)+1)
jo(j,prem)=(vec(prem)+r*v(j,prem))+...
(-2*v(j,prem)+v(j,prem-1))/12*(vec(prem)+r);
else jo(j,prem)=(vec(prem)+r)*v(j,prem)+...
(v(j,prem-1)-2*v(j,prem)+v(j,prem+1))/12*...
```

```
(vec(prem)+r);
end
end
H(:,j+1)=linsolve(A,B*H(:,j)+k*jo(j,:)');
else
for prem=1:1:(floor(2*L/h)+1)
if prem==1
jo(j,prem)=(vec(prem)+r)*v(j,prem)+...
(-2*v(j,prem)+v(j,prem+1))/12*(vec(prem)+r);
elseif prem==(floor(2*L/h)+1)
jo(j,prem)=(vec(prem)+r)*v(j,prem)+...
(-2*v(j,prem)+v(j,prem-1))/12*(vec(prem)+r);
else jo(j,prem)=(vec(prem)+r)*v(j,prem)+...
(v(j,prem-1)-2*v(j,prem)+...
v(j,prem+1))/12*(vec(prem)+r);
end
end
H(:,j+1)=linsolve(A,B*H(:,j)+k/2*...
((jo(j,:)'+jo(j-1,:)')));
end
for(i=1:(floor(2*L/h)+1))
C(i,:)=C_1(i,:).*(H(:,j)<=0)'+...
C_2(i,:).*(H(:,j)>0)';
D(i,:)=D_1(i,:).*(H(:,j)<=0)'+...
D_2(i,:).*(H(:,j)>0)';
E(i,:)=E_1(i,:).*(H(:,j)<=0)'+...
E_2(i,:).*(H(:,j)>0)';
end
v(j+1,:)=linsolve(C,D*v(j,:)'+...
E*(H(:,j)+H(:,j+1)));
end
```