Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

Cross Sectional Forecasts of the Equity Premium

Master's Thesis

Katarína Beláková

Bratislava 2013

Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

Cross Sectional Forecasts of the Equity Premium

Master's Thesis

Katarína Beláková

Department of Applied Mathematics and Statistics
9.1.9 Applied Mathematics 1114
Mathematics of Economics and Finance

Supervisor:

Mgr. Juraj Katriak

Bratislava, 2013

Univerzita Komenského v Bratislave
Fakulta Matematiky, Fyziky a Informatiky

Prierezové Prognózy Prémie Akcií

Diplomová práca

Katarína Beláková

Katedra aplikovanej matematiky a štatistiky
9.1.9 Aplikovaná matematika 1114
Ekonomická a finančná matematika

Vedúci diplomovej práce:

Mgr. Juraj Katriak

Bratislava, 2013

65969892

Univerzita Komenského v Bratislave
Fakulta matematiky, fyziky a informatiky

ZADANIE ZÁVEREČNEJ PRÁCE

Meno a priezvisko študenta: Bc. Katarína Beláková
Študijný program: ekonomická a finančná matematika (Jednoodborové štúdium, magisterský II. st., denná forma)
Študijný odbor: 9.1.9. aplikovaná matematika
Typ záverečnej práce: diplomová
Jazyk záverečnej práce: anglický
Sekundárny jazyk: slovenský

Názov: Cross sectional forecasts of the equity premium

Cieľ: Cross sectional forecasts of the equity premium

Vedúci: Mgr. Juraj Katriak
Katedra: FMFI.KAMŠ - Katedra aplikovanej matematiky a štatistiky

Dátum zadania: 13.01.2011

Dátum schválenia: 14.01.2011

prof. RNDr. Daniel Ševčovič, CSc.
garant študijného programu

....
študent

....
vedúci práce

I declare on my honour that this work is based only on my knowledge, references and consultations with my supervisor.

Katarína Beláková

Acknowledgement

I would like to express my gratitude and appreciation to my supervisor Mgr.Juraj Katriak for his guidance, ideas and patience. I am also grateful to my family and friends for their support and help during my studies.

Abstract

BELÁKOVÁ, Katarína: The Cross Sectional Forecasts of the Equity Premium [Master's Thesis], Comenius University in Bratislava, Faculty of Mathematics, Physics and Informatics, Department of Applied Mathematics and Statistics; Supervisor: Mgr. Juraj Katriak, Bratislava, 2013, 39 s.

The main purpose of the Master's thesis is to reconstruct one of the cross-sectional measures of the equity premium from the Polk, Thompson and Vuolteenaho paper and further test its predictive ability against data that are different (S&P 500 Index). We also select a different period according to the data availability.
The results can be summarized by the simple finding that the predictability of our constructed measure is very low and therefore not significant.

Key words: CAPM model, Expected equity premium, Linear regression

Abstrakt

BELÁKOVÁ, Katarína: Prierezové prognózy prémie akcií [Diplomová práca], Univerzita Komenského v Bratislave, Fakulta matematiky, fyziky a informatiky, Katedra aplikovanej matematiky a štatistiky; školiteľ: Mgr. Juraj Katriak, Bratislava, 2013, 39 s.

Hlavným cieľom diplomovej práce je zrekonštruovať jednu z prierezových mier prémie akcií z článku Polk, Thompson a Vuolteenaho a ďalej otestovať jej predikčnú schopnosť na iných dátach (S&P 500 index). Taktiež vyberieme odlišnú periódu, pričom berieme ohľad na dostupnosť dát.
Výsledky môžeme zhrnúť do jednoduchého zistenia, že predikčná schopnosť nami vytvorenej miery je veľmi malá a preto nie je signifikantná.

Kľúčové slová: CAPM model, očakávaná prémia akcií, lineárna regresia

Contents

Introduction

The Capital asset pricing model (CAPM) offers intuitive and powerful predictions about the relation between the equity return and risk. Until lately it was widely used in evaluating portfolio returns and measuring the risk. However in the most recent studies it appears that CAPM does a poor job describing more recent equity return-risk relation.
We challenge CAPM predictive ability using purely cross-sectional data to predict the equity premium realizations in US stock market during the period of last 21 years.

In the first chapter we derive the efficient portfolio frontier and continue to review the main issues in CAPM theory in the second chapter. The purpose of the third chapter is to link the CAPM model and the expected equity premium. In the last chapter we summarize our empirical results from predicting the equity premium realizations using our constructed cross-sectional measure.

Chapter 1

The Efficient Portfolio Frontier

In this chapter we analyze and derive the efficient portfolio frontier. In the first section we look at the meaning and main characteristics. The next section provides an analytical derivation of the efficient frontier in the matrix notation.

1.1 Characteristics

According to the model of portfolio choice developed by Harry Markowitz in 1959, investors that select a portfolio among the set of portfolios are risk averse and their decision is based exclusively on the value of mean and variance of the expected return. They minimize the variance of portfolio return, given expected return and at the same time maximize expected return, given variance. As a result, investors choose portfolios on the efficient portfolio frontier.

The concept of efficient portfolio frontier implies a boundary of the set of feasible portfolios that have the highest expected return for a given level of risk represented by the variance or standard deviation. Any portfolios that are situated above the efficient frontier (i.e. portfolios with higher expected return given variance) cannot be achieved. By contrast, portfolios beyond the frontier are dominated by those situated on the efficient frontier.

1.2 Analytical Derivation of the Efficient Frontier

The analytical derivation is based mainly on Merton (1970). He used a sum notation in his calculations. Here we apply the more modern matrix notation.

We use the following notation.

- i, j the subscripts denoting individual securities in a portfolio
- m the number of portfolio securities (i.e. $i, j \in \{1, ..., m\}$)
- γ_i the expected return on the security i
- σ_{ij} the covariance of returns between the securities i and j
- $\sigma_{ii} = \sigma_i^2$ the variance of the return on the security i
- $\Omega = [\sigma_{ij}]$ the variance-covariance matrix of returns
- w_i the weight of the security i in the portfolio (i.e. the percentage of the value of the portfolio invested in the security i)

The derivation is based on the following assumptions.

- $\sigma_i^2 > 0$ for all $i \in \{1, ..., m\}$
  Thus all securities in the portfolio are risky.
- $\sum_{i=1}^{m} w_i = 1$
  The total sum of weights assigned to securities in the portfolio is equal to one.
- $w_i > 1$ or $w_i < 0$ is possible for all $i \in \{1, ..., m\}$
  Borrowing and short-selling of all securities is allowed.
- Ω is a non-singular matrix
  That means a vector of return covariances between the security i ($i \in \{1, ..., m\}$) and all m securities in the portfolio cannot be represented as a linear combination of other such vectors. Hence these vectors are linearly independent.

From these m securities we can construct several portfolios differing by the weights of individual securities. The efficient frontier of all portfolios following the assumptions (i.e. the feasible portfolios) is defined as the locus of feasible portfolios with the smallest variance given expected return. Thus the efficient frontier is a set of portfolios which satisfy the constrained minimization problem

$$
\begin{aligned}
min \quad & \frac{1}{2} w^T \Omega w \\
subject\ to \quad & \gamma^T w = \alpha \\
& \mathbf{1}^T w = 1
\end{aligned}
\tag{1.1}
$$

where $\sigma^2 \equiv w^T \Omega w$ is the variance of the portfolio on the frontier[1] and α denotes its expected return. $\gamma = [\gamma i], i \in \{1, ..., m\}$ is the vector of expected returns of the portfolio securities and $w = [w_i], i \in \{1, ..., m\}$ the vector of weights assigned to each security. Symbol T as a superior index refers to a transposition (of a vector in this case). The last equation is equivalent to $\sum_1^m w_i = 1$. $\mathbf{1}$ is a vector of ones with a dimension equal to m.

To find the minimum of the given function we use Lagrange multipliers. The method of Lagrange multipliers provides a strategy for finding the extrema of a multivariate function subject to the constraints. Hence it is exploited in mathematical optimization. Using Lagrange multipliers, (1.1) can be rewritten as

$$
min \quad \{\frac{1}{2} w^T \Omega w + \lambda_1[\alpha - \gamma^T w] + \lambda_2[1 - \mathbf{1}^T w]\} \tag{1.2}
$$

where we minimize the Lagrangian of the problem and λ_1, λ_2 are the multipliers. A critical point occurs where the partial derivatives of Lagrangian with respect to w, λ_1 and λ_2 are equal to zero. Therefore the standard first order conditions for a critical point are

$$
\frac{\partial L}{\partial w} = \Omega w - \lambda_1 \gamma - \lambda_2 \mathbf{1} = \mathbf{0} \tag{1.3}
$$

$$
\frac{\partial L}{\partial \lambda_1} = \alpha - \gamma^T w = 0 \tag{1.4}
$$

$$
\frac{\partial L}{\partial \lambda_2} = 1 - \mathbf{1}^T w = 0 \tag{1.5}
$$

The solution of vector w that figures in equations (1.3), (1.4) and (1.5) is important to find by reason that it minimizes σ^2. w is unique by the assumption on Ω (the variance-covariance matrix of returns is regular, i.e. it is the square matrix that has an inverse). System of equations (1.3), (1.4) and (1.5) is linear in w, therefore it is simple to express from (1.3).

[1] $\frac{1}{2}$ can be omitted because the minimization of $\frac{1}{2} w^T \Omega w$ will minimize $w^T \Omega w$. Nevertheless, $\frac{1}{2}$ is beneficial during the differentiation with respect to w.

$$w = \lambda_1 \Omega^{-1}\gamma + \lambda_2 \Omega^{-1}\mathbf{1} \tag{1.6}$$

where Ω^{-1} is the matrix inverse of the variance-covariance matrix Ω. Multiplying (1.6) by γ^T from the left we obtain

$$\gamma^T w = \lambda_1 \gamma^T \Omega^{-1}\gamma + \lambda_2 \gamma^T \Omega^{-1}\mathbf{1} \tag{1.7}$$

and analogously multiplying (1.6) by $\mathbf{1}^T$ from the left

$$\mathbf{1}^T w = \lambda_1 \mathbf{1}^T \Omega^{-1}\gamma + \lambda_2 \mathbf{1}^T \Omega^{-1}\mathbf{1} \tag{1.8}$$

At this point in order to simplify we define

$$\begin{aligned} A &\equiv \gamma^T \Omega^{-1}\mathbf{1} = \mathbf{1}^T \Omega^{-1}\gamma \\ B &\equiv \gamma^T \Omega^{-1}\gamma \\ C &\equiv \mathbf{1}^T \Omega^{-1}\mathbf{1} \end{aligned}$$

where A, B and C are constants.[2]

From (1.4), (1.5), (1.7) and (1.8) we obtain a simple system of linear equations for λ_1 and λ_2.

$$\begin{aligned} \alpha &= B\lambda_1 + A\lambda_2 \\ 1 &= A\lambda_1 + C\lambda_2 \end{aligned} \tag{1.9}$$

Notice that $B > 0$ and $C > 0$, because Ω and Ω^{-1} are square, non-singular, symmetric and positive definite matrices (the variances of the returns on the portfolio securities are positive). Seeing that B and C are quadratic forms of Ω^{-1}, they are strictly positive (with the only exception of γ being a zero vector).

Solving this simple linear system for λ_1 and λ_2 we obtain the following solution.

$$\begin{aligned} \lambda_1 &= \frac{C\alpha - A}{D} \\ \lambda_2 &= \frac{B - A\alpha}{D} \end{aligned} \tag{1.10}$$

[2] The equality of mathematical terms in A results from the symmetry of Ω and Ω^{-1}. By the transposition of $\gamma^T\Omega^{-1}\mathbf{1}$ we obtain $\mathbf{1}^T\Omega^{-1}\gamma$. Since a constant transposition is applied, the expressions are equal.

where $D \equiv BC - A^2$ is positive.[3]

We substitute λ_1 and λ_2 in (1.6) by the value of λ_1 and λ_2 from (1.10).

$$w = \frac{C\alpha - A}{D}\Omega^{-1}\gamma + \frac{B - A\alpha}{D}\Omega^{-1}\mathbf{1}$$

After the exemption of $\frac{1}{D}$, merging components and joining them back together according to whether they contain α or not, we obtain the following result.

$$w = \frac{1}{D}[\alpha\Omega^{-1}(C\gamma - A\mathbf{1}) + \Omega^{-1}(B\mathbf{1} - A\gamma)] \tag{1.11}$$

The equation represents the solution of the proportions of risky assets held in the frontier portfolio.

We multiply (1.3) by w^T from the left to derive

$$w^T\Omega w = \lambda_1 w^T\gamma + \lambda_2 w^T\mathbf{1} \tag{1.12}$$

Connecting the definition of σ^2 and (1.12) with (1.4) and (1.5) we calculate $\sigma^2 = w^T\Omega w = \lambda_1 w^T\gamma + \lambda_2 w^T\mathbf{1} = \lambda_1\alpha + \lambda_2 1$. Fundamental is the fact that $w^T\mathbf{1} = \mathbf{1}^T w$ and also $w^T\gamma = \gamma^T w$ and thus

$$\sigma^2 = \lambda_1\alpha + \lambda_2 \tag{1.13}$$

Substituting for λ_1 and λ_2 from (1.10) into (1.13) we obtain the equation for the variance of a portfolio on the frontier. We can observe that the variance of a frontier portfolio is a function of its expected return by the means

$$\sigma^2 = \frac{1}{D}(C\alpha^2 - 2A\alpha + B) \tag{1.14}$$

Actual presentation of the frontier is situated in the mean-variance plane. Obviously, the frontier takes form of a parabola. Through the examination of the first and second derivatives of σ^2 with respect to α we capture information concerning an extreme point and convexity respectively.

[3] Ω^{-1} is positive definite and because $A\mathbf{1} - C\gamma$ is a non-zero vector, $(A\mathbf{1} - C\gamma)^T\Omega^{-1}(A\mathbf{1} - C\gamma) > 0$. After simple modifications we obtain $CD > 0$. But $C > 0$, hence $D > 0$.

$$\begin{aligned} \frac{\mathrm{d}\sigma^2}{\mathrm{d}\alpha} &= 2\ \frac{C\alpha - A}{D} \\ \frac{\mathrm{d}^2\sigma^2}{\mathrm{d}\alpha^2} &= 2\ \frac{C}{D} \end{aligned} \tag{1.15}$$

The second derivative of σ^2 with respect to α is positive by reason of C and D being positive as well. Hence, σ^2 is a strictly convex function of α. The first derivative of σ^2 with respect to α is equal to zero provided that $\alpha = \frac{A}{C}$. Thus, a unique minimum point of the frontier parabola has position data $\alpha = \frac{A}{C}$ and $\sigma^2 = \frac{1}{C}$ (from substitution of $\frac{A}{C}$ for α in (1.14)).

We denote $\bar{\alpha} \equiv \frac{A}{C}$ and $\bar{\sigma}^2 \equiv \frac{1}{C}$. $\bar{\alpha}$ and $\bar{\sigma}^2$ represent the expected return and variance of the minimum-variance portfolio, videlicet the portfolio with the minimum variance given expected return. Consequently we define $\bar{w}$ to be the vector of weights assigned to each security in the minimum-variance portfolio. In order to express a formula for $\bar{w}$ we substitute $\frac{A}{C}$ for α in (1.11).

$$w = \frac{1}{C}\Omega^{-1}\mathbf{1} \tag{1.16}$$

Figure 1.1 depicts the frontier in the form of parabola (see equation (1.14)). The graph is designed in MATLAB using the specific values of γ and Ω for the portfolio composed of two securities. However, to preserve a generality of the derivation, Figure 1.1 does not contain any specific values except for zero. Notice that the unique minimum point with co-ordinates $[\frac{A}{C} = \bar{\alpha}, \frac{1}{C} = \bar{\sigma}^2]$ represents the minimum-variance portfolio. The point of vertical axis intersection is $[0, \frac{B}{D}]$. This point appertains to a portfolio with the expected return equal to zero and variance equal to $\frac{B}{D}$.[4]

In comparison with the mean-variance plane, the mean-standard deviation plane is more usual. A formulation of the frontier is slightly different and we conceive the standard deviation of a frontier portfolio as a function of its expected return using a simple modification of equation (1.14).

$$\sigma = \sqrt{\frac{1}{D}(C\alpha^2 - 2A\alpha + B)} \tag{1.17}$$

Further we calculate and review the first and second derivatives of σ with respect to α.

[4]Foreseeing investors would not even choose a portfolio situated to the left of the minimum point. For the identical level of variance there always exists a portfolio with a higher expected return.

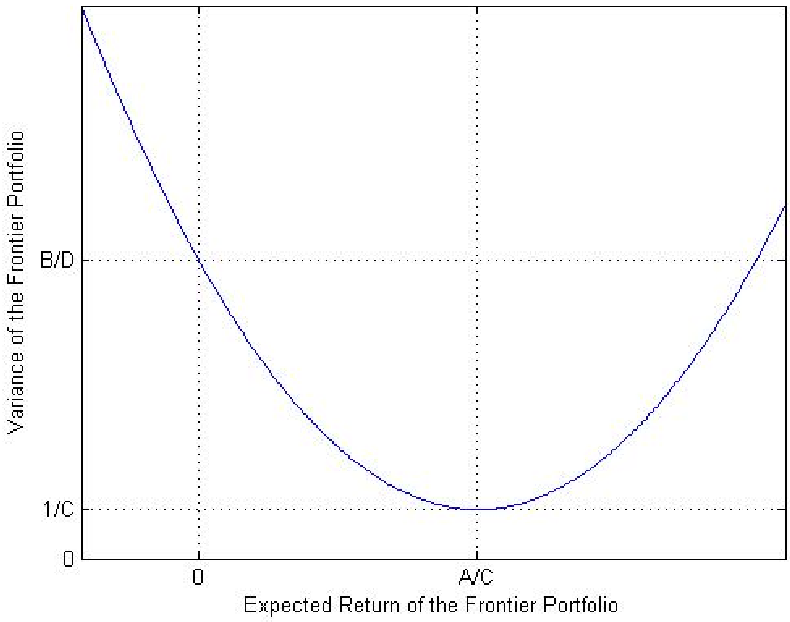

Figure 1.1: The Frontier Parabola in the Mean-Variance Plane

As regards the input data, suppose there exists a portfolio composed of two securities, i.e. $m = 2$. We assign $\gamma = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\Omega = \begin{pmatrix} 0,5 & 0,2 \\ 0,2 & 1,5 \end{pmatrix}$. That means the covariance of returns between the securities is $0,2$ and the variance of the returns on the securities is $0,5$ and $1,5$. The values of A, B, C and D calculated by MATLAB can be found in the following overview.

A	2,6761	**A/C**	1,1875
B	3,8028	**1/C**	0,4438
C	2,2535	**B/D**	2,7
D	1,4085		

$$\begin{aligned} \frac{\mathrm{d}\sigma}{\mathrm{d}\alpha} &= \frac{(C\alpha - A)}{D\sigma} \\ \frac{\mathrm{d}^2\sigma}{\mathrm{d}\alpha^2} &= \frac{1}{D\sigma^3} \end{aligned} \tag{1.18}$$

The second derivative is positive and hence σ is a strictly convex function of α. Considering the relation between a variance σ^2 and a standard deviation σ, the minimum-variance portfolio is equivalent to the minimum-standard

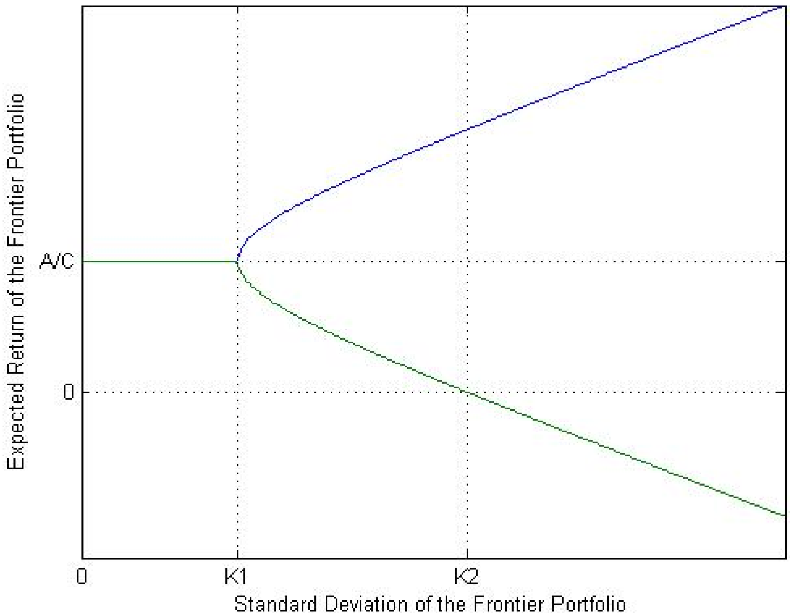

Figure 1.2: The Frontier in the Mean-Standard Deviation Plane

deviation portfolio.

Figure 1.2 depicts a graph of the frontier in the standard form. Notice the axes labels: the standard deviation σ on the abscissa axis and the expected return α on the vertical axis. The input data so as values of A, B, C and D are identical with the previous values calculated for the parabola in the mean-variance plane. Another important values are $\frac{A}{C} = 1,1875$, $K1 \equiv \sqrt{\frac{1}{C}} = 0,6661$ and $K2 \equiv \sqrt{\frac{A^2}{CD} + \frac{1}{C}} = 1,6432$. The minimum-variance portfolio is situated in the point with co-ordinates $[K1, \frac{A}{C}]$ and the point of intersection with the abscissa axis is $[K2, 0]$.

The equation for α as a function of σ is obtained from (1.17). It represents a formula for the expected return of a portfolio on the frontier regarding its standard deviation.

$$\alpha = \frac{A}{C} \pm \frac{1}{C}\sqrt{DC\left(\sigma^2 - \frac{1}{C}\right)} \tag{1.19}$$

Substituting $\bar{\alpha}$ for $\frac{A}{C}$ and $\bar{\sigma}^2$ for $\frac{1}{C}$ under the radix we obtain

$$\alpha = \bar{\alpha} \pm \frac{1}{C}\sqrt{DC(\sigma^2 - \bar{\sigma}^2)} \tag{1.20}$$

From all feasible portfolios, only those with the highest expected return for a given standard deviation are of our interest. The set of feasible portfolios possessing this characteristic is defined as the efficient portfolio frontier. It is situated in the upper blue part of the frontier in Figure 1.2 starting with the minimum-variance portfolio. As a consequence we can specify the final form of the equation for the efficient portfolio frontier as follows.

$$\alpha = \bar{\alpha} + \frac{1}{C}\sqrt{DC(\sigma^2 - \bar{\sigma}^2)} \tag{1.21}$$

Chapter 2

The Capital Asset Pricing Model

The capital asset pricing model (CAPM) developed by William Sharpe and John Lintner was the first boundary mark in the theory of asset pricing. The main idea consists in the expression of the relation between the expected return and risk (represented as the variance of the expected return) of a certain portfolio. According to Fama and French (2004), in spite of the fact that CAPM is still widely used, intuitive and powerful model, it faces failings in empirical implementation. This can be caused by many simplifying assumptions or difficulties in implementing valid tests of the model.

We begin by explaining the logic of the CAPM. We distinguish between the investment opportunities including exclusively risky securities (i.e. the variance of the return on the securities is positive) and investment opportunities with a risk-free security (risk-free borrowing and lending is allowed). Applying the assumptions we introduce Sharpe-Lintner CAPM equation.

2.1 The basis and key assumptions

Harry Markowitz's model of portfolio choice introduces and defines a concept of mean-variance-efficient portfolios. A risk averse investor selects a portfolio that produces a stochastic return at the end of the period. This model assumes that the investor's decision is based purely on the mean and the variance of his investment return. Hence he minimizes the variance of portfolio return, given expected return and at the same time maximizes the expected return, given variance. The outcome of this optimization process is that he always chooses a mean-variance-efficient portfolio, because it satisfies his requirements (there does not exist any portfolio providing higher expected return, given return variance or lower return variance, given expected return).

The consequent task is to identify the portfolio that must be mean-variance-efficient. It cannot be achieved without defining assumptions. Sharpe and Lintner add two key assumptions to the Markowitz model:

- complete agreement
  Given market clearing security prices, investors agree on the joint distribution of security returns during the next period.

- borrowing and lending at a risk-free rate
  It applies for all investors and does not depend on the amount borrowed or lent, i.e. it is unlimited.

2.2 Investment opportunities without a risk-free security

Investment opportunities including only risky securities are described in Figure 1.2. The portfolio risk is measured by the standard deviation of portfolio return (horizontal axis). We derived the equation for the efficient portfolio frontier in Chapter 1. It represents the set of combinations of expected return and risk for portfolios of risky securities. The common characteristics of these combinations is that they minimize the variance of portfolio returns at different given levels of expected return.

Regarding investors, they choose the level of expected return they want and must accept the corresponding volatility of returns (return risk) that is the lowest possible for this level of return. The higher return he wants, the higher volatility he must accept. If there is no risk-free borrowing or lending, only portfolios located in the upper part of the frontier are mean-variance efficient (these portfolios maximize expected return given their return variance).

2.3 Investment opportunities with risk-free borrowing and lending allowed

Adding risk-free borrowing and lending, investors combine risky securities with a risk-free security. Suppose that proportion x of portfolio p funds is invested in a risk-free security f and $1 - x$ in some risky portfolio g. That means

$$R_p = xR_f + (1-x)R_g, \quad x \leq 1 \tag{2.1}$$

where R_p denotes the return on the portfolio p, R_f the return on the risk-free security f and R_g the return on the risky portfolio g. Then the expected return on the portfolio p and the standard deviation of portfolio return can be expressed

$$\begin{aligned} E(R_p) &= xR_f + (1-x)E(R_g) \\ \sigma(R_p) &= (1-x)\sigma(R_g) \end{aligned} \tag{2.2}$$

as R_f is known beforehand and therefore $\sigma(R_f) = 0$.

Equations (2.2) imply that the portfolios combining risk-free lending or borrowing with some risky portfolio g plot along a straight line from R_f through g.[1] The position of the portfolio p depends on the proportion x.

- $x = 1$
  All funds are invested in the risk-free security f (loaned at the risk-free rate of interest). Then the portfolio p has zero variance, the risk-free rate of return and is located in the point $[0, R_f]$.

- $x \in (0, 1)$
  Portfolio funds are divided between f (risk-free lending) and g (positive investment in the risky portfolio). In this case, p is located somewhere on the straight line between R_f and g.

- 0
  This case implies that investors do not use a possibility of risk-free borrowing or lending (see Section 2.2).

- $x < 0$
  The result is a point to the right of g on the line. These points represent borrowing at the risk-free rate. The proceeds from the borrowing is used to increase investment in g.

After this manner, several portfolios p can be obtained by combining a risk-free security with some risky portfolio. To find the mean-variance-efficient portfolios we swing a line from R_f through different feasible risky portfolios. The result of a simple observation is following. With higher slope comes higher expected return, given variance and lower variance, given expected return. To obtain the portfolios with the best tradeoff between expected return and risk, we design a line from R_f through portfolio T,

[1] That means from the point with co-ordinates $[0, R_f]$ through the $[\sigma(R_g), E(R_g)]$.

which is the tangency portfolio to the efficient frontier. Hence mean-variance-efficient portfolios are combinations of the risk-free security (either risk-free borrowing or lending) and a single risky tangency portfolio T (Fama and French, 2004).

With the assumption of complete agreement, all investors see the same investment opportunities, combine risk-free borrowing or lending with the same risky tangency portfolio T and therefore T is the value-weight market portfolio M (see Figure (2.1)). The weight of each risky security in the market portfolio is calculated as the total market value of all outstanding units of the security divided by the total market value of all risky securities. The prices of risk-free securities and the value of risk-free rate must clear the market for risk-free borrowing and lending.

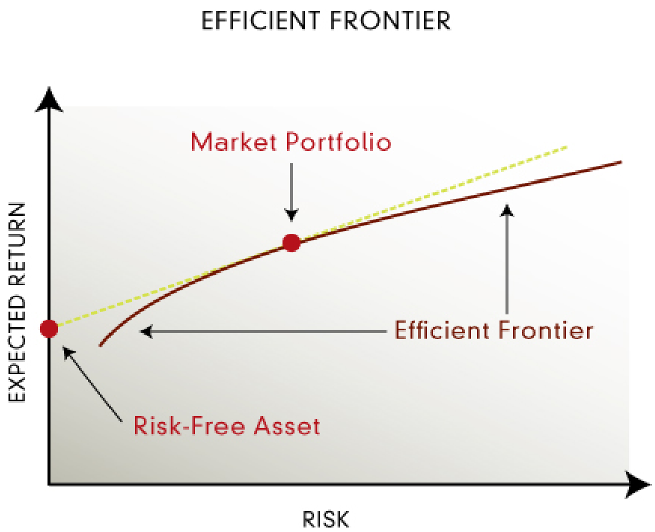

Figure 2.1: Investment Opportunities

Since the market portfolio M is located on the efficient frontier, the relation for any minimum variance portfolio holds for the market portfolio as well. The minimum variance condition for M is

$$E(R_i) = E(R_{ZM}) + [E(R_M) - E(R_{ZM})]\, \beta_{iM}, \quad i \in \{1, ..., N\} \tag{2.3}$$

where $E(R_i)$ is the expected return on security i and there are N risky assets. β_{iM} is the market beta of security i equal to the covariance of its return with the market return divided by the variance of the market return.

$$\beta_{iM} = \frac{cov(R_i, R_M)}{\sigma^2(R_M)} \tag{2.4}$$

$E(R_{ZM})$ is the expected return on securities that have market betas equal to zero (their returns are uncorrelated with the market return). The term $[E(R_M) - E(R_{ZM})]\ \beta_{iM}$ represents a risk premium and $E(R_M) - E(R_{ZM})$ is a premium per unit of beta. The market beta of security i has more than one interpretation.

1. β_{iM} measures the sensitivity of the security return to variation in the market return.

   It results from the fact that beta of security i is the slope in the regression of the security return on the market return.

2. β_{iM} represents the covariance risk of security i in the market portfolio relative to the average covariance risk of all securities (the variance of the market return). That means β_{iM} is proportional to the risk each dollar invested in security i contributes to the market portfolio.

   The variance of the market return can be rewritten as follows (with x_{iM} denotative the weight of security i in the market portfolio).

   $$\sigma^2(R_M) = Cov(R_M, R_M) = Cov\left(\sum_{i=1}^{N} x_{iM}R_i, R_M\right) = \sum_{i=1}^{N} x_{iM}Cov(R_i, R_M)$$

   We see that the risk of the market portfolio (the denominator of β_{iM}) is equal to a weighted average of the covariance risks of the securities in M (the numerators of β_{iM}).

Equation (2.3) resembles Sharpe-Lintner CAPM. The only difference is related to the term $E(R_{ZM})$. The beta of a security is equal to zero when its return is uncorrelated with the market return (see equation (2.4))[2]. Hence

[2]The average of the covariances between the return on the security i and the return on other securities in the market portfolio just offsets the variance of the return on the security i.

the security contributes nothing to the variance of the market return and is riskless in the market portfolio.

Under the opportunity of risk-free borrowing and lending $E(R_{ZM})$ must equal the risk-free rate R_f. Thus we obtain Sharpe-Lintner CAPM equation.

$$E(R_i) = R_f + [E(R_M) - R_f]\ \beta_{iM}, \quad i \in \{1, ..., N\} \tag{2.5}$$

The expected return on security i equals the sum of the risk-free interest rate and a risk premium (premium per unit of beta risk times the market beta of security i).

Chapter 3

The Equity Premium

3.1 The link between CAPM and the expected equity premium

The capital asset pricing model predicts that a level of risk determines the expected return of a stock - risky stock should have higher expected returns than less risky stocks. The beta of a stock (the regression coefficient of a stock's return on the market return) is specified as the relevant measure of risk. The expected return premium per one unit of beta is the expected equity premium and equals to the expected return on the value-weight market portfolio less the risk-free rate. The Sharpe-Lintner CAPM holds for every period. After the addition of a time dimension, CAPM can link the time series and cross-section (Polk, Thompson and Vuolteenaho, 2006).

$$E_{t-1}(R_{i,t}) = R_{f,t-1} + [E_{t-1}(R_{M,t}) - R_{f,t-1}]\,\beta_{i,t-1}, \quad i \in \{1, ..., N\} \tag{3.1}$$

where $R_{i,t}$ is the return on asset i during the period t. $R_{f,t-1}$ is the risk-free rate during the period t. It is known beforehand, at the end of period $t-1$. $\beta_{i,t-1}$ is the beta of asset i known at time $t-1$. $E_{t-1}(R_{M,t}) - R_{f,t-1}$ represents the expected market premium.

The expected return on a stock should be negatively related with its price. The high expected return can be caused by the high equity premium, the high beta of stock i or both and should result in the low price of the stock. According to Gordon (1962), risk premium can be forecasted using a following stock-valuation model.

$$\frac{D_i}{P_i} - R_f + E(g_i) = E(R_i) - R_f \tag{3.2}$$

where $\frac{D_i}{P_i}$ is the dividend yield of stock i and $E(g_i)$ the expected dividend growth of stock i. After the substitution for $E(R_i)$ from CAPM and assuming that betas and the risk-free rate are constant yields we obtain

$$\frac{D_{i,t}}{P_{i,t-1}} \approx E_{t-1}[R_{M,t} - R_f]\beta_i - E(g_i - R_f) \tag{3.3}$$

There exist three reasons for the dividend yield on stock i to be high:

- The expected equity premium $E_{t-1}[R_{M,t} - R_f]$ is high.
- Stock i has a high beta β_i.
- The dividends of stock i are expected to grow slowly.

Regressing the cross-section of dividend yields on betas and expected dividend growth we obtain

$$\frac{D_{i,t}}{P_{i,t-1}} \approx \lambda_{0,t-1} + \lambda_{1,t-1}\beta_i + \lambda_{2,t-1}E(g_i) \tag{3.4}$$

Polk, Thompson and Vuolteenaho (2006) measure $\lambda_{1,t-1}$ for each period using cross-sectional data and subsequently forecast the next period's equity premium. They propose a number of cross-sectional risk premium measures along with the construction and results summary. In the next section we provide a construction of their λ^{SRC}.

3.2 Lambda SRC - λ^{SRC}

In this section we introduce our version of λ^{SRC} as a proxy for the risk premium. There is an intention for λ^{SRC} to be a valid cross-sectional variable in regression forecasting the equity premium. Hence we have to be careful not to include any look-ahead information.

Our proxy is based on the ordinal association measure between a stock's beta and its valuation ratios. By using the ordinal measure and ranking procedures during the calculation we avoid an outlier impact leading to robustness. This pertains not only to the possible outliers in the underlying data but also to extreme values of the proxy itself. On the other hand there is a possible loss of information about the magnitude of the spread in valuation multiples.

The first step before implementing λ^{SRC} is to obtain required data. Secondly we transform these row data sets into the eligible condition to serve as the input data for further calculations. The construction of λ^{SRC} then consists of three parts.

3.2.1 Data and input summary

For our purposes we select US stock data for firms included in S&P 500 index and compute λ^{SRC} predictions from June 1991 to May 2012. The data required in calculations come from Datastream database[1] and are summarized below. The source data are weekly (for Set 1) or monthly (for Set 2) values of corresponding variables. Index i represents individual firms of S&P 500 Index ($i \in 1, ..., 500$).

- $D_{i,t}$
  represents the total dividends paid by the firm i from June year $t-1$ to May year t (included)
- $BE_{i,t}$
  represents the book value of the firm i for fiscal year end in year $t-1$
- $E_{i,t}$
  represents the earnings per share of the firm i for fiscal year end in year $t-1$
- $C_{i,t}$
  represents the cash flow of the firm i for fiscal year end in year $t-1$
- $P_{i,t}$
  represents the price per share of the firm i for the end of May year t
- $ME_{i,t}$
  represents the market equity value of the firm i for the end of May year t calculated as a product of price and number of shares

Since we do not have the information about the exact date of reporting BE, E and C values for previous fiscal year ends, as a compromise we take these values from specific date - the fiscal year ends (of previous year) + 100 days. For each year we assume these values to be updated within 100 days after fiscal year end.

[1]Datastream database provides historical financial statistics for different securities, including stock data and interest rates.

- $r_{i,t}$
  represents the monthly return of the individual stock i at time t

- $r_{M,t}$
  represents the monthly market portfolio return at time t, which in our case is the monthly return of S&P 500 Index

- $R_{f,t}$
  represents the risk-free rate[2] at time t

3.2.2 Construction of λ^{SRC}

Lambda SRC construction is a process that contains three steps. The input data are prepared using Excel. However, in order to effectively handle further calculations within λ_{SRC} construction we continue using MATLAB, which is a perfect tool for our purposes (all functions are enclosed).

Step 1 - $VALRANK$

Every year t during the period from 1991 to 2011 we select firms that are components of S&P 500 Index at the end of May year t. For these firms we construct up to four valuation ratios D/P, BE/ME, E/P and C/P as follows. We match actual $D_{i,t}$, $ME_{i,t}$ and $P_{i,t}$ known at the end of May with $BE_{i,t}$, $E_{i,t}$ and $C_{i,t}$ for all fiscal year ends in calendar year $t-1$ that we assume to be known by this time.
Each year we transform these ratios into a relative percentile rank, which is the rank divided by the number of firms for which the data are available. Subsequently we average the available valuation ratio percentile ranks for each firm and re-rank this average across firms.[3]
After this manner we calculate $VALRANK_{i,t}$ as our expected return measure with the values from interval zero to one. $VALRANK_{i,t}$ is negatively correlated with the price level of stocks,i.e. low values of $VALRANK_{i,t}$ correspond to high prices and also to low expected returns. Since this annual composite measure is constructed from several considerable firm-level indicators,we could consider it to be closely connected to firm valuation.

[2]As a risk-free rate we take yields on Treasury nominal securities at "constant maturity" (in this case 1 year) interpolated by the US Treasury from the daily yield curve for non-inflation-indexed Treasury securities. This curve relates the yield of a security to its time to maturity.

[3]Notice that due to the repeated ranking there is no reason to be concerned about the units or whether the valuation ratios are per share.

Step 2 - $BETA$

At this point we estimate the monthly measure of risk for each stock - the market beta $\beta_{i,t}$ by OLS regression of monthly returns $r_{i,t}$ on a constant and the contemporaneous monthly return on the S&P 500 Index:

$$r_{i,t} = \beta_{0,i} + \beta_{1,i} r_{M,t} + \epsilon_t \tag{3.5}$$

Each month we use three years of previous monthly returns skipping months in which a firm is missing returns.

Step 3 - Spearman rank correlation coefficient

Our cross-sectional proxy λ_t^{SRC} is the Spearman rank correlation coefficient between $VALRANK_{i,t}$ and $\beta_{i,t}$ at time t. Notice that the same value of $VALRANK_{i,t}$ belongs to twelve values of $\beta_{i,t}$. λ_t^{SRC} is updated monthly and the time series begins in June 1991 and ends in May 2012.

Chapter 4

Empirical results

For the purpose of evaluation the predictive ability of our risk premium measure Lambda SRC, we estimate descriptive statistics not only for λ^{SRC}, but also for R^e_M which is the excess return on the S&P 500 Index. We use R^e_M as a measure of the realized eqity premium and is computed as the difference between the simple return on S&P 500 Index and a risk-free rate $r_{f,t}$. The results are following:

λ^{SRC}	Mean	Med	SD	Min	Max
Full period	-0,2118	-0,202	0,1307	-0,4958	0,08355
Before	-0,2298	-0,2105	0,1183	-0,4958	-0,018
After	-0,1289	-0,1577	0,1531	-0,3313	0,0836

R^e_M	Mean	Med	SD	Min	Max
Full period	0,0132	0,0159	0,1286	-0,7214	0,5995
Before	0,0101	0,012	0,0814	-0,3074	0,3083
After	0,0272	0,0564	0,2511	-0,7214	0,5995

where Med denotes median, SD is the standard deviation, Min is a minimum and Max a maximum value in the sample.
Notice that we included three rows, each row representing the different sample range:

1. Full period
   - consists of the whole period from June 1991 to May 2012

2. Before
   - period from June 1991 to August 2008

3. After
   - period from September 2008 to May 2011

The observation period is divided into two parts. This way we can compare the results for each period. The Before row represents the period before the financial crisis and the After row belongs to the period during and after the crisis. We can see that the mean and median for both λ^{SRC} and R^e_M are significantly higher in the second subsample.

Furthermore we calculate correlations between $R^e_{M,t}$, λ^{SRC}_t,$R^e_{M,t-1}$ and λ^{SRC}_{t-1} in this exact order.

Full period:

1	0,0177	-0,1021	0,034
0,0177	1	0,0046	0,9505
-0,1021	0,0046	1	0,0159
0,034	0,9505	0,0159	1

Before:

1	0,038	0,2047	0,0504
0,038	1	0,0295	0,9623
0,2047	0,0295	1	0,0296
0,0504	0,9623	0,0296	1

After:

1	-0,0404	-0,2577	-0,0144
-0,0404	1	-0,0639	0,8988
-0,2577	-0,0639	1	-0,0367
-0,0144	0,8988	-0,0367	1

We can observe that the correlation of λ^{SRC}_t and $R^e_{M,t}$ is weak in the full sample (only 0,02). In the "before-crisis" subsample this correlation is higher (0,04). However in "after-crisis" subsample it is surprisingly negative. This result implies that especially in the "after-crisis subsample" our λ^{SRC}_t is a very poor predictor of the realized equity premium.

The last table containing θ and *pvalue* represents the results from the model

$$R^e_{M,t} = \mu_1 + \theta\lambda^{SRC}_{t-1} + u_t \tag{4.1}$$

	θ	p value
Full period	0,0206	0,1852
Before	0,0185	0,1391
After	0,0227	0,6544

These results reveal that λ_t^{SRC} does not forecast the future excess market returns. Because of the high p values we cannot reject the hypothesis of a zero coefficient in favor of a positive coefficient. The measureλ_t^{SRC} is not a significant predictor in neither of the samples.

Conclusion

Our empirical results differ from the results of Polk, Thompson and Vuolteenaho significantly. Their cross-sectional measure of the equity premium is a good predictor of the future market returns. However, we discovered that the λ_t^{SRC} does a poor job predicting the future excess market returns.

There are many reasons for the results to be different, although it is difficult to tell which one is most significant. On the other it is obvious that some of the factors play an important role in modifying the result to a great extent. Firstly, it is the result of the data choice - we use the different sample also selecting S&P 500 Index as our market portfolio instead of choosing CRSP value-weight Index. Secondly, the period is biased. Polk, Thompson and Vuolteenaho use period from 1927 to 2002 whereas our period lies between 1991 to 2012 and has experienced the financial instability during the financial crisis.

The opportunities to explore this issue further are wide. One can construct a number of different variables with a range of techniques and assumptions trying to construct the equity premium forecasts. However, each variable has to be subsequently tested against data to reveal the predictive skills.

Bibliography

[1] C. MERTON, R.:*An Analytic Derivation of the Efficient Portfolio Frontier*, 493-70, October 1970

[2] FAMA, E.; FRENCH, K. : *The Capital Asset Pricing Model: Theory and Evidence*, The Journal of Economic Perspectives, Vol. 18, No.3, Summer 2004

[3] POLK, CH.; THOMPSON, S.; VUOLTEENAHO, T. : *Cross-sectional forecasts of the equity premium*, Journal of Financial Economics 81 (2006) 101-141

Appendix

VALRANK

```
function [VALRANK] = vypocet valrank (DP,BM,EP,CP)

   rozmer=zeros(4,2);
   [rozmer(1,1),rozmer(1,2)] =size(DP);
   [rozmer(2,1),rozmer(2,2)]=size(BM);
   [rozmer(3,1),rozmer(3,2)]=size(EP);
   [rozmer(4,1),rozmer(4,2)]=size(CP);

   - DIMENSION CONTROL

   for i=1:4
   if rozmer(i,1) =500
   error('The number of firms is not correct.')
   end
   if rozmer(i,2) =21
   error('The number of years is not correct.')
   end end

   - RELATIVE PERCENTILE RANKINGS OF FIRMS (EACH YEAR)

   for i=1:21
   a=isnan(DP(:,i));
   b=500-sum(a);
   DP(:,i)=tiedrank(DP(:,i))/b;
   a=isnan(BM(:,i));
   b=500-sum(a);
   BM(:,i)=tiedrank(BM(:,i))/b;
   a=isnan(EP(:,i));
   b=500-sum(a);
```

```
EP(:,i)=tiedrank(EP(:,i))/b;
a=isnan(CP(:,i));
b=500-sum(a);
CP(:,i)=tiedrank(CP(:,i))/b;
end
```

- THE AVERAGE OF PERCENTILE RANKS FOR EACH FIRM

```
for i=1:500
for j=1:21
v=[DP(i,j) BM(i,j) EP(i,j) CP(i,j)];
poc=4-sum(isnan(v));
Average(i,j)=nansum(v)/poc;
end
end
```

- RE-RANK ACROSS FIRMS

```
for i=1:21
VALRANK(:,i)=tiedrank(Average(:,i))/500;
end
end
```

BETA

```
function [BetaRok] = vypocet beta (r,rM)

b=zeros(500,12);

for i=1:500
for j=1:12
Y=r(i,j:(j+35))';
x2=rM(j:(j+35),1);
```

-MODIFICATION TO INCLUDE ONLY DATA AVAILABLE

```
a=isnan(Y);
indexy=find(a==0);
Y mod=Y( isnan(Y));
```

```
x1=ones(length(Y mod),1);
x2 mod=x2(indexy,1);

-REGRESSION

X=[x1,x2 mod];
pomb=regress(Y mod,X);
b(i,j)=pomb(2,1);
end
end

BetaRok=b;
end
```

LAMBDA SRC

function [SRC] = lambda SRC (VALRANK,BETA)

```
-VALRANK EXTENSION

k=0;
for i=1:21
v=VALRANK(:,i);
for j=1:12
pomVALRANK(:,j+k)=v;
end
k=k+12;
end

-SPEARMAN RANK CORRELATION COEFFICIENT

for i=1:252
rankedVALRANK(:,i)=tiedrank(pomVALRANK(:,i));
rankedBETA(:,i)=tiedrank(BETA(:,i));
end

dif=rankedVALRANK-rankedBETA;
d=dif.^2;
dsum=sum(d);
```

```
n=500;

for i=1:252
SRC(1,i)=1-((6*dsum(1,i))/(n*(n2̂-1)));
end
end
```

DESCRIPTIVE STATISTICS

```
function [DES,COR] = descriptive (rMe,SRC)

DES=zeros(2,5);
DES(1,:)=[mean(rMe),median(rMe),std(rMe),min(rMe),max(rMe)];
DES(2,:)=[mean(SRC),median(SRC),std(SRC),min(SRC),max(SRC)];

if length(rMe) =length(SRC)
error('The dimensions must agree!')
end
n=length(rMe);

rMet=rMe(2:n);
rMet 1=rMe(1:(n-1));
SRCt=SRC(2:n);
SRCt 1=SRC(1:(n-1));
X=[rMet,SRCt,rMet 1,SRCt 1];
COR=corrcoef(X);

end
```

REGRESSION COEFFICIENTS

```
function [beta,pval] = predictor (rMe,SRC)

if length(rMe) =length(SRC)
error('The dimensions must agree!')
end
```

```
n=length(rMe);

y=rMe(2:n);
X=SRC(1:(n-1));
stats=regstats(y,X,'linear','beta','tstat');

beta=stats.beta;
tstat=stats.tstat;
pval=tstat.pval;

end
```