COMENIUS UNIVERSITY IN BRATISLAVA
Faculty of Mathematics, Physics and Informatics

NUMERICAL TREATMENT OF OPTIMAL LIQUIDATION OF A LARGE TRADING POSITION

Master's thesis

Bratislava 2014 **Bc. Ján Komadel**

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Study program:	Ekonomic and Financial Mathematics
Branch of Study:	1114 Applied Mathematics
Supervisor:	prof. RNDr. Pavel Brunovský, DrSc.

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UNIVERZITA KOMENSKÉHO V BRATISLAVE
FAKULTA MATEMATIKY, FYZIKY A INFORMATIKY

NUMERICKÉ SPRACOVANIE ÚLOHY OPTIMÁLNEJ LIKVIDÁCIE MASÍVNEJ OBCHODNEJ POZÍCIE

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Abstract

KOMADEL, Ján: Numerical Treatment of Optimal Liquidation of a Large Trading Position [Master's Thesis], Comenius University in Bratislava, Faculty of Mathematics, Physics and Informatics, Department of Applied Mathematics and Statistics; Supervisor: prof. RNDr. Pavel Brunovský, DrSc., Bratislava, 2014

The main contribution of this Master's thesis is an alternative approach to solving the problem of optimal liquidation of a large trading position. The problem is formulated in [4] where the author mentions that standard numerical methods fail in solving it due to instability of solutions. We propose an alternative approach based on truncating the problem to a finite time horizon and finding the solution to the original problem as a limit of solutions to the finite horizon problems. We use this approach in three numerical methods which we demonstrate on numerical examples. We compare the methods and conclude that the explicit Euler method is the most suitable one for this problem. The experiments confirm that our suggested procedure leads to good approximations of the searched solution. We also provide proofs of monotonicity of the value function for the finite horizon problem with respect to the length of the time interval.

Keywords: Optimal liquidation, numerical treatment, singular problem, Hamilton-Jacobi-Bellman equation

Abstrakt

KOMADEL, Ján: Numerické spracovanie úlohy optimálnej likvidácie masívnej obchodnej pozície [Diplomová práca], Univerzita Komenského v Bratislave, Fakulta matematiky, fyziky a informatiky, Katedra aplikovanej matematiky a štatistiky; Vedúci práce: prof. RNDr. Pavel Brunovský, DrSc., Bratislava, 2014

Hlavným prínosom tejto diplomovej práce je alternatívny prístup k riešeniu problému optimálnej likvidácie veľkej obchodnej pozície. Tento problém je formulovaný v [4], kde autor uvádza, že štandardné metódy zlyhávajú v jeho riešení z dôvodu nestability riešení. My navrhujeme alternatívny postup založený na obmedzení úlohy na konečný časový horizont a hľadaní riešenia pôvodnej úlohy ako limity riešení úloh na konečnom horizonte. Tento prístup použijeme v troch numerických metódach, ktoré predstavíme na numerických príkladoch. Metódy porovnáme a usúdime, že explicitná Eulerova metóda je najvhodnejšia pre tento problém. Experimenty potvrdzujú, že nami navrhovaný postup vedie k dobrým aproximáciám hľadaného riešenia. Tiež uvedieme dôkazy monotónnosti hodnotovej funkcie pre úlohu na konečnom horizonte s ohľadom na dĺžku časového intervalu.

Kľúčové slová: Optimálna likvidácia, numerické riešenie, singulárna úloha, Hamilton-Jacobi-Bellmanova rovnica

CONTENTS

INTRODUCTION

One of the many problems which one encounters in finance is the problem of an optimal liquidation of a trading position. The aim is to sell certain amount of an asset in a way which maximizes the expected revenue. We consider a model which assumes stochastic underlying dynamics for the price of the asset and the actual market price is affected by the amount sold by the investor.

The considered optimal liquidation problem is inspired by [4] where it is applied to a speculative attack. The relevant part of the attack for our work is when the speculator already owns a large amount of foreign currency and wishes to convert it optimally back to domestic currency. The asset in this case is the foreign currency and the price is the exchange rate.

In the first chapter, we formulate this problem as an optimal control problem and we derive the Hamilton-Jacobi-Bellman (HJB) equation for its value function $w(y, z)$. It is a singular second order partial differential equation for a function of two variables which can be transformed to a one-dimensional problem. The transformed ordinary differential equation is the area of interest in our work as well as in [3], where the authors prove the existence and uniqueness of the solution $u(x)$ under certain conditions for the parameters, and they also prove some properties of this solution. We list these results and we add a proof of the upper bound for the solution.

In chapter 2, we present an alternative approach to the problem which leads to solving a parabolic partial differential equation the solutions to which tend to $u(x)$. We also describe an alternative initial condition and we show monotonicity of the value function with respect to the length of the considered time interval.

The third chapter is dedicated to the numerical treatment of the problem. We argue why standard methods cannot be used and we refer to specific examples from literature. Then we describe three numerical methods which are based on the alternative approach, the explicit Euler method, the predictor-corrector and the orthogonal collocation.

In the final chapter, we present the three methods on four numerical examples. The first example is the non-stochastic case where we compare our results to the analytical solution presented in [4]. The other three examples are stochastic with different values of parameters. We include graphical illustrations of the resulting solutions which help us verify whether the solution has the properties proven in [3] and listed in chapter 1. We also analyze the approximation error.

The aim of our work is to find a way of solving the formulated problem of optimal liquidation of a large trading position by treating the corresponding HJB equation numerically. According to [4], straightforward numerical treatment of the equation proved problematic which is why we propose an alternative approach. This approach may be used to solve similar problems with singularities where standard methods fail.

CHAPTER 1

THE OPTIMAL LIQUIDATION PROBLEM

In this chapter we formulate the optimal liquidation problem where the aim is the expected revenue-maximizing sale of a large trading position. We derive the corresponding Hamilton-Jacobi-Bellman equation and transform this partial differential equation to an ordinary differential equation. Finally, we describe the properties of the solution which are proven in [3].

1.1 Problem Formulation

The problem we consider in this work is a generalization of one of the problems formulated in [4]. We consider a major investor who owns a certain amount of an asset $z(0)$ and wishes to maximize his expected revenue from selling it. We denote by $z(t)$ the amount of the asset which the investor owns at time t, and by $y(t)$ the so called shadow price of the asset. It is the price which would prevail if there were no interventions. The total expected revenue from the sale of the asset is given by

$$E_0 \left(\int_0^{T(z=0)} e^{-\rho s} f(s) \Big[y(s) - \eta f\big(y(s), z(s)\big) \Big] ds \right) , \tag{1.1}$$

where $T(z=0)$ is the first time t when $z(t)=0$, ρ is the discount rate, and $f(s)$ is the amount of the asset sold at time s. The amount sold f is not multiplied only by the shadow price y but it is reduced by ηf as well. This takes into account the fact that the real price which the investor receives by selling, is negatively influenced by the amount which he decides to sell. The constant η has the role of Kyle's lambda which is a measure of market impact named from the well known paper [6] by A. Kyle. It describes the sensitivity of the price to market interventions which are in this model represented only by the investor's actions.

The discount factor ρ in the expected revenue (1.1) can be interpreted in two ways. The investor either does not have his own funds and ρ is the interest rate at which he borrows the resources, or he owns the money which could alternatively be invested with an interest rate of ρ.

With a certain abuse of notation, we will write the amount sold $f(s)$ as $f\big(y(s),z(s)\big)$ to emphasize that it is the the investor's strategy which he chooses based on the values of the shadow price y and the owned amount z at time s. The problem which he is facing can then be formulated as the optimal control problem

$$w(y(0),z(0)) = \max_f E_0\left(\int_0^{T(z=0)} e^{-\rho s} f\big(y(s),z(s)\big)\Big[y(s)-\eta f\big(y(s),z(s)\big)\Big]ds\right) \tag{1.2}$$

subject to

$$dy(t) = \lambda y(t)dt + \sigma y(t)dW(t), \tag{1.3}$$

$$dz(t) = \Big[r^* z(t) - f\big(y(t),z(t)\big)\Big]dt, \tag{1.4}$$

with the initial values $y(0)$ a $z(0)$.

The shadow price $y(t)$ follows the geometric Brownian motion (1.3), where $W(t)$ is the standard Wiener process and σ is a volatility parameter. It is a stochastic process and the parameter λ represents its expected growth rate. The amount of the asset owned by the investor follows the dynamics (1.4), where r^* is the rate at which the amount of the asset grows. This growth is reduced by the amount f which the investor sells.

In the speculator's case, ρ can be thought of as domestic interest rate while r^* is foreign interest rate and λ is the average rate of depreciation of domestic currency.

1.2 Derivation of the HJB Equation

The value function $w(y(0), z(0))$ defined by (1.2) represents the expected revenue from the optimal disposal of the amount $z(0)$ of the asset under the current shadow price $y(0)$. The Hamilton-Jacobi-Bellman (HJB) equation for the value function can be derived by use of the dynamic programming equation (DPE) which is used for discrete problems. By the Bellman optimality principle, the value function $w(y(t), z(t))$ should, by the transition from time t to time $t + \Delta t$, satisfy

$$\begin{aligned} w\big(y(t), z(t)\big) &= \max_f E_t\Big[\int_t^{t+\Delta t} e^{-\rho(s-t)} f\big(y(s) - \eta f\big) ds + e^{-\rho \Delta t} w\big(y(t+\Delta t), z(t+\Delta t)\big)\Big] \\ &= \max_f \Big\{ f\big(y(t) - \eta f\big)\Delta t + O(\Delta t) + E_t\Big[e^{-\rho\Delta t} w\big(y(t+\Delta t), z(t+\Delta t)\big)\Big]\Big\}. \end{aligned} \tag{1.5}$$

The second equality comes from the fact that we only consider small values of Δt as we will take the limit $\Delta t \to 0$. We replace the term $w\big(y(t+\Delta t), z(t+\Delta t)\big)$ by the first order Taylor expansion

$$\begin{aligned} w\big(y(t+\Delta t), z(t+\Delta t)\big) &= w\big(y(t), z(t)\big) + \lambda y(t) w_y\big(y(t), z(t)\big)\Delta t \\ &\quad + \sigma y(t) w_y\big(y(t), z(t)\big) \left[W(t+\Delta t) - W(t)\right] \\ &\quad + \left[r^* z(t) - f\right] w_z\big(y(t), z(t)\big)\Delta t \\ &\quad + \frac{1}{2}\sigma^2 y(t)^2 w_{yy}\big(y(t), z(t)\big)\Delta t + O(\Delta t). \end{aligned} \tag{1.6}$$

Then we substitute this expansion into (1.5) and we use y instead of $y(t)$ a z instead of $z(t)$

$$\begin{aligned} w(y,z) = \max_f \Big\{ & f(y - \eta f)\Delta t + e^{-\rho\Delta t} w(y,z) + O(\Delta t) \\ & + e^{-\rho\Delta t}\Big[\lambda y w_y(y,z) + \big[r^* z - f\big] w_z(y,z) + \frac{1}{2}\sigma^2 y^2 w_{yy}(y,z)\Big]\Delta t \\ & + e^{-\rho\Delta t}\sigma y w_y(y,z) E_t\big[W(t+\Delta t) - W(t)\big]\Big\}. \end{aligned} \tag{1.7}$$

Since the expected value of an increment of the Wiener process is zero, the last term can be dropped. We subtract $e^{-\rho\Delta t} w(y,z)$ from both sides and divide the whole equation by Δt

$$\begin{aligned} w(y,z)\frac{1-e^{-\rho\Delta t}}{\Delta t} &= \max_f \Bigg\{ f(y-\eta f) + \frac{O(\Delta t)}{\Delta t} \\ &\quad + e^{-\rho\Delta t}\left[\lambda y w_y(y,z) + \big[r^* z - f\big] w_z(y,z) + \frac{1}{2}\sigma^2 y^2 w_{yy}(y,z)\right]\Bigg\}. \end{aligned} \tag{1.8}$$

By taking the limit for $\Delta t \to 0$ we obtain

$$\rho w(y,z) = \max_f \left\{ f(y-\eta f) + \lambda y w_y(y,z) + \big[r^* z - f\big] w_z(y,z) + \frac{1}{2}\sigma^2 y^2 w_{yy}(y,z)\right\}. \tag{1.9}$$

An even more clearly arranged form of the optimality condition can be attained by dropping the arguments y, z of the function w and its derivatives

$$\begin{aligned} 0 &= \max_f \left\{ f(y-\eta f) + \lambda y w_y + \frac{1}{2}\sigma^2 y^2 w_{yy} + (r^* z - f) w_z \right\} - \rho w \\ &= \max_f \left\{ f(y-\eta f) - f w_z \right\} + \lambda y w_y + \frac{1}{2}\sigma^2 y^2 w_{yy} + r^* z w_z - \rho w. \end{aligned} \tag{1.10}$$

From (1.10) it is clear that the optimal control f is given as

$$f = \frac{y - w_z}{2\eta}. \tag{1.11}$$

Substituting this back to (1.10) we obtain

$$\begin{aligned} 0 &= \frac{1}{2}y^2\sigma^2 w_{yy} + \lambda y w_y + r^* z w_z - \rho w + f\,(y - w_z - \eta f) \\ &= \frac{1}{2}y^2\sigma^2 w_{yy} + \lambda y w_y + r^* z w_z - \rho w + \frac{y-w_z}{2\eta}\left(y - w_z - \eta\frac{y-w_z}{2\eta}\right) \\ 0 &= \frac{1}{2}y^2\sigma^2 w_{yy} + \lambda y w_y + r^* z w_z - \rho w + \frac{(y-w_z)^2}{4\eta}. \end{aligned} \tag{1.12}$$

Relationship (1.12) with the initial condition

$$w(y,0) = 0 \tag{1.13}$$

is called the Hamilton-Jacobi-Bellman partial differential equation for the value function $w(y,z)$ defined by (1.2). The condition (1.13) says that the expected revenue from the optimal disposal of a zero amount of the asset is zero under any arbitrary shadow price y.

In this work, we assume without proof that the HJB equation is not only a necessary condition for the value function of the optimal liquidation problem, but that it is also sufficient. This can also be proven and the proof is included in the article [2] which has not yet been published.

1.3 Dimension Reduction

The homogeneity of $w(y,z)$ allows us to aggregate the amount of the asset z and the shadow price y in one variable by use of the substitution

$$w(y,z) = \frac{y^2}{\eta}u(x), \qquad x = \eta\frac{z}{y}. \tag{1.14}$$

The variable x corresponds to the amount z in that sense that if the latter is zero, then also $x = 0$, and if z increases, then x increases as well. The derivatives of w given as

$$w_y = 2\frac{y}{\eta}u(x) - zu'(x), \tag{1.15}$$

$$w_{yy} = \frac{2}{\eta}u(x) - 2\frac{z}{y}u'(x) + \eta\frac{z^2}{y^2}u''(x), \tag{1.16}$$

$$w_z = yu'(x), \tag{1.17}$$

can be substituted into (1.12) which yields the ordinary differential equation

$$x^2u'' - \frac{2}{\sigma^2}\left(\sigma^2 + \lambda - r^*\right) x\, u' - \frac{2}{\sigma^2}\left(\rho - \sigma^2 - 2\lambda\right) u + \frac{1}{2\sigma^2}\left(u' - 1\right)^2 = 0, \tag{1.18}$$

for $x > 0$. The initial condition (1.13) becomes

$$\frac{y^2}{\eta}u(0) = 0$$

or, equivalently,

$$u(0) = 0. \tag{1.19}$$

To obtain the investor's optimal policy g for the reduced problem, we need to change the unit of measurement of f the same way as we did with the amount z, i.e.

$$g = \eta\frac{f}{y}. \tag{1.20}$$

Combining this with (1.11) and (1.17) one obtains

$$g = \eta\frac{f}{y} = \frac{y - w_z}{2y} = \frac{1 - u'}{2}. \tag{1.21}$$

1.4 Properties of the Solution

Equation (1.18) was examined in [3] and in this section we will list some of the properties of the solution which were proven there. The authors used the parameters

$$a = \frac{2}{\sigma^2}\left(\sigma^2 + \lambda - r^*\right), \tag{1.22}$$

$$b = \frac{2}{\sigma^2}\left(\rho - \sigma^2 - 2\lambda\right), \tag{1.23}$$

$$c = \frac{1}{2\sigma^2}. \tag{1.24}$$

The equation then takes the form

$$x^2 u'' = a\,x\,u' + b\,u - c\left(u' - 1\right)^2. \tag{1.25}$$

It is an ordinary differential equation of second order with a singularity in the point $x = 0$. Moreover, the initial condition (1.19) also refers to this singular point. By a solution to (1.25) on the interval $[0, x_0]$ we understand a function u continuous on $[0, x_0]$ and twice continuously differentiable on $(0, x_0)$ which satisfies (1.25) for $x > 0$.

In the beginning of the article the authors list expected properties of the solution of (1.2)-(1.4) which could in case of multiple solutions help identify the relevant one. First of all, the solution should be increasing because a higher amount of the asset should lead to a higher revenue from sale. Moreover, $u(x)$ should be concave to reflect decreasing return to scale because a greater amount sold leads to a greater decrease in real price. Also, the solution should be bounded between 0 and x.

Non-negativity is obvious because not selling anything would ensure zero revenue and the upper bound corresponds to an immediate sale without any negative effect on price. Since the sale is immediate, the revenue will not be discounted, and because also the price is not reduced, it is clear that this value is the upper estimate of the revenue from the sale of the asset.

Let us now now present the proof of the upper bound $u(x) \leq x$ which is not included in [3]. First, observe that

$$\int_0^T e^{-\rho s}\Big[y(s) - \eta f(s)\Big] f(s) ds \leq \int_0^T e^{-\rho s} y(s) f(s) ds, \tag{1.26}$$

because f is non-negative. Now use (1.4) to substitute for $f(s)ds$ in the right-hand side integral

$$\begin{aligned}\int_0^T e^{-\rho s} y(s) f(s) ds &= \int_0^T e^{-\rho s} y(s) \Big[r^* z(s) ds - dz(s) \Big] \\ &= \int_0^T e^{-\rho s} r^* y(s) z(s) ds - \int_0^T e^{-\rho s} y(s) dz(s).\end{aligned} \tag{1.27}$$

Integrating the last integral by parts we obtain

$$\int_0^T e^{-\rho s} y(s) dz(s) = e^{-\rho T} y(T) z(T) - y(0) z(0) + \int_0^T e^{-\rho s} \rho\, y(s) z(s) ds - \int_0^T e^{-\rho s} z(s) dy(s), \tag{1.28}$$

because $e^{-\rho s} y(s) z(s) = e^{(-\rho+\lambda)s} z(s) e^{-\lambda s} y(s)$, where $e^{(-\rho+\lambda)s} z(s)$ is a process with finite variation and $e^{-\lambda s} y(s)$ is a local martingale. Therefore, their quadratic variation is zero and (1.28) holds. Now we substitute (1.28) into (1.27) and, at the same time, we substitute (1.3) for $dy(s)$ in the last term

$$\begin{aligned}\int_0^T e^{-\rho s} f(s) y(s) ds &= \int_0^T e^{-\rho s} r^* y(s) z(s) ds - e^{-\rho T} y(T) z(T) + y(0) z(0) \\ &\quad - \int_0^T e^{-\rho s} \rho\, y(s) z(s) ds + \int_0^T e^{-\rho s} z(s) \Big[\lambda y(s) ds + \sigma y(s) dW(s) \Big] \\ &= y(0) z(0) - e^{-\rho T} y(T) z(T) + \int_0^T e^{-\rho s} \Big[r^* + \lambda - \rho \Big] y(s) z(s) ds \\ &\quad + \int_0^T e^{-\rho s} \sigma y(s) z(s) dW(s)\end{aligned} \tag{1.29}$$

Let us now look at the individual terms in (1.29). The term $e^{-\rho T} y(T) z(T)$ is clearly non-negative. Furthermore, if $\rho \geq r^* + \lambda$, then

$$\int_0^T e^{-\rho s} \Big[r^* + \lambda - \rho \Big] y(s) z(s) ds \leq 0.$$

Therefore, from (1.29) we obtain

$$\int_0^T e^{-\rho s} f(s) y(s) ds \leq y(0) z(0) + \int_0^T e^{-\rho s} \sigma y(s) z(s) dW(s). \tag{1.30}$$

Now, taking expectations on both sides

$$E_0\left(\int_0^T e^{-\rho s} f(s)y(s)ds\right) \leq y(0)z(0) + E_0\left(\int_0^T e^{-\rho s}\sigma y(s)z(s)dW(s)\right) = y(0)z(0), \tag{1.31}$$

because $e^{-\rho s}\sigma y(s)z(s)$ is a continuous process and for $\phi(s)$ continuous

$$E_0\left(\int_0^T \phi(s)dW(s)\right) = 0. \tag{1.32}$$

Combining (1.26) and (1.31)

$$E_0\left(\int_0^T e^{-\rho s}\Big[y(s) - \eta f(s)\Big] f(s)ds\right) \leq y(0)z(0) \tag{1.33}$$

and by taking maximum, one obtains the upper bound for w

$$w(y(0), z(0)) = \max_f E_0\left(\int_0^T e^{-\rho s}\Big[y(s) - \eta f(s)\Big] f(s)ds\right) \leq y(0)z(0). \tag{1.34}$$

To see the upper bound for $u(x)$, we simply use the substitution (1.14)

$$u(x) = \frac{\eta}{y^2}w(y,z) \leq \frac{\eta}{y^2}yz = \eta\frac{z}{y} = x. \tag{1.35}$$

This proves that the value function $u(x)$ is bounded from above by x (and also $w(y,z)$ by yz) as we argued at the beginning of this section.

Because of singularity of the HJB equation (1.25) it does not follow from standard theory that there needs to exist a solution with these properties, which are expected from the value function. We will now summarize the properties of the solutions to this equation which were proven by the authors of [3].

1.4.1 Existence

The existence of the solution depends on whether $a + b \geq 0$ or $a + b < 0$. Let us first examine what this expression means

$$a + b = \frac{2}{\sigma^2}\left(\sigma^2 + \lambda - r^* + \rho - \sigma^2 - 2\lambda\right) = \frac{2}{\sigma^2}\left(\rho - \lambda - r^*\right). \tag{1.36}$$

Since $\frac{2}{\sigma^2}$ is positive, $a+b \geq 0$ if and only if $\rho \geq r^* + \lambda$ which means that the discounting is greater than the sum of appreciation of the asset and the average growth of shadow price.

When applied to the speculator's problem from [4], the interpretation is following. The discount rate ρ can be interpreted as domestic interest rate or yield from investment in domestic currency. The sum $r^* + \lambda$ represents the expected yield from investment in foreign currency for a domestic investor. The condition $a + b \geq 0$ therefore says that, for a domestic investor, the expected yield from investment abroad is not higher than in domestic economy.

Now we formulate two propositions which are proven in [3].

Proposition 1.1. *Let $a + b < 0$. Then for all $x_0 > 0$ problem (1.25),(1.19) has no solution u in $[0, x_0]$.*

Proposition 1.2. *(i) For $a + b > 0$ and any $x_0 > 0$ there exists a continuum of solutions to the problem (1.25),(1.19) on $[0, x_0]$ which satisfy $0 \leq u(x) \leq x$.*

(ii) For $a + b \geq 0$ problem (1.25),(1.19) has at least one solution on $[0, \infty)$ which satisfies $0 \leq u(x) \leq x$.

Proposition 1.1 says that if the discounting is lower than the expected yield from investment in foreign currency, then the problem has no solution. This is a situation when the investor borrows for a lower interest than the expected value of his yields. It is therefore reasonable that in this case his revenue is unlimited and the solution does not exist.

In the other case when $a + b \geq 0$, i.e. discounting is at least as big as the expected yield from investment in foreign currency, according to 1.2(ii) there is at least one solution, and if $a + b > 0$, then according to the first part of this proposition there is are infinitely many solutions. It is this last case, when $\rho > r^* + \lambda$, that we will be dealing with.

1.4.2 Uniqueness, Monotonicity and Concavity

In this part we assume $a + b > 0$ and we state a result about uniqueness of the solution the existence is claimed in proposition 1.2(i).

Proposition 1.3. *There is one, and only one, solution u of the problem (1.25),(1.19) in $[0, \infty)$ which has the additional property $0 \leq u(x) \leq x$ for all $x > 0$. This solution necessarily satisfies $u > 0$, $u' > 0$, $u'' < 0$ and $u''' > 0$ on $(0, \infty)$.*

Proposition 1.3 ensures uniqueness of the solution u to (1.25) and, in addition, it also says that this solution is increasing and concave as we demanded.

1.4.3 Properties of the Derivative

So far, we know about the derivative of solution u that it is positive on $(0, \infty)$. The authors have also proven in [3] what happens to $u'(x)$ when x tends to the borders of this interval.

Lemma 1.4. *Let for some $x_0 > 0$ function $u \in C^0\big([0, x_0]\big) \cap C^2\big((0, x_0)\big)$ be the solution of (1.25),(1.19). Then*

$$\lim_{x \searrow 0} u'(x) = 1. \tag{1.37}$$

Lemma 1.5. *Let u be the solution of (1.25),(1.19) with $x_0 = \infty$ such that $0 \leq u(x) \leq x$ for all $x > 0$. Then*

$$\lim_{x \to \infty} u'(x) = 0. \tag{1.38}$$

These two lemmas give us more information about the demanded solution. According to lemma 1.4 we know that the slope of the solution tends to 1 for $x \to 0$, and according to lemma 1.5 we know that as x tends to infinity, the slope goes to zero.

1.4.4 The Limit

Lemma 1.6. *Let u be a non-constant solution to (1.25),(1.19) on $(0, \infty)$.*

(i) If $b > 0$, $u \geq 0$ a $u'(x) > 0$ for all x, then $u(x) \to \frac{c}{b}$ as $x \to \infty$.

(ii) If $b \leq 0$ and $u \geq 0$, then $u'(x) > 0$ for all x and u is unbounded.

This lemma tells us that for positive b we know the limit of $u(x)$ for x going to infinity and this limit is $\frac{c}{b}$. This can be seen from (1.25) when we take into account (1.38) and assume that the second derivative will also be zero in the limit. What is left is

$$0 = bu - c,$$

from where it is evident that $u = \frac{c}{b}$. The second claim of the lemma is that if $b \leq 0$, then the solution u increases without bound with increasing x.

1.4.5 Summary of Properties

Based on the findings from [3] we know that if the condition $a+b>0$ holds, then there exists a unique solution $u(x)$ to the problem (1.25),(1.19), such that $0 \leq u(x) \leq x$, and hence also to the problem (1.2)-(1.4). Moreover, we know these properties of the solution $u(x)$

- u is increasing and concave,
- $\lim\limits_{x \searrow 0} u'(x) = 1$,
- $\lim\limits_{x \to \infty} u'(x) = 0$,
- if $b > 0$, then $\lim\limits_{x \to \infty} u(x) = \frac{c}{b}$.

The existence and uniqueness allow us to use a suitable numerical scheme to search for the solution and the remaining properties can be used to verify the correctness of a found solution.

We only have one initial condition (1.19) for equation (1.25) which is not enough to find a precise solution of a second order differential equation. As a second condition one of the properties (1.37) or (1.38) can be used.

CHAPTER 2

THEORETICAL PRELIMINARIES

This chapter is dedicated to theoretical preliminaries for the alternative approach which we use for numerical treatment of the optimal liquidation problem. We define the problem on a finite time horizon and the original problem is then the limit when increasing the time interval. Then we reduce the problem's dimension by a similar substitution as was used in the previous chapter. Finally, we formulate an alternative initial condition which corresponds to a different scenario, where the investor is allowed to sell the remaining quantity of the asset at the end of the considered time interval. We also show monotonicity of the value function with respect to the length of the time interval. For the original initial condition, the value function is non-decreasing, whereas for the alternative condition, it is non-increasing.

2.1 Parabolic PDE

We limit time in the original problem (1.2)-(1.4) to a finite interval from t to $\tau \geq t$ and we define

$$w^\tau\Big(t, y(0), z(0)\Big) = \max_f E_t \left(\int_t^{T(z=0)\wedge\tau} e^{-\rho(s-t)} f\big(y(s), z(s)\big)\Big[y(s) - \eta f\big(y(s), z(s)\big)\Big] ds \right), \tag{2.1}$$

with preserved dynamics for $y(t)$ and $z(t)$

$$dy(t) = \lambda y(t)dt + \sigma y(t)dW(t), \tag{1.3}$$

$$dz(t) = \Big[r^* z(t) - f\big(y(t), z(t)\big)\Big]dt. \tag{1.4}$$

From (2.1) we obtain (1.2) if we let $t = 0$ and $\tau \to \infty$. Therefore we expect that the original value function $w(y, z)$ is the limit of $w^\tau(0, y, z)$

$$w(y, z) = \lim_{\tau \to \infty} w^\tau(0, y, z). \tag{2.2}$$

The argument is that w^τ is non-decreasing with increasing τ because giving the investor more time, i.e. relaxing the constraint, cannot make him worse off. This can also be shown mathematically. Let $0 < \tau_1 < \tau_2$. Then

$$\begin{aligned} w^{\tau_1}\Big(0, y(0), z(0)\Big) &= \max_f E_0 \left(\int_0^{T(z=0)\wedge\tau_1} e^{-\rho s} f\big[y - \eta f\big] ds \right) \\ &\le \max_f E_0 \left(\int_0^{T(z=0)\wedge\tau_1} e^{-\rho s} f\big[y - \eta f\big] ds + \int_{\tau_1}^{T(z=0)\wedge\tau_2} e^{-\rho s} f\big[y - \eta f\big] ds \right) \\ &= \max_f E_0 \left(\int_0^{T(z=0)\wedge\tau_2} e^{-\rho s} f\big[y - \eta f\big] ds \right) = w^{\tau_2}\Big(0, y(0), z(0)\Big) \end{aligned} \tag{2.3}$$

The inequality in the second line follows from the fact that equality can be achieved by letting f in the right-hand side be the maximizing control from the left-hand side for $s \in [0, \tau_1]$ and zero for $s \in (\tau_1, \tau_2]$.

Moreover, for every p

$$w^{\tau - p}(t - p, y, z) = w^\tau(t, y, z), \tag{2.4}$$

since only the length of the considered time interval and the initial values y for $y(t)$ and z for $z(t)$ are important for the problem and not the actual values which the time variable takes. It follows

$$w(y, z) = \lim_{t \to -\infty} w^0(t, y, z). \tag{2.5}$$

By assumption, $w^0(t, y, z)$ is the value function of the problem

$$\max_f E_t \left(\int_t^{T(z=0)\wedge 0} e^{-\rho(s-t)} f\big(y(s), z(s)\big) \Big[y(s) - \eta f\big(y(s), z(s)\big)\Big] ds \right), \tag{2.6}$$

with dynamics (1.3),(1.4). We drop the index 0 for simplicity so instead of $w^0(t,y,z)$ we write merely $w(t,y,z)$.

As we did in part 1.2, we will use the dynamic programming equation to derive the Hamilton-Jacobi-Bellman equation for this problem. By transition from time t to time $t+\Delta t$ DPE takes the form

$$\begin{aligned} w\big(t,y(t),z(t)\big) &= \max_f E_t\Big[\int_t^{t+\Delta t} e^{-\rho(s-t)} f\big(y(s)-\eta f\big)ds \\ &\qquad + e^{-\rho\Delta t} w\big(t+\Delta t, y(t+\Delta t), z(t+\Delta t)\big)\Big] \\ &= \max_f \Big\{ f\big(y(t)-\eta f\big)\Delta t + O(\Delta t) \\ &\qquad + E_t\Big[e^{-\rho\Delta t} w\big(t+\Delta t, y(t+\Delta t), z(t+\Delta t)\big)\Big]\Big\}. \end{aligned} \tag{2.7}$$

Again, we use first order Taylor expansion for the term $w\big(t+\Delta t, y(t+\Delta t), z(t+\Delta t)\big)$ which is given as

$$\begin{aligned} w\big(t+\Delta t,&y(t+\Delta t), z(t+\Delta t)\big) \\ &= w\big(t,y(t),z(t)\big) + w_t\big(t,y(t),z(t)\big)\Delta t + \lambda y(t) w_y\big(t,y(t),z(t)\big)\Delta t \\ &+ \sigma y(t) w_y\big(t,y(t),z(t)\big)\left[W(t+\Delta t) - W(t)\right] \\ &+ \left[r^* z(t) - f\right] w_z\big(t,y(t),z(t)\big)\Delta t \\ &+ \frac{1}{2}\sigma^2 y(t)^2 w_{yy}\big(t,y(t),z(t)\big)\Delta t + O(\Delta t). \end{aligned} \tag{2.8}$$

Substituting (2.8) into (2.7) the term with the Wiener process increment drops out as in (1.7). For better readability we drop the argument t of y and z

$$\begin{aligned} w(t,y,z)\frac{1-e^{-\rho\Delta t}}{\Delta t} &= \max_f \Big\{ f(y-\eta f) + e^{-\rho\Delta t}\Big[w_t(t,y,z) + \lambda y w_y(t,y,z) \\ &\quad + \big[r^* z - f\big] w_z(t,y,z) + \frac{1}{2}\sigma^2 y^2 w_{yy}(t,y,z)\Big] + \frac{O(\Delta t)}{\Delta t}\Big\}. \end{aligned} \tag{2.9}$$

Letting $\Delta t \to 0$ we obtain

$$\rho w = \max_f \Big\{ f(y-\eta f) + w_t + \lambda y w_y + \big[r^* z - f\big] w_z + \frac{1}{2}\sigma^2 y^2 w_{yy} \Big\} \tag{2.10}$$

which is standardly written as

$$-w_t = \max_f \Big\{ f(y-\eta f) - f w_z \Big\} + \lambda y w_y + \frac{1}{2}\sigma^2 y^2 w_{yy} + r^* z w_z - \rho w. \tag{2.11}$$

The right-hand side is the same as in (1.10) and hence the optimal control f is again given by (1.11). Substituting this back into (2.11) we obtain the HJB equation

$$-w_t = \frac{1}{2}y^2\sigma^2 w_{yy} + \lambda y w_y + r^* z w_z - \rho w + \frac{(y - w_z)^2}{4\eta}. \tag{2.12}$$

The conditions are

$$w(t, y, 0) = 0, \tag{2.13}$$

which has a similar meaning as (1.13) and says that the expected revenue from optimal disposal of a zero amount of the asset is zero for any shadow price y and any length of the time interval $|t|$, and

$$w(0, y, z) = 0, \tag{2.14}$$

which can be interpreted as the expected revenue from optimal disposal of any amount of the asset z being zero for any shadow price y as long as the length of the time interval is zero.

Equation (2.12) is a parabolic partial differential equation with a singularity in the point $y = 0$. Note that the right-hand sides of equations (1.12) and (2.12) are the same.

2.2 Dimension Reduction

Just like in case of equation (1.12) the dimension of (2.12) can be reduced by use of a substitution which aggregates variables z and y into one new variable x

$$w(t, y, z) = \frac{y^2}{\eta}u(t, x), \qquad x = \eta\frac{z}{y}. \tag{2.15}$$

The derivatives of $w(t, y, z)$ are then given as

$$w_t(t, y, z) = \frac{y^2}{\eta}u_t(t, x), \tag{2.16}$$

$$w_y(t, y, z) = 2\frac{y}{\eta}u(t, x) - z u_x(t, x), \tag{2.17}$$

$$w_{yy}(t, y, z) = \frac{2}{\eta}u(t, x) - 2\frac{z}{y}u_x(t, x) + \eta\frac{z^2}{y^2}u_{xx}(t, x), \tag{2.18}$$

$$w_z(t, y, z) = y u_x(t, x). \tag{2.19}$$

Substituting them into (2.12) yields

$$-\frac{2}{\sigma^2}u_t = x^2\, u_{xx} - a\, x\, u_x - b\, u + c(u_x - 1)^2, \tag{2.20}$$

where coefficients a, b, c are given by (1.22)-(1.24). Another substitution

$$\tau = -\frac{\sigma^2}{2}t \tag{2.21}$$

simplifies equation (2.20) to

$$u_\tau = x^2\, u_{xx} - a\, x\, u_x - b\, u + c(u_x - 1)^2, \tag{2.22}$$

where $u = u(\tau, x)$ and $\tau, x \in [0, \infty)$.

Boundary conditions (2.13), (2.14) become

$$u(\tau, 0) = 0, \tag{2.23}$$

$$u(0, x) = 0. \tag{2.24}$$

As was the case for equations (1.12) and (2.12), equations (1.25) and (2.22) only differ in the presence of the time derivative in the latter equation. The investor's optimal policy g for the reduced problem can be calculated, similarly as on the infinite horizon, as

$$g = \eta\frac{f}{y} = \frac{1 - u_x}{2}. \tag{2.25}$$

2.3 Alternative Initial Condition

Initial condition (2.14) says that if the considered time interval is zero, the expected revenue from the an optimal disposal of any amount z of the asset under any shadow price y is zero. In other words, it says that once the time runs out, the investor can no longer sell any of the asset. An alternative scenario at the end of the time interval would be allowing the speculator to sell everything he has left with no negative effect on the price, i.e. selling for the shadow price y. The corresponding initial condition is

$$w(0, y, z) = yz. \tag{2.26}$$

After substitution (2.15) this condition becomes

$$u(0, x) = \frac{\eta}{y^2}yz = \eta\frac{z}{y} = x \tag{2.27}$$

and it can be used instead of (2.24) as the initial condition in numerical treatment.

The use of the original zero initial condition produces a series of solutions to (2.22) which tend to the solution to (1.25) from below (as (2.3) and numerical experiment suggest), whereas the use of (2.27) for the initial condition produces a series of solutions to (2.22) which tend to the solution to (1.25) from above.

It is intuitive that in this case a longer time interval should lead to a lower revenue for the investor. The reason for that is that it delays the time when he can sell without the negative effect on the exchange rate. Monotonicity of $w^{\tau}(0,y,z)$ can again be shown mathematically. Let $0<\tau_1<\tau_2$. Then

$$
\begin{aligned}
w^{\tau_2}\Big(0,y(0),z(0)\Big) &= \max_f E_0\left(\int_0^{T(z=0)\wedge\tau_2} e^{-\rho s} f\big[y-\eta f\big]ds + e^{-\rho\tau_2}y(\tau_2)z(\tau_2)\right)\\
&= \max_f E_0\Bigg(\int_0^{T(z=0)\wedge\tau_1} e^{-\rho s} f\big[y-\eta f\big]ds + e^{-\rho\tau_1}y(\tau_1)z(\tau_1) - e^{-\rho\tau_1}y(\tau_1)z(\tau_1)\\
&\qquad + \int_{\tau_1}^{T(z=0)\wedge\tau_2} e^{-\rho s} f\big[y-\eta f\big]ds + e^{-\rho\tau_2}y(\tau_2)z(\tau_2)\Bigg)\\
&\le \max_f E_0\left(\int_0^{T(z=0)\wedge\tau_1} e^{-\rho s} f\big[y-\eta f\big]ds + e^{-\rho\tau_1}y(\tau_1)z(\tau_1)\right)\\
&\quad + \max_f E_0\left(\int_{\tau_1}^{T(z=0)\wedge\tau_2} e^{-\rho(s-\tau_1)} f\big[y-\eta f\big]ds + e^{-\rho\tau_2}y(\tau_2)z(\tau_2) - e^{-\rho\tau_1}y(\tau_1)z(\tau_1)\right)
\end{aligned}
\tag{2.28}
$$

The inequality is due to the facts that the maximum of a sum is not greater than the sum of maxima and $e^{\rho\tau_1}\ge 1$. Let us now examine the last term. We would like to show that it is not greater than zero. Analogically to $w(y,z)\le yz$, which is proven by (1.26)-(1.34), we can write

$$
w^{\tau_2}(\tau_1,y,z)\le e^{-\rho\tau_1}y(\tau_1)z(\tau_1) \tag{2.29}
$$

which is nothing else than

$$
\max_f E_{\tau_1}\left(\int_{\tau_1}^{T(z=0)\wedge\tau_2} e^{-\rho(s-\tau_1)} f\big[y-\eta f\big]ds + e^{-\rho\tau_2}y(\tau_2)z(\tau_2)\right)\le e^{-\rho\tau_1}y(\tau_1)z(\tau_1). \tag{2.30}
$$

Taking the conditional expectations at the time zero from both sides and using the law of iterated expectations on the left-hand side we obtain

$$\max_f E_0\left(\int_{\tau_1}^{T(z=0)\wedge\tau_2} e^{-\rho(s-\tau_1)} f\big[y-\eta f\big]\,ds + e^{-\rho\tau_2} y(\tau_2) z(\tau_2)\right) \le E_0\left(e^{-\rho\tau_1} y(\tau_1) z(\tau_1)\right). \tag{2.31}$$

Expressions (2.28) and (2.31) yield monotonicity of $w^{\tau}(0,y,z)$ with respect to τ

$$w^{\tau_2}\Big(0, y(0), z(0)\Big) \le w^{\tau_1}\Big(0, y(0), z(0)\Big). \tag{2.32}$$

CHAPTER 3

NUMERICAL METHODS

In this chapter, we focus on methods of numerical solution of the HJB equations (1.25), which we have derived in chapter 1, and (2.20), derived in chapter 2. In addition to the natural conditions, we will use condition (1.38) which is a property of the solution proven in [3]. First, we mention some methods used to solve singular differential equations and we state the reasons why they cannot be applied in the treatment of equation (1.25). Then we describe three methods, the explicit Euler method, the predictor-corrector method and the orthogonal collocation, based on the alternative approach presented in chapter 2.

Standard numerical methods fail in solving (1.25) due to the singularity in zero as stated in [4]. Therefore, we tried to find methods designed for solving singular differential equations in literature. This has been the area of interest for numerous authors in the past (examples include [1, 5, 9, 10]) but the assumptions about the problem imposed in the works have always ruled out our problem.

In [1] the authors apply shooting method to second order problems of the type

$$
\begin{aligned}
&u''(t) = \frac{A_1(t)}{t}u'(t) + \frac{A_0(t)}{t^2}u(t) + f\big(t, u(t)\big), \quad t \in (0,1],\\
&g\big(u(0), u'(0), u(1), u'(1)\big) = 0,\\
&u \in C([0,1]),
\end{aligned}
\tag{3.1}
$$

where $u : [0,1] \to \mathbb{R}^n$ and $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ are n-dimensional functions, A_0 and

A_1 are $n \times n$ matrices, and $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$, $p \leq 2n$. In our case, the variable x is from $[0, \infty)$ but to solve the problem numerically we have to truncate this interval to a finite interval $[0, L]$. Then x could be scaled into $[0, 1]$ by a simple substitution $t = \frac{x}{L}$. The problem is, however, that in (3.1) the first derivative $u'(t)$ is only present linearly whereas in equation (1.25) it appears also in the quadratic term. For this reason, our second order equation cannot be transformed to a two-dimensional system of first order equations as they do in this work. Hence the methods presented in [1] are not suitable for us and for similar reasons methods from [5] and [9] cannot be used either.

The problem considered in [10] is of type

$$\begin{aligned} &u''(t) - \frac{A_1}{t}u'(t) - \frac{A_0}{t^2}u(t) = f\big(t, u(t), u'(t)\big), \quad t \in (0, 1], \\ &B\big(u(0), u(1), u'(1)\big) = 0, \end{aligned} \tag{3.2}$$

where u and f are n-dimensional functions, A_0 and A_1 are constant $n \times n$ matrices, and B is an m-dimensional function, $m \leq 2n$. It seems that in this formulation the problem we had with the formulation from (3.1) is no longer present. However, one of the assumptions about f is that it is continuous on $[0, 1] \times \mathbb{R}^n \times \mathbb{R}^n$. In our case f is given as

$$f(t, u, u') = \frac{c\,(u' - 1)^2}{t^2} \tag{3.3}$$

and so it is not even defined for $t = 0$. Therefore our problem does not satisfy the assumptions from [10] either and we again cannot use the methods presented in this work.

These findings lead us to try an alternative approach which is described in chapter 2 and which we implement in following sections.

3.1 Explicit Euler Method

The first numerical method we use for the HJB equation (2.22) is the explicit Euler method. We need to approximate the continuous variables τ and x by their discrete counterparts. The variable x is from the interval $[0, \infty)$ which we truncate to a finite interval $[0, L]$ and we consider its partition

$$0 = x_0 < x_1 < x_2 \cdots < x_N = L. \tag{3.4}$$

Since time t is from $(-\infty, 0]$, the new time variable τ is from $[0, \infty)$. Again, we approximate this interval by a finite interval $[0, T]$ and we use the partition points

$$0 = \tau_0 < \tau_1 < \tau_2 \cdots < \tau_M = T, \tag{3.5}$$

where $\tau_i = ih$, $h = \frac{T}{M}$. To ensure stability of the explicit scheme, the time step h needs to be sufficiently small. We denote by $u_{i,j}$ the numerical approximation of $u(\tau, x)$ at the point (τ_i, x_j), i.e.

$$u_{i,j} \approx u(\tau_i, x_j), \qquad i = 0, 1, \dots, M,\ j = 0, 1, \dots, N. \tag{3.6}$$

We discretize the right-hand side of (2.22) by approximating the first derivative u_x by the central difference

$$\frac{\partial u(\tau_i, x_j)}{\partial x} \approx \frac{u_{i,j+1} - u_{i,j-1}}{x_{j+1} - x_{j-1}} \tag{3.7}$$

and the second derivative u_{xx} by the difference

$$\frac{\partial^2 u(\tau_i, x_j)}{\partial x^2} \approx \frac{\frac{u_{i,j+1}-u_{i,j}}{x_{j+1}-x_j} - \frac{u_{i,j}-u_{i,j-1}}{x_j-x_{j-1}}}{\frac{x_{j+1}-x_{j-1}}{2}} = \frac{2}{x_{j+1} - x_{j-1}} \left(\frac{u_{i,j+1} - u_{i,j}}{x_{j+1} - x_j} - \frac{u_{i,j} - u_{i,j-1}}{x_j - x_{j-1}} \right). \tag{3.8}$$

Equation (2.22) with discretized right-hand side is

$$\begin{aligned} u_\tau(\tau_i, x_j) = {} & \frac{2x_j^2}{x_{j+1} - x_{j-1}} \left(\frac{u_{i,j+1} - u_{i,j}}{x_{j+1} - x_j} - \frac{u_{i,j} - u_{i,j-1}}{x_j - x_{j-1}} \right) \\ & - a x_j \frac{u_{i,j+1} - u_{i,j-1}}{x_{j+1} - x_{j-1}} - b u_{i,j} + c \left(\frac{u_{i,j+1} - u_{i,j-1}}{x_{j+1} - x_{j-1}} - 1 \right)^2 \end{aligned} \tag{3.9}$$

for $i = 1, 2, \dots, M$ a $j = 1, 2, \dots, N-1$. For $i = 0$, which corresponds to $\tau = 0$, we employ the initial condition (2.24) by defining

$$u(0, x_j) = u_{0,j} = 0, \qquad j = 0, 1, \dots, N. \tag{3.10}$$

For $j = 0$, which corresponds to $x = 0$, we employ the boundary condition (2.23) by defining

$$u(\tau_i, 0) = u_{i,0} = 0, \qquad i = 0, 1, \dots, M. \tag{3.11}$$

As the other boundary condition we use property (1.38) which gives information about the derivative at infinity and which we conjecture to be valid also for $u(\tau, x)$. In the discretized case, the point x_N represents an approximation of infinity and so we let the backward difference at this point to vanish

$$\frac{u_{i,N} - u_{i,N-1}}{x_N - x_{N-1}} = 0, \tag{3.12}$$

which yields $u_{i,N} = u_{i,N-1}$. We, therefore, obtain a problem with mixed boundary conditions, where at $x = 0$ we have a condition for the function $u(\tau, x)$ itself and at $x = L$ we have a condition for its derivative $u_x(\tau, x)$.

For better clarity we switch to vector notation which we also use in the predictor-corrector method. By V_i we denote the profile at the time layer τ_i consisting only of the internal points

$$V_i = (u_{i,1}, u_{i,2}, \ldots, u_{i,N-1})^T \tag{3.13}$$

and by U_i we will understand the complete profile containing all the points at this layer

$$U_i = \left(u_{i,0}, V_i^T, u_{i,N}\right) = (u_{i,0}, u_{i,1}, \ldots, u_{i,N})^T. \tag{3.14}$$

The linear part of the right side of (3.9) can be written by help of matrix A which is defined as

$$A = \begin{pmatrix} A_{1,0} & A_{1,1} & A_{1,2} & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & A_{j,j-1} & A_{j,j} & A_{j,j+1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & A_{N-1,N-2} & A_{N-1,N-1} & A_{N-1,N} \end{pmatrix},$$

where for $j = 1, 2, \ldots, N-1$

$$A_{j,j-1} = \frac{2\,x_j^2}{(x_{j+1}-x_{j-1})(x_j-x_{j-1})} + \frac{a\,x_j}{x_{j+1}-x_{j-1}}, \tag{3.15}$$

$$A_{j,j} = -\frac{2\,x_j^2}{x_{j+1}-x_{j-1}}\left(\frac{1}{x_{j+1}-x_j} + \frac{1}{x_j-x_{j-1}}\right) - b, \tag{3.16}$$

$$A_{j,j+1} = \frac{2\,x_j^2}{(x_{j+1}-x_{j-1})(x_{j+1}-x_j)} - \frac{a\,x_j}{x_{j+1}-x_{j-1}}. \tag{3.17}$$

In case of equidistant partition (3.4) the expressions are simpler because then $x_{j+1} - x_j = k$ for $j = 0, 1, \ldots, N-1$.

The quadratic term from equation (3.9) will be approximated by the function

$$F(U_i) = \begin{pmatrix} c\left(\frac{u_{i,2}-u_{i,0}}{x_2-x_0} - 1\right)^2 \\ \vdots \\ c\left(\frac{u_{i,j+1}-u_{i,j-1}}{x_{j+1}-x_{j-1}} - 1\right)^2 \\ \vdots \\ c\left(\frac{u_{i,N}-u_{i,N-2}}{x_N-x_{N-2}} - 1\right)^2 \end{pmatrix}. \tag{3.18}$$

Equations (3.9) for the internal points can be written in vector notation as

$$\frac{\partial V_i}{\partial \tau} = AU_i + F(U_i). \tag{3.19}$$

To compute the internal points V_{i+1} of the new time layer U_{i+1} from the current layer U_i we will use the explicit Euler method. We approximate the derivative by the forward difference

$$\frac{\partial V_i}{\partial \tau} \approx \frac{V_{i+1} - V_i}{\tau_{i+1} - \tau_i} = \frac{V_{i+1} - V_i}{h} \tag{3.20}$$

to obtain

$$\frac{V_{i+1} - V_i}{h} = AU_i + F(U_i), \tag{3.21}$$

or, rearranged,

$$V_{i+1} = V_i + h\,[AU_i + F(U_i)]. \tag{3.22}$$

The values at the boundary points are given by (3.11), (3.12) as

$$u_{i+1,0} = 0, \qquad u_{i+1,N} = u_{i+1,N-1}. \tag{3.23}$$

Using (3.22) and (3.23) we are able to calculate the new layer U_{i+1} from the current known time layer U_i .

We are interested in the profile U_∞ which, of course, cannot be actually computed. Numerical experiments, however, indicate that the profiles settle down to a limit case after sufficiently many time steps. Therefore we assume that this limit case is a good approximation of the desired solution $u(x)$ to the problem (1.25).

3.2 Predictor-Corrector

The predictor-corrector method has a similar idea as the explicit Euler method. It also approximates the solution $u(\tau, x)$ on a discrete grid given by points (τ_i, x_j). The difference from the explicit method is that the predictor-corrector is a two-step method where the first step, the predictor, is actually an estimate made by the explicit Euler method, and this estimate is then improved by the second step, the corrector.

This method is a compromise between the explicit and implicit scheme. The implicit scheme would require to solve a system of nonlinear equations in every step which can be avoided by use of the predictor-corrector method.

In the predictor step, we make a first estimate $\tilde{U}_{i+1}$ of the values at the new time layer. To estimate the internal points V_{i+1} we use the explicit scheme

$$\tilde{V}_{i+1} = V_i + h\,[AU_i + F(U_i)], \tag{3.24}$$

with the boundary points are given as $\tilde{u}_{i+1,0} = 0$, resp. $\tilde{u}_{i+1,N} = \tilde{u}_{i+1,N-1}$.

The second step is the corrector where we improve the estimate from the first step by the use of an augmented trapezoidal rule. The trapezoidal rule combines the explicit and implicit schemes as

$$V_{i+1} = V_i + \tfrac{1}{2}h\,[AU_i + F(U_i) + AU_{i+1} + F(U_{i+1})]\,, \tag{3.25}$$

but in predictor-corrector, the estimate from predictor $\tilde{U}_{i+1}$ is used instead of the profile U_{i+1} on the right-hand side. This makes it an explicit scheme without the need to solve a system of equations. The rule to compute the internal points of the new time layer is

$$\begin{aligned} V_{i+1} &= V_i + \tfrac{1}{2}h\left[AU_i + F(U_i) + A\tilde{U}_{i+1} + F(\tilde{U}_{i+1})\right] \\ &= V_i + \tfrac{1}{2}h\left[A(U_i + \tilde{U}_{i+1}) + F(U_i) + F(\tilde{U}_{i+1})\right] \end{aligned} \tag{3.26}$$

and the boundary points are again given by (3.11), (3.12) as $u_{i+1,0} = 0$ and $u_{i+1,N} = u_{i+1,N-1}$.

The predictor-corrector scheme (3.24), (3.26) enables us to use the current known time layer U_i to calculate the new layer U_{i+1}.

Similarly as in the explicit Euler method, numerical experiments indicate that the profiles settle down to a limit case U_∞ after sufficiently many time steps.

3.3 Orthogonal Collocation

In this section we explain an ideologically different numerical method which can be used to calculate the solution of (2.22), (2.23), (2.24). The method is called orthogonal collocation and it is most widely used in chemical engineering. It was introduced by Villadsen and Stewart in [8] and the method described and used in this work is inspired by [7]. Unlike the two previously described methods, orthogonal collocation ultimately leads to a system ordinary differential equations which can then be solved by standard methods.

This method requires the spatial variable, which is in our case x, to be in the interval $[0,1]$. As in the case of predictor-corrector, we use a finite interval $[0,L]$ to approximate the original interval $[0,\infty)$. Then we make the substitution

$$z = \frac{x}{L} \tag{3.27}$$

so that the new spatial variable z is in $[0,1]$. The equation (2.22) then becomes

$$u_\tau = z^2\, u_{zz} - a\, z\, u_z - b\, u + c\left(\frac{u_z}{L} - 1\right)^2 \tag{3.28}$$

for the function $u(\tau, z)$. The conditions (2.23), (2.24) become

$$u(\tau, 0) = 0, \tag{3.29}$$

$$u(0, z) = 0. \tag{3.30}$$

As in the previous section, we also use condition (1.38) which now translates to

$$\frac{\partial u}{\partial z}(\tau, 1) = 0. \tag{3.31}$$

In orthogonal collocation the discretization of the spatial variable z is given by the so called collocation points. They consist of the two boundary points 0 and 1 and the N internal points which are given as the roots of an N-th order orthogonal polynomial. There are different sets of orthogonal polynomials but the most used for this method are the shifted Legendre polynomials. The N-th order shifted Legendre polynomial $P_N(x)$ can be expressed as

$$P_N(x) = (-1)^N \sum_{k=0}^{N} \binom{N}{k}\binom{N+k}{k}(-x)^k. \tag{3.32}$$

It has N roots between 0 and 1 and they are symmetric around the point 0.5.That gives us a total of $N+2$ collocation points

$$0 = z_0 < z_1 < \cdots < z_{N+1} = 1. \tag{3.33}$$

For each of these points z_j, $j = 0, 1, \ldots, N+1$, we define a function $u_j(\tau)$ which describes the solution $u(\tau, z)$ at the collocation point z_j for all times τ

$$u_j(\tau) = u(\tau, z_j). \tag{3.34}$$

We then approximate the solution $u(\tau, z)$ by the expression

$$u(\tau, z) \approx \hat{u}(\tau, z) = \sum_{j=0}^{N+1} \ell_j(z) u_j(\tau), \tag{3.35}$$

where $\ell_j(z)$ are the Lagrange basis interpolation polynomials for the collocation points $z_0, z_1, \ldots, z_{N+1}$

$$\ell_j(z) = \frac{(z - z_0) \cdots (z - z_{j-1})(z - z_{j+1}) \cdots (z - z_{N+1})}{(z_j - z_0) \cdots (z_j - z_{i-1})(z_j - z_{i+1}) \cdots (z_j - z_{N+1})} = \prod_{\substack{i=0 \\ i \neq j}}^{N+1} \frac{z - z_i}{z_j - z_i}. \tag{3.36}$$

The polynomials $\ell_j(z)$ have the property

$$\ell_j(z_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \tag{3.37}$$

Based on (3.35) the first spatial derivatives of $u(\tau, z)$ at the collocation points can be approximated as

$$\frac{\partial \hat{u}}{\partial z}(\tau, z_i) = \sum_{j=0}^{N+1} \ell_j'(z_i) u_j(\tau) \tag{3.38}$$

for $i = 0, 1, \ldots, N+1$ and similarly the second derivatives are

$$\frac{\partial^2 \hat{u}}{\partial z^2}(\tau, z_i) = \sum_{j=0}^{N+1} \ell_j''(z_i) u_j(\tau). \tag{3.39}$$

This means that both the first and the second spatial derivatives can be expressed as linear combinations of the functions $u_j(\tau)$. We can write the vector of first derivatives at the collocation points defined by (3.38) as

$$\begin{pmatrix} \frac{\partial \hat{u}}{\partial z}(\tau, z_0) \\ \frac{\partial \hat{u}}{\partial z}(\tau, z_1) \\ \vdots \\ \frac{\partial \hat{u}}{\partial z}(\tau, z_{N+1}) \end{pmatrix} = \begin{pmatrix} \ell_0'(z_0) & \ell_1'(z_0) & \cdots & \ell_{N+1}'(z_0) \\ \ell_0'(z_1) & \ell_1'(z_1) & \cdots & \ell_{N+1}'(z_1) \\ \vdots & \vdots & \ddots & \vdots \\ \ell_0'(z_{N+1}) & \ell_1'(z_{N+1}) & \cdots & \ell_{N+1}'(z_{N+1}) \end{pmatrix} \begin{pmatrix} u_0(\tau) \\ u_1(\tau) \\ \vdots \\ u_{N+1}(\tau) \end{pmatrix} \tag{3.40}$$

or in a more compact form

$$\hat{u}_z(\tau) = \frac{\partial}{\partial z} \hat{u}(\tau) = A u(\tau), \tag{3.41}$$

where A is an $N+2 \times N+2$ matrix with elements $A_{i,j} = \ell_j'(z_i)$. So the derivation $\frac{\partial}{\partial z}$ can be represented by the matrix A.

The second spatial derivatives from 3.39 can be expressed in a similar way as

$$\hat{u}_{zz}(\tau) = \frac{\partial^2}{\partial z^2}\hat{u}(\tau) = \frac{\partial}{\partial z}\frac{\partial}{\partial z}\hat{u}(\tau). \tag{3.42}$$

Replacing $\frac{\partial}{\partial z}$ by A we obtain

$$\hat{u}_{zz}(\tau) = AAu(\tau) = A^2u(\tau) = Bu(\tau), \tag{3.43}$$

where B is defined as A^2. The matrix B could alternatively be computed as $B_{i,j} = \ell_j''(z_i)$ but having calculated the matrix A already, it is simpler to just square it.

Now we can write equation (3.28) at internal collocation points using functions $u_j(\tau)$ defined by (3.34) and replacing the spatial derivatives by matrices A and B

$$\frac{\mathrm{d}u_j}{\mathrm{d}\tau} = z_j^2 \sum_{i=0}^{N+1} B_{j,i}u_i - a\, z_j \sum_{i=0}^{N+1} A_{j,i}u_i - b\, u_j + c\left(\frac{1}{L}\sum_{i=0}^{N+1} A_{j,i}u_i - 1\right)^2, \tag{3.44}$$

$j = 1, 2, \ldots, N$. Furthermore, condition (3.29) tells us that

$$u_0(\tau) \equiv 0 \tag{3.45}$$

so all the sums in (3.44) can start at $i = 1$ instead of $i = 0$. Moreover, from condition (3.31) we know that

$$\frac{\mathrm{d}u_{N+1}}{\mathrm{d}z} = \sum_{i=1}^{N+1} A_{N+1,i}u_i = 0, \tag{3.46}$$

which allows us to express u_{N+1} in terms of the other functions

$$u_{N+1} = -\frac{A_{N+1,1}\,u_1 + A_{N+1,2}\,u_2 + \cdots + A_{N+1,N}\,u_N}{A_{N+1,N+1}}. \tag{3.47}$$

Equations (3.44),(3.47) give us a system of $N+1$ ordinary differential equations for $N+1$ unknown functions $u_j(\tau)$ and (3.30) gives us the initial conditions

$$u_j(0) = 0. \tag{3.48}$$

This system can be solved by standard numerical methods for ODEs, such as the Runge-Kutta method. After solving for $u_j(\tau)$ using some grid points τ_i the solution $u(\tau, z)$ of (3.28) can be approximated as

$$u(\tau_i, z_j) \approx u_{i,j} = u_j(\tau_i). \tag{3.49}$$

Again, we are interested in the solution for $\tau \to \infty$ and just like in the predictor-corrector case, according to numerical experiments the solutions appear to converge to a stationary solution. This solution is assumed to be a good approximation of the desired solution $u(x)$ to the problem (1.25).

CHAPTER 4

NUMERICAL EXAMPLES

In this chapter we demonstrate the use of the three numerical methods, described in the previous chapter, by solving numerical examples with chosen values of the parameters. We have implemented the methods in Matlab and we include graphical representations of the solutions in text. We compare our numerical solutions to the analytical solution for the non-stochastic case which is presented in [4]. Then we compare the solutions found by the three methods for different cases where we do not know the analytical solution. Also, we verify whether the proven properties from section 1.4 are satisfied by the resulting solutions.

We would like to compare how the resulting solutions settle down to the limit for different methods. For this reason we define a measure of change of value of the approximation of $u(x_j)$ at time τ_i as

$$\varepsilon_{i,j} = |u_{i,j} - u_{i-1,j}| \tag{4.1}$$

for $i = 1, \ldots, M$, $j = 0, \ldots, N$. This measure tells us how much the approximate value of u at the partition point x_j changes from τ_{i-1} to τ_i. Furthermore, we define a measure of the distance between two succeeding approximations of $u(x)$

$$\varepsilon_i = \max_{j=0,\ldots,N} \varepsilon_{i,j} = \max_{j=0,\ldots,N} |u_{i,j} - u_{i-1,j}| \tag{4.2}$$

for $i = 1, \ldots, M$. This measure allows us to determine a suitable number of time steps

M as the smallest value for which

$$\varepsilon_M < \varepsilon^0 \tag{4.3}$$

for a given parameter $\varepsilon^0 > 0$.

To make use of the fact that the original zero initial condition (2.24) produces lower approximation of the limit solution $u(x)$ and the alternative initial condition (2.27) leads to upper approximations, we define the error of approximation at the partition point x_j and time τ_i as

$$e_{i,j} = u^U_{i,j} - u^L_{i,j}, \tag{4.4}$$

where $u^U_{i,j}$ are the numerical solutions for the alternative initial condition and $u^L_{i,j}$ the ones for the zero condition. Again, we also define a measure of the approximation error of the time profile at τ_i as

$$e_i = \max_{j=0,\dots,N} e_{i,j} = \max_{j=0,\dots,N} \left\{ u^U_{i,j} - u^L_{i,j} \right\}. \tag{4.5}$$

4.1 Example 1: Non-Stochastic Case

The stochasticity of the problem (1.2)-(1.4) is given by the parameter σ in the dynamics of $y(t)$. A completely non-stochastic case would therefore correspond to letting $\sigma = 0$. That would, however, change the problem significantly as parameters a, b, c could not be defined by (1.22), (1.23), (1.24) because they have σ^2 in the denominator. To preserve this notation and to be able to use the methods described in the previous chapter, we will only approximate the non-stochastic case by setting the value of σ close to zero.

As stated in [4], the parametric solution of

$$\frac{1}{4}\left(u' - 1\right)^2 - \left(\lambda - r^*\right) x\, u' - \left(\rho - 2\lambda\right) u = 0, \tag{4.6}$$

which is (1.18) first multiplied by $\frac{\sigma^2}{2}$ and then σ is being made, is given by

$$x(s) = \frac{1}{2}\left(\frac{e^{(\lambda - r^*)s} - 1}{\lambda - r^*} - e^{(r^* + \lambda - \rho)s}\frac{e^{(\rho - 2r^*)s} - 1}{\rho - 2r^*}\right), \tag{4.7}$$

$$u\big(x(s)\big) = \frac{1}{4}\left(\frac{e^{(2\lambda - \rho)s} - 1}{2\lambda - \rho} - e^{2(r^* + \lambda - \rho)s}\frac{e^{(\rho - 2r^*)s} - 1}{\rho - 2r^*}\right). \tag{4.8}$$

For numerical solution, we set the parameters to be $\sigma^2 = 0.001$, $\lambda = 0.7$, $r^* = 0.5$, $\rho = 2$. The values of a, b and c are

$$a = \frac{2}{\sigma^2}\left(\sigma^2 + \lambda - r^*\right) = 402, \tag{4.9}$$

$$b = \frac{2}{\sigma^2}\left(\rho - \sigma^2 - 2\lambda\right) = 1\,198, \tag{4.10}$$

$$c = \frac{1}{2\sigma^2} = 500, \tag{4.11}$$

so the condition $a + b > 0$ is satisfied. The analytical solution for $\sigma = 0$ is depicted by the pink color in figure 4.2.

4.1.1 Explicit Euler Method

Now we solve this problem by the explicit Euler method. We set L to be 15, so we consider values of x form $[0, 15]$. The number of considered partition points for x is $N + 1 = 31$.

We partition the interval $[0, 15]$ by first dividing the interval equidistantly and then squaring the points so that the points are denser close to zero. This is advantageous for capturing the increase of u for small values of x. A comparison of solutions for this partition to the solution for equidistant partition can be seen in figure 4.1. The red line represents $u(x) = x$ with slope 1, the black curves are the solutions for the non-equidistant partition, and the blue curves are the ones for the equidistant partition. It is clear from the picture that non-equidistant partition leads to solutions which are closer to fulfilling property (1.37) saying that the derivative at zero is one.

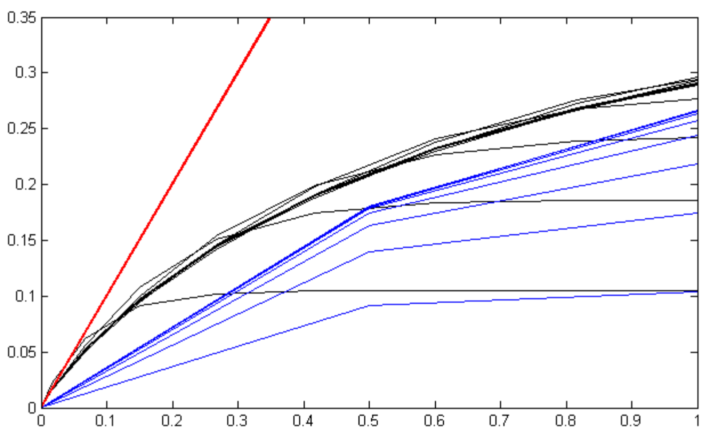
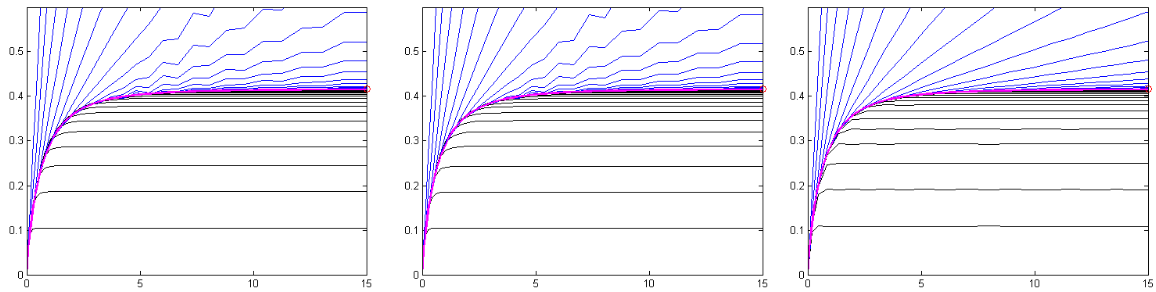


Figure 4.1: Comparison of solutions for non-equidistant and equidistant partitions.

The time step h is set to 10^{-5} and we also choose $\varepsilon^0 = 10^{-5}$ for the parameter of required precision in (4.3). This leads to the number of time steps $M = 487$ which means that the considered values of τ are $\tau \in [0, 0.00487]$. To obtain time in years, we must make an inverse substitution to (2.21) and then we obtain that it takes 9.74 years for the solutions to settle down. This suggests that any considered amount of the asset can be optimally liquidated in at most 9.74 years.

The resulting explixit Euler method solutions are depicted by the black color in figure 4.2a. For better clarity we only plot 20 iterations. We observe that they indeed appear to converge to the analytical solution. Table 4.1 shows settling down of the solutions represented by values of $\varepsilon_{i,j}$. We chose partition points x_j for $j \in 1, 8, 15, 23, 30$ to represent the whole interval of vales of x. We used x_1 rather than x_0 because the value at 0 is set to zero in every iteration. We show 10 values of τ which are evenly distributed in $[0, T]$. We observe that the order of $\varepsilon_{i,j}$ decreases with i and in the last iteration all the shown values are of the order 10^{-6} or smaller.



(a) Explicit method. **(b)** Predictor-corrector. **(c)** Orthogonal collocation.

Figure 4.2: Numerical solutions tend to the analytical solution (pink) in the non-stochastic case. Black solutions correspond to the original initial condition and the blue ones to the alternative condition.

To demonstrate the use of the alternative initial condition (2.27) we include the blue curves in figure 4.2. The black curves are the solutions computed from the initial condition (2.24) and the blue curves are the solutions from the alternative initial condition (2.27). We observe that these solutions converge to the same stationary solution from opposite sides as we claimed in section 2.3. Some of the solutions are not concave, which is probably just a numerical artifact, but the resulting limit case has the required properties. The number of time steps required in case of the alternative initial

	$x_1 = 0.017$	$x_8 = 1.067$	$x_{15} = 3.75$	$x_{23} = 8.82$	$x_{30} = 15$
$\tau_{49} =$ 4.90e-004	1.31e-004	2.79e-003	2.80e-003	2.80e-003	2.80e-003
$\tau_{97} =$ 9.70e-004	2.42e-005	1.19e-003	1.57e-003	1.57e-003	1.57e-003
$\tau_{146} =$ 1.46e-003	1.20e-005	1.82e-005	8.70e-004	8.71e-004	8.71e-004
$\tau_{195} =$ 1.95e-003	9.14e-006	1.03e-004	4.61e-004	4.83e-004	4.83e-004
$\tau_{244} =$ 2.44e-003	8.24e-006	4.58e-005	1.81e-004	2.67e-004	2.67e-004
$\tau_{292} =$ 2.92e-003	7.43e-006	1.08e-005	9.98e-006	1.49e-004	1.50e-004
$\tau_{341} =$ 3.41e-003	5.58e-006	2.28e-006	2.47e-005	7.77e-005	8.30e-005
$\tau_{390} =$ 3.90e-003	3.68e-006	4.34e-006	1.78e-006	3.12e-005	4.56e-005
$\tau_{438} =$ 4.38e-003	1.91e-006	3.00e-006	4.00e-006	3.67e-006	2.40e-005
$\tau_{487} =$ 4.87e-003	6.56e-007	1.26e-006	2.03e-006	4.06e-006	9.80e-006

Table 4.1: Example 1, explicit method: Development of $\varepsilon_{i,j}$ for the plotted values of τ and chosen points x_j.

condition is 500 which is slightly higher than the 487 steps for the original condition.

4.1.2 Predictor-Corrector

We use the same values of parameters for the predictor-corrector method as we used in the previous method. This method requires $M = 489$ time steps for the solutions to settle down which is only marginally more than 487 steps in the explicit method. The corresponding time interval is 9.78 years.

	$x_1 = 0.017$	$x_8 = 1.067$	$x_{15} = 3.75$	$x_{23} = 8.82$	$x_{30} = 15$
$\tau_{49} =$ 4.90e-004	1.06e-004	2.78e-003	2.80e-003	2.80e-003	2.80e-003
$\tau_{98} =$ 9.80e-004	1.76e-005	1.14e-003	1.55e-003	1.55e-003	1.55e-003
$\tau_{147} =$ 1.47e-003	8.40e-006	7.70e-006	8.62e-004	8.64e-004	8.64e-004
$\tau_{196} =$ 1.96e-003	6.13e-006	8.30e-005	4.55e-004	4.81e-004	4.81e-004
$\tau_{245} =$ 2.45e-003	5.23e-006	3.20e-005	1.78e-004	2.67e-004	2.67e-004
$\tau_{293} =$ 2.93e-003	4.42e-006	5.68e-006	1.65e-005	1.49e-004	1.50e-004
$\tau_{342} =$ 3.42e-003	3.10e-006	2.34e-006	1.90e-005	7.76e-005	8.35e-005
$\tau_{391} =$ 3.91e-003	1.88e-006	2.87e-006	2.30e-008	3.14e-005	4.59e-005
$\tau_{440} =$ 4.40e-003	8.95e-007	1.61e-006	2.86e-006	4.77e-006	2.38e-005
$\tau_{489} =$ 4.89e-003	2.52e-007	5.47e-007	1.10e-006	2.80e-006	9.79e-006

Table 4.2: Example 1, predictor-corrector: Development of $\varepsilon_{i,j}$ for the plotted values of τ and chosen points x_j.

Predictor-corrector solutions are shown in figure 4.2b together with the analytical solution. Again, we also present the solutions for the alternative initial condition which are depicted by the blue curves. In this case, the required M is 507, which is again a little higher than for the zero condition. We observe that the predictor-corrector solutions appear practically identical to the explicit Euler solutions. Table 4.2 shows the development of $\varepsilon_{i,j}$ with increasing i. The results are similar as in the case of the explicit method.

4.1.3 Orthogonal Collocation

Now we use orthogonal collocation to solve this problem. We set the number of internal collocation points N to 24 so the total number of collocation points, including the two boundary points, is 26. To solve the system of the $N+1 = 25$ ODEs for the 25 unknown functions $u_j(\tau)$ we use the Matlab solver `ode45` which implements the Runge-Kutta method. We consider the time interval $0 \leq \tau \leq 0.00489$ which was required by the predictor-corrector for the solutions to converge to the stationary solution. In figure 4.2c we observe that the orthogonal collocation solutions also tend to the analytical solution.

Also for this method we include a table with the development of $\varepsilon_{i,j}$ (table 4.3). We observe that the values in the last row are of the order 10^{-7} or higher which could suggest that the settling down is higer for this method. However, the table only shows values for some of the collocation points so it is not conclusive. We offer a more thorough comparison in the following subsection.

	$x_1 = 0.036$	$x_6 = 1.95$	$x_{13} = 7.98$	$x_{19} = 13.05$	$x_{25} = 15$
$\tau_{49} =$ 4.90e-004	6.85e-006	2.88e-003	2.78e-003	2.77e-003	2.77e-003
$\tau_{98} =$ 9.80e-004	2.22e-006	1.57e-003	1.52e-003	1.54e-003	1.65e-003
$\tau_{147} =$ 1.47e-003	7.28e-007	6.43e-004	8.44e-004	8.52e-004	9.65e-004
$\tau_{196} =$ 1.96e-003	1.89e-007	1.14e-005	4.85e-004	4.75e-004	4.34e-004
$\tau_{245} =$ 2.45e-003	5.27e-008	1.48e-006	2.72e-004	2.67e-004	3.25e-004
$\tau_{293} =$ 2.93e-003	9.59e-009	5.69e-007	1.47e-004	1.53e-004	8.57e-005
$\tau_{342} =$ 3.42e-003	8.23e-009	9.39e-008	8.02e-005	8.63e-005	1.39e-004
$\tau_{391} =$ 3.91e-003	9.85e-010	3.14e-008	1.03e-005	4.48e-005	3.74e-005
$\tau_{440} =$ 4.40e-003	8.55e-009	1.41e-007	6.96e-007	3.02e-005	2.55e-005
$\tau_{489} =$ 4.89e-003	2.76e-009	3.38e-008	9.36e-008	2.78e-006	3.34e-006

Table 4.3: Example 1, orthogonal collocation: Development of $\varepsilon_{i,j}$ for the plotted values of τ and chosen points x_j.

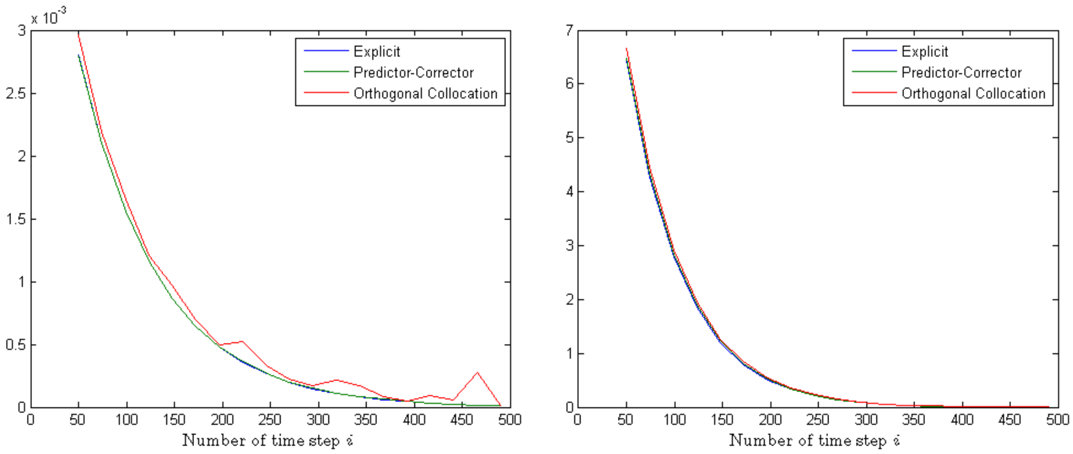
4.1.4 Comparison for Example 1

Let us now look at the properties from section 1.4. The solution is increasing and concave and we have already shown in figure 4.1 that the derivative at zero can be 1 as required. Moreover, the function flattens out as x increases and it is reasonable to believe that the limit of the derivative for $x \to \infty$ is zero. This was to be expected as we used this property as a boundary condition. The last property says that if $b > 0$, which is satisfied, then

$$\lim_{x\to\infty} u(x) = \frac{c}{b} \doteq 0.417.$$

The limit value is depicted by the red circle in figure 4.2a and we can see that the solutions also have this property.

From the above facts we conclude that our solutions correspond well to the analytical solution and they also meet the requirements given by the expected properties of the solution listed in section 1.4. Hence we believe that the methods give correct results.



(a) Development of ε_i.

(b) Development of e_i.

Figure 4.3: Comparison of the development of error for the three methods in ex. 1.

Now we compare the three methods in terms of ε_i and e_i defined by (4.2) and (4.5) respectively. The former is a measure of settling down of the solutions at given τ_i whereas the latter is a measure of precision of approximation at given τ_i.

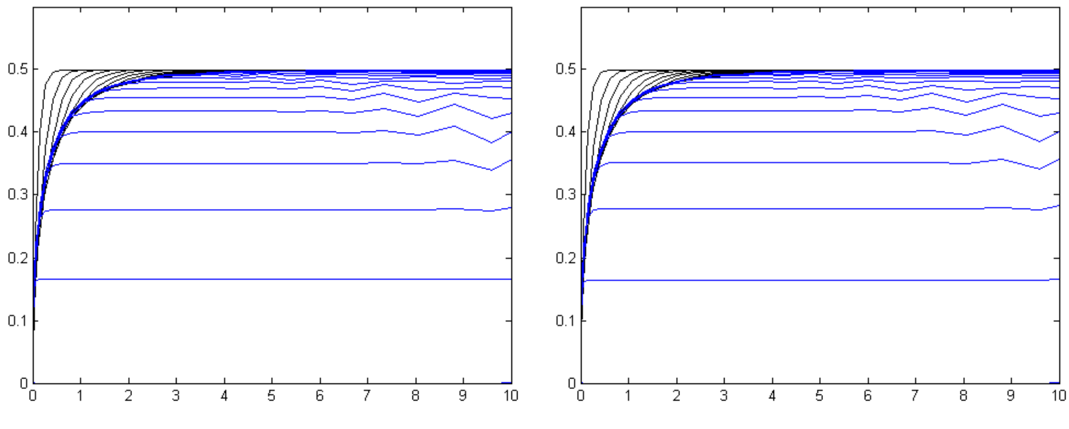
Table 4.4 shows the development of ε_i and e_i for the three numerical methods as the number of time layer i increases. Figure 4.3 illustrates the same values graphically. We observe that ε_i decreases monotonically for the explicit method and the predictor-

	ε_i			e_i		
	Explicit	Pred-Corr	O.Colloc	Explicit	Pred-Corr	O.Colloc
τ_{49} = 4.90e-004	2.80e-003	2.80e-003	2.96e-003	6.44e+000	6.48e+000	6.67e+000
τ_{98} = 9.80e-004	1.55e-003	1.55e-003	1.65e-003	2.79e+000	2.83e+000	2.93e+000
τ_{147} = 1.47e-003	8.61e-004	8.64e-004	9.69e-004	1.21e+000	1.23e+000	1.27e+000
τ_{196} = 1.96e-003	4.77e-004	4.81e-004	4.98e-004	5.08e-001	5.23e-001	5.41e-001
τ_{245} = 2.45e-003	2.64e-004	2.67e-004	3.25e-004	2.10e-001	2.18e-001	2.26e-001
τ_{293} = 2.93e-003	1.48e-004	1.50e-004	1.73e-004	8.50e-002	8.93e-002	9.26e-002
τ_{342} = 3.42e-003	8.20e-005	8.35e-005	1.75e-004	3.21e-002	3.42e-002	3.59e-002
τ_{391} = 3.91e-003	4.50e-005	4.59e-005	5.19e-005	1.09e-002	1.19e-002	1.27e-002
τ_{440} = 4.40e-003	2.33e-005	2.38e-005	6.11e-005	3.01e-003	3.46e-003	3.57e-003
τ_{489} = 4.89e-003	9.34e-006	9.79e-006	1.57e-005	4.05e-004	6.06e-004	5.89e-004

Table 4.4: Example 1: Development of ε_i and e_i for the plotted values of τ.

corrector while it is not monotonic for orthogonal collocation. The figure shows more values because it includes also a value of ε_i between each of the listed τ's. The values are very similar for the first two methods and they are slightly higher for the last method. The qualitative difference may be related to the fact that, unlike the other two methods, orthogonal collocation does not work with time layers in computation.

The values of e_i are very similar for all three methods. We observe that the precision of approximation increases with increasing i which corresponds to the fact that the upper and lower solutions tend to the same limit as we can see in figure 4.2.



(a) Explicit method. **(b)** Predictor corrector.

Figure 4.4: The optimal policy g for the two initial conditions in example 1.

Figure 4.4 shows approximations of the investor's optimal strategy g which is related to the spatial derivative of u and given by (2.25). Solutions for the zero initial condition are depicted by black color and the ones for the alternative condition are blue.

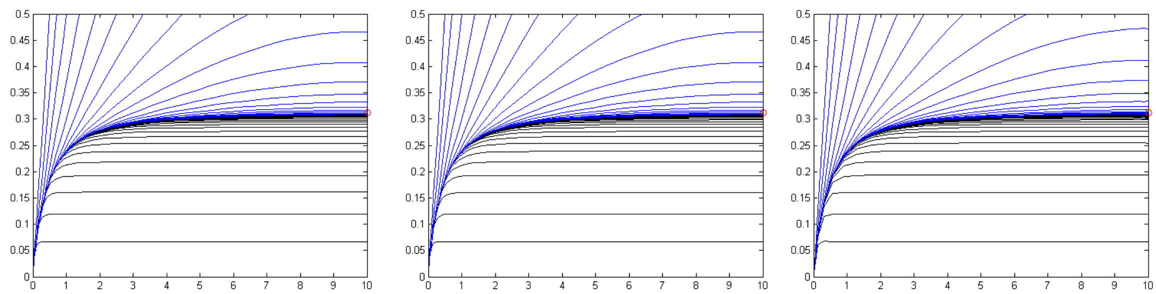
For the original initial condition the policy decreases with increasing τ. An intuitive interpretation could be that the investor wants to sell as much as he can in the allowed time interval because he cannot sell afterwards. When τ increases, he is given more time so he does not need to sell so quickly.

The alternative initial condition results in a strategy which increases with τ. This is again reasonable because this time the investor is allowed to sell the remaining amount of the asset at the end of the time interval without the negative effect. Therefore, if the time interval is short, i.e. τ is small, it is beneficial to delay the sale. However, as the interval gets longer, the situation at the end has less and less influence on the strategy during the interval. That is the reason why the policies tend to the same limit for both initial conditions as we can see in the figure.

4.2 Example 2: $a > 0$, $b > 0$

In this example we set the parameters to $\sigma^2 = 0.2$, $\lambda = 1$, $r^* = 1$, $\rho = 3$ so we have $a = 2 > 0$, $b = 8 > 0$, $c = 2.5$. The limit value of u is $\frac{c}{b} = 0.3125$. We set the maximal considered value of x to $L = 10$.

For the explicit method and the predictor-corrector, we use $N + 1 = 31$ partition points for x. We partition the interval in the same way as we did in the non-stochastic case, i.e. squaring the equidistant points. The time step is set to $h = 10^{-3}$ and we use the same $\varepsilon^0 = 10^{-5}$ as in example 1.



(a) Explicit method. **(b)** Predictor-corrector. **(c)** Orthogonal collocation.

Figure 4.5: Numerical solutions for example 2. Black solutions correspond to the original initial condition and the blue ones to the alternative condition.

4.2.1 Explicit Euler Method

The explicit method with the zero initial condition requires $M = 597$ time steps to meet (4.3). This means that the time interval needed for the solutions to settle down is 5.97 years. The explicit method solutions with the zero initial condition are shown in figure 4.5a.

The same figure also shows solutions for the alternative initial condition (blue curves). The number of time steps required in this case is 734 which seems significantly higher than for the original condition.

Table 4.5 shows the development of the development $\varepsilon_{i,j}$ with increasing i for chosen points x_j. The numbers confirm that the differences between the approximations of $u(x_j)$ vanish as τ increases.

	$x_1 = 0.011$	$x_8 = 0.71$	$x_{15} = 2.5$	$x_{23} = 5.88$	$x_{30} = 10$
$\tau_{60} = 6.00\text{e-}002$	8.64e-005	1.55e-003	1.56e-003	1.56e-003	1.56e-003
$\tau_{119} = 1.19\text{e-}001$	2.80e-005	7.39e-004	9.69e-004	9.69e-004	9.69e-004
$\tau_{179} = 1.79\text{e-}001$	1.31e-005	1.44e-004	5.86e-004	5.98e-004	5.98e-004
$\tau_{239} = 2.39\text{e-}001$	3.16e-006	1.15e-005	3.19e-004	3.68e-004	3.69e-004
$\tau_{299} = 2.99\text{e-}001$	1.54e-006	3.89e-006	1.43e-004	2.21e-004	2.27e-004
$\tau_{358} = 3.58\text{e-}001$	2.07e-006	4.59e-007	5.47e-005	1.26e-004	1.37e-004
$\tau_{418} = 4.18\text{e-}001$	8.20e-007	3.20e-007	1.89e-005	6.59e-005	7.80e-005
$\tau_{478} = 4.78\text{e-}001$	1.60e-008	1.79e-007	6.55e-006	3.21e-005	4.16e-005
$\tau_{537} = 5.37\text{e-}001$	1.92e-007	4.79e-008	2.35e-006	1.50e-005	2.10e-005
$\tau_{597} = 5.97\text{e-}001$	1.01e-007	2.93e-008	8.66e-007	6.70e-006	9.97e-006

Table 4.5: Example 2, explicit method: Development of $\varepsilon_{i,j}$ for the plotted values of τ and chosen points x_j.

4.2.2 Predictor-Corrector

The number of time steps for the predictor-corrector with the original initial condition is 601 and for the alternative condition it is $M = 738$. Just like in example 1, these values are very similar to the explicit Euler method. The predictor-corrector solutions are shown in figure 4.5b.

Table 4.6 shows the values of $\varepsilon_{i,j}$ for this method with zero initial condition and chosen i and j.

	$x_1 = 0.011$	$x_8 = 0.71$	$x_{15} = 2.5$	$x_{23} = 5.88$	$x_{30} = 10$
τ_{60} = 6.00e-002	7.66e-005	1.54e-003	1.55e-003	1.55e-003	1.55e-003
τ_{120} = 1.20e-001	2.48e-005	7.20e-004	9.60e-004	9.61e-004	9.61e-004
τ_{180} = 1.80e-001	1.10e-005	1.49e-004	5.81e-004	5.95e-004	5.95e-004
τ_{240} = 2.40e-001	2.42e-006	5.08e-006	3.16e-004	3.66e-004	3.68e-004
τ_{301} = 3.01e-001	1.25e-006	3.37e-006	1.42e-004	2.18e-004	2.24e-004
τ_{361} = 3.61e-001	1.55e-006	2.69e-007	5.43e-005	1.23e-004	1.34e-004
τ_{421} = 4.21e-001	5.72e-007	2.08e-007	1.92e-005	6.48e-005	7.66e-005
τ_{481} = 4.81e-001	3.87e-008	1.17e-007	6.74e-006	3.18e-005	4.10e-005
τ_{541} = 5.41e-001	1.33e-007	3.01e-008	2.40e-006	1.48e-005	2.06e-005
τ_{601} = 6.01e-001	6.12e-008	2.09e-008	8.87e-007	6.68e-006	9.88e-006

Table 4.6: Example 2, predictor-corrector: Development of $\varepsilon_{i,j}$ for the plotted values of τ and chosen points x_j.

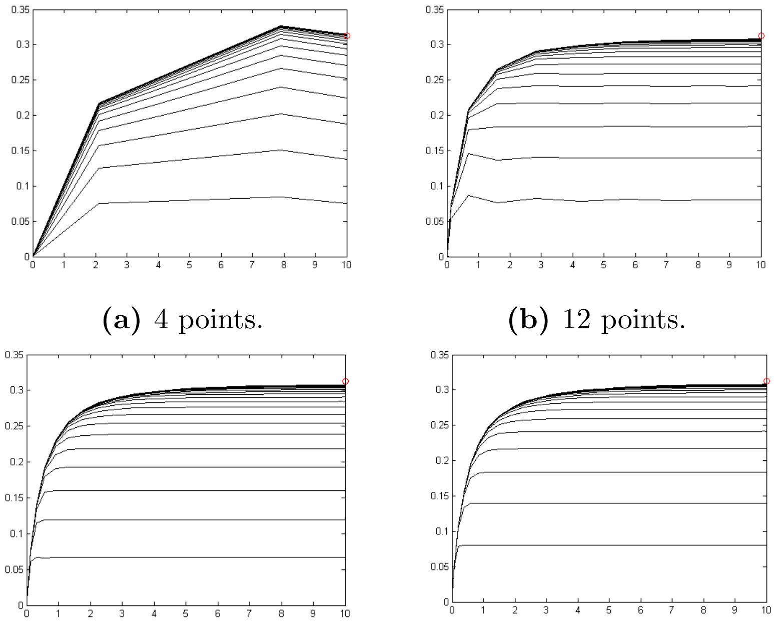
4.2.3 Orthogonal Collocation

Again, we use $N = 24$ internal collocation points in the orthogonal collocation. This time the time interval $\tau \in [0, 0.601]$ is used to match the interval needed by the predictor-corrector. The resulting solutions for both initial conditions are depicted in figure 4.5c. The development of $\varepsilon_{i,j}$ in time is shown in table 4.7

	$x_1 = 0.024$	$x_6 = 1.30$	$x_{13} = 5.32$	$x_{19} = 8.70$	$x_{25} = 10$
τ_{60} = 6.00e-002	1.07e-005	1.56e-003	1.55e-003	1.55e-003	1.53e-003
τ_{120} = 1.20e-001	1.98e-006	9.48e-004	9.61e-004	9.61e-004	1.03e-003
τ_{180} = 1.80e-001	5.06e-007	4.33e-004	5.94e-004	5.94e-004	6.80e-004
τ_{240} = 2.40e-001	1.51e-008	1.33e-004	3.65e-004	3.67e-004	4.05e-004
τ_{301} = 3.01e-001	3.01e-008	3.28e-005	2.14e-004	2.23e-004	3.67e-004
τ_{361} = 3.61e-001	1.77e-009	8.22e-006	1.17e-004	1.34e-004	9.75e-005
τ_{421} = 4.21e-001	3.40e-010	2.14e-006	5.94e-005	7.55e-005	1.98e-004
τ_{481} = 4.81e-001	3.70e-010	6.00e-007	2.82e-005	4.06e-005	6.94e-005
τ_{541} = 5.41e-001	2.97e-011	1.78e-007	1.29e-005	2.02e-005	5.13e-005
τ_{601} = 6.01e-001	2.43e-010	6.01e-008	5.73e-006	1.00e-005	8.98e-005

Table 4.7: Example 2, orthogonal collocation: Development of $\varepsilon_{i,j}$ for the plotted values of τ and chosen points x_j.

Let us now compare orthogonal collocation solutions for different numbers of collocation points. In figure 4.6 we present the results for 2, 10, 24 and 30 internal collocation points or 4, 12, 26 and 32 total collocation points respectively. We observe that 4 points are not sufficient to achieve an increasing and concave stationary solution but it is interesting that even in this case the limit value of $\frac{c}{b} = 0.3125$ is met (red circle in the graph). The other three cases produce a stationary solution which has the required properties.



(a) 4 points. **(b)** 12 points.

(c) 26 points. **(d)** 32 points.

Figure 4.6: Solutions of example 2 for different numbers of collocation points.

Using more points leads to a smoother solution than fewer points but the computation time is significantly higher (computation time for 26 points was almost 60 times as long as for 12 points; for 32 points it was more than 170 times longer) due to the higher dimension of the system of ODEs. We decide to work with 26 collocation points as this seems to be a good compromise between accuracy and time consumption.

4.2.4 Comparison for Example 2

Let us check whether the resulting limit solutions have the properties listed in section 1.4. Clearly, they are increasing and concave and they flatten out and approach the limit at infinity $\frac{c}{b} = 0.3125$.

Figure 4.7 shows the stationary solutions by the three methods together with the identity line $u(x) = x$ (red). It is zoomed to show only the interval $x \in [0, 0.5]$ and we can see that for all methods the slope at zero is close to 1.

Now we compare the development of ε_i and e_i for example 2. Figure 4.8 illustrates the development graphically while table 4.8 lists the numerical values.

For ε_i we observe a similar development as in example 1. For the explicit method

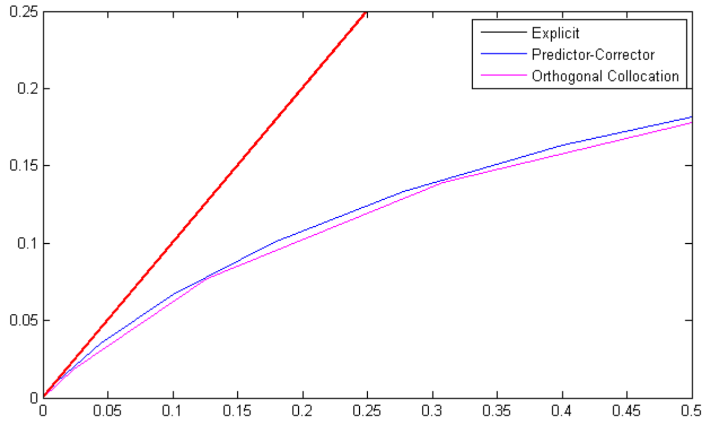
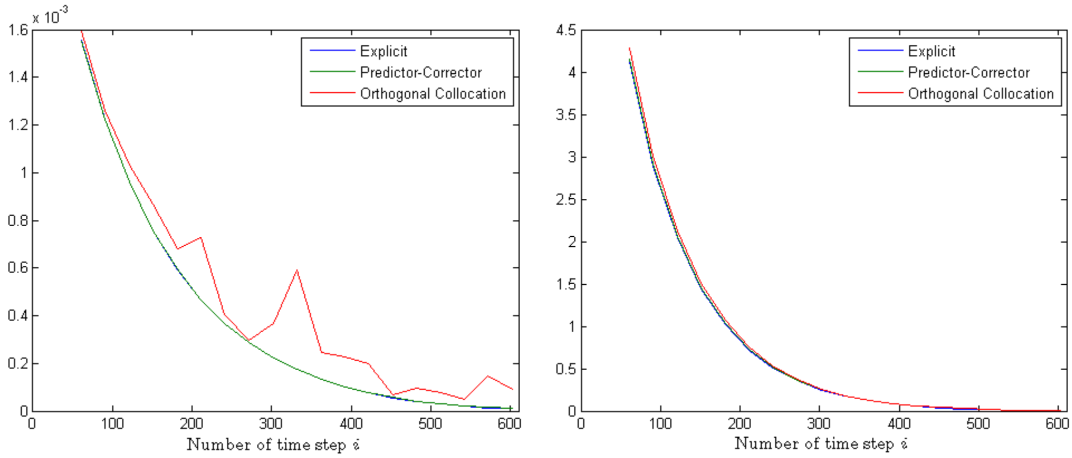


Figure 4.7: Stationary solutions of example 2 together with the identity line.

and the predictor-corrector the values are practically the same and they decrease monotonically, while for orthogonal collocation the development is not monotonic and the values are slightly higher than for the other two methods. The development of e_i is similar for all three methods.



(a) Development of ε_i.

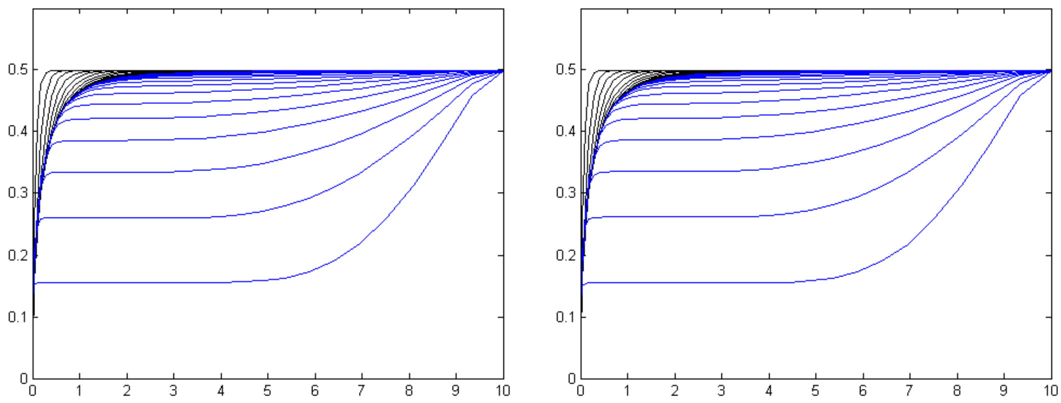
(b) Development of e_i.

Figure 4.8: Comparison of the development of error for the three methods in ex. 2.

Figure 4.9 shows approximations of the investor's optimal policy g where the black solutions are for the zero initial condition and the blue ones are for the alternative condition. As it was in example 1, the original condition leads to g decreasing with τ and the alternative condition to increasing strategies. We reason why this is the case at the end of section 4.1.4.

	ε_i			e_i		
	Explicit	Pred-Corr	O.Colloc	Explicit	Pred-Corr	O.Colloc
τ_{60} = 6.00e-002	1.56e-003	1.55e-003	1.59e-003	4.13e+000	4.14e+000	4.29e+000
τ_{120} = 1.20e-001	9.61e-004	9.61e-004	1.03e-003	2.04e+000	2.06e+000	2.13e+000
τ_{180} = 1.80e-001	5.94e-004	5.95e-004	6.80e-004	1.02e+000	1.03e+000	1.07e+000
τ_{240} = 2.40e-001	3.66e-004	3.68e-004	4.05e-004	5.10e-001	5.18e-001	5.35e-001
τ_{301} = 3.01e-001	2.23e-004	2.24e-004	3.67e-004	2.48e-001	2.53e-001	2.62e-001
τ_{361} = 3.61e-001	1.33e-004	1.34e-004	2.47e-004	1.20e-001	1.23e-001	1.28e-001
τ_{421} = 4.21e-001	7.57e-005	7.66e-005	1.98e-004	5.68e-002	5.86e-002	6.12e-002
τ_{481} = 4.81e-001	4.02e-005	4.10e-005	9.47e-005	2.61e-002	2.72e-002	2.86e-002
τ_{541} = 5.41e-001	2.00e-005	2.06e-005	5.13e-005	1.18e-002	1.23e-002	1.30e-002
τ_{601} = 6.01e-001	9.47e-006	9.88e-006	8.98e-005	5.19e-003	5.48e-003	5.83e-003

Table 4.8: Example 2: Development of ε_i and e_i for the plotted values of τ.



(a) Explicit method. **(b)** Predictor corrector.

Figure 4.9: The optimal policy g for the two initial conditions in example 2.

4.3 Example 3: $a < 0$, $b > 0$

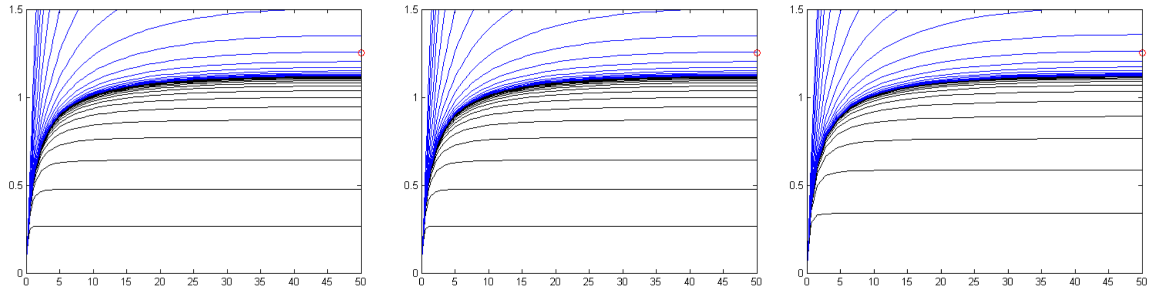
Now we present an example with a negative and b positive. To achieve that we choose the values of parameters to be $\sigma^2 = 0.5$, $\lambda = 0.6$, $r^* = 1.2$, $\rho = 1.9$ so the considered equation (2.20) becomes

$$u_\tau = x^2\, u_{xx} + 0.4\, x\, u_x - 0.8\, u + (u_x - 1)^2. \tag{4.12}$$

The value of a is -0.4 but the condition $0.4 = a + b > 0$ is still fulfilled. Parameter $b = 0.8$ is positive so, according to lemma 1.6, the limit of u is

$$\lim_{x\to\infty} u(x) = \frac{c}{b} = 1.25.$$

This time we use a higher value of L equal to 50 and we also use more partition points for x in the explicit and predictor-corrector methods, $N + 1 = 71$. The time step is set to $h = 10^{-4}$ and $\varepsilon^0 = 10^{-7}$.

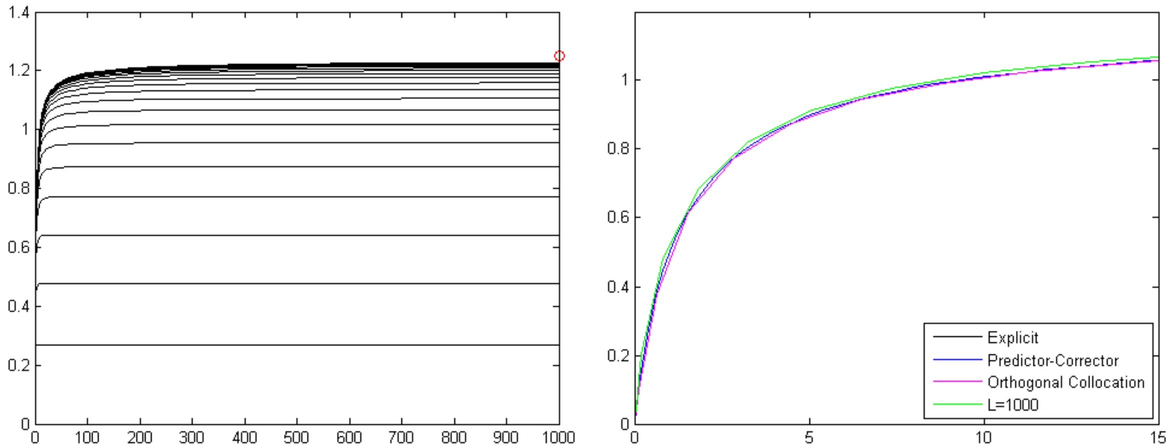


(a) Explicit method. **(b)** Predictor-corrector. **(c)** Orthogonal collocation.

Figure 4.10: Numerical solutions for example 3. Black solutions correspond to the original initial condition and the blue ones to the alternative condition.

4.3.1 Explicit Euler Method

The number of time steps required by the explicit Euler method is $M = 60\,088$ for the original initial condition and $M = 78\,773$ for the alternative one. These correspond to 24 and 31.5 years respectively. Once again, the number is higher for the alternative condition.



(a) Solution approach the limit value. **(b)** Comparison of solutions for $x \in [0, 15]$.

Figure 4.11: Explicit method solutions of example 3 for $L = 1\,000$.

The solutions are plotted in figure 4.10a. It is not clear from the figure whether the solutions approach the theoretical limit value of 1.25 in infinity (red circle in the figure). We include figure 4.11a to demonstrate that for $L = 1\,000$ the explicit method produces solutions which suggest that the limit is correct. Figure 4.11b compares limit solutions found by the three methods with $L = 50$ to the explicit method solution with $L = 1\,000$ for values of $x \in [0, 15]$. We observe that the higher value of L does not change the approximation of $u(x)$ significantly. For this reason we believe that it is sufficient to use lower values of L.

For $L = 50$ and the original initial condition, table 4.9 shows the development of $\varepsilon_{i,j}$ for increasing i and chosen j.

	$x_1 = 0.010$	$x_{18} = 3.31$	$x_{35} = 12.5$	$x_{53} = 28.66$	$x_{70} = 50$
$\tau_{6009} = 6.01\text{e-}001$	6.14e-007	5.68e-005	6.17e-005	6.18e-005	6.18e-005
$\tau_{12018} = 1.20\text{e+}000$	1.14e-008	2.03e-005	3.45e-005	3.73e-005	3.77e-005
$\tau_{18026} = 1.80\text{e+}000$	1.30e-009	7.08e-006	1.67e-005	2.03e-005	2.10e-005
$\tau_{24035} = 2.40\text{e+}000$	3.12e-011	2.74e-006	7.65e-006	1.00e-005	1.06e-005
$\tau_{30044} = 3.00\text{e+}000$	1.43e-012	1.14e-006	3.48e-006	4.74e-006	5.05e-006
$\tau_{36053} = 3.61\text{e+}000$	5.58e-012	4.96e-007	1.58e-006	2.19e-006	2.34e-006
$\tau_{42062} = 4.21\text{e+}000$	1.04e-012	2.19e-007	7.14e-007	1.00e-006	1.07e-006
$\tau_{48070} = 4.81\text{e+}000$	6.50e-013	9.82e-008	3.23e-007	4.55e-007	4.87e-007
$\tau_{54079} = 5.41\text{e+}000$	2.54e-013	4.42e-008	1.46e-007	2.06e-007	2.21e-007
$\tau_{60088} = 6.01\text{e+}000$	1.17e-013	1.99e-008	6.61e-008	9.33e-008	1.00e-007

Table 4.9: Example 3, explicit method: Development of $\varepsilon_{i,j}$ for the plotted values of τ and chosen points x_j.

4.3.2 Predictor-Corrector

For the predictor-corrector method the required numbers of time steps were $M = 60\,092$ and $M = 78\,777$ for the zero and alternative initial conditions respectively which mean 24 and 31.5 years. Both values are slightly higher than for the explicit method. The solutions are shown in figure 4.10b.

4.3.3 Orthogonal Collocation

For orthogonal collocation we again work with 26 collocation points and the time interval $\tau \in [0, 6.0092]$ is used. The resulting solutions are shown in figure 4.10c and they look practically the same as for the other two methods.

The considerably longer time interval, which is in this case necessary for the solutions to settle down to the stationary solution, caused the computation time for this method

	$x_1 = 0.010$	$x_{18} = 3.31$	$x_{35} = 12.5$	$x_{53} = 28.66$	$x_{70} = 50$
$\tau_{6009} = 6.01\text{e-}001$	6.06e-007	5.68e-005	6.17e-005	6.18e-005	6.18e-005
$\tau_{12018} = 1.20\text{e+}000$	9.93e-009	2.03e-005	3.45e-005	3.73e-005	3.77e-005
$\tau_{18028} = 1.80\text{e+}000$	1.40e-009	7.08e-006	1.67e-005	2.03e-005	2.10e-005
$\tau_{24037} = 2.40\text{e+}000$	4.50e-011	2.74e-006	7.65e-006	1.00e-005	1.06e-005
$\tau_{30046} = 3.00\text{e+}000$	5.75e-013	1.14e-006	3.48e-006	4.74e-006	5.05e-006
$\tau_{36055} = 3.61\text{e+}000$	5.35e-012	4.96e-007	1.58e-006	2.19e-006	2.34e-006
$\tau_{42064} = 4.21\text{e+}000$	1.06e-012	2.20e-007	7.14e-007	1.00e-006	1.07e-006
$\tau_{48074} = 4.81\text{e+}000$	6.47e-013	9.82e-008	3.23e-007	4.54e-007	4.87e-007
$\tau_{54083} = 5.41\text{e+}000$	2.54e-013	4.42e-008	1.46e-007	2.06e-007	2.21e-007
$\tau_{60092} = 6.01\text{e+}000$	1.17e-013	1.99e-008	6.61e-008	9.33e-008	1.00e-007

Table 4.10: Example 3, predictor-corrector: Development of $\varepsilon_{i,j}$ for the plotted values of τ and chosen points x_j.

to be significantly longer than for the other methods. Hence, it seems that in this case orthogonal collocation is less suitable than the explicit method or the predictor-corrector.

	$x_1 = 0.12$	$x_6 = 6.50$	$x_{13} = 26.60$	$x_{19} = 43.50$	$x_{25} = 50$
$\tau_{6009} = 6.01\text{e-}001$	5.54e-007	6.09e-005	6.18e-005	6.09e-005	3.04e-004
$\tau_{12018} = 1.20\text{e+}000$	8.11e-008	2.91e-005	3.70e-005	3.62e-005	4.43e-004
$\tau_{18028} = 1.80\text{e+}000$	2.12e-008	1.23e-005	2.01e-005	2.20e-005	2.69e-004
$\tau_{24037} = 2.40\text{e+}000$	3.87e-009	5.21e-006	9.96e-006	1.12e-005	1.61e-004
$\tau_{30046} = 3.00\text{e+}000$	7.46e-010	2.27e-006	4.72e-006	5.44e-006	9.39e-005
$\tau_{36055} = 3.61\text{e+}000$	2.34e-010	1.01e-006	2.27e-006	4.52e-006	5.81e-004
$\tau_{42064} = 4.21\text{e+}000$	2.55e-011	4.55e-007	9.62e-007	3.39e-007	2.02e-004
$\tau_{48074} = 4.81\text{e+}000$	7.01e-011	2.11e-007	3.26e-007	2.37e-006	7.73e-004
$\tau_{54083} = 5.41\text{e+}000$	1.16e-010	1.02e-007	2.61e-008	3.82e-006	1.09e-003
$\tau_{60092} = 6.01\text{e+}000$	5.25e-011	3.80e-008	1.67e-007	1.76e-006	4.47e-004

Table 4.11: Example 3, orthogonal collocation: Development of $\varepsilon_{i,j}$ for the plotted values of τ and chosen points x_j.

Tables 4.10 and 4.11 show the development of $\varepsilon_{i,j}$ with increasing τ. In both cases we observe that the differences between the approximations of $u(x_j)$ diminish with increasing time variable.

4.3.4 Comparison for Example 3

The found stationary solutions for all methods are increasing and concave and, as we argued earlier, they are likely to satisfy the limit property. The zero derivative at infinity also holds as it was again used as a boundary condition. Figure 4.12 shows a close up of the stationary solutions around zero together with the identity line (red). The derivative of the solutions for all methods is again close to one as expected. Thus the properties listed in section 1.4 are met also in this example. Figure 4.12 also includes the limit solution with $L = 1\,000$ and the graph shows that its slope is also very close to 1.

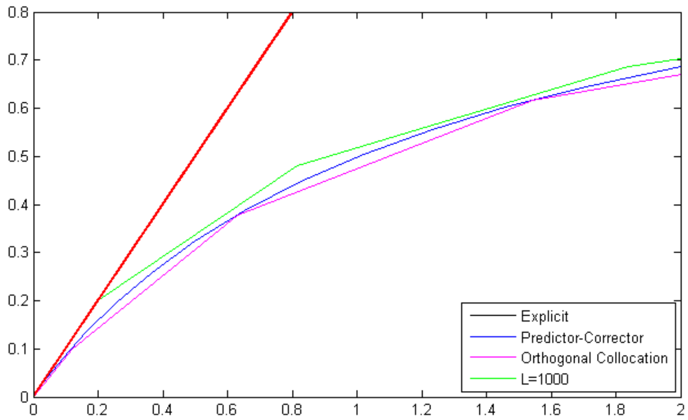
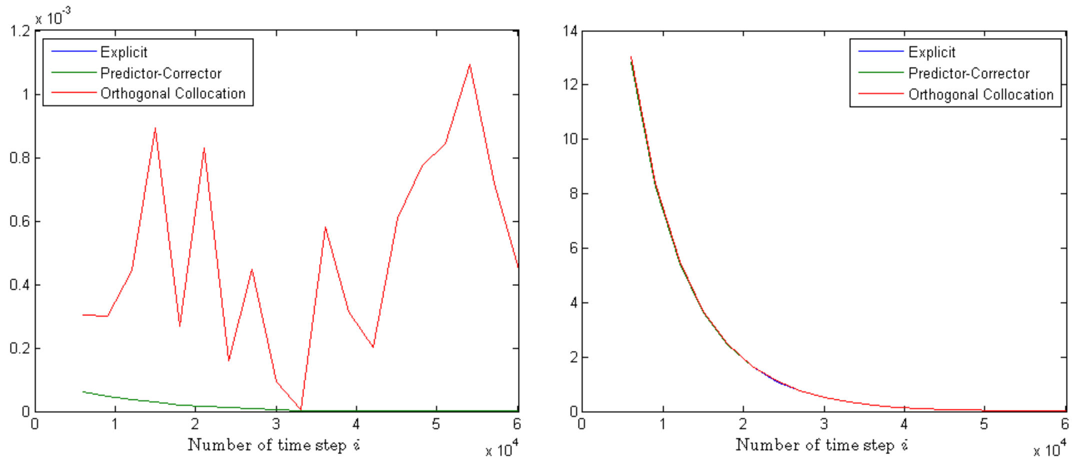


Figure 4.12: Stationary solutions of example 3 together with the identity line.

Let us now examine the settling down of the solutions for the three methods. The values of ε_i are listed in table 4.12 and depicted in figure 4.13a. Progressive settling down is apparent for the explicit method and the predictor corrector but in case of orthogonal collocation ε_i does not seem to decrease any further than to the order 10^{-4}.

The development of the approximation error e_i is very similar for all three methods and the precision in the last iteration is around 10^{-2}.

Figure 4.14 shows the investor's optimal strategy as it changes with τ and the results are similar to those from previous examples. For the original initial condition g decreases to the limit while it increases for the alternative initial condition.

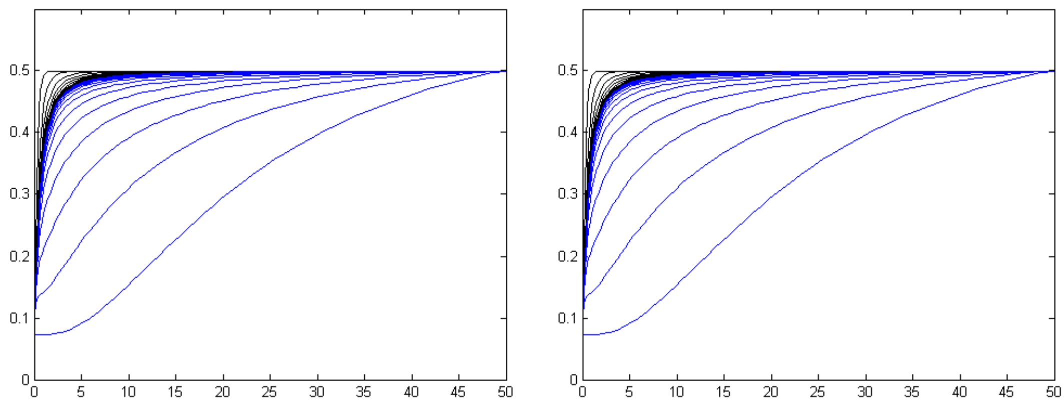


(a) Development of ε_i. **(b)** Development of e_i.

Figure 4.13: Comparison of the development of error for the three methods in ex. 3.

	ε_i			e_i		
	Explicit	Pred-Corr	O.Colloc	Explicit	Pred-Corr	O.Colloc
τ_{6009} = 6.01e-001	6.18e-005	6.18e-005	3.04e-004	1.28e+001	1.28e+001	1.30e+001
τ_{12018} = 1.20e+000	3.77e-005	3.77e-005	4.43e-004	5.43e+000	5.43e+000	5.52e+000
τ_{18028} = 1.80e+000	2.10e-005	2.10e-005	2.69e-004	2.43e+000	2.43e+000	2.47e+000
τ_{24037} = 2.40e+000	1.06e-005	1.06e-005	1.61e-004	1.11e+000	1.11e+000	1.13e+000
τ_{30046} = 3.00e+000	5.05e-006	5.05e-006	9.39e-005	5.04e-001	5.04e-001	5.14e-001
τ_{36055} = 3.61e+000	2.34e-006	2.34e-006	5.81e-004	2.29e-001	2.29e-001	2.36e-001
τ_{42064} = 4.21e+000	1.07e-006	1.07e-006	2.02e-004	1.04e-001	1.04e-001	1.07e-001
τ_{48074} = 4.81e+000	4.87e-007	4.87e-007	7.73e-004	4.72e-002	4.72e-002	4.85e-002
τ_{54083} = 5.41e+000	2.21e-007	2.21e-007	1.09e-003	2.14e-002	2.14e-002	2.21e-002
τ_{60092} = 6.01e+000	9.99e-008	1.00e-007	4.47e-004	9.67e-003	9.68e-003	1.01e-002

Table 4.12: Example 3: Development of ε_i and e_i for the plotted values of τ.

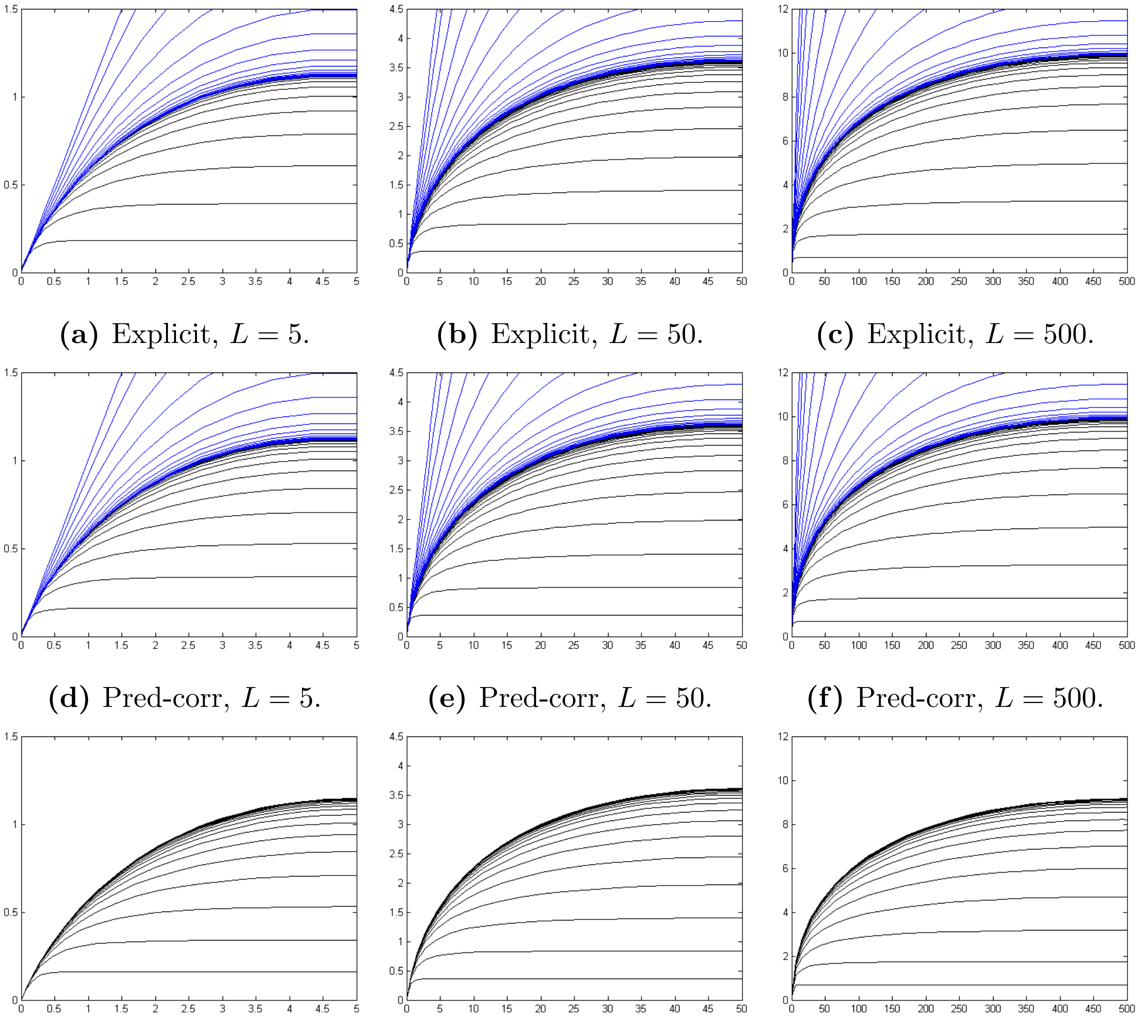


(a) Explicit method. **(b)** Predictor corrector.

Figure 4.14: The optimal policy g for the two initial conditions in example 3.

4.4 Example 4: $a > 0$, $b < 0$

This time we choose the values of parameters in such a way that a is positive and b negative. These values are $\sigma^2 = 0.8$, $\lambda = 0.7$, $r^* = 1.2$, $\rho = 2$ and hence $a = 0.75$, $b = -0.5$, $c = 0.625$. The condition of existence of solution, $a + b > 0$, holds. This is the first example where the second part of lemma 1.6 is valid. It says that if $b \leq 0$, then $u(x)$ is unbounded.



(a) Explicit, $L = 5$. **(b)** Explicit, $L = 50$. **(c)** Explicit, $L = 500$.

(d) Pred-corr, $L = 5$. **(e)** Pred-corr, $L = 50$. **(f)** Pred-corr, $L = 500$.

(g) Collocation, $L = 5$. **(h)** Collocation, $L = 50$. **(i)** Collocation, $L = 500$.

Figure 4.15: Numerical solutions for example 4. Black solutions correspond to the original initial condition and the blue ones to the alternative condition.

We solve this example with three different values of L: 5, 50 and 500. Now we specify the settings used in the explicit and predictor-corrector methods. For $L = 5$ we use $N + 1 = 16$ partition points for x and the time step $h = 5 \cdot 10^{-3}$. For the other two values of L, $N+1 = 61$ and $h = 5 \cdot 10^{-4}$ are used. In all three cases we work with $\varepsilon^0 = 10^{-6}$.

4.4.1 Explicit Euler Method

The numbers of steps M needed for the solutions to stabilize in the explicit Euler method as well as in the predictor-corrector are listed in table 4.13. As it was in previous examples, the use of the alternative initial condition leads to a higher number time steps required. The times in years are shown in the same table. They range for this method from less than 14 years when $L = 5$ up to almost 53 years when $L = 500$.

Number of time steps M						
	$L = 5$		$L = 50$		$L = 500$	
	$u(0,x) = 0$	$u(0,x) = x$	$u(0,x) = 0$	$u(0,x) = x$	$u(0,x) = 0$	$u(0,x) = x$
Explicit	1099	1196	20 423	23 642	35 052	42 156
Pred-Corr	963	1200	20 462	23 646	35 057	42 161
Corresponding time in years						
	$L = 5$		$L = 50$		$L = 500$	
	$u(0,x) = 0$	$u(0,x) = x$	$u(0,x) = 0$	$u(0,x) = x$	$u(0,x) = 0$	$u(0,x) = x$
Explicit	13.74	14.95	25.53	29.55	43.82	52.70
Pred-Corr	12.04	15.00	25.58	29.56	43.82	52.70

Table 4.13: Resulting values of M and the corresponding time in years for example 4.

The resulting solutions for both initial conditions are shown in the first row of figure 4.15. According to theory (lemma 1.6), the limit solutions should be strictly increasing for all x and we observe that they do not flatten out as it was the case in previous examples. They have similar shape for different values of L and they only stop increasing at the end of the considered intervals which is caused by the boundary condition (3.12) saying that the derivative at the end point is zero.

Unlike the cases when $b > 0$, there is no upper bound for values of $u(x)$ and therefore choosing a higher L allows it to attain higher values. For $L = 5$ it reaches to around 1.1, for $L = 50$ it is 3.6, and for $L = 500$ the values go up to 10.

We include table 4.14 which shows how $\varepsilon_{i,j}$ decreases with τ for the explicit method used on example 4 with $L = 5$ and the zero initial condition.

	$x_1 = 0.022$	$x_4 = 0.36$	$x_8 = 1.42$	$x_{11} = 2.69$	$x_{15} = 5$
$\tau_{110} = 5.50\text{e-}001$	1.81e-005	7.80e-004	3.17e-003	3.81e-003	3.97e-003
$\tau_{220} = 1.10\text{e+}000$	3.14e-005	5.59e-005	1.27e-003	2.31e-003	2.83e-003
$\tau_{330} = 1.65\text{e+}000$	2.82e-005	2.74e-005	4.24e-004	9.33e-004	1.21e-003
$\tau_{440} = 2.20\text{e+}000$	1.40e-005	3.67e-005	1.19e-004	3.16e-004	4.26e-004
$\tau_{550} = 2.75\text{e+}000$	2.38e-006	2.51e-005	1.34e-005	7.97e-005	1.18e-004
$\tau_{659} = 3.29\text{e+}000$	8.45e-007	9.92e-006	1.15e-005	3.57e-006	1.35e-005
$\tau_{769} = 3.85\text{e+}000$	4.00e-007	1.52e-006	9.18e-006	1.05e-005	1.03e-005
$\tau_{879} = 4.40\text{e+}000$	6.67e-008	3.70e-007	3.54e-006	6.63e-006	8.08e-006
$\tau_{989} = 4.95\text{e+}000$	9.36e-008	1.41e-007	9.24e-007	2.42e-006	3.24e-006
$\tau_{1099} = 5.50\text{e+}000$	2.12e-008	7.11e-008	2.93e-007	7.30e-007	9.91e-007

Table 4.14: Example 4, explicit method: Development of $\varepsilon_{i,j}$ for the plotted values of τ and chosen points x_j.

4.4.2 Predictor-Corrector

The values of M and the corresponding times in years for the predictor-corrector method are included in table 4.13. In previous examples this method required a slightly higher number of time steps than the explicit Euler method and that is also true for all variations of example 4 except the first one with $L = 5$ and the original initial condition. In this one case the predictor-corrector requires only 963 time steps while $M = 1099$ for the explicit method.

The resulting predictor-corrector solutions for both initial conditions are depicted in the middle row of figure 4.15. We observe that these solutions are very much the same as the ones from the explicit method.

The approximations of $u(x_j)$ settle down to their limits as τ increases, cf. table 4.15.

	$x_1 = 0.022$	$x_4 = 0.36$	$x_8 = 1.42$	$x_{11} = 2.69$	$x_{15} = 5$
$\tau_{96} = 4.80\text{e-}001$	2.33e-005	1.06e-003	3.34e-003	3.80e-003	3.90e-003
$\tau_{193} = 9.65\text{e-}001$	3.13e-005	1.20e-004	1.63e-003	2.75e-003	3.26e-003
$\tau_{289} = 1.45\text{e+}000$	2.81e-005	1.34e-005	6.46e-004	1.34e-003	1.70e-003
$\tau_{385} = 1.93\text{e+}000$	1.54e-005	3.44e-005	2.34e-004	5.57e-004	7.35e-004
$\tau_{482} = 2.41\text{e+}000$	3.88e-006	2.78e-005	6.71e-005	2.00e-004	2.75e-004
$\tau_{578} = 2.89\text{e+}000$	5.96e-007	1.41e-005	9.01e-006	5.64e-005	8.42e-005
$\tau_{674} = 3.37\text{e+}000$	7.52e-007	3.97e-006	4.21e-006	7.87e-006	1.58e-005
$\tau_{770} = 3.85\text{e+}000$	1.33e-007	4.44e-008	3.32e-006	2.60e-006	1.61e-006
$\tau_{867} = 4.34\text{e+}000$	1.10e-007	4.33e-007	1.02e-006	2.11e-006	2.57e-006
$\tau_{963} = 4.82\text{e+}000$	8.26e-008	1.05e-007	1.20e-007	6.77e-007	9.95e-007

Table 4.15: Example 4, predictor-corrector: Development of $\varepsilon_{i,j}$ for the plotted values of τ and chosen points x_j.

4.4.3 Orthogonal Collocation

We solve this problem by orthogonal collocation with 26 collocation points as well. The results can be seen in the last row of figure 4.15. The considered intervals for τ are $[0, 4.82]$, $[0, 10.23]$ and $[0, 17.53]$ for $L = 5$, $L = 50$ and $L = 500$ respectively. This again means very long computation times compared to the other two methods. The resulting orthogonal collocation solutions are very similar to the ones from the other methods for $L = 5$ and $L = 50$.

For the last case, $L = 500$, the solutions also look similar at first sight but the limit solution only achieves values up to around 9 while for the other two methods it reaches 10. The reason for this is a significantly finer partition of the interval used in the first two methods. The explicit method and the predictor-corrector solutions increase rapidly from zero whereas the orthogonal collocation solution needs to cover a wider interval there and therefore the slope is not quite as steep. A finer partition in the last method would mean solving a higher dimension system of ODEs which would make the computation even longer.

We believe the explicit method and the predictor-corrector solutions to be better approximations of the true solution $u(x)$ and for the reasons explained we favor this methods over orthogonal collocation.

Table 4.16 shows the decreasing values of $\varepsilon_{i,j}$ with increasing i for chosen j.

	$x_1 = 0.012$	$x_6 = 0.65$	$x_{13} = 2.66$	$x_{19} = 4.35$	$x_{25} = 5$
$\tau_{96} = 4.80\text{e-}001$	5.06e-007	1.83e-003	3.83e-003	3.94e-003	3.91e-003
$\tau_{193} = 9.65\text{e-}001$	4.53e-008	4.62e-004	2.57e-003	3.20e-003	3.22e-003
$\tau_{289} = 1.45\text{e+}000$	2.13e-008	1.60e-004	1.25e-003	1.66e-003	2.02e-003
$\tau_{385} = 1.93\text{e+}000$	3.40e-009	6.57e-005	5.66e-004	7.63e-004	7.01e-004
$\tau_{482} = 2.41\text{e+}000$	7.65e-010	2.85e-005	2.51e-004	3.40e-004	2.81e-004
$\tau_{578} = 2.89\text{e+}000$	2.70e-010	1.27e-005	1.12e-004	1.52e-004	1.59e-004
$\tau_{674} = 3.37\text{e+}000$	4.22e-010	5.73e-006	5.03e-005	6.46e-005	1.17e-003
$\tau_{770} = 3.85\text{e+}000$	1.04e-011	2.58e-006	2.26e-005	3.12e-005	1.77e-004
$\tau_{867} = 4.34\text{e+}000$	3.19e-011	1.15e-006	1.01e-005	1.42e-005	1.76e-004
$\tau_{963} = 4.82\text{e+}000$	3.45e-010	5.15e-007	4.45e-006	2.60e-006	1.12e-003

Table 4.16: Example 4, orthogonal collocation: Development of $\varepsilon_{i,j}$ for the plotted values of τ and chosen points x_j.

4.4.4 Comparison for Example 4

The resulting limit solutions found by the three methods, shown in figure 4.15, are all increasing and concave with zero slope on at $x = L$. In this example b is negative so there is no limit for $u(x)$. The only property from section 1.4 which remains to be verified is the derivative at zero. Figure 4.16 shows a close up of the neighborhood of zero for the stationary solutions for all methods with $L = 5$ together with the identity line. We observe that the slope is again close to 1 as it was in the other examples.

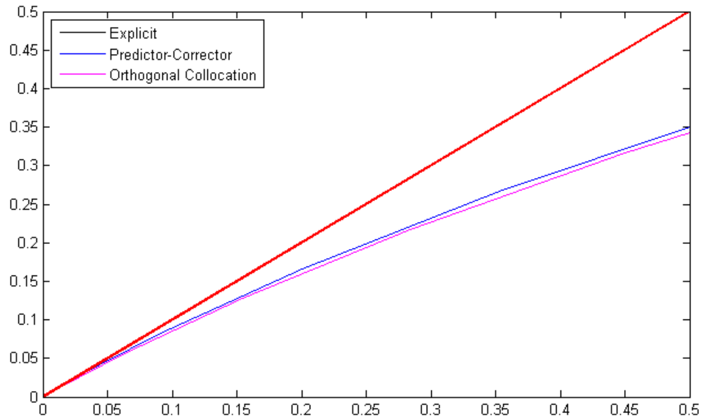
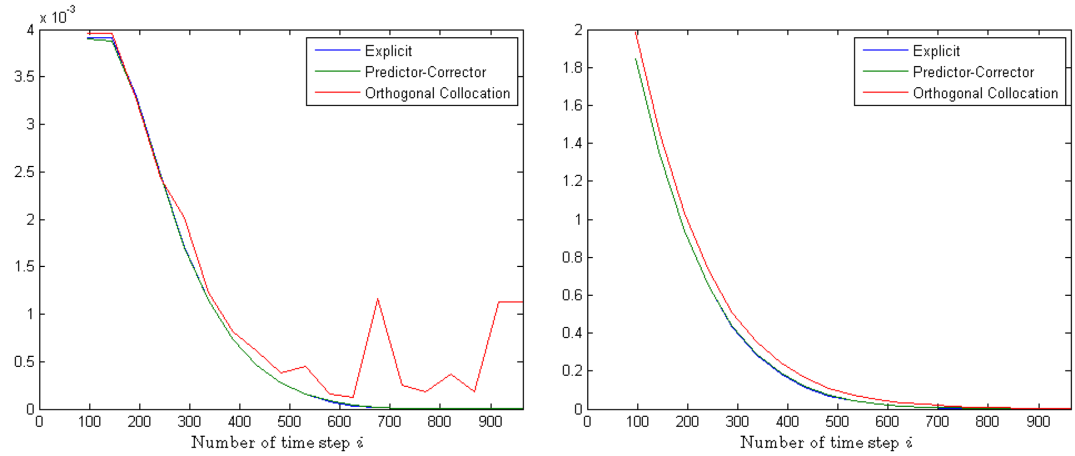


Figure 4.16: Stationary solutions of example 4 together with the identity line.

The rate of settling down ε_i is depicted in figure 4.17a and the numerical values are listed in table 4.17. In this part $L = 5$ and the original initial condition are used. We observe similar results as in previous examples. For orthogonal collocation the process is not monotonic like it is for the other two methods.

	ε_i			e_i		
	Explicit	Pred-Corr	O.Colloc	Explicit	Pred-Corr	O.Colloc
$\tau_{96} =$ 4.80e-001	3.91e-003	3.90e-003	3.96e-003	1.85e+000	1.85e+000	1.99e+000
$\tau_{193} =$ 9.65e-001	3.29e-003	3.26e-003	3.26e-003	9.36e-001	9.39e-001	1.03e+000
$\tau_{289} =$ 1.45e+000	1.71e-003	1.70e-003	2.02e-003	4.32e-001	4.37e-001	5.08e-001
$\tau_{385} =$ 1.93e+000	7.32e-004	7.35e-004	8.16e-004	1.83e-001	1.88e-001	2.41e-001
$\tau_{482} =$ 2.41e+000	2.71e-004	2.75e-004	3.78e-004	7.00e-002	7.42e-002	1.11e-001
$\tau_{578} =$ 2.89e+000	7.84e-005	8.42e-005	1.59e-004	2.49e-002	2.84e-002	5.05e-002
$\tau_{674} =$ 3.37e+000	1.41e-005	1.58e-005	1.17e-003	8.40e-003	1.10e-002	2.34e-002
$\tau_{770} =$ 3.85e+000	1.05e-005	3.32e-006	1.77e-004	2.92e-003	4.53e-003	1.04e-002
$\tau_{867} =$ 4.34e+000	8.67e-006	2.57e-006	1.76e-004	1.15e-003	1.98e-003	5.12e-003
$\tau_{963} =$ 4.82e+000	4.17e-006	9.95e-007	1.12e-003	5.04e-004	8.74e-004	2.08e-003

Table 4.17: Example 4: Development of ε_i and e_i for the plotted values of τ.



(a) Development of ε_i.

(b) Development of e_i.

Figure 4.17: Comparison of the development of error for the three methods in ex. 4.

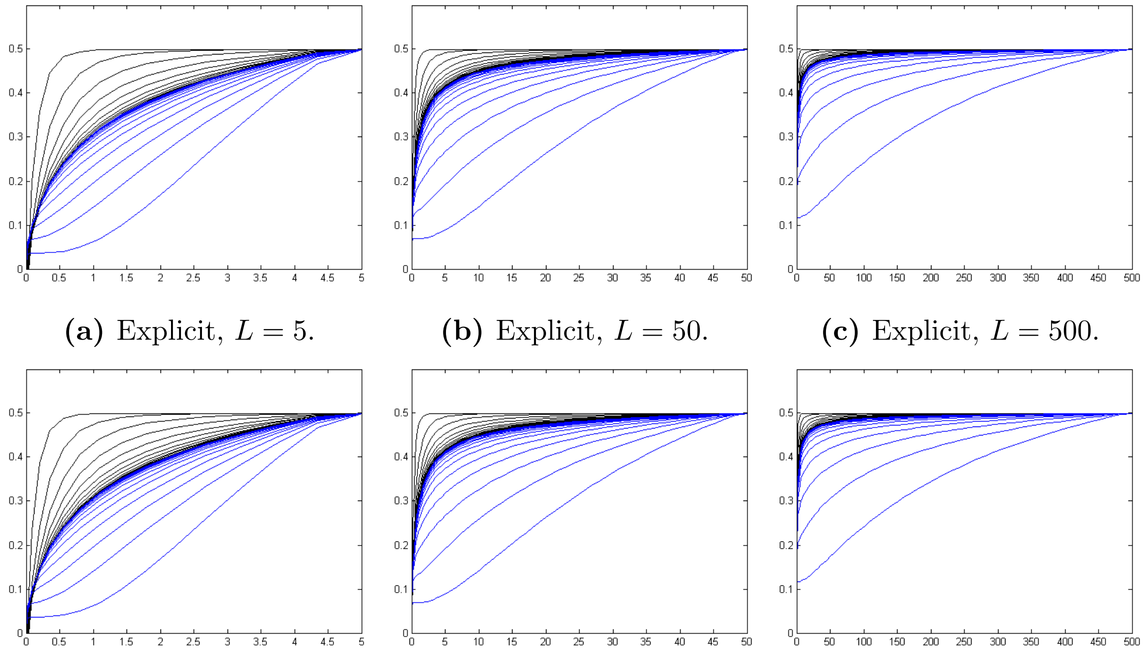
The imprecision of approximation at time τ_i, denoted by e_i and shown in figure 4.17b and table 4.17, decreases monotonically for all methods and we observe that the values for orthogonal collocation are slightly higher than for the other two methods but they are of the same order.

Figure 4.18 shows numerical approximations of the investor's optimal strategy for example 4 with the three different values of L. The policies are again increasing w.r.t. τ for the alternative initial condition (blue curves) while they are decreasing for the original initial condition (black curves) and they tend to the same limit.

4.5 Comparison of the Methods

Based on numerical examples we conclude that the explicit Euler method, the predictor-corrector and orthogonal collocation all produce similar results. In all cases, the limit solution satisfies the properties of the solution $u(x)$ proven in [3] and listed in section 1.4 of this work. Therefore we believe that the solution $u(x)$ suggested by the methods is a good approximation of the solution to the HJB equation (1.25) and that the value function $w(y, z)$ of problem (1.2)-(1.4) can be approximated by use of $u(x)$ and substitution (1.14).

The explicit Euler method and the predictor-corrector produce very similar results. In fact, the corrector step of the latter method does not seem to be an improvement



(a) Explicit, $L = 5$. **(b)** Explicit, $L = 50$. **(c)** Explicit, $L = 500$.

(d) Pred-corr, $L = 5$. **(e)** Pred-corr, $L = 50$. **(f)** Pred-corr, $L = 500$.

Figure 4.18: The optimal policy g for the two initial conditions in example 4.

to the predictor step, which is an estimate from the explicit Euler method. With the exception of one case in example 4, the predictor-corrector needed slightly more time steps for the solutions to settle down. Even though the differences were minimal, it is a reason to favor the explicit method over the predictor-corrector because the former is simpler which means that its computation times are lower.

The orthogonal collocation method proved to be significantly more time-consuming than the other two methods. As we argued earlier, this is due to the fact that it requires to solve a system of $N + 1$ ordinary differential equations, where N is the number of internal collocation points. It clearly limits the number of collocation points which can lead to a poor approximation of the solution especially when a longer interval for x is considered, i.e. higher value of L is used. Also, working with a longer time interval proved to lead to a significantly longer computation time for orthogonal collocation than for the other methods.

These reasons lead us to the conclusion that even though all three methods produce similar results, the explicit method is to be favored over predictor-corrector and orthogonal collocation for this problem.

CONCLUSION

The aim of this work was to find a suitable numerical method for the Hamilton-Jacobi-Bellman (HJB) equation corresponding to the problem of optimal liquidation of a large trading position. We have not found a method which would allow to find the solution directly but we have been successful in developing an alternative approach to the problem which leads to the searched solution. The approach was based on truncating the problem to a finite time horizon. The outcome was a series of lower solutions (or upper solutions if the alternative initial condition is used) to the original problem which tend monotonically to the solution. We presented and compared three methods based on this approach all of which proved successful in finding the searched solution

In chapter 1, we introduce the problem and derive the HJB equation. Except listing known results about the problem and its solution, we also add a proof for the upper bound of the solution $u(x) \leq x$.

In the second chapter, we deal with theoretical preliminaries for the alternative approach which we use for numerical treatment of the optimal liquidation problem. We present an alternative initial condition which corresponds to a scenario where, at the end of the considered time interval, the speculator can sell the remaining amount of foreign currency with no negative effect on the exchange rate, as opposed to the original scenario, where no selling after the time time interval is allowed. Finally, we show monotonicity of the value function with respect to the length of the considered time interval for the original and the alternative initial conditions.

We present in the third chapter three numerical methods based on the alternative approach. The explicit Euler and the predictor-corrector methods both work with discretized time and spatial variables and a new time layer is computed from a previous one. In orthogonal collocation only the spatial variable is discretized and one eventually arrives at a system of $N + 2$ ordinary differential equations. The solution to this system are functions describing the solution $u(\tau, x)$ at the collocation points for different values of the time variable. The system of ODEs can be solved by standard numerical methods.

In the final chapter, we present four numerical examples where we demonstrate the use of the three methods described in the previous chapter. The first example is a non-stochastic case for which there is an analytical solution and we confirm graphically that the solutions suggested by both of our methods are very close to the analytical one. The other three examples are for different feasible combinations of the signs of parameters a and b. In all examples we find that the solutions to the parabolic PDE settle down to a stationary solution which we believe to be the solution to the original equation.

We also analyze the development of the approximation error with increasing τ. Finally, we compare the three methods and conclude that the explicit Euler method is the most suitable for numerical treatment of this kind of problems. It is computationally simpler than the other two methods and produces good approximations of the searched solution.

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APPENDIX

Appendix A - Source Code for the Explicit Euler Method

```
function [x,u,g,M,eps] = explicit(sigma2,lambda,rStar,rho,L,N,h,Mmax,eps0,alt)
% explicit euler with non-equidistant partition of [0,Xmax]
% we consider the equation u_t = x^2 u_{xx} - a x u_x - b u + c(u_x-1)^2

tic
a = 2/sigma2*(sigma2 + lambda - rStar);
b = 2/sigma2*(rho - sigma2 - 2*lambda);
c = 1/(2*sigma2);

x = linspace(0,sqrt(L),N+1);
x = x.^(2);

u = zeros(Mmax+1,N+1);  % solution
g = zeros(Mmax+1,N+1);  % policy
F = zeros(N-1,1);
eps = zeros(Mmax,1);

% alternative initial condition
if(alt==1)
    u(1,:) = x;
end

A = zeros(N-1,N+1);
for j=2:N
```

```
    A(j-1,j-1) = 2*x(j)^2/((x(j+1)-x(j-1))*(x(j)-x(j-1))) + a*x(j)/(x(j+1)-x(j-1));
    A(j-1,j) = -2*x(j)^2/(x(j+1)-x(j-1))*(1/(x(j)-x(j-1))+1/(x(j+1)-x(j))) - b;
    A(j-1,j+1) = 2*x(j)^2/((x(j+1)-x(j-1))*(x(j+1)-x(j))) - a*x(j)/(x(j+1)-x(j-1));
end

M = 1;
err = 1;
while(err>eps0 && M<Mmax)
    % solution
    for j=2:N
        F(j-1) = c*( (u(M,j+1)-u(M,j-1))/(x(j+1)-x(j-1)) -1 )^2;
    end
    u(M+1,2:N) = u(M,2:N) + h*(A*u(M,:)' + F)';
    u(M+1,N+1) = u(M+1,N);
    % policy
    for j=2:N
        g(M+1,j) = (1-(u(M+1,j+1)-u(M+1,j-1))/(x(j+1)-x(j-1)))/2;
    end
    g(M+1,1) = (1-(u(M+1,2)-u(M+1,1))/(x(2)-x(1)))/2;
    g(M+1,N+1) = 1/2;

    err = norm(u(M+1,:)-u(M,:),Inf);
    eps(M) = err;
    M = M+1;
end
u = u(1:M,:);
g = g(1:M,:);
eps = eps(1:M-1);
toc
fprintf('The number of time steps is %d.\n',M-1);
fprintf('Tau is from [0, %g].\n',(M-1)*h);
fprintf('The considered time is %g years.\n',(M-1)*h*2/sigma2);
```

Appendix B - Source Code for the Predictor-Corrector

```
function [x,u,g,M,eps] = pred_corr(sigma2,lambda,rStar,rho,L,N,h,Mmax,eps0,alt)
% predictor-corrector with non-equidistant partition of [0,Xmax]
% we consider the equation u_t = x^2 u_{xx} - a x u_x - b u + c(u_x-1)^2

tic
a = 2/sigma2*(sigma2 + lambda - rStar);
b = 2/sigma2*(rho - sigma2 - 2*lambda);
c = 1/(2*sigma2);

x = linspace(0,L^(1/2),N+1);
x = x.^(2);

u = zeros(Mmax+1,N+1);  % solution
g = zeros(Mmax+1,N+1);  % policy
F = zeros(N-1,1);
F2 = zeros(N-1,1);
uTilde = zeros(N+1,1);
eps = zeros(Mmax,1);

% alternative initial condition
if(alt==1)
    u(1,:) = x;
end

A = zeros(N-1,N+1);
for j=2:N
    A(j-1,j-1) = 2*x(j)^2/((x(j+1)-x(j-1))*(x(j)-x(j-1))) + a*x(j)/(x(j+1)-x(j-1));
    A(j-1,j) = -2*x(j)^2/(x(j+1)-x(j-1))*(1/(x(j)-x(j-1))+1/(x(j+1)-x(j))) - b;
    A(j-1,j+1) = 2*x(j)^2/((x(j+1)-x(j-1))*(x(j+1)-x(j))) - a*x(j)/(x(j+1)-x(j-1));
end

M = 1;
err = 1;
while(err>eps0 && M<Mmax)
    % predictor:
    for j=2:N
        F(j-1) = c*( (u(M,j+1)-u(M,j-1))/(x(j+1)-x(j-1)) -1 )^2;
    end
    uTilde(2:N) = u(M,2:N)' + h*(A*u(M,:)' + F);
    uTilde(N+1) = uTilde(N);
    % corrector:
    for j=2:N
        F2(j-1) = c*( (uTilde(j+1)-uTilde(j-1))/(x(j+1)-x(j-1)) -1 )^2;
```

```
    end
    u(M+1,2:N) = u(M,2:N) + h/2*( A*(u(M,:)'+uTilde) + F + F2 )';
    u(M+1,N+1) = u(M+1,N);
    % policy
    for j=2:N
        g(M+1,j) = (1-(u(M+1,j+1)-u(M+1,j-1))/(x(j+1)-x(j-1)))/2;
    end
    g(M+1,1) = (1-(u(M+1,2)-u(M+1,1))/(x(2)-x(1)))/2;
    g(M+1,N+1) = 1/2;

    err = norm(u(M+1,:)-u(M,:),Inf);
    eps(M) = err;
    M = M+1;
end
u = u(1:M,:);
g = g(1:M,:);
eps = eps(1:M-1);
toc
fprintf('The number of time steps is %d.\n',M-1);
fprintf('Tau is from [0, %g].\n',(M-1)*h);
fprintf('The considered time is %g years.\n',(M-1)*h*2/sigma2);
```

Appendix C - Source Code for Orthogonal Collocation

We include a script for orthogonal collocation with two internal collocation points. The reason is that the matrices A and B have a low dimension (4×4) in this case which makes the code easier to read. Scripts for more collocation points are analogical.

```
function [x,u,M,eps] = collocation2(sigma2,lambda,rStar,rho,L,Tmax,M,alt)
% orthogonal collocation with 24 internal points
% we consider the equation u_t = x^2 u_{xx} - a x u_x - b u + c(u_x-1)^2

a = 2/sigma2*(sigma2 + lambda - rStar);
b = 2/sigma2*(rho - sigma2 - 2*lambda);
c = 1/(2*sigma2);

param = [a,b,c,L];
% pre-calculated collocation points
z = [0.000000  0.211325  0.788675  1.000000];
% pre-calculated matrix of collocation weights
A = [-7.0000000      8.1961524     -2.1961524      1.0000000
     -2.7320508      1.7320508      1.7320508     -0.7320508
      0.7320508     -1.7320508     -1.7320508      2.7320508
     -1.0000000      2.1961524     -8.1961524      7.0000000];

x = L*z;
if alt==0
    init = zeros(1,2);
else
    init = x(2:3);
end
% solving the system of ODEs
tic
[T,U] = ode45(@derivatives2,linspace(0,Tmax,M),init,[],param);
toc

% M = length(T);
u = [zeros(M,1),U,-(A(4,2)*U(:,1) + A(4,3)*U(:,2))/A(4,4)];

eps = zeros(M-1,1);
for i=1:(M-1)
    eps(i) = norm(u(i+1,:)-u(i,:),Inf);
end
```

```
function du = derivatives2(t,u,param)
a = param(1);
b = param(2);
c = param(3);
L = param(4);
% pre-calculated collocation points
z = [0.000000  0.211325  0.788675  1.000000];
% pre-calculated collocation weights for first derivatives
A = [-7.0000000      8.1961524     -2.1961524      1.0000000
     -2.7320508      1.7320508      1.7320508     -0.7320508
      0.7320508     -1.7320508     -1.7320508      2.7320508
     -1.0000000      2.1961524     -8.1961524      7.0000000];
% pre-calculated collocation weights for second derivatives
B = [24.0000000    -37.1769145     25.1769145    -12.0000000
     16.3923048    -24.0000000     12.0000000     -4.3923048
     -4.3923048     12.0000000    -24.0000000     16.3923048
    -12.0000000     25.1769145    -37.1769145     24.0000000];

du = zeros(2,1);
for i=1:2
    du(i) = z(i+1)^2 * B(i+1,:)*[0;u;-(A(4,2:3)*u)/A(4,4)]...
        - a*z(i+1)* A(i+1,:)*[0;u;-(A(4,2:3)*u)/A(4,4)] - b*u(i)...
        + c*(1/L*A(i+1,:)*[0;u;-(A(4,2:3)*u)/A(4,4)] - 1)^2;
end
```