Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

Higher Order Finite Difference Schemes for Solving Path Dependent Options

Master's Thesis

Bratislava 2012

Bc. Michal Takáč

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Study Programme:	Economical and Financial Mathematics
Branch of Study:	Applied Mathematics 1114
Department:	Department of Applied Mathematics and Statistics
Supervisor:	prof. RNDr. Daniel Ševčovič, CSc.
Consultant:	Prof. Dr. Matthias Ehrhardt

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Univerzita Komenského v Bratislave
Fakulta Matematiky, Fyziky a Informatiky

Konečno - diferenčné schémy vyššieho rádu pre riešnie dráhovo závislých opcií

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Cieľ: V práci sa budeme zaoberať návrhom a analýzou konečno-diferenčných numerických schém ne riešenie parciálnych diferenciálnych rovníc opisujúcich cenu dráhovo závislej opcie. Dráhovo závislé opcie majú svoj výplatný diagram závislý od histórie vývoja podkladového aktíva. Matematický model je vyjadrený ako parabolická parciálna diferenciálna rovnica s viacerými premennými reprezentujucími nielen podkladové aktívum (cenu akcie) ale aj jeho časovo spriemernenú hodnotu. V práci sa pokúsime ukázať, ze použitím konečno diferenčnej schémy vyššieho (štvrtého) rádu dosiahneme vyššiu presnosť a časovú úsporu pri riešení problému ocenenia opcie.

Vedúci: prof. RNDr. Daniel Ševčovič, CSc.
Katedra: FMFI.KAMŠ - Katedra aplikovanej matematiky a štatistiky

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garant študijného programu

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študent vedúci práce

Declaration on World of Honour

I declare on my honour that this work is based only on my own knowledge, references and consultation with my supervisor(s).

. . . .

Michal Takáč

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Abstract

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In our work we investigate path dependent American options . We focus mainly on the position of the free boundary. We derive the corresponding pricing equation. Then, using several method, we estimate the early exercise boundary in order to make the estimation faster. We explore the advantages of the new method in comparison to the old ones. To demonstrate the usage of these options we also introduce a hedging example.

Keywords: path dependent options • free boundary • numerical methods • splitting algorithms • early exercise • financial derivatives.

Abstrakt

TAKÁČ, Michal: Konečno - diferenčné schémy vyššieho rádu pre riešnie dráhovo závislých opcií [Diplomová práca].
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V práci sa zaoberáme dráhovo závislými Americkými opciami. Odvodíme príslušnú oceňovaciu funckiu. Potom sa sústredíme hlavne na pozíciu voľnej hranice. Použijeme viacero metód pre aproximáciu tejto hranice so zámerom zrýchlenia algoritmu. Ďalej skúmame výhody novej metódy v porovnaní s aktuálne používanými. Uvedieme aj krátky 'hedging' príklad.

Kľúčové slová: dráhovo závislé opcie • voľná hranica • numerické metódy • rozdeľovacie algoritmy • predčasné uplatnenie • finančné deriváty.

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Introduction

In terms of derivatives in financial context, one can refer to a contract which price at given time depends on the value of the underlying asset i.e. any financial contract. An example of an underlying asset can be stocks, exchange rates, commodities such as crude oil, gold, etc. or interest rates. In the last decades, there was a huge expansion of derivative trading on financial markets. Derivative securities have became a successful trading instrument all over the world.

In this thesis we investigate path dependent options. We particularly focus on options with early exercise - American options. This type of options are very lucrative to the end-users of commodities or energies who are tend to be exposed to the average prices over time. Asian options are also very popular with corporations, who have ongoing currency exposures. The main idea of the pricing is to examine the free boundary position, ([21], [16], [10]) on which the value of the option is depending. We focus on developing a efficient and fast numerical algorithm for this boundary.

In the first Chapter, we give an informative description of the financial derivatives. The second Chapter is devoted to the analytical derivation of the corresponding partial differential equation coming from the original Black - Scholes equation. In the third Chapter we describe important numerical methods and discretize the problem. We introduce a new Improved - Strang splitting method and compare it to other used methods. Finally, in the fourth and fifth Chapter we make numerical experiments with the free boundary and we also perform a hedging example.

Chapter 1

The World of Financial Markets

"Sometimes your best investments are the ones you don't make."

Donald Trump

1.1 Financial Derivatives

Financial derivatives are used as a main securing tool against unpredictable movements of financial markets. Examples of derivatives are *forwards*, *futures*, *swaps* and *options*. In the case of forward and futures the asset must be exercised, while in the case of the option this is just a right. A swap is a derivative in which counter-parties exchange certain benefits from their financial instruments for a predefined period of time. Combination of these types are also possible. They might include *compound options*, which are options on options; or *futures options*, where the underlying is a future contract. We follow Takáč [15], Hull [9] and Wilmott [18].

1.1.1 Options

An European call (put) on an underlying asset gives the holder the right, but not the obligation, to buy (sell) the underlying at a predefined price E (strike price or exercise price) at a certain future date T (the maturity). At this time the *writer* of the options is obliged to sell (buy) the underlying from the holder of the options. The purchase value of the option is called the *premium*, and it is payed by the holder to a writer when the contract is sold. The European option can be exercised only at the maturity time. Mathematically, it can be expressed by the following *payoff*[1] function

$$V(S,T) = \begin{cases} V^{CE} = [S(T) - E]^{+}, \\ V^{PE} = [E - S(T)]^{+}, \end{cases}$$

where V^{CE} (V^{PE}) denotes the European call (put) option and by $[S(T)-E]^{+}$ we define $\max[S(T) - E, 0]$.

An American option is an option which can be exercised at any time up to maturity. In the case of the American options the payoff functions, when exercised, are identical with the European type. Because of that, the prices of the American call (V^{CA}) and put (V^{PA}) are bounded from below

$$V(S,t) = \begin{cases} V^{CA} \geq [S(t) - E]^{+}, \\ V^{PA} \geq [E - S(t)]^{+}. \end{cases}$$

Option writers and buyers also, called *option traders*, complete the market together. Therefore, one can take four different positions, see Figure 1.1, on the market namely:

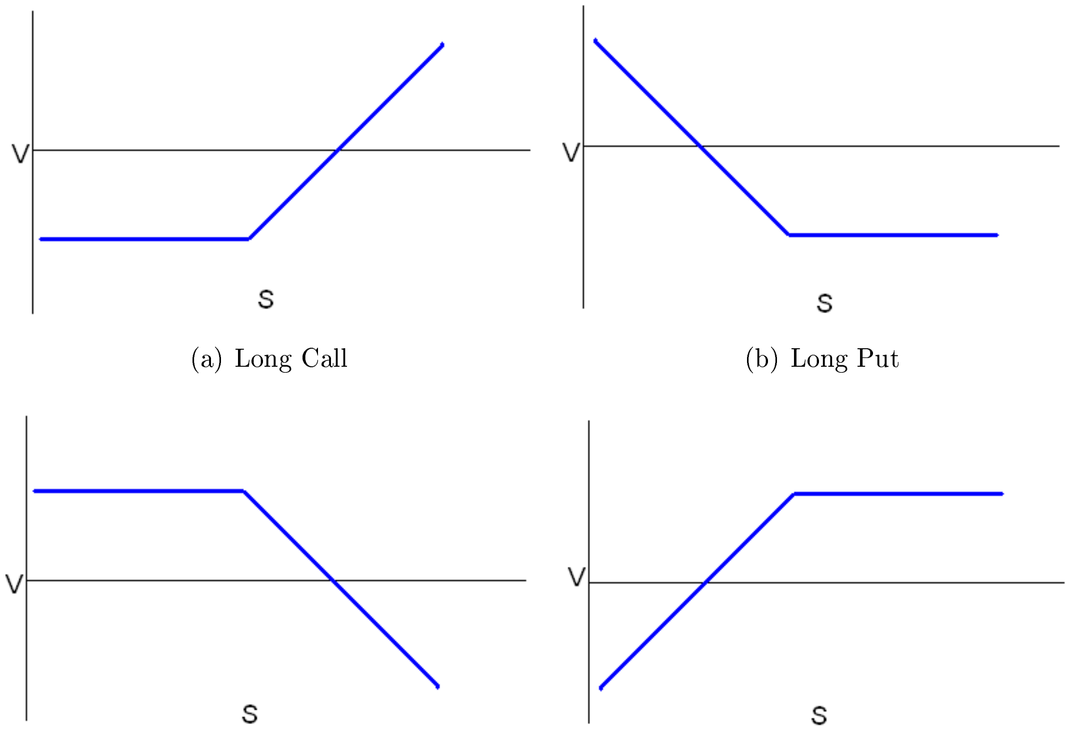
- long call - buy call option,
- short call - sell call option,
- long put - buy put option,
- short put - sell put option.

Simple options as call and put are commonly called *plain vanilla options*. Even though, the plain vanilla options are widely known and used, there are also many different types of options demanded on the market called by the common name **exotic options**.

Exotic option are commonly traded *over-the-counter*[2] (OTC) and their features are making them more complex compared to the plain vanilla options. They can differ in many ways such as they can depend on more underlying assets (*basket options*), the price can depend not just on the the current asset price (*path-dependent options*) etc. From above mentioned, the most commonly used are the path-dependent options.

[1]Payoff in a financial context is the income or profit arising from a certain transaction.

[2]OTC trading is to trade financial instruments between two parties.



(a) Long Call (b) Long Put

(c) Short Call (d) Short Put

Figure 1.1: Graphical illustration of option positions. In order : long call, long put, short call and short put.

The most frequent are

- Asian options - the price of the option depends on the averaged asset price during the lifetime of the option,
- Barrier option - the option is either activated or extinguished upon the occurrence of the event of the underlying price reaching a predefined barrier,
- Chooser option - gives the holder a predefined time to decide the type of the option (call or put),
- Lookback options - the price of the option depends on the maximum (minimum) of asset price through the options lifetime.

In this work we focus on Asian and Lookback options which are briefly described in Sections 1.2 and 1.3.

1.1.2 Option Pricing

An inseparable part of derivative products is their pricing procedure. The model developed by Black and Scholes [2] and independently by Merton [12] has brought a completely new perspective to the financial world. In spite of the strict assumptions the model and its variations are widely used as a main mathematical model of financial markets and derivative instruments.

Assuming that the movement of the underlying asset follows the *Geometrical Brownian Motion*[3] (GBM)

$$\frac{dS}{S} = \mu dt + \sigma dX,$$

where S is the asset price, μ is the drift term and σ the volatility of the stock return. By the term X we denote the standard Wiener process. Taking into consideration the following assumptions

- the risk-free interest rate r and the volatility σ are known functions,
- there are no transaction costs associated with hedging a portfolio,
- the underlying asset pays no dividends during the life of the option,
- there are no arbitrage possibilities,
- trading of the underlying asset can bee take place continuously,
- short selling is permitted,
- the assets are infinitely divisible,

the following backward-in-time partial differential equation can be derived, Wilmott [18],

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0, \quad 0 \le t \le T.$$

The solution of this PDE using the final condition $V(S,T) = (S-E)^+$, i.e. the price of the European non-dividend paying call option is given in the explicit form Wilmott [18]

[3] GMB is a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion, also called a Wiener process.

$$V^{PE}(S,t) = SN(d_1) - Xe^{-r(T-t)}N(d_2),$$

$$d_{1,2} = \frac{\ln\frac{S}{E} + (r \pm \frac{\sigma^2}{2})T - t}{\sigma\sqrt{T-t}},$$

where $N(.)$ is a cumulative normal distribution function with $\mu = 0$ and $\sigma = 1$. The price of the European non-dividend paying put is calculated similarly with the final condition $V(S,T) = (E-S)^+$.
In the last years many extensions have been made to the model. The model is versatile and capable to adapt for the case of the dividend paying underlying asset, variable interest rates and volatilities,transaction costs and also for the American case, even though their valuation is different.

1.2 Asian Options

Asian options are path-dependent options. The payoff of these options depends not only on the current price of underlying asset, but also on some of its average over a specified time period. The main advantage of Asian options is their price, which is less than its plain vanilla alternative. Asian options are often used as a hedge tool against unexpected movements in asset prices i.e averaging reduces the susceptibility to price manipulation. An example could be a crude oil consumer who is afraid of price increase in future. He prefers to have his crude oil supplies for the price equal to the average of last few weeks. His requirements can be satisfied by a special type of Asian options. These option were first introduced in Tokyo of Banker's Trust in 1987 issued on already mentioned crude oil contracts Zhang [20].

As is it was already mentioned Asian options are perfect hedging tools for the energy derivatives especially for the crude oil market. As for other use, these options can also be a good equivalent for the traditional *FX options*[4]. The floating strike Asian option is particularly appropriate when the foreign currency cashflows being hedged are regular and expected over a defined period of time. It provides a general, rather then

[4]FX option is a derivative financial instrument that gives the owner the right but not the obligation to exchange money denominated in one currency into another currency at a pre-agreed exchange rate on a specified date.

a specific hedge against adverse currency moves. They usually come in hand where an investor or a company want to hedge a series of cashflows and individual options are to expensive to manage. For its low premium the floating strike Asian option can also be considered as position taking tool, when the view is that the currency will be higher or lower then its average at the end of the period. As a short example consider a holder of a currency payable. He would purchase an average strike call options to hedge against currency appreciation. By this, he is allowing potential to benefit from the currency's depreciation. Once all the exchange are given and the strike has been determined, the option buyer has unlimited protection. If the option expire out-of-the-money, the holder's maximum loss is limited to the premium paid.

There are many variations of Asian options depending on how the payoff function is defined and what input variables are used. In the first case the type of averaging should be discussed which can be either arithmetic or geometric. It is convenient to use *geometric average* if the underlying asset behaves according to the geometrical Brownian motion. In this case the problem can be transformed into the classical heat equation. On the other hand it is the *arithmetic average* which is used in real world, even though its valuation is more difficult. The sampling of both arithmetic and geometric average can be either continuous or discrete. While the discrete type of sampling is used in the real world, it is more convenient to use the continuous from a mathematical point of view. Then the average in the case of continuous-time models for geometric and arithmetic average are given respectively by

$$\begin{aligned}
\overline{S}^A(t) &= \frac{1}{t}\int_0^t S(u)du, \\
\overline{S}^G(t) &= \exp\left[\frac{1}{t}\int_0^t \ln S(u)du\right].
\end{aligned}$$

For the discrete average, where N denotes the number of equidistant averaging points, the average process are expressed respectively by

$$\begin{aligned}
\overline{S}_d^A(N) &= \frac{1}{N}\sum_{n=1}^{N} S(t_n), \\
\overline{S}_d^G(N) &= \exp\left[\frac{1}{N}\sum_{n=1}^{N} \ln(S(t_n))\right].
\end{aligned}$$

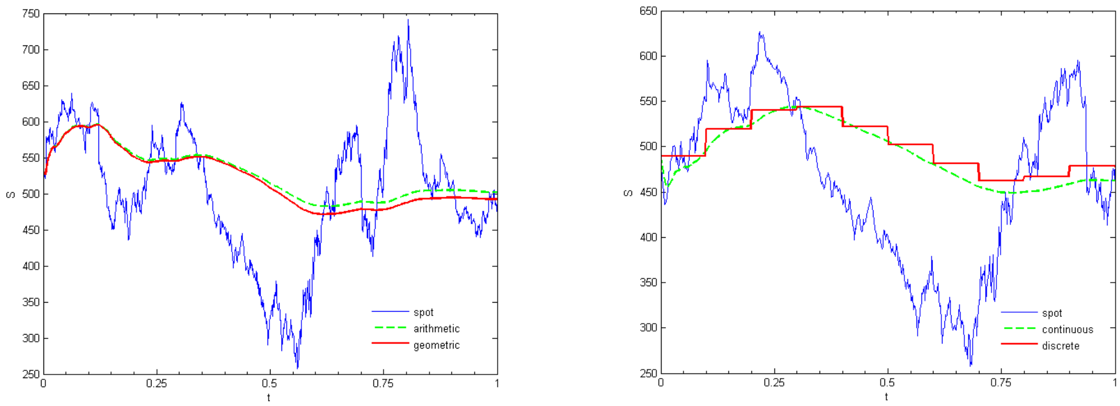


Figure 1.2: The Example of the time development of *Microsoft Corporation* stock price and corresponding type of continuous average (on the left) and the difference between continuous and discrete average (on the right). Source: http://finance.yahoo.com.

We can particularly divide Asian options depending on the form of the payoff function into two main categories:

- **the average strike** options, also known as **the floating strike** options, payoff is given by a difference between the spot price at maturity time and the strike price calculated as an average of the underlying during the specified time interval

$$V(S,A,T) = \begin{cases} V^{CS} = [S(T) - \overline{S}(T)]^+, \\ V^{PS} = [\overline{S}(T) - S(T)]^+, \end{cases}$$

- **the average rate (fixed strike)** options payoff is defined as a difference between the average price of underlying and a predefined strike price

$$V(S,A,T) = \begin{cases} V^{CS} = [\overline{S}(T) - E]^+, \\ V^{PS} = [E - \overline{S}(T)]^+. \end{cases}$$

As for the plain vanilla European option, there exist as well an American alternative for the European Asian options. **Hawaiian options** are options with the early-exercise feature, also called as American-style Asian options. The holder of these options can exercise not just in the maturity time but at any time during the lifetime of the contract.

The received payoff is derived from the average up to the exercise time. Unsurprisingly, all the characteristic features introduced for the European case can be adapted to the American style options.

1.3 Lookback Options

A Lookback option is a derivative product whose payoff depends on the maximum or minimum realised price during the life time of the option. The price is calculated as the difference between the maximum price and the spot price at expiry for the case of the put option. The case of the call is slightly different and the payoff is given as the difference between the spot price and the minimum price that takes place during its lifetime. As in the other cases, here also, the maximum and minimum is most commonly measured discretely. This options gives the holder an incredible advantageous payoff. Because such options enables the investor to buy low and sell high for the put option (vice versa for the call), their cost is relatively high. However, this type of options may not always be the distinct advantage to buy, because of their price. Assume the spot price S and the maximum (minimum) realised price J, then the payoff *V(S,J,T)* of a put and call respectively can be given in the form

$$V(S,J,T) = \begin{cases} V^{PB} = [J-S]^+, \\ \\ V^{CB} = [S-J]^+, \end{cases}$$

where $J = \max S(\tau)$ and $\tau \in [0,T]$.
Here the maximum (minimum) value of the spot price S can be interpreted in the integral from

$$J = \begin{cases} J^{max} = \lim_{p\to\infty} \left(\int_0^T (S(\tau))^p \right)^{\frac{1}{p}}, \\ \\ J^{min} = \lim_{p\to-\infty} \left(\int_0^T (S(\tau))^p \right)^{\frac{1}{p}}. \end{cases}$$

To give some credit to the discrete measurement of the maximum and minimum it is necessary to mention two advantages. It is easier to measure by this method and the contract can became cheaper as we reduce the frequency at which the spot price is measured. The obtained pricing formula for Lookback options is the same for the

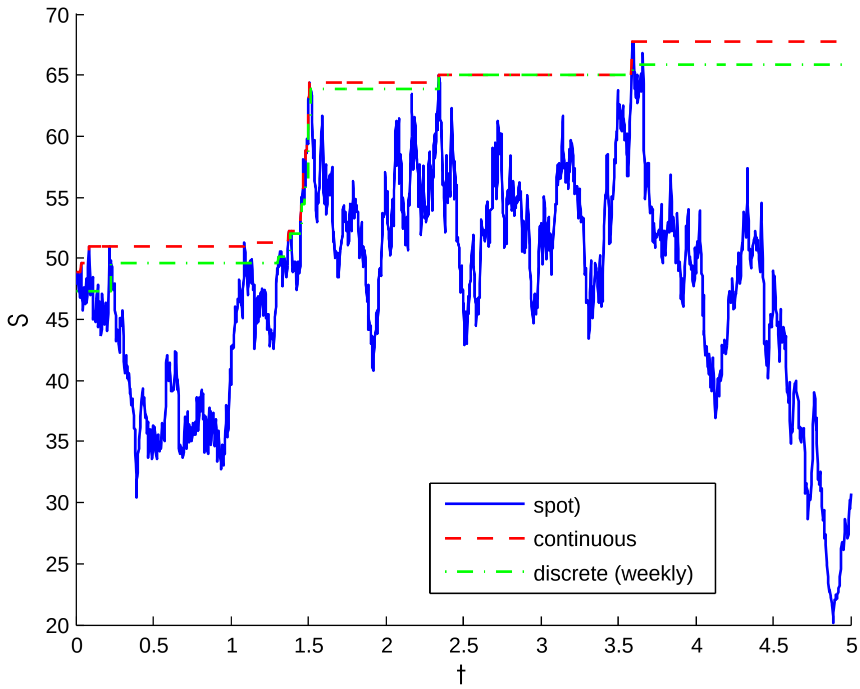


Figure 1.3: The development of the spot price and the corresponding maximum sampled continuously and discretely (own simulation).

discrete and continuous sampling (see Figure 1.3). The same payoff holds for the American type of Lookback options with the difference that option can be exercised at any time of its lifetime. Hence the payoff is given at the time t, $t \in [0, T]$, where t is the time of the early exercise.

Chapter 2

The Analytical Derivation

In this Chapter we discuss transformation methods for pricing Asian options proposed by Ševčovič & Bokes & Takáč([5], [16], [13], [17]). We particularly focus on path-dependent options such as arithmetically and geometrically averaged Asian and Lookback options. Asian with arithmetic and geometric average and also on the Lookback options[1]. We explore the free boundary problem arising from the equation.

2.1 Transformational Method

In this section we shall consider the price dynamics driven by the GBM in the following form

$$dS = (r - q)Sdt + \sigma SdX, \tag{2.1}$$

where r is the risk free interest rate, $q > 0$ is the continuous dividend yield and σ denotes the volatility. By the term X we denote the standard Wiener process. As we already discussed, the floating strike Asian option with arithmetic average is a financial instrument, which depends not only on the stock price S and maturity time T, but also on the average A. Thus, the price can be written as a function $V(S, A, t)$. Applying Itô's lemma (see Appendix 1) we get the following expression

$$dV = \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial A}dA + \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt. \tag{2.2}$$

[1]We refer to the Lookback option as a floating strike Asian option in further, since their pricing equation can be interpreted in common, general form.

The arithmetic average $A = \frac{1}{t}\int_0^t S(u)du$ yields the differential equation

$$\frac{dA}{dt} = \frac{dA_t}{dt} = \frac{1}{t}S_t - \frac{1}{t^2}\int_0^t S_\tau d\tau = \frac{S_t - A_t}{t} = \frac{S - A}{t} = A\frac{\frac{S}{A} - 1}{t}. \tag{2.3}$$

For the geometric average, $A = \exp\left[\frac{1}{t}\int_0^t \ln S(u)du\right]$, on the other hand, we the differential equation:

$$\frac{dA}{dt} = \frac{dA_t}{dt} = \frac{1}{t}\ln S_t - \frac{1}{t^2}\int_0^t \ln S_\tau d\tau A = \frac{\ln S_t - \ln A_t}{t}A = A\frac{\ln\frac{S}{A}}{t}. \tag{2.4}$$

Special is the case of the looback options. Here as was already mentioned the maximum (minimum) price throughout the life time of the option is observed. Notice that the maximum (minimum) A, can be calculated as $A = \left(\frac{1}{t}\int_0^t S^p(u)du\right)^{\frac{1}{p}}$, as $p \to \infty(-\infty)$. Hence:

$$\frac{dA}{dt} = \frac{dA_t}{dt} = \left(\frac{1}{t}\int_0^t S_\tau^p d\tau\right)^{\frac{1}{p}-1}\left(\frac{1}{t}S_t^p - \frac{1}{t^2}\int_0^t S_\tau^p d\tau\right) = A\frac{(\frac{S}{A})^p - 1}{pt}. \tag{2.5}$$

As a matter of fact, if we take the limit of (2.5) as $p \to \infty(-\infty)$ when A represents the maximum (minimum), the expression $\left(\frac{S}{A}\right)^p \to 0$. Hence, the whole expression (2.5) limits to 0 for both the maximum and minimum case. As a conclusion, one can see that for every single case regarding the averaging and a Lookback type dA can be transformed and written as general function of two variables

$$\frac{dA}{dt} = Af(\frac{S}{A}, t). \tag{2.6}$$

Substituting (2.1) and (2.6) into equation (2.2) we obtain the differential equation for the price process $V(S, A, t)$

$$dV = \left(\frac{\partial V}{\partial t} + (r - q)S\frac{\partial V}{\partial S} + \frac{\partial^2 V}{\partial S^2} + Af\left(\frac{S}{A}, t\right)\frac{\partial V}{\partial A}\right)dt + \sigma S\frac{\partial V}{\partial S}dX, \tag{2.7}$$

where $0 < t < T$ and S, $A > 0$. We consider now a portfolio Π, which consists of one derivative (option) and $-\Delta$ of underlyings. The derivative $d\Pi$ of this portfolio, so the one time step change in the case of the dividend paying underlying asset is

$$d\Pi = dV - \Delta dS - q\Delta S dt. \tag{2.8}$$

We consider here Δ being a constant during one time step. Now, finally putting (2.1), (2.7), (2.8) together we obtain

$$d\Pi = \left[\frac{\partial V}{\partial t} + (r-q)S\left(\frac{\partial V}{\partial S} - \Delta\right) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + Af\left(\frac{S}{A}, t\right)\frac{\partial V}{\partial A} - q\Delta S\right]dt + \sigma S\left[\frac{\partial V}{\partial S} - \Delta\right]dX. \tag{2.9}$$

In order to get rid off the uncertainty caused by the term dX in our portfolio we shall choose $\Delta = \frac{\partial V}{\partial S}$. By this setting we can eliminate the randomness present in our portfolio through the asset price process, which is driven by the Brownian motion. This move is a so called *delta hedging*. Because of the fact of arbitrage opportunities we shall consider a risk-free investment into a riskless asset. An investment of the amount Π into this asset would bring a growth

$$d\Pi = r\Pi dt \tag{2.10}$$

in one time step. Any other deterministic growth would arise in an arbitrage opportunity. Thus (2.9) and (2.10) should be equal. Using this equality, and dividing by dt we obtain a PDE for the floating strike Asian option

$$\frac{\partial V}{\partial t} + (r-q)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + Af\left(\frac{S}{A}, t\right)\frac{\partial V}{\partial A} - rV = 0. \tag{2.11}$$

For the American type of options we have to develop also boundary conditions. According to Kwok [11] - [6] we denote the early exercise boundary of the call option as $S_f(A,t)$ and describe the early exercise region by

$$\varepsilon = \{(S,A,t) \in [0,\infty) \times [0,\infty) \times [0,T), S \geq S_f(A,t)\}. \tag{2.12}$$

For the call option the first two conditions arise from the European types. The terminal condition at time T and the homogeneous Dirichlet condition at $S = 0$

$$V(S,A,T) = (S-A)^+, \quad V(0,A,t) = 0. \tag{2.13}$$

As the option price reaches the early exercise (free) boundary one can determine the price of the option from the payoff function at that moment. The slope of the option with respect to the price S, $\frac{\partial C}{\partial S}$ at the free boundary $S_f(A,t)$ should be equal 1.

This guarantees that the option value is connected to the payoff function arising from the early exercise of the option smoothly, ensuring us no arbitrage opportunity. The boundary conditions following this arguments can be written as

$$V(S_f(A,t),A,t) = S_f(A,t) - A, \quad \frac{\partial V}{\partial S}(S_f(A,t),A,t) = 1. \tag{2.14}$$

Thus we obtain a two-dimensional PDE. Fortunately, there exist a transformation method using similarities for floating strike Asian option, which reduces the dimension of this problem. Using the new variables

$$x = \frac{S}{A}, \quad \tau = T - t, \quad W(x,\tau) = \frac{1}{A}V(S,A,t), \tag{2.15}$$

the equation (2.11) can be transformed to the following parabolic PDE:

$$\frac{\partial W}{\partial \tau} - (r-q)x\frac{\partial W}{\partial x} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 W}{\partial x^2} - f(x, T-\tau)\left(W - x\frac{\partial W}{\partial x}\right) + rW = 0. \tag{2.16}$$

The early exercise boundary S_f can be also reduced to a one dimensional variable $x_f(t) = S_f(A,t)/A$. To obtain a spatial domain for the equation (2.16) we introduce a new variable $\rho(\tau) = x_f(T-\tau)$. Further $W(x,\tau)$ is the solution of this equation for $x \in (0, \rho(\tau))$, $\tau \in (0,T)$. From (2.13) and (2.14) we can determine the new initial and boundary conditions

$$W(x,0) = (x-1)^+, \quad \forall x > 0, \tag{2.17}$$

respectively

$$W(0,\tau) = 0, \quad W(\rho(\tau),\tau) = \rho(\tau) - 1, \quad \frac{\partial W}{\partial x}(\rho(\tau),\tau) = 1. \tag{2.18}$$

For the equation one can simply derive the $f(x, T-\tau)$ using the dimension reduction $x = \frac{S}{A}$ and $\tau = T - t$:

$$f(x, T-\tau) = \begin{cases} \frac{x-1}{T-\tau}, \\ \\ \frac{\ln x}{T-\tau}, \\ \\ \frac{x^p - 1}{p(T-\tau)}. \end{cases} \tag{2.19}$$

To estimate the limit of the early exercise boundary close to the expiry $\rho(0^+)$ we shall use the linear complementarity problem and variational inequality for the American types of options following Kwok [11]

For the case of a floating strike Asian call option suppose the the holder receives the payoff $\phi(x) = (x-1)^+$. Assume that $W(x,\tau)$ solves the PDE for the floating strike Asian option. Then we have:

$$\frac{\partial W}{\partial \tau} - (r-q)x\frac{\partial W}{\partial x} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 W}{\partial x^2} - f(x,T-\tau)\left(W - x\frac{\partial W}{\partial x}\right) + rW \geq 0,$$

$$\tau \in [0,T), x \in [0,\infty). \tag{2.20}$$

$$W - \phi \geq 0, \tau \in (0,T), x \in (0,\infty) \tag{2.21}$$

$$\Bigg\{\underbrace{\frac{\partial W}{\partial \tau} - (r-q)x\frac{\partial W}{\partial x} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 W}{\partial x^2} - f(x,T-\tau)\left(W - x\frac{\partial W}{\partial x}\right) + rW}_{\mathcal{L}_{x,\tau}W}\Bigg\}$$

$$\times \Bigg\{\underbrace{W - (x-1)}_{g(x,\tau)}\Bigg\} = 0, \tag{2.22}$$

while $\mathcal{L}_{x,\tau}W \geq 0$ and $g(x,\tau) \geq 0$ in $(0,\infty) \times (0,T)$.

From the condition $W(0,\tau) = (x-1)^+$ it is straightforward that in the exercise region $W = x-1$. Substituting this to the equation (2.16), we obtain an inequality for the stopping region

$$\frac{\partial (x-1)}{\partial \tau} - (r-q)x\frac{\partial (x-1)}{\partial x} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 (x-1)}{\partial x^2} - f(x,T-\tau)\left(x-1-x\frac{\partial (x-1)}{\partial x}\right) + r(x-1)$$

$= qx + f(x,T-\tau) - r \geq 0.$

Along with the non-negativity of the final exercise payoff $x \geq 1$ we have

$$x(0^+) \geq \max\{\hat{x}, 1\},$$

where $\hat{x}$ is the solution of the function :

$$qx + f(x,T-\tau) - r = 0.$$

Now, to reduce the inequality to an equality, we assume, that there exist an x in the continuation region such that $x > \max\{\hat{x}, 1\}$. In the continuation region $W(x, 0^+) = x - 1$ and

$$\left.\frac{\partial W}{\partial \tau}\right|_{\tau \to 0^+} = -\left[qx + f(x, T - \tau) - r\right] < 0,$$

but this leads to a contradiction with $\left.\frac{\partial W}{\partial \tau}\right|_{\tau \to 0^+} > 0$ from $g(x, \tau) \geq 0$ and $W(x, 0) = x - 1$ ($W(x, \tau)$non-decreasing in τ). We finally deduce, that

$$x(0^+) = \max\{\hat{x}, 1\},$$

where $\hat{x}$ solves the equation

$$qx + f(x, T - \tau) - r = 0.$$

For each different case we can derive the form of the exercise boundary close to expiry $\rho(0^+)$ and conclude

- Arithmetic average

$$\rho(0^+) = \max\left\{\frac{1 + rT}{1 + qT}, 1\right\}. \tag{2.23}$$

- Geometric average

$$\rho(0^+) = \max\{\hat{x}, 1\}, \tag{2.24}$$

  where $\hat{x}$ solves the equation $qxT + \ln x - rT = 0$.

- Lookback type (minimum)

$$\rho(0^+) = \max\left\{\frac{r}{q}, 1\right\}. \tag{2.25}$$

Bokes in [3] used a different approach to determine the $\rho(0^+)$. Using the pricing of American type of derivatives with an approach of summing the value of the European type and the American bonus function, he determined a general analytic form of the the early exercise boundary at expiry. This results correspond with the values in this thesis.

2.1.1 Fixed Domain Transformation

In this section, we present a fixed domain transformation of the free boundary problem. The idea is to transform the problem into a nonlinear parabolic equation on a fixed domain. Following Ševčovič [16] and Bokes [5], we use a new variable ξ and a new auxiliary function representing a synthetic portfolio

$$\xi = \ln \frac{\rho(\tau)}{x}, \qquad \Pi(\xi,\tau) = W(x,\tau) - x\frac{\partial W}{\partial x}(x,\tau). \tag{2.26}$$

Now, if we assume that $W(x,\tau)$ is a smooth solution of (2.16) we can differentiate it with respect to x and multiply the result by x. In the following we substract the result from (2.16) and obtain

$$\frac{\partial W}{\partial \tau} - x\frac{\partial^2 W}{\partial \tau \partial x} - (r-q+\frac{1}{2}\sigma^2)x^2\frac{\partial^2}{\partial x^2} - f(x,T-\tau)x^2\frac{\partial^2 W}{\partial x^2} + \frac{1}{2}\sigma^2 x^3\frac{\partial^3 W}{\partial x^3}$$

$$+\frac{1}{T-\tau}\left(W - x\frac{\partial W}{\partial x}\right) + r\left(W - x\frac{\partial W}{\partial x}\right) = 0. \tag{2.27}$$

From the used new variables (2.26), we can derive the following equations

$$\frac{\partial \Pi}{\partial \xi} = x^2\frac{\partial^2 W}{\partial x^2}, \quad \frac{\partial^2 \Pi}{\partial \xi^2} + 2\frac{\partial \Pi}{\partial \xi} = -x^3\frac{\partial^3 W}{\partial x^3}, \quad \frac{\partial \Pi}{\partial \tau} + \frac{\dot\rho}{\rho}\frac{\partial \Pi}{\partial \xi} = \frac{\partial W}{\partial \tau} - x\frac{\partial^2 W}{\partial x \partial \tau}.$$

Substituting into equation (2.26) we finally obtain the parabolic PDE in terms of $\Pi(\xi,\tau)$

$$\frac{\partial \Pi}{\partial \tau} + a(\xi,\tau)\frac{\partial \Pi}{\partial \xi} - \frac{1}{2}\sigma^2\frac{\partial^2 \Pi}{\partial \xi^2} + b(\xi,\tau)\Pi = 0, \tag{2.28}$$

where $\xi \in (0,\infty)$, $\tau \in (0,T)$ and $a(\xi,\tau)$ is the function of the ρ in the form

$$a(\xi,\tau) = \frac{\dot\rho(\tau)}{\rho(\tau)} + (r-q) - \frac{1}{2}\sigma^2 - f(\rho e^{-\xi}, T-\tau). \tag{2.29}$$

Moreover, the function $b(\xi,\tau)$ is represented as:

$$b(\xi,\tau) = r + x\frac{\partial f}{\partial x} - f(x,T-\tau)|_{x=\rho e^{-\xi}}. \tag{2.30}$$

In the process of determining initial conditions we use (2.17) and obtain

$$\Pi(\xi,0)=\begin{cases}-1, & \xi<\ln\rho(0),\\[2ex] 0, & \xi>\ln\rho(0).\end{cases} \tag{2.31}$$

For the case of boundary conditions we use our knowledge from (2.18) and impose Dirichlet conditions for $\Pi(\xi,\tau)$

$$\Pi(0,\tau)=-1, \qquad\qquad \Pi(\infty,\tau)=0. \tag{2.32}$$

Since $W(\rho(\tau),\tau)=\rho(\tau)-1$ and $\frac{\partial W}{\partial x}(\rho(\tau),\tau)=1$, we can easily conclude, that $\frac{\partial W}{\partial \tau}(\rho(\tau),\tau)=0$. Assuming C^2-continuity of the function $\Pi(\xi,\tau)$ up to the boundary $\xi=0$ we obtain

$$x^2\frac{\partial^2 W}{\partial x^2}(x,\tau)\longrightarrow\frac{\partial\Pi}{\partial\xi}(0,\tau),\qquad x\frac{\partial W}{\partial x}(x,\tau)\longrightarrow\rho(\tau)\quad as\quad x\longrightarrow\rho(\tau). \tag{2.33}$$

Passing to the limit $x\longrightarrow\rho(\tau)$ in equation (2.16), we end up with an algebraic relation between the free boundary position $\rho(\tau)$ and the boundary trace $\frac{\partial\Pi}{\partial\xi}(0,\tau)$

$$-(r-q)\rho(\tau)-\frac{1}{2}\sigma^2\frac{\partial\Pi}{\partial\xi}(0,\tau)+f(\rho(\tau),T-\tau)+r[\rho(\tau)-1]=0. \tag{2.34}$$

2.1.2 An Equivalent Form of the Free Boundary

Ševčovič [16] used the expression (2.34) to determine a nonlocal algebraic formula for the free boundary position. This result contains the value of $\frac{\partial\Pi}{\partial\xi}(0,\tau)$, which causes in case of small inaccuracy an computational error in the whole domain of $\xi\in(0,\infty)$. Therefore, this equation is not suitable for a robust numerical approximation scheme. Bokes and Ševčovič [5] suggested an equivalent form of the free boundary $\rho(\tau)$, which was proved to be a more robust scheme from the numerical point of view. They integrated the equation (2.28) with respect to ξ on the domain $\xi\in(0,\infty)$

$$\frac{d}{d\tau}\int_0^\infty\Pi d\xi+\int_0^\infty a(\xi,\tau)\frac{\partial\Pi}{\partial\xi}d\xi-\frac{1}{2}\sigma^2\int_0^\infty\frac{\partial^2\Pi}{\partial\xi^2}d\xi+\int_0^\infty b(\xi,\tau)\Pi d\xi.$$

Now, using boundary conditions (2.32) and the algebraic equation (2.34) they derived the following differential equation

$$\frac{d}{d\tau}\left[\ln\rho(\tau)+\int_0^\infty\Pi d\xi\right]+q\rho(\tau)-q-\frac{1}{2}\sigma^2+\int_0^\infty\left[r-f(\rho(\tau)e^{-\xi},T-\tau))\Pi d\xi\right]=0. \tag{2.35}$$

2.1.3 The Backward Transformation

The pricing equation for the American type of Asian call and Lookback can be derived using a backward transformation of the equation (2.26). This equation can be modified to

$$\frac{\partial}{\partial x}\left(\frac{W(x,\tau)}{x}\right) = -x^2\Pi(\xi,\tau). \tag{2.36}$$

Integrating this equation with respect to x on the domain $[x, \rho(\tau)]$ yields

$$\frac{\rho(\tau)-1}{\rho\tau} - \frac{W(x,\tau)}{x} = -\int_x^{\rho(\tau)} x^{-2}\Pi(\xi,\tau)dx. \tag{2.37}$$

Let us recall, that a transformation $\xi = \ln\frac{\rho(\tau)}{x}$ was used from where $x = e^{-\xi}\rho(\tau)$. Substituting back we have

$$W(x,\tau) = \frac{1}{\rho(\tau)}\left[\rho(\tau) - 1 + \int_0^{\ln\frac{\rho(\tau)}{x}} e^{\xi}\Pi(\xi,\tau)d\xi\right]. \tag{2.38}$$

Finally, applying the series of transformations (2.15), the price of the contract depending on the position of the free boundary $\rho(T-t)$ follows the equation

$$V(S,A,t) = \frac{A}{\rho(T-t)}\left[\rho(T-t) - 1 + \int_0^{\ln\frac{A\rho(T-t)}{S}} e^{\xi}\Pi(\xi,\tau)d\xi\right]. \tag{2.39}$$

2.2 Put Option

Following the same logic as in the section 2.1 but with the boundary conditions for the put option one can derive the corresponding pricing equation for the American type floating strike Asian put option. As the procedure does not differ at all in the fundamental way, we show the final obtained PDE with corresponding boundary and initial conditions.

$$\frac{\partial\Pi}{\partial\tau} + a(\xi,\tau)\frac{\partial\Pi}{\partial\xi} - \frac{1}{2}\sigma^2\frac{\partial^2\Pi}{\partial\xi^2} + b(\xi,\tau)\Pi = 0, \tag{2.40}$$

where the coefficient $a(\xi,\tau)$ and $b(\xi,\tau)$ are given in the form

$$a(\xi,\tau) = \frac{\dot\rho(\tau)}{\rho(\tau)} + (r-q) - \frac{1}{2}\sigma^2 - f(\rho e^{-\xi}, T-\tau),$$

$$b(\xi,\tau) = r + x\frac{\partial f}{\partial x} - f(x, T-\tau)|_{x=\rho e^{-\xi}}.$$

The initial and boundary conditions for the put options:

$$\begin{aligned}
\Pi(\xi,0) &= \begin{cases} 1, & \xi > \ln\rho(0), \\ 0, & \xi < \ln\rho(0), \end{cases} \\
\Pi(0,\tau) &= \quad 1, \\
\Pi(\xi,\tau) &= \quad 0, \;\; \xi \longrightarrow \infty.
\end{aligned} \tag{2.41}$$

The equivalent form of the free boundary also depends on the boundary conditions, i.e. for the case of the put options it takes a different form:

$$\begin{aligned}
0 = \frac{d}{d\tau}\left[\ln\rho(\tau) - \int_0^{-\infty}\Pi d\xi\right] + q\rho(\tau) - q - \frac{1}{2}\sigma^2 \\
- \int_0^{-\infty}\left[r - f(\rho(\tau)e^{-\xi}, T-\tau))\Pi d\xi\right]\Pi d\xi.
\end{aligned} \tag{2.42}$$

The procedure of the linear complementarity problem (2.20 -2.22) for the call option can be used also for the the case of the American type of floating strike Asian put option. The aforementioned variational inequality with the final payoff $W(x,0) = (1-x)^+$ result for different cases similarly, but with minimum condition. Hence for the $\rho(0^+)$ in the case of the put options holds:

- Arithmetic average

$$\rho(0^+) = \min\left\{\frac{1+rT}{1+qT}, 1\right\}. \tag{2.43}$$

- Geometric average

$$\rho(0^+) = \min\{\hat{x}, 1\}, \tag{2.44}$$

  where $\hat{x}$ solves the equation $qxT + \ln x - rT = 0$.

- Lookback type (maximum)

$$\rho(0^+) = \min\left\{\frac{r}{q}, 1\right\}. \tag{2.45}$$

Chapter 3

The Numerical Treatment of The Problem

In Chapter 2 we reduce the dimension of the corresponding pricing equation and also eliminate the dependence on the free boundary on computational domain. In this chapter, we introduce the used numerical techniques i.e. the finite difference method, splitting techniques and numerical integration. Then, the numerical treatment of the pricing equation is performed with the described methods (see also Takáč [15]).

3.1 Numerical Methods

The Finite Difference Methods

In general, the finite difference methods are used to solve differential equations using finite difference quotients to numerically approximate the derivative terms. These techniques are used especially for boundary values problems. The finite-differences can be obtained either form the limiting behaviour or from Taylor's expansion of the function. To construct and solve a finite-difference scheme for a differential equation we need to define and generate a set of points, where the numerical approximation will exist. It is usually done by dividing the domain $-\infty < a < b < \infty$ into $N+1$ subintervals as following: $a = x_0 < x_1 \ldots < x_N = b$. The set $\{x_0, x_1 \ldots x_N\}$ is called the **grid**. We denote the step size between two points by $h_i = x_i - x_{i-1}$. If all step size have the same length we refer to the discrete uniform grid of the interval $[a, b]$.

Therefore, we can write $h = (b-a)/N$. In this work, all the discretizations are uniform.

The error of the solution is defined as a difference between the exact and numerical solution. The error term is caused either by computer rounding (*round-off error*) or the discretization procedure (*truncation error*). We are particularly interested in the local truncation error which refers to the error arising from a single application of the method. To determine the truncation error the reminding term from the Taylor's expansion can be used. Usually, it is written in terms of $O(h^i)$ where $i = 1, 2, \dots, N$ is the order of the truncation error.

The most commonly used first order finite-difference quotients to approximate the first order derivatives of the function $u(x)$ are:

- The forward finite-difference
$$D_+u(x) = \frac{u(x+h) - u(x)}{h} + O(h), \tag{3.1}$$
- The backward finite-difference
$$D_-u(x) = \frac{u(x) - u(x-h)}{h} + O(h), \tag{3.2}$$
- The central finite-difference
$$D_0u(x) = \frac{u(x+h) - u(x-h)}{2h} + O(h^2). \tag{3.3}$$

From the family of higher order derivatives we mention just the most common finite difference formula of the second order derivative. The formula can be derived using (3.1) and (3.2):

- The central finite-difference
$$D_0^2u(x) = D_+D_-u(x) = \frac{u(x-h) - 2u(x) + u(x+h)}{h^2} + O(h^2). \tag{3.4}$$

The Operator Splitting Methods

The idea of the method is to solve complex models by splitting it into a sequence of sub-models, which are comparably simpler to solve. Physical processes like convection or diffusion are usually simulated. As every numerical treatment the operator splitting produces an error term as well. By increasing the order of the splitting we can obtain higher numerical precision linked with higher computational time. In this we work refer to a time splitting techniques often called as fractional steps method Yanenko [19]. About splitting methods for pricing American type of options see also Ehrhardt [7].

The Lie - Trotter Splitting Method

The Lie - Trotter splitting method is a first order splitting which solves two subproblems sequentially. Suppose we have given the Cauchy problem

$$\frac{\partial u(t)}{\partial t} = Au(t) + B(t), \quad t \in [0, T], \quad u(0) = u_0. \tag{3.5}$$

Splitting techniques assume that the problem can be split into two or more subproblems. By these assumptions we can introduce the Lie splitting on the interval $[t^n, t^{n+1}]$ in the following way:

$$\frac{\partial u(t)}{\partial t} = Au(t), \quad t \in [t^n, t^{n+1}], \quad u(t^n) = u^n, \tag{3.6}$$

$$\frac{\partial v(t)}{\partial t} = Bv(t), \quad t \in [t^n, t^{n+1}], \quad v(t^n) = u(t^{n+1}), \tag{3.7}$$

for $n = 0, 1, \cdots, N-1$ and u^n is given as a initial condition for time step n from (3.5). Then, we refer to $u^{n+1} = v(t^{n+1})$ as the solution and a new starting point for $t \in [t^{n+1}, t^{n+2}]$. One can show using Taylor series that the Lie splitting method gives first order accuracy.

The Strang Splitting

One of the widely used and very popular operator splitting technique is the second-order Strang splitting Strang [14]. The idea is to solve (3.6) for time step $\Delta t/2$, then to solve (3.7) for a full time step Δt and finally a half time step solution $\Delta t/2$ for the equation (3.6). The algorithm is given in this way:

$$\frac{\partial u(t)}{\partial t} = Au(t), \quad t \in [t^n, t^{n+\frac{1}{2}}], \quad u(t^n) = u^n, \tag{3.8}$$

$$\frac{\partial v(t)}{\partial t} = Bv(t), \quad t \in [t^n, t^{n+1}], \quad v(t^n) = u(t^{n+\frac{1}{2}}), \tag{3.9}$$

$$\frac{\partial u(t)}{\partial t} = Au(t), \quad t \in [t^{n+\frac{1}{2}}, t^{n+1}], \quad u(t^{n+\frac{1}{2}}) = v(t^{n+1}), \tag{3.10}$$

for $n = 0, 1, \cdots, N-1$ and u^n is given as a initial condition for time step n from (3.5). Again, $u^{n+1} = v(t^{n+1})$ is used as starting point for the next approximation interval $[t^{n+1}, t^{n+2}]$. The order of the accuracy is two. This can be shown using Taylor series.

The Numerical Integration

In this work, the numerical integration of the definite integral based on the Newton-Cotes method is used. We use the first order method based on linear interpolation often called as *trapezoidal method*. The integral on the spatial domain $[x_n, x_{n+1}]$ is calculated as follows

$$\begin{aligned} \int_{x_n}^{x_{n+1}} f(x)dx &\approx \int_{x_n}^{x_{n+1}} \Big[f(x_n) + \frac{x - x_n}{x_{n+1} - x_n}(f(x_{n+1}) - f(x_n)) \Big] dx \\ &= \frac{f(x_n) + f(x_{n+1})}{2}(x_{n+1} - x_n). \end{aligned} \tag{3.11}$$

3.2 The Numerical Treatment

Hence, we have all the tools now, we can move to the numerical treatment of the model. To sum up, we present for convenience the problem with respective boundary conditions for the call option once more

$$\frac{\partial \Pi}{\partial \tau} + a(\xi, \tau)\frac{\partial \Pi}{\partial \xi} - \frac{1}{2}\sigma^2 \frac{\partial^2 \Pi}{\partial \xi^2} + b(\xi, \tau)\Pi = 0, \tag{3.12}$$

where the coefficient $a(\xi, \tau)$ and $b(\xi, \tau)$ are given in the form

$$\begin{aligned} a(\xi, \tau) &= \frac{\dot{\rho}(\tau)}{\rho(\tau)} + (r - q) - \frac{1}{2}\sigma^2 - f(\rho e^{-\xi}, T - \tau), \\ b(\xi, \tau) &= r + x\frac{\partial f}{\partial x} - f(x, T - \tau)|_{x = \rho e^{-\xi}}. \end{aligned}$$

The set of initial and boundary conditions have been derived for the call option contract as follows

$$\begin{aligned} \Pi(\xi, 0) &= \begin{cases} -1, & \xi < \ln \rho(0), \\ 0, & \xi > \ln \rho(0), \end{cases} \\ \Pi(0, \tau) &= \quad -1, \\ \Pi(\xi, \tau) &= \quad 0, \quad \xi \longrightarrow \infty. \end{aligned} \tag{3.13}$$

Our problem is also closely connected with the equivalent form of the free boundary position $\rho(\tau)$ and with a value of this boundary close to expiry:

$$\begin{aligned} 0 &= \frac{d}{d\tau}\left[\ln\rho(\tau) + \int_0^\infty \Pi d\xi\right] + q\rho(\tau) - q - \frac{1}{2}\sigma^2 \\ &+ \int_0^\infty \left[r - f(\rho(\tau)e^{-\xi}, T-\tau))\Pi d\xi\right]\Pi d\xi, \\ \rho(0^+) &= \max\left[\frac{1+rT}{1+qT}, 1\right], \quad \textit{arithmetic average}, \\ \rho(0^+) &= \max[\hat{x}, 1], \qquad \textit{geometric average}, \\ \rho(0^+) &= \max\left[\frac{r}{q}, 1\right], \qquad \textit{lookback}. \end{aligned} \tag{3.14}$$

Instead of the spatial domain $x \in (0, \infty)$ we consider a finite range $x \in (0, R)$, where R is sufficiently large for our purposes. This artificial boundary limits the computation domain and speeds up the numerical computation. We work with the parameter $R = 3$ as this number is sufficient for the numerical approximations. The time domain $\tau \in (0, T)$ is finite as well and depends on the maturity time of the option contract.

The finite difference method is used in the discretization process of the equation (3.12). We use the time step $k = \Delta\tau$ for the time domain $\tau \in (0, T)$ and $h = \Delta\xi$ correspondingly for the spatial domain $\xi \in (0, R)$. We may define $N = \frac{T}{k}$ and $M = \frac{R}{h}$ as a finite amount of time and space steps in our discretization. Hence, $\tau_j = jk, j \in [0, N]$ and $\xi_i = ih$, where $i \in [0, M]$. The abbreviations $\Pi_i^j = \Pi(\xi_i, \tau_j)$ and $\rho(\tau_j) = \rho^j$ are used throughout all the thesis.

Using the foregoing notations and the backward-in-time finite difference (3.2) the equation (3.12) is discretized such as

$$\frac{\Pi^j - \Pi^{j-1}}{k} + c^j \frac{\partial \Pi^j}{\partial \xi} - \left(\frac{\sigma^2}{2} + \frac{\rho^j e^{-\xi} - 1}{T - \tau_j}\right)\frac{\partial \Pi^j}{\partial \xi} - \frac{1}{2}\sigma^2 \frac{\partial^2 \Pi^j}{\partial \xi^2} + \left(r + \frac{1}{T - \tau_j}\right)\Pi^j = 0, \tag{3.15}$$

where c^j is the approximation of $c(\tau_j) = \frac{\dot{\rho}(\tau_j)}{\rho(\tau_j)} + (r - q)$. In the following we apply splitting techniques to this equation separating it into two nonlinear parts, to the convection and diffusive part presented first by Ševčovič (for further see Ševčovič [16]).

3.3 The Strang Splitting Procedure

The classical Lie splitting was presented in the work by Ševčovič [16] and many others [5],[13],[4]. Thus we ignore it and concentrate on the second order Strang - splitting methods Takáč [15]. Now, using two auxiliary portfolios $\overline{\Pi}, \overline{\overline{\Pi}}$, the finite differences and procedures (3.8)-(3.10) we obtain a three step method:

$$\frac{\overline{\Pi}_i^{j+\frac{1}{2}} - \overline{\Pi}_i^j}{\frac{k}{2}} + c_i^j(\rho^j)\frac{\overline{\Pi}_{i+1}^{j+\frac{1}{2}} - \overline{\Pi}_{i-1}^{j+\frac{1}{2}}}{2h} = 0, \quad \overline{\Pi}_i^j = \Pi_i^j, \tag{3.16}$$

$$\begin{aligned} \frac{\overline{\overline{\Pi}}_i^{j+1} - \overline{\overline{\Pi}}_i^j}{k} &- \left(\frac{\sigma^2}{2} + f(\rho^j e^{-\xi_i}, T-\tau_i)\right)\frac{\overline{\overline{\Pi}}_{i+1}^{j+1} - \overline{\overline{\Pi}}_{i-1}^{j+1}}{2h} - \frac{1}{2}\sigma^2\frac{\overline{\overline{\Pi}}_{i+1}^{j+1} - 2\overline{\overline{\Pi}}_i^{j+1} + \overline{\overline{\Pi}}_{i-1}^{j+1}}{h^2} \\ &+ \left(r + x\frac{\partial f}{\partial x} - f(\rho^j e^{-\xi}, T-\tau_i)|_{x=\rho e^{-\xi}}\right)\overline{\overline{\Pi}}_i^{j+1} = 0, \quad \overline{\overline{\Pi}}_i^j = \overline{\Pi}_i^{j+\frac{1}{2}}, \end{aligned} \tag{3.17}$$

$$\frac{\Pi_i^{j+1} - \Pi_i^{j+\frac{1}{2}}}{\frac{k}{2}} + c_i^j(\rho^j)\frac{\Pi_{i+1}^{j+1} - \Pi_{i-1}^{j+1}}{2h} = 0, \quad \Pi_i^{j+\frac{1}{2}} = \overline{\overline{\Pi}}_i^{j+1}. \tag{3.18}$$

We expect higher computational time, but higher precision since we talk about a second order method. We work with the initial and boundary conditions ρ^0, Π^0. Defining p as the order of the inner loop for all $j = 1, 2, \ldots, N$ we proceed to the following procedure. Supposing the pair $(\Pi^{j,p}, \rho^{j,p})$ as $p \longrightarrow \infty$ converges to the value $(\Pi^{j,\infty}, \rho^{j,\infty})$. The computation of the pair $(\Pi^{j,p+1}, \rho^{j,p+1})$ for all $p = 0, 1, \ldots, N-1, \ldots$ follows now a four step algorithm:

(I.) Using the forward finite differences (3.1) we discretize the time step in the equivalent form of the free boundary

$$\begin{aligned} \ln \rho^{j,p+1} &= \ln \rho^{j,0} - \int_0^\infty \Pi^{j,0} d\xi + \int_0^\infty \Pi^{j,p} d\xi \\ +k&\left[q + \frac{1}{2}\sigma^2 - q\rho^{j,0} - \int_0^\infty \left(r - f(\rho^{j,0}e^{-\xi}, T-\tau_{j,0})\right)\Pi^{j,0} d\xi\right]. \end{aligned} \tag{3.19}$$

We use the trapezoidal method to approximate the expressions $\int_0^\infty \Pi^n d\xi$.

(II.) The transport equation $\partial_\tau\overline{\Pi}+c(\tau)\partial_\xi\overline{\Pi}$ can be solved analytically with the difference to the Lie splitting that only a half-time step is performed. Therefore,the convection part, the transport equation, changes to

$$\overline{\Pi}_i^{j,p+\frac{1}{2}} = \begin{cases} \Pi_i^{j,0}(\eta_i), & \text{if } \eta_i = \xi_i - \ln\frac{\rho^{j,0}}{\rho^{j,p+\frac{1}{2}}} - (r-q)\frac{k}{2} > 0, \\ -1, & \text{otherwise.} \end{cases} \tag{3.20}$$

Since the value $\rho^{j,p+\frac{1}{2}}$ is not known we obtain it using interpolation.

(III.) Next, equation (3.17) is solved. With $\overline{\Pi}_i^{j,p+\frac{1}{2}}$ we enter the set of equations

$$\begin{bmatrix} \beta_0^j & \gamma_0^j & 0 & 0 & \cdots & 0 \\ \alpha_1^j & \beta_1^j & \gamma_1^j & 0 & \cdots & 0 \\ 0 & \alpha_2^j & \beta_2^j & \gamma_2^j & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \alpha_{n-1}^j & \beta_{n-1}^j & \gamma_{n-1}^j \\ 0 & \cdots & \cdots & 0 & \alpha_n^j & \beta_n^j \end{bmatrix} \overline{\overline{\Pi}}^{j,p+1} = \overline{\overline{\Pi}}^{j,p} + \begin{bmatrix} \alpha_0^j \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \tag{3.21}$$

where $\overline{\overline{\Pi}}^{j,p} = \overline{\Pi}^{j,p+\frac{1}{2}}$, we recall the boundary conditions $\Pi(0,\tau)=-1$, $\Pi(M,\tau)=0$,

$$\begin{aligned} \alpha_i^j &= \alpha_i^j(\rho^{j,p+1}) = -\frac{k}{2h^2}\sigma^2 + \frac{k}{2h}\left(\frac{1}{2}\sigma^2 + f(\rho^{j,p+1}e^{-\xi_i}, T-\tau_j)\right), \\ \gamma_i^j &= \gamma_i^j(\rho^{j,p+1}) = -\frac{k}{2h^2}\sigma^2 - \frac{k}{2h}\left(\frac{1}{2}\sigma^2 + f(\rho^{j,p+1}e^{-\xi_i}, T-\tau_j)\right), \\ \beta_i^j &= \beta_i^j(\rho^{j,p+1}) = 1 + b(\xi_i,\tau_j)k - \alpha_i^j(\rho^{j,p+1}) - \gamma_i^j(\rho^j), \end{aligned} \tag{3.22}$$

where $b(\xi_i,\tau_j) = r + x\frac{\partial f}{\partial x} - f(x, T-\tau_j)|_{x=\rho e^{-\xi_i}}$.

(IV.) Repeating the step (II.) with the auxiliary portfolio $\overline{\overline{\Pi}}^{j,p+1}$

$$\Pi_i^{j,p+1} = \begin{cases} \overline{\overline{\Pi}}^{j,p+1}(\eta_i), & \text{if } \eta_i = \xi_i - \ln\frac{\rho^{j,p+\frac{1}{2}}}{\rho^{j,p+1}} - (r-q)\frac{k}{2} > 0, \\ -1, & \text{otherwise.} \end{cases} \tag{3.23}$$

We set $p = p + 1$ and repeat step I. - IV. Once we have an acceptable tolerance for $p \longrightarrow \infty$ we set $\Pi^j = \Pi^{j,\infty}$ and $\rho^j = \rho^{j,\infty}$ and we move on to the next time step $j + 1$.

3.4 The Improved - Strang Splitting Procedure

Fast and precise decisions are the integral parts of the financial world. To be and stay competitive in this cruel world one must have the tools to react quickly and accurate to this fast changing environment. Different tools were created to achieve this goal. One of the fields where precise and fast algorithms are necessary is the field of the option pricing. This holds also for the pricing of Asian options. The calculation of the price of the floating strike Asian option i.e. the evaluation of the free boundary position is a very time consuming procedure. The dependence of the evaluation time and the gird of the numerical estimation is an increasing function. Undoubtedly, to have more precise results, we need to increase the number of time steps i.e. decrease *dt*. However this leads to an evaluation time increase which is not acceptable in the 'real' world. To overcome this shortcoming, one may try to use all available methods. Since the mathematical theory and analytical background are very strong for the problem of the floating strike Asian options, in this work, we try to have a look at this problem from the numerical point of view. The final goal is not only to speed up the procedure but also to keep the already archived accuracy at a standard level. In the process of understanding the already established numerical procedure we noticed an unused potential the Strang - splitting enables us.

We propose a new algorithm which may improve the above mentioned Strang splitting procedure from section 3.3. The idea of this algorithm is based on the mentioned splitting itself. For a better and faster convergence, we insert the numerical approximation of the ρ to our steps one more time. Although, this can be done in different ways, we present here the best working algorithm for our case. Using the two auxiliary portfolios $\overline{\Pi}, \overline{\overline{\Pi}}$, the finite differences and procedures (3.8)-(3.10) we obtain a three step method. The same three steps as in the case of the standard Strang - splitting. The difference here will appear in the numerical evaluation of the steps (3.16 - 3.18).
Since we use the Strang method we can talk about the second order accuracy. However, we expect better computational time as it was in the case of the Standard Strang - splitting procedure. The implementation of the extra calculation of ρ may give us this

advantage. We work with the same initial and boundary conditions ρ^0, Π^0. Defining p as the order of the inner loop for all $j = 1, 2, ...N$ we proceed to the successive iteration procedure. Supposing the pair $(\Pi^{j,p}, \rho^{j,p})$ as $p \longrightarrow \infty$,converges to the value $(\Pi^{j,\infty}, \rho^{j,\infty})$ we set $\Pi^{j,0} = \Pi^{j-1}$ and $\rho^{j,0} = \rho^{j-1}$ .The computation of the pair $(\Pi^{j,p+1}, \rho^{j,p+1})$ for all $p = 0, 1, \ldots, N-1, \ldots$ follows this five step algorithm:

(I.) To the discretize the time step in the equivalent form of the free boundary we use the forward-finite difference (3.1)

$$\begin{aligned} \ln \rho^{j,p+1} = \ln \rho^{j,0} - \int_0^\infty \Pi^{j,0} d\xi + \int_0^\infty \Pi^{j,p} d\xi \\ +k\left[q + \frac{1}{2}\sigma^2 - q\rho^{j,0} - \int_0^\infty \Big(r - f(\rho^{j,0}e^{-\xi}, T - \tau_{j,0})\Big)\Pi^{j,0} d\xi\right]. \end{aligned} \tag{3.24}$$

(II.) As in the step (II.) dedicated to the standard Strang - splitting the transport equation $\partial_\tau \overline{\Pi} + c(\tau)\partial_\xi \overline{\Pi} = 0$ can be solved analytically. Hence:

$$\overline{\Pi}_i^{j,p+\frac{1}{2}} = \begin{cases} \Pi_i^{j,0}(\eta_i), & \text{if } \eta_i = \xi_i - \ln \frac{\rho^{j,0}}{\rho^{j,p+\frac{1}{2}}} - (r-q)\frac{k}{2} > 0, \\ -1, & \text{otherwise.} \end{cases} \tag{3.25}$$

Since the value $\rho^{j,p+\frac{1}{2}}$ is not known we obtain it using interpolation technique.

(III.) Next, the equation (3.17) is solved. With $\overline{\Pi}_i^{j,p+\frac{1}{2}}$ we enter the set of equations

$$\begin{bmatrix} \beta_0^j & \gamma_0^j & 0 & 0 & \cdots & 0 \\ \alpha_1^j & \beta_1^j & \gamma_1^j & 0 & \cdots & 0 \\ 0 & \alpha_2^j & \beta_2^j & \gamma_2^j & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \alpha_{n-1}^j & \beta_{n-1}^j & \gamma_{n-1}^j \\ 0 & \cdots & \cdots & 0 & \alpha_n^j & \beta_n^j \end{bmatrix} \overline{\overline{\Pi}}^{j,p+1} = \overline{\overline{\Pi}}^{j,p} + \begin{bmatrix} \alpha_0^j \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \tag{3.26}$$

where $\overline{\overline{\Pi}}^{j,p} = \overline{\Pi}^{j,p+\frac{1}{2}}$, we recall the boundary conditions $\Pi(0,\tau) = -1$, $\Pi(M,\tau) = 0$ and

$$\alpha_i^j = \alpha_i^j(\rho^{j,p+1}) = -\frac{k}{2h^2}\sigma^2 + \frac{k}{2h}\left(\frac{1}{2}\sigma^2 + f(\rho^{j,p+1}e^{-\xi_i}, T-\tau_j)\right),$$
$$\gamma_i^j = \gamma_i^j(\rho^{j,p+1}) = -\frac{k}{2h^2}\sigma^2 - \frac{k}{2h}\left(\frac{1}{2}\sigma^2 + f(\rho^{j,p+1}e^{-\xi_i}, T-\tau_j)\right), \tag{3.27}$$
$$\beta_i^j = \beta_i^j(\rho^{j,p+1}) = 1 + b(\xi_i,\tau_j)k - \alpha_i^j(\rho^{j,p+1}) - \gamma_i^j(\rho^j),$$

where $b(\xi_i,\tau_j) = r + x\frac{\partial f}{\partial x} - f(x, T-\tau_j)|_{x=\rho e^{-\xi_i}}$.

(IV.) Here we proceed to the calculation of the $\rho^{j,p+1}$ once again. This time we use our knowledge of the value of $\overline{\overline{\Pi}}^{j,p+1}$. Using finite differences we work with the following equation:

$$\ln\rho^{j,p+1} = \ln\rho^{j,0} - \int_0^\infty \overline{\overline{\Pi}}^{j,0} d\xi + \int_0^\infty \overline{\overline{\Pi}}^{j,p} d\xi$$
$$+k\left[q + \frac{1}{2}\sigma^2 - q\rho^{j,0} - \int_0^\infty \left(r - f(\rho^{j,0}e^{-\xi}, T-\tau_{j,0})\right)\overline{\overline{\Pi}}^{j,0} d\xi\right]. \tag{3.28}$$

(V.) Repeating the step (II.) with the auxiliary portfolio $\overline{\overline{\Pi}}^{j,p+1}$

$$\Pi_i^{j,p+1} = \begin{cases} \overline{\overline{\Pi}}^{j,p+1}(\eta_i), & \text{if } \eta_i = \xi_i - \ln\frac{\rho^{j,p+\frac{1}{2}}}{\rho^{j,p+1}} - (r-q)\frac{k}{2} > 0, \\ -1, & \text{otherwise.} \end{cases} \tag{3.29}$$

We set $p = p+1$ and repeat step I. - step V. Once we have an acceptable tolerance for $p \longrightarrow \infty$ we set $\Pi^j = \Pi^{j,\infty}$ and $\rho^j = \rho^{j,\infty}$ and we move on to the next time step $j+1$.

As one could have already noticed all the steps except of step IV. are the same as for the traditional Strang - splitting. Hence the application of this improvement is simple either from the mathematical or programming point of view. Thus the basic idea of the step IV. is to **update** the position of the free boundary $\rho^{j,p+1}$ once we have a new information about the value of $\overline{\overline{\Pi}}^{j,p+1}$.

Chapter 4

The Numerical Experiments

To understand better the results of this thesis we present the practical part where all the theoretical models are implemented. This implantation is done in software *MATLAB*. We mainly focus on the free boundary, because it affects also the option price. To observe the behaviour of the free boundary we explore the changing of the time steps m. Yearly, monthly, weekly and daily approximations are performed what corresponds to $m = \{1, 12, 52, 252\}$ for a one year option. The number of spatial steps on the grid is chosen according to the value of the parameter R. We work with the parameter $R = 3$. Hence the number of spatial steps $n = 300$ in every case. In the following we analyse the computational (CPU[1]) time, convergence in each inner loop and option pricing. We illustrate the position of the free boundary for different types of options and we also perform analysis of this position with respect to the input parameters.

4.1 Input Parameters

The structure of the free boundary, so in fact the option price depends on many parameters. Therefore we should not forget to mention what type of parameters we use and how they affect the analysis. The Table 4.1 contains all the parameters which enters the models and values which are used throughout the simulations.

[1]The central processing unit (CPU) is the portion of a computer system that carries out the instructions of a computer program.

parameter	definition	value range
m	number of time steps on the grid,	$\{1, 12, 52, 252\}$ p.a.
n	number of spatial steps	300
r	riskfree interest rate	0.06
q	continuous dividend yield	0.04
σ	volatility	0.20
eps	minimum tolerance for the ρ	10^{-7}
$p - max$	maximum number of iteration in every inner loop	500
T	maturity of the option	50

Table 4.1: Input parameters of the model, they meaning and values.

4.2 Computational Time

In this part we make a numerical comparison of the computational times for different methods and different types of options. This comparison is made for different number of time steps for a call option with *T=50*. As we already mentioned the CPU time is crucial part of the option pricing and we shall always try to minimize it with all available methods. With higher CPU performance one can always achieve this goal. However, the new discretization method is independent on this factor i.e. the speed increase rates will hold also for better CPU. The Table 4.2 shows the computational time and the percentage difference from the benchmark which represents the Lie and Strang splitting procedure.

One can see that the Improved - Strang splitting algorithm is a less time consuming out of all. It also outperform the classical Lie splitting methods which is only a first order method. By the new algorithm one can get a more precise second order results and also achieve better computational time.

4.3 Convergence - Number of Inner Loops

The convergence of each inner loop is an important part of the whole algorithm. The faster the convergence the better computational time we get. We obtain a convergence when the difference of the ρ obtained in two consecutive approximations is less the acceptable tolerance eps. The number of iterations needed to fulfil this condition is

$r = 0.06,\ q = 0.04,\ \sigma = 0.2,\ T = 50$			
Arithmetic average			
frequency	Lie splitting	Strang splitting	Improved - Strang splitting
yearly	2.87	7.87	2.58
monthly	76.83	155.35	38.09
weekly	424.32	785.93	154.13
daily	2589.52	4207.93	726.51
Geometric average			
frequency	Lie splitting	Strang splitting	Improved - Strang splitting
yearly	3.09	8.54	2.67
monthly	83.05	176.52	38.72
weekly	482.04	884.00	159.25
daily	2388.54	3831.85	612.29
Lookback			
frequency	Lie splitting	Strang splitting	Improved - Strang splitting
yearly	3.22	10.08	3.12
monthly	85.80	179.98	38.86
weekly	426.76	773.14	157.24
daily	1522.54	2291.20	547.94

Table 4.2: CPU time for different methods and different time discretization for the call options in seconds.

then the number of inner loop p.

The Table 4.3 shows the average number of inner loops needed for the approximation of the free boundary for the arithmetically and geometrically averaged Asian and Lookback option, and for different methods.

The comparison of the number of inner loops needed for the calculation of each individual ρ are presented on the Figure 4.3. We compare the Lie, Strang and Improved - Strang splitting method for the arithmetic average call option with a daily averaging. The improved methods requires less inner iterations. This is making the calculation faster.

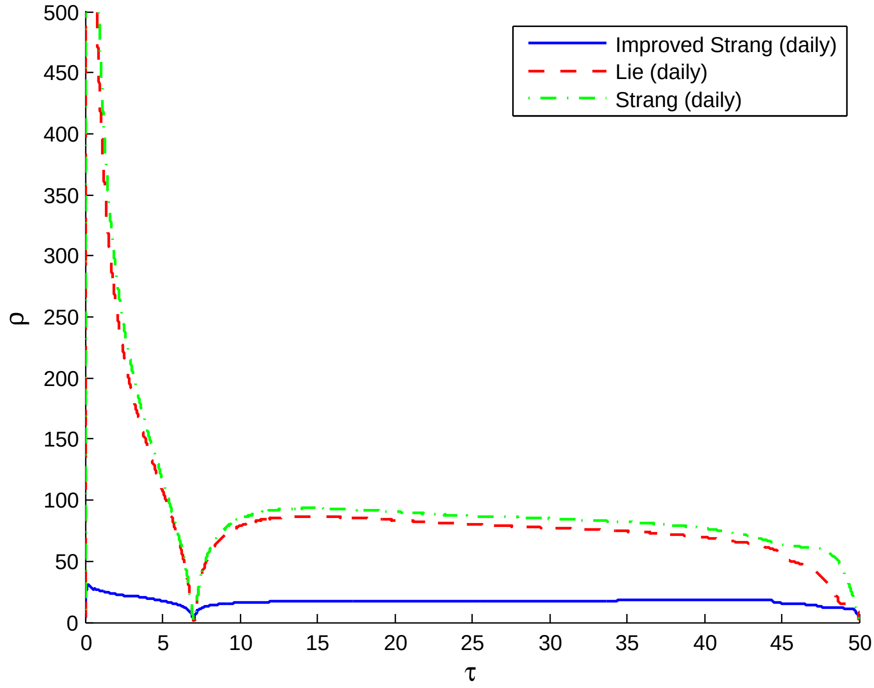


Figure 4.1: A graphical comparison of the number of the inner loops need for the convergence at every time step for the Lie, Strang and Improved Strang methods.

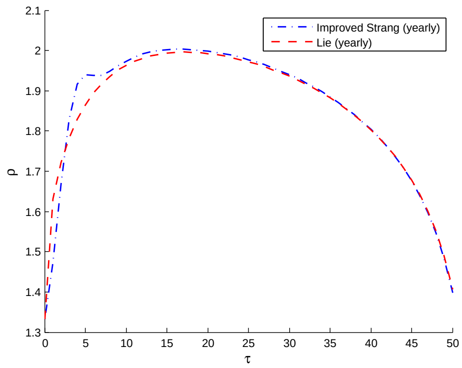
$r = 0.06,\ q = 0.04,\ \sigma = 0.2,\ T = 50$			
Arithmetic average			
frequency	Lie splitting	Strang splitting	Improved - Strang splitting
yearly	28.72	54.80	16.70
monthly	65.34	91.31	20.940
weekly	84.96	104.11	20.10
daily	91.41	101.91	17.47
Geometric average			
frequency	Lie splitting	Strang splitting	Improved - Strang splitting
yearly	32.28	61.84	18.82
monthly	72.73	103.90	22.29
weekly	95.12	117.12	21.13
daily	101.79	115.45	17.98
Lookback			
frequency	Lie splitting	Strang splitting	Improved - Strang splitting
yearly	33.32	68.46	20.30
monthly	73.60	105.35	22.01
weekly	84.56	103.38	21.13
daily	60.35	67.73	15.39

Table 4.3: The mean of the inner loops needed for the convergence of the ρ at each time step.

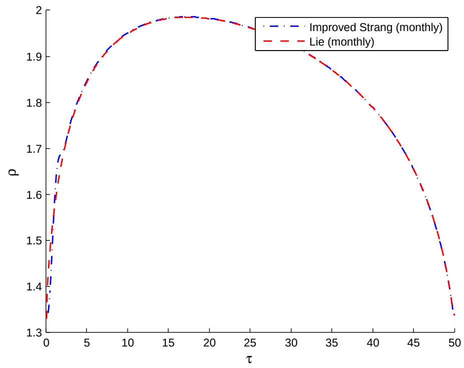
4.4 Comparison of the Methods

In this section we make a graphical presentation of the free boundary position for different methods and different option types. By this, one can see the difference of the boundaries calculated with all presented methods. We try to enumerate the the difference of this two methods for different time step discretizations. We use the *maximum* ($||.||_\infty$)and the *Euclidean* norm ($||.||_2$). The Table 4.4 shows the difference of the Improved - Strang splitting with the Lie and Strang splitting procedure.

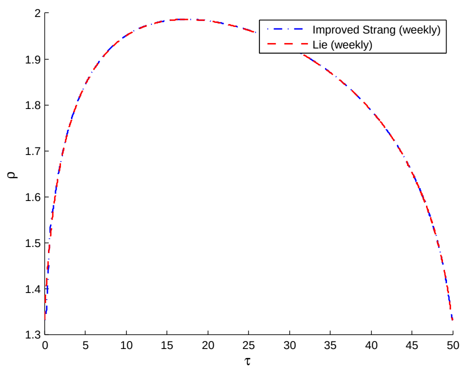
The Figure 4.4 shows the comparison of the free boundary position for the Lie splitting and Improved - Strang splitting methods. To see the convergence of the two curves we plot the free boundary for several time step discretization. The numerical



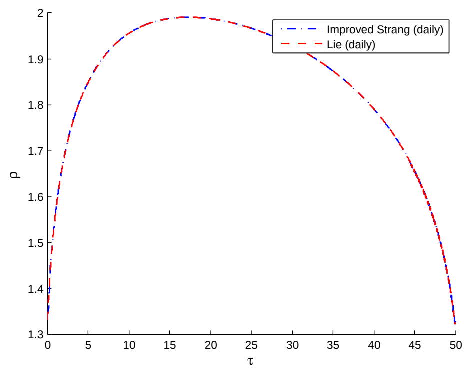
(a) yearly average



(b) monthly average



(c) weekly average



(d) daily average

Figure 4.2: A comparison of the free boundary for the arithmetic average call option for different discretization types.

experiment was done with the parameters $r = 0.06$, $q = 0.04$, $\sigma = 0.2$, $T = 50$.

In the Figure 4.4 one can see the comparison of the free boundary position for the geometric average call option and Lookback call options. One can see that those types of options also converge when we use an adequate time step discretization.
Using appropriate time discretization steps ensure the convergence of the Improved - Strang splitting method. Even-though, the grid is denser, the better computational time still defines the new method as an effective one.

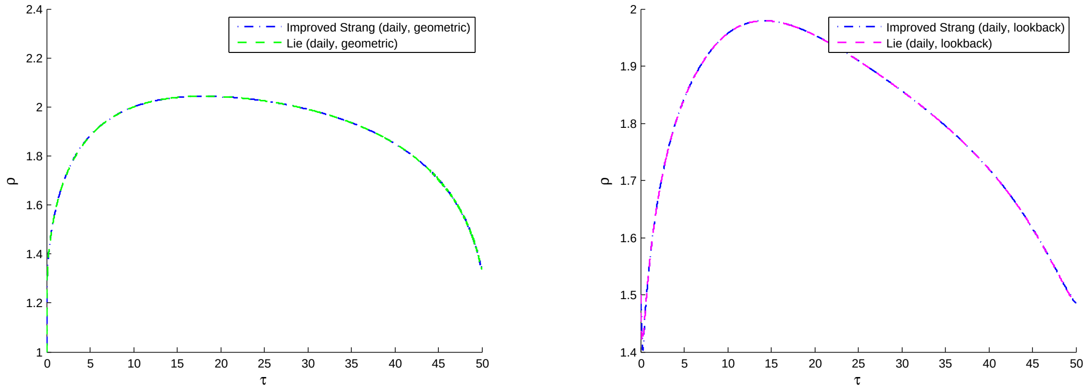


Figure 4.3: A comparison of the free boundary of the geometric average call option (left) and a lookback call option (right) with daily discretization.

4.5 Option Pricing

Once the position of the free boundary is located one can proceed to the calculation of the non-arbitrage price of each particular option. The backward transformation presented in Section 2.1.3 is used. We compare the option price obtained from the classical Lie splitting, Strang splitting and the Improved - Strang splitting with other results. The Table 4.5 compares the call option prices with arithmetic average with the values from Hansen & Jørgensen [8]. On the other side the Table 4.6 compares our results with the *FSG method* Barraquand & Puder [1]. It is important to mention that the method presented by Hansen & Jørgensen uses the position of the free boundary close to expiry $\rho(0^+)$ =1. This was proved to be wrong in further research (Kwol [11] and Ševčovič [16]). Therefore, the results may biased.

$r = 0.06,\ q = 0.04,\ \sigma = 0.2,\ T = 50$				
Arithmetic average				
frequency	IS Strang vs Lie		IS Strang vs Lie	
	$\|\|\cdot\|\|_2$	$\|\|\cdot\|\|_\infty$	$\|\|\cdot\|\|_2$	$\|\|\cdot\|\|_\infty$
yearly	0.2168	0.1566	0.2408	0.4399
monthly	0.2577	0.0925	0.2313	0.1107
weekly	0.2351	0.0561	0.2771	0.0544
daily	0.1436	0.0207	0.2732	0.0194
Geometric average				
frequency	IS Strang vs Lie		IS Strang vs Strang	
	$\|\|\cdot\|\|_2$	$\|\|\cdot\|\|_\infty$	$\|\|\cdot\|\|_2$	$\|\|\cdot\|\|_\infty$
yearly	0.2784	0.2448	0.2375	0.3897
monthly	0.2456	0.1047	0.2147	0.1248
weekly	0.2798	0.0654	0.2654	0.0751
daily	0.1671	0.0247	0.2194	0.0201
Lookback				
frequency	IS Strang vs Lie		IS Strang vs Strang	
	$\|\|\cdot\|\|_2$	$\|\|\cdot\|\|_\infty$	$\|\|\cdot\|\|_2$	$\|\|\cdot\|\|_\infty$
yearly	0.2443	0.3451	0.2645	0.2784
monthly	0.2397	0.0785	0.2121	0.1043
weekly	0.2145	0.0499	0.2045	0.0513
daily	0.1742	0.0134	0.2145	0.0146

Table 4.4: A comparison of the Improved - Strang splitting with the Lie and Strang splitting in terms of difference. We use the Euclidean and maximum norm.

r	σ	T	Hansen Jørgensen	Lie splitting	Strang splitting	I - Strang splitting
0.03	0.2	$\frac{1}{12}$	1.950	1.8598	1.8344	1.8660
		$\frac{4}{12}$	4.000	4.2346	4.1994	4.2201
		$\frac{7}{12}$	5.370	5.5393	5.5177	5.5457
	0.3	$\frac{1}{12}$	2.910	2.8788	2.8442	2.8147
		$\frac{4}{12}$	5.900	6.1553	6.1281	6.1451
		$\frac{7}{12}$	7.880	7.9936	7.9784	8.0145
	0.4	$\frac{1}{12}$	3.860	3.8573	3.8224	3.8614
		$\frac{4}{12}$	7.800	8.0275	8.0068	8.0451
		$\frac{7}{12}$	10.390	10.3745	10.3657	10.3451
0.05	0.2	$\frac{1}{12}$	1.990	1.9482	1.9223	1.9789
		$\frac{4}{12}$	4.130	4.2228	4.1958	4.2417
		$\frac{7}{12}$	5.600	5.8107	5.7941	5.8174
	0.3	$\frac{1}{12}$	2.940	2.9603	2.9253	2.9492
		$\frac{4}{12}$	6.020	6.1575	6.1399	6.1484
		$\frac{7}{12}$	8.090	8.2532	8.2429	8.2621
	0.4	$\frac{1}{12}$	3.890	3.8908	3.8541	3.9174
		$\frac{4}{12}$	7.920	8.0513	8.0430	8.0424
		$\frac{7}{12}$	10.600	10.6221	10.6177	10.6145
0.07	0.2	$\frac{1}{12}$	2.020	2.0402	2.0137	2.0145
		$\frac{4}{12}$	4.260	4.3118	4.2896	4.3284
		$\frac{7}{12}$	5.830	5.8362	5.8257	5.8247
	0.3	$\frac{1}{12}$	2.970	3.0429	3.0076	3.0101
		$\frac{4}{12}$	6.150	6.2175	6.2011	6.2145
		$\frac{7}{12}$	8.310	8.2951	8.2971	8.2945
	0.4	$\frac{1}{12}$	3.920	4.0221	3.9870	4.0012
		$\frac{4}{12}$	8.040	8.0777	8.0676	8.0142
		$\frac{7}{12}$	10.810	10.6542	10.6578	10.6451

Table 4.5: The comparison of the call option values with arithmetic average obtained using our methods and the method of Hansen and Jørgensen for various values of T, σ and r.

r	σ	T	FSG	Lie splitting	Strang splitting	I - Strang splitting
0.1	0.1	1/4	2.142	2.155	2.137	2.134
		1/2	3.404	3.603	3.596	3.589
		1	5.67	5.975	6	5.994
	0.2	1/4	3.689	3.856	3.833	3.857
		1/2	5.514	5.886	5.878	5.879
		1	8.463	8.811	8.821	8.841
	0.3	1/4	6.817	7.039	7.040	7.041
		1/2	9.86	10.405	10.421	10.441
		1	14.446	14.703	14.729	14.778

Table 4.6: A numerical comparison of the call option values with arithmetic average obtained using the *FSG method* and our methods for different input parameters.

Chapter 5

Hedging Example

In this section, we make an qualitative comparison of the behaviour of the American and European type floating strike Asian option. The goal of this experiment is to analyse the advantage of the early exercise. We simulate a behaviour of an international investor who wants to be secured and wants to avoid a risk coming from the appreciation or depreciation of the exchange rate.

Imagine an multinational *USD* based company making an investment with a budget of *EUR 25 000 000* in and *EUR* based country in the following one year. The cashflow of the investment is dynamic one i.e. the investor needs to deal with several receipts during this year which are difficult to forecast. Despite this *'random'* cashflow the company has the possibility to manage his loss or profit coming from the exchange rate change. Assume the current *EUR/USD*[1] FX rate. In case of the dollar depreciation against (increasing *EUR/USD*) the company's cost in *USD* will be higher as it was with the FX rate at the begging of the period. For this reason they decide to hedge their investment with and floating strike Asian call option. Taking in account the current exchange and the historical volatility, so the risk free interest rate, one can calculate the price of the European and American price according to the presented methods in Section 2.3 - 2.4.

We use simulation methods for different scenarios of the *FX rate* behaviour. According to this results we quantify the values of the investment taking in account also the option prices. We generate a random vector of the cashflow for each scenario. The

[1] *EUR/USD* currency pair is the quotation of the relative value of a currency unit *EUR* against the unit of another currency *USD* in the foreign exchange market

simulation of the *FX rate* path is done using a simple Monte Carlo method based on the Black - Scholes equation and GBM. By this, one can evaluate the intrinsic value of the investment and also see the usage of Asian option in real life.

To simulate the one year free boundary we use a weekly arithmetic average. In case the the value $x = \frac{S}{A}$ touches the boundary we execute the option. Since the cashflow amount is not know in advance, we generate a vector, where every week a random percentage of the whole budget is charged. As for the fact, that we price the option using the Black - Scholes equation under a risk neutral principle and also simulate the path of the underlying we generally expect that the mean of the simulation using an American and European type should equal in some level with the increasing number of simulations. However, the standard deviation of this intrinsic values may be different. The early exercise of the option from the mathematical point of view says us that at the moment t, when the the exercise occurs we get back our invested capital. This, in the case of the European option is missing and may happen that this disadvantage will cause a loss or even a higher profit. Nevertheless, we expect the standard deviation of both hedges, either with American type or European to be less as the case of the unhedged investment.

The table 5.1 and 5.2 respectively represents the mean and the standard deviation of the intrinsic value of the investment for an unhedged, a hedged with an European type of option and hedged investment with an American type of option. The evaluation is done with a different number of simulations. As one can see the mean for all the three types of the investment converges to the same value. A difference can be seen in the standard deviation of the final intrinsic values. The hedging with an American type of option shows a better performance in comparison to the European type and to the unhedged investment. In other words the intrinsic value of the investment hedged with an American type of option is not that *'volatile'* as the other two cases. The percentage comparison is presented in table and 5.3.
The same fact is visible in the table 5.4 where the 2.5% and 97.5% quantiles differences are shown for a different numbers of simulation. In all the cases the quantiles are located closer to the mean either for the lower and upper one. It is also worth to mention that the European type of hedging shows a better performance comparing to the unhedged investment. The investor should consider in any case a hedging strategy. In this case when the cashflow is not predicable the American type of average strike Asian option can be a good choice. Even though, the price of the American type of

$r = 0.0138$, $q = 0$, $\sigma = 0.108$, $T = 1$			
n	without hedging	European hedging	American edging
1 000	32 963 118	32 907 148	32 980 272
10 000	32 920 284	32 917 079	32 920 228
50 000	32 887 315	32 889 537	32 902 319
100 000	32 896 312	32 888 108	32 899 892
1 000 000	32 890 451	32 888 014	32 910 421

Table 5.1: The comparison of the mean of different type of hedging for different numbers of simulations. The values are in USD

$r = 0.0138$, $q = 0$, $\sigma = 0.108$, $T = 1$			
n	without hedging	European hedging	American edging
1 000	1 834 844	1 653 314	1 424 663
10 000	1 744 276	1 604 123	1 381 529
50 000	1 751 273	1 605 179	1 379 636
100 000	1 754 358	1 599 258	1 375 957
1 000 000	1 754 481	1 598 475	1 394 951

Table 5.2: The comparison of the standard deviation of different type of hedging for different numbers of simulations. The values are in USD.

this option is more expensive the simulations shows better behaviour. In average the intrinsic value of the investment is in a 'thinner' range which is undoubtedly more suitable for any kind of hedged investment.

$r = 0.0138$, $q = 0$, $\sigma = 0.108$, $T = 1$				
	standard deviation in $\Delta\%$		mean in $\Delta\%$	
n	Amer vs. Euro	Amer vs. $\emptyset$	Amer vs. Euro	Amer vs. $\emptyset$
1 000	-13.83	-22.35	-0.29	0.00
10 000	-13.87	-20.79	-0.22	-0.05
50 000	-14.05	-21.22	-0.01	0.00
100 000	-13.97	-21.57	-0.04	-0.05
1 000 000	-13.98	-21.63	0.00	0.00

Table 5.3: The percentage comparison of the mean and the standard deviation of the investment between the American and the European type of hedging and the American type and the unhedged investment. The $\emptyset$ represents the unhedged investment.

$r = 0.0138$, $q = 0$, $\sigma = 0.108$, $T = 1$				
	2.5% quantile in $\Delta\%$		97.5% quantile in $\Delta\%$	
n	Amer vs. Euro	Amer vs. $\emptyset$	Amer vs. Euro	Amer vs. $\emptyset$
1 000	1.20	2.23	-0.91	-1.86
10 000	1.38	2.21	-1.11	-2.04
50 000	1.48	2.25	-1.14	-2.10
100 000	1.42	2.26	-1.12	-2.17
1 000 000	1.41	2.26	-1.12	-2.19

Table 5.4: The percentage comparison of the 2.5% and 97.5% quantile got from the simulations between the American and the European type of hedging and the American type and the unhedged investment. The $\emptyset$ represents the unhedged investment.

Chapter 6

Conclusion

In this thesis we dealt with American type of path dependent options with a floating strike. The main goal of the work was to develop efficient numerical methods for the evaluation of the free boundary, which is strongly connected to the pricing of these options.

In chapter 1, we gave an short, but precise introduction to the world of financial markets, option pricing and a brief derivative overview. Chapter 2 describes the transformational method for the pricing equation. We derived the value of the early exercise boundary close to the expiry, using the linear complementarity problem, for the arithmetic and geometric Asian option and also for the Lookback options. The backward transformation derived in this chapter allows us to price the options, once the position of the free boundary is determinated.

Using the introduced numerical methods in Chapter 3, we presented the Strang - splitting procedure for the unified pricing equation. This second order method gave us a more precise but also a more time consuming algorithm. Then, we introduced the "Improved - Strang" splitting scheme, which overcame this disadvantage of the classical Strang splitting by updating the position of the free boundary more often during the numerical evaluation. This allowed faster convergence and generally speed up the method.

The behaviour of this new method was the object of research in Chapter 4. We studied the difference, speed and convergence of the methods. Once a acceptable discretization was used, the Improved - Strang splitting method showed a perfect fit to the classical one. Generally, the new method was faster the the Strang splitting and even faster than the first order Lie splitting method. It requires lower number of inner loops for convergence, this makes the calculation time faster. The option pricing procedure also confirmed the conservation of the accuracy to the benchmark, which is in our case represented by the Lie Splitting. The hedging example from Chapter 5 explored the advantages of the hedging using American type of options. The intrinsic value of the simulated investment in one year horizon showed better behaviour for the hedging with American type of options.

List of Symbols

t	Time.
T	Expiration time.
E	Strike price.
$S(t)$	Spot price - Price of the underlying asset at time t.
$A(t)$	Average at time t
$V(S, A, t)$	Option value - Price of the financial derivative depending on time t, asset's price S and average A.
$[S(T) - E]$	Payoff function at T $(= \max[S(T) - E, 0])$.
r	Interest rate.
σ	Volatility.
q	Dividend yield.
$x(t)$	Transformed spatial variable$(= \frac{S(t)}{A(t)})$.
τ	Time variable $(= T - t)$.
$\rho(\tau)$	Transformed free boundary $(= x(T - \tau))$.
$W(x, \tau)$	Transformed payoff function $(= \frac{1}{A} V(S, A, t))$.
ξ	Transformed spatial variable $(= \frac{\rho(\tau)}{x})$.
Π	Synthetic portfolio $(= W(x, \tau) - x \frac{\partial W(x,\tau)}{\partial x})$.
k	Time step.
h	Spatial step.
j	Index for time step.
i	Index for spatial step.

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Appendix

1. A Multidimensional Version of Itô's lemma

Suppose $f(x_1, \dots, x_n, t)$ is a multidimensional differentiable function, the stochastic process Y_n is defined by $Y_n = f(X_1, \dots, X_n, t)$, where the process X_j follows

$$dX_j(t) = \mu(t)dt + \sigma_j(t)dW_j(t), \quad æ = 1, 2, \dots, n,$$

where $W_j(t)$ is a standard Wiener's process. $W_j(t) \mathrm{and} W_i(t)$ are assumed to be correlated so that $dW_j dW_i = \rho_{ij}$, then we define the multidimensional Itô's lemma:

$$\begin{aligned} dY_n = &\bigg[\frac{\partial f}{\partial t}(X_1, \dots, X_n, t) + \sum_{j=1}^{n} \mu_j(t) \frac{\partial f}{\partial x_j}(X_1, \dots, X_n, t) \\ &+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_i(t)\sigma_j(t)\rho_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(X_1, \dots, X_n, t)\bigg] dt \\ &+ \sum_{j=1}^{n} \sigma_j(t) \frac{\partial f}{\partial x_j}(X_1, \dots, X_n, t) dW_j. \end{aligned}$$