COMENIUS UNIVERSITY IN BRATISLAVA
Faculty of Mathematics, Physics and Informatics

OPTIMIZATION OF STABLE BUDGET

Master's thesis

2014 Bc. Michaela Rošková

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Master's thesis

Study programme:	Economic and Financial Mathematics
Field of Study:	1114. Applied Mathematics
Department:	Department of Applied Mathematics and Statistics
Supervisor:	Mgr. Richard Kollár, PhD.

Bratislava 2014

Bc. Michaela Rošková

UNIVERZITA KOMENSKÉHO V BRATISLAVE
Fakulta Matematiky, Fyziky a Informatiky

OPTIMALIZÁCIA STABILNÉHO ROZPOČTOVANIA

Diplomová práca

Študijný program:	Ekonomická a finančná matematika
Študijný odbor:	1114. Aplikovaná matematika
Školiace pracovisko:	Katedra aplikovanej matematiky a štatistiky
Vedúci práce:	Mgr. Richard Kollár, PhD.

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Study programme:	Economic and Financial Mathematics (Single degree study, master II. deg., full time form)
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Title: Optimization of Stable Budget

Aim: The main goal of the diploma thesis is an optimization of government decisions in the model of stable budgeting proposed in Bachelor thesis of M. Rošková (Comenius University, 2012). The optimal control problem in economy is solved by analytical and numerical methods.

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Approved:	04.02.2013

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ZADANIE ZÁVEREČNEJ PRÁCE

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Cieľ: Práca bude nadväzovať na bakalársku prácu M. Roškovej (FMFI UK 2012), v ktorej bol zostavený model stabilného priebežného rozpočtovania. Cieľom diplomovej práce je optimalizácia nastavenia tohto modelu prepojeného s modelom ekonomiky. Zostavený (singulárny) optimalizačný problém sa bude riešiť analytickými a numerickými metódami.

Vedúci: Mgr. Richard Kollár, PhD.
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Dátum zadania: 25.01.2013

Dátum schválenia: 04.02.2013 prof. RNDr. Daniel Ševčovič, CSc.
garant študijného programu

....
študent vedúci práce

Declaration on Word of Honour

I declare on my honour that this thesis was written on my own, with the only help provided by my supervisor and the referred-to literature and sources.

In Bratislava, April 20, 2014 Bc. Michaela Rošková

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I would like to express my gratitude and appreciation to my supervisor Mgr. Richard Kollár, PhD. for his guidance, help, ideas, patience, and corrections of my writings. I am also grateful to my family and friends for their support and help during my studies.

Abstract

The government decisions on public budgeting and redistribution of public revenue between social spending, public investments and paying off public debts should optimally reflect a current state of economy. However financial decisions of politicians have often rather populistic motivations that drive them far from an optimal policy. We study optimal economy decisions of governments in a model of an isolated state economy with expenditure rules restricted to weighted redistribution of public revenue in time.

The results offer some interesting insights: (i) the policies naturally show discrepancies of the government promises and real policy characterized by preferences of the government and demonstrate high importance of (unknown) future depreciation of government objectives, (ii) the amount of social spending in optimal decisions is limited by its influence on the natural rate of unemployment and on the public investments rather than by budgetary restrictions, (iii) in the optimum even in a period of economical stability the main component of the public budget in optimum is composed of the past revenues rather than of future estimated revenues.

Keywords : state budgeting, social spending, state debts, redistribution of public revenue, budget optimization

Abstrakt v štátnom jazyku

Súčasná ekonomika štátu sa odzrkadľuje na vládnych rozhodnutiach, ktoré ovplyvňujú tvorbu verejného rozpočtu, splácanie štátneho dlhu, či prerozdelenie štátnych príjmov medzi vyplatené sociálne dávky a verejné investície. Tieto rozhodnutia by mali prezentovať optimálny vplyv vlády na ekonomiku, no väčšinou sa jedná len o rozhodnutia, ktoré sú populisticky motivované. V práci vytvárame model izolovanej ekonomiky štátu, v ktorom definujeme tvorbu štátneho rozpočtu pomocou príjmov prerozdelených v čase a analyzujeme optimálne ekonomické rozhodnutia vlády.

Výsledky práce ponúkajú niekoľko zaujímavých postrehov: (i) vývoj vnútroštátnej ekonomiky zvýrazňuje dôležitosť (neznámeho) budúceho znehodnotenia vládnych cieľov a prirodzene poukazuje na rozdiely medzi vládnymi prísľubmi a reálnymi rozhodnutiami vlády, ktoré sú ovplyvnené vládnymi preferenciami, (ii) vládne rozhodnutia ohľadne navýšenia sociálnych dávok sú limitované predovšetkým úrovňou zamestnanosti v krajine a verejnými investíciami a nie výškou obmedzeného štátneho rozpočtu, (iii) tvorba štátneho rozpočtu sa dokonca aj v štádiu ekonomickej stability zameriava skôr na výšku minulých vládnych príjmov ako na výšku odhadovaných budúcich príjmov.

Kľúčové slová: tvorba štátneho rozpočtu, sociálne dávky, štátny dlh, prerozdelenie príjmov, optimalizácia štátneho rozpočtu

Contents

Introduction

The current world debt crisis comes with the urgent need for a change in an economy development. The suitable idea of the change of the expenditure funding could be based on the unique idea of a creation of the available budget used in various universities in the USA. The university publishes the projects for improving the quality of the education and gives the opportunity for investors to support a development of the university. The management of the university yearly creates the spending rule which characterizes the amount of the annual yield from investments used for the university expenditure which exact value is defined in advance. Using this spending rule the university may offer the same conditions for the education to several generations of the students and can easily cope with a funding of needed innovations.

The idea of the expenditure funding by the revenues redistributed in time is summarized in the bachelor thesis of Michaela Rošková [1]. The thesis speaks of the creation of an available budget using the selected levels of the government revenues for several years. Using this method of the time redistribution the available budget could consist of the previous revenues representing the expenditure funding by the money saved, of the current revenues, and of the estimated future revenues corresponding to increase of the government lend which would be repaid by the real future revenues. The revenues of each year have a different impacts on the budget assignment which are represented by the constant vector of coefficients $\tilde{\alpha}$.

The bachelor thesis shows the method of an estimation of these coefficients on the real data of selected countries for the time redistribution of revenues of three years. We assume that the government defines the available budget for expenditure funding by the exact values of the previous, current and estimated future revenues and by the set values of the corresponding coefficients. The estimations of the vector $\tilde{\alpha}$ show the differences among budget assignments of the selected countries. The countries with unstable economy hopefully use the higher value of the coefficient corresponding to future revenue for speeding up the development of the internal economy of the state. In the contrary, the states with developed economy possibly raise the impact of the previous revenues for creation of the current budget. This attitude of the government expenditure funding by the previous revenues represents the effort for the stabilization of the state economy.

Each country has a different redistribution of the revenues in time which defines the trend of economy development and a different sums of the selected coefficients $\tilde{\alpha}$ characterize the level of the spending rule and a utilization of the money gain from the revenues. If the sum of $\tilde{\alpha}$ is greater than one the government cannot finance the expenditure by the amount of the selected revenues and thus the state debt deepens. On the other hand, the sum less than one represents stabilization or a compensation of the state debt at the expense of the restrictive economy development.

The bachelor thesis comes with the possible estimation of the coefficients already used for the budget assignment, but the estimation does not necessary define the optimal choice of the revenue time redistribution for the particular economy. The optimal choice of the values of the vector $\tilde{\alpha}$ is affected by the effort of the government for improving living conditions for people and for stabilization of the state economy.

Organization of Thesis

In the first chapter we define a model of the isolated economy characterized by the equations of development of selected economical quantities. Particular constant parameters used in the difference equations and the initial values of the selected quantities in the model are estimated from Slovak Republic data. Therefore, the simulation of the economy development should characterize the possible development of isolated Slovak Republic economy considering no external influences. We define an objective function used for an optimization of government decisions. The function represents the value of the populistic decisions of the government related to a depression of the state debt and to an increase of a social benefits level.

For avoiding the computation errors we have to define the dimensionless equivalent of the model. The second chapter shows the derivation of the nondimensional model and the definition of the initial conditions for the system of the difference equations. For correct simulation of the model and for correct analysis of the optimal solutions we have to nondimensionalize the impacts of the particular constant parameters used in the difference equations on the economy development. If the government wants to create the same living and working conditions for future generations, it can possibly change the values of some parameter to cause the constant growth of the isolated economy of the state.

The results defining the difference between government promises and real government policy are summarized in the third chapter. The parameters in the objective function define the optimal government decisions considering a type of its policy and a discount factor used for diminishing of the impact of the government decisions. We analyze the optimal value of the constant social benefits level growth for solely liberal and conservative governments and its impact on the economy development. The research specifies the short interval of values of these parameters for government with non-extreme type of the policy.

The fourth chapter shows the optimal solution for the maximization of the populistic decisions of the government. The optimization process offers various suggestions for the restricted budget assignment affected by the revenue redistribution in time defined by the optimal vector $\tilde{\alpha}$ and for the social benefits increase.

1 Mathematical Model of State Economy

The main idea of this thesis is to create a complex model that quantitatively describes economical and mathematical processes in an isolated economy. A realistic model of state capital and cash flow would be highly complex as it would need to characterize state revenue and expenditure, social benefits, state investments and an entire prosperity of the country. Such a model should also include government decisions, adaptation to the new situations and control of state budgeting. Therefore the complete economical model would necessary contain a high number of equations that capture possible state difficulties and offer solutions to various financial situations. On the other hand, such a very complex model would represent a risk of extreme application complexity and misinterpretations of the model parameters and thus would possibly lead to wrong results. These factors need to be considered while managing the balance between complex real processes, simplicity, and accessibility of the model. Thus our goal is to design a simple state economy model with equations for selected key factors that

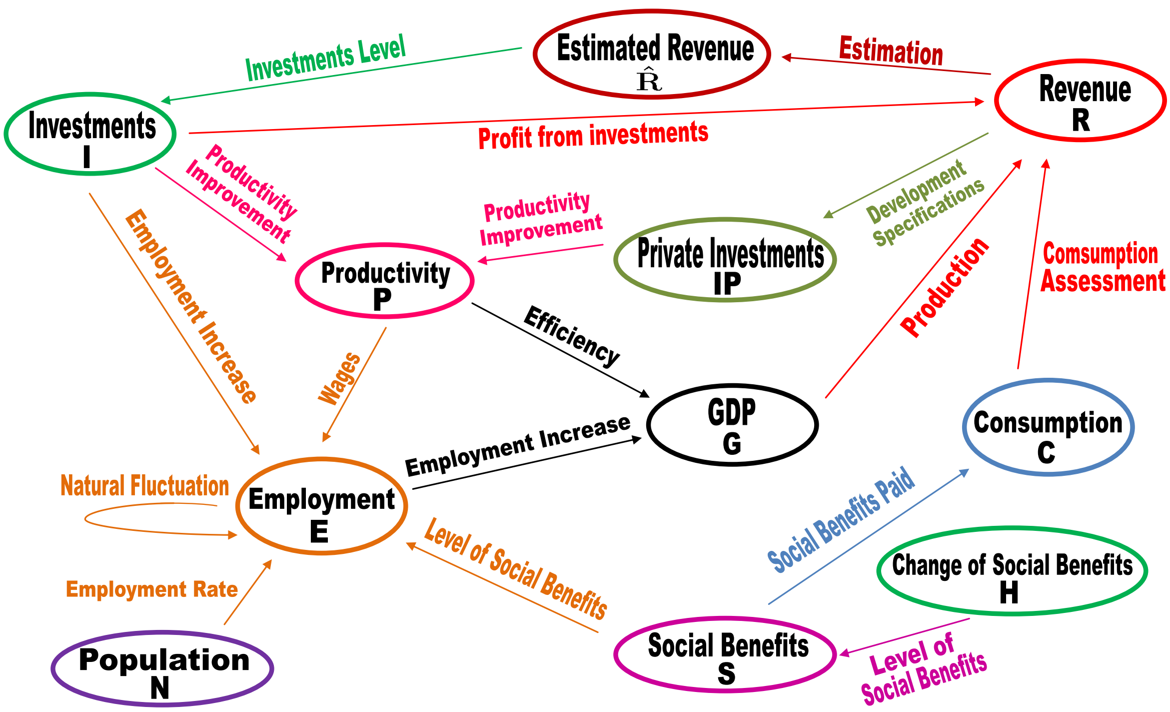

Figure 1: A visual scheme of key economic quantities, interactions, and dependencies amongst them. The quantities are displayed in the circles, the arrows represent the causality.

would describe underlying processes in the country with sufficient reliability.

We also want to consider a sustainable state budgeting scheme based on "The Story about Three Coppers" analyzed in detail in the bachelor thesis [1]. Such an approach should improve long term sustainability of the state budgeting and alleviate an accrual of state debt. The Three Coppers story speaks of consumption and investments funding through a sum of partial revenues from several years, i.e. one part of recent revenue is used for debt funding, other for recent expenditure and other for savings needed for funding of unpredictable expenditures in the future. This financing concept could help government to take use of all possible resources and affect state debt and economy by selection of the right time distribution for fund consumption. For example, by choosing fund consumption to be covered by future revenue, the economy trend would accelerate. In the contrary, spending covered from past revenue would stabilize the economy and provide higher security in the case of future financial instabilities.

In order to create a model we need to select key economic quantities and understand relations amongst them. The visual scheme of these processes is captured in figure 1. Dependent state variables are displayed in circles and each arrow characterizes a process in economy and state governing. Our mathematical model contains all the quantities and processes.

1.1 Populism in Government Policy (Objective Function)

The most important goal of each state government seems to be a reelection for another election term. That is often the reason why governments try to create suitable life conditions and make populistic decisions right before new elections. The government uses a reduction of state debt or improvements of social benefits to affect voters, since both these factors are important for life conditions in the country. These economy indicators are affected by government decisions and its ability to improve living standards. The trick is that government is not able to reduce the state debt and raise the level of the social benefits at the same time. These parameters are dependent, i.e. the state debt increases with an increase of the social benefits but the state debt does not necessary decrease with a decrease of the social benefits. The objective function in our model that contains these two factors should ensure the right government decisions. It

Variable	Explanation
N_k	a population size at time $k\Delta t$
S_k	social benefits per one fiscal period per capita
H_k	change of social benefits per one fiscal period per capita
C_k	total government consumption per one fiscal period
P_k	productivity per one fiscal period per capita
E_k	a number of employed people at time $k\Delta t$
R_k	total government revenue per one fiscal period
$\widehat{R_k}$	estimated total government revenue for one fiscal period
G_k	gross domestic product per one fiscal period
I_k	total public investments per one fiscal period
IP_k	total private investments per one fiscal period
D_k	state debt at time $k\Delta t$
$\widetilde{\nu}$	annual rate of population growth
$\widetilde{\pi}_I/\widetilde{\pi}_{IP}$	public/private investments impact on productivity
$\widetilde{\lambda}_P$	level of investments necessary for stabilization of productivity
$\widetilde{\lambda}_E$	level of investments necessary for stabilization of employment
$\widetilde{\zeta}_I/\widetilde{\zeta}_{IP}$	public/private investments impact on employment
$\widetilde{\mu}$	level of productivity used for wages for employees
$\widetilde{\kappa}$	level of GDP used for government expenditure funding
$\widetilde{\iota}$	level of public investments return
$\widetilde{\gamma}$	level of consumption return
$\widetilde{\rho}$	impact of economy development on private investments
$\widetilde{\delta}$	annual real debt interest rate
$\widetilde{\omega}$	parameter used for next year revenue estimation
$\tilde{\theta}$	time preference of government
$\tilde{\eta}$	impact of social benefits changes on government objective

Table 1: Summary of all selected key quantities and parameters of the model.

would balance the level of the state debt and the social benefits and would eventually maximize an electorate gain from provided suggestions.

The objective function for economic and mathematic optimal control problem captures the performance of both indicators during whole target period. Thus the government should choose the fixed time horizon T for optimizing its decisions and their impact on the state economy. The selection of the fixed time horizon depends on the expected governing period length or on the time horizon of projects realization and it changes the optimal control problem solutions, i.e. it affects the government populistic decisions.

The created economic model reflects a behavior of all quantities according to each government decisions. Therefore one of the most important parameters in the model is length of the fiscal period Δt, which should correspond to the typical time frame of the frequency of government decisions. Thus we set

$$\Delta t = \frac{1}{12} \text{ year} = 1 \text{ month} \quad \text{and} \quad T = 10 \text{ years} \frac{1}{\Delta t} = 120 \ .$$

We describe the optimal choice of government decisions as an argument of the solution of optimal control problem with objective function (1)

$$\max_{H} \sum_{j=0}^{T} (1 + \tilde{\theta}\Delta t)^{-j} \frac{1}{\Delta t} \left[-(D_{j+1} - D_j) + \tilde{\eta}(S_{j+1} - S_j) \right] \ . \tag{1}$$

By making decisions on social benefits and state debt accrual of payment the government may affect state economy and future life standards. The value of current decisions is typically maximized in the present time and diminishes in the future. Therefore, we discount the impact by the factor $(1 + \tilde{\theta}\Delta t)$ over a single fiscal period. Low values of $\tilde{\theta}$ mean that the government wants to gain electorate by promoting an idea of future improvements of living standards.

The objective function includes the parameter $\tilde{\eta}$ which characterizes emphasis of change of social benefits on government objective. Liberal governments highlight level of social benefits and their changes because they are trying to give people social guarantees and high living standard. In this case parameter $\tilde{\eta}$ should reach high values and should reflect the liberal type of governing. On the other hand, the low $\tilde{\eta}$ emphasizes the change of the state debt and its impact on utility function. Therefore, one may expect that the conservative government is represented by low values of $\tilde{\eta}$.

1.2 Population Growth N_k

The classical economic theory approximates the discrete population growth in time by the simple difference equation with the initial condition N_0 equal to the population at time zero and

$$N_{k+1} = (1 + \Delta t \tilde{\nu}) N_k \ ,$$

where N_{k+1} represents the population at time $(k+1)\Delta t$ and the annual rate of population growth is expressed by the parameter $\tilde{\nu}$.

Figure 2 displays the relative population growth over 30 years with 0.2 percent annual rate of population growth. This level of population growth rate is approximately equal to actual population growth rate in Slovak Republic [2].

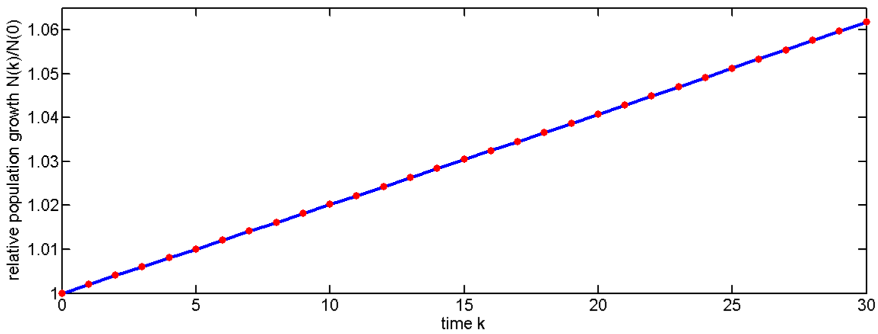

Figure 2: The figure shows the development of population over 30 years with annual rate of population growth equal to 0.2 percent, where the continuous time is characterized by the solid blue line and the red dots represent the discrete time approximation.

1.3 Public Investments I_k, Consumption C_k and Social Benefits S_k

One of the government's assignments is managing reallocation of available budget between consumption and investments. The consumption covers all social benefits S_k paid to people in the country included benefits for unemployment, maternity benefits, family allowances, benefits for children or rents. This quantity could be defined as a multiplication of the social benefit level, the population, and a parameter $\tilde{\beta}$ (see eq. (2)). This parameter characterizes the amount of consumption used for social contributions, for education and foreign debt funding, etc. The value of nondimensional parameter $\tilde{\beta} = 1.5$ reflects the average of proportion of the real annual government expenditure and all annual social benefits paid to people in Slovak Republic within years 2007 - 2012 [3].

$$C_k = \tilde{\beta} N_k S_k \tag{2}$$

In contrary, the investments include projects for improving living conditions, increasing productivity, creating new job openings or investments with other beneficial intentions. Unlike consumption, entire profit from investments can be reached after many years, when financed projects are finished. Every year the government would choose right allocation of the constricted budget between investments and consumption (see eq. (3)).

$$\text{Available Budget}_k = C_k + I_k \tag{3}$$

The allocation depends on the offered projects, living conditions, time to elections, the ideology of the government or the amount of recent budget. Depending on the current consumption level the government can restrict the planned investments but never reduces the level of social benefits paid at the expense of investments.

1.4 Great Domestic Product G_k, Productivity P_k and Private Investments IP_k

Great domestic product G_k summarizes the value of all final goods and services produced within a country over one fiscal period. We can approximate a variable of GDP by multiplication of the productivity per employee P_k and the number of people employed E_k (see eq. (20)), i.e. in our case the GDP represents whole productivity within a country.

$$G_k = E_k P_k \tag{4}$$

Private investments IP_k are characterized by the investments of owners and shareholders into their companies including investments to assets, technologies and investments for improving living standard. The level of the private investments is affected by the overall economic trend (see eq. (24)) captured in the state revenue changes.

$$IP_{k+1} = IP_k + \tilde{\rho}(R_{k+1} - R_k) \tag{5}$$

The parameter $\tilde{\rho}$ defines the rate of the private investments growth considering the state economy acceleration. The Eurostat data tables [3] show the amount of private investments including foreign investments which affect the internal economy of state. We approximate the parameter $\tilde{\rho} = 0.1$ as a multiplication of net profit margin and level of profit used for corporate investing [4, 5].

$$\tilde{\rho} = \text{(net profit margin)x(level of profit used for investments)} = 10\%.100\%$$

Productivity per employee is affected by the public and private investments used for an improvement of working conditions, an increase of personal abilities and an introduction of new technologies in every fiscal period. New investments can be divided into two groups, one part is used for stabilizing and the other for raising level of the productivity. The stabilization level characterizes a minimal investment needed for maintenance of the productivity level. For example, a company invests money to new software which improves the productivity of the employees. The productivity stabilization is ensured just by the investments to licenses for usage of this software. Therefore, the productivity could be expressed by equation (6)

$$P_{k+1} = P_k + \tilde{\pi}_{IP}\Delta t \left[\frac{IP_k - \tilde{\lambda}_P IP_{k-1}}{E_k} \right] + \tilde{\pi}_I \Delta t \left[\frac{I_k - \tilde{\lambda}_P I_{k-1}}{E_k} \right] , \tag{6}$$

where the quantities IP_k and I_k indicate the entire private and public investments, the constant parameter $\tilde{\lambda}_P$ denotes a level of the investments used for the productivity stabilization and the constant parameters $\tilde{\pi}_{IP}$ and $\tilde{\pi}_I$ represent the annual influence of the private and public investments per employee on the productivity changes.

We estimate the constants $\tilde{\pi}_{IP} = 10\%$ *p.a.*, $\tilde{\pi}_I = 12\%$ *p.a.* and $\tilde{\lambda} = 10.6\%$ as a solution of a system of the equations (6) expressed for years 2008, 2011 and 2012 and for one year fiscal period. The specific values of quantities used in the system could be found on Eurostat web-page in financial and economic tables [3].

1.5 Employment E_k

The key factors influencing a level of employment in the country are the natural level of unemployment, the amount of job opportunities, the level of the productivity and average wages, the level of the social benefits, etc. We represent the employment change by equation (7)

$$E_{k+1} = E_k \left[1 + \left(\frac{N_k - E_k}{N_k} \right) \left\{ \tilde{\zeta}_{IP} \Delta t \frac{IP_k - \tilde{\lambda}_E IP_{k-1}}{E_k} + \tilde{\zeta}_I \Delta t \frac{I_k - \tilde{\lambda}_E I_{k-1}}{E_k} \right\} \left\{ 1 - \frac{E_k}{(1 - \frac{S_k}{\tilde{\mu} P_k} \frac{N_k}{N_k - E_k}) N_k} \right\} \right] \quad (7)$$

where the quantities IP_k and I_k measure the value of all private and public investments in the country at time $k\Delta t$ and the expressions $\tilde{\lambda}_E IP_{k-1}$ and $\tilde{\lambda}_E I_{k-1}$ reflect necessary investments for the stabilization of the employment.

All investments used for improving work opportunities $[IP_k - \tilde{\lambda}_E IP_{k-1}]$ and $[I_k - \tilde{\lambda}_E I_{k-1}]$ are multiplied by $\frac{1}{E_k}$ for the investments per employee and multiplied by $\frac{N_k - E_k}{N_k}$ for the investments per the unemployed. The constants $\tilde{\zeta}_{IP}$ and $\tilde{\zeta}_I$ define the impacts of the private and public investment changes per the unemployed on the employment growth in the country. We approximate the value of the parameters $\tilde{\zeta}_{IP} = 0.00013 \frac{number\ of\ employees}{private\ investments\ amount}\ p.a.$, $\tilde{\zeta}_I = 0.000035 \frac{number\ of\ employees}{public\ investments\ amount}\ p.a.$ and $\tilde{\lambda}_E = 86.33\%$ as solution of a system of the equations (7) for years 2008, 2009 and 2012 using the Slovak Republic data for the selected quantity values needed in the system [3].

The level of the current investments determines the employment changes, if the investments are lower than the stabilization level, the employment decreases, if the investments reach the stabilization level the employment maintains constant.

The natural level of the employment is given by an expression

$$E_k = (1 - \frac{S_k}{\tilde{\mu} P_k} \frac{N_k}{N_k - E_k}) N_k$$

and it is affected by the social benefits for the unemployed

$$S_k \frac{N_k}{N_k - E_k} \ ,$$

the average wages $\tilde{\mu} P_k$ and the population level N_k. The expression

$$\frac{S_k}{\tilde{\mu} P_k} \frac{N_k}{N_k - E_k}$$

measures the level of attractivity of being unemployed and it is well defined considering the level of average wages is never less than the level of the social benefits for the unemployed. The unemployment attractivity measure has values in an interval $(0, 1)$. People are the most attracted to work when the parameter is equal to zero, on the other hand if the parameter is equal to one, they are not attracted at all.

We estimate the parameter $\tilde{\mu} = \frac{W_k}{P_k} = 0.29$ as a proportion of the average wages and the average productivity per employee using the Eurostat data for Slovak Republic within years 2007 - 2012 [3].

1.6 Government Revenue R_k and Estimated Government Revenue $\hat{R}_k$

The main part of state revenue corresponds to payroll taxes and to all corporate taxes. The level of the taxation including health and social insurance taxes is determined by the laws of a particular country. We characterize all tax levies and contributions by a parameter $\tilde{\kappa}$ which defines a percentage of the gross domestic product used for the government consumption and investments funding ($\tilde{\kappa} = 0.295 = 29.5\%$ [6]).

The level of the government revenues is also affected by a return from the previous investments and consumption (see eq. (8)). We approximate the consumption return at a level of the value added tax $\tilde{\gamma} = 20\%$ used in Slovak Republic. We assume that recipient of the social benefits uses the most of them for periodical living expenses or short term investments that are subject to the value added tax. Therefore the time horizon for the consumption return could be defined by the short time period k_C.

In contrary, the investments return could not be limited by a fixed time horizon. The government finances many different projects where a realization time is not a priori defined and their time horizons differ from few days to several years. Thus we should choose a typical time horizon for the investments return k_I.

The level of the investments return included in the government revenue could be represented by a constant $\tilde{\imath} = 7\%\ p.a.$ [8], which defines the average annual gain from the previous investments.

$$R_k = \tilde{\kappa} G_k + \sum_{j=k-k_I}^{k-1} \tilde{\imath} \Delta t I_j + \sum_{j=k-k_C}^{k-1} \tilde{\gamma} C_j \tag{8}$$

Each year the government must estimate the revenues for next year, set a level of the social benefits and choose projects for financing. The government's estimation is influenced by the previous revenues of several years. In general it could be approximated by a function f (see eq. (9)).

$$\widehat{R}_{k+1} = f(R_{k-m}, \cdots, R_k, I_k, C_k, ...) \tag{9}$$

For the simplicity we assume that the estimated revenues for next fiscal period solely depend on the revenues from previous and current year and on the parameter $\tilde{\omega}$, which characterizes an expected revenue acceleration (see eq. (10)).

$$\widehat{R}_{k+1} = (1 - \tilde{\omega})R_{k-1} + \tilde{\omega}R_k \tag{10}$$

1.7 Redistribution of Revenues in Time

The initial idea of the revenues redistribution in time is proposed in a bachelor thesis of M. Rošková [1] or in "The Story about Three Coppers" [9]. This story speaks of the financing of recent expenditures or investments by revenues of several years, i.e. the expenditures should be financed by current year yield, by loans or by money saved. The attitude of debt financing is based on an ideology of a country government. Some countries fund the expenditure by loans for improvement of an internal economy growth rate. In contrary, the financing by money saved represents an ideology of restrained economy growth and a stabilization of state economy.

$$\text{Available Budget}_k = \sum_{j=-k_{-\alpha}}^{0} \tilde{\alpha}_j R_{k+j} + \sum_{j=1}^{k_\alpha} \tilde{\alpha}_j \widehat{R}_{k+j} \tag{11}$$

The fixed time horizons k_α and $k_{-\alpha}$ define the amount of the previous revenues and the estimated revenues used for a creation of the current available budget. The parameters $\tilde{\alpha}$ represent the time distribution of the government expenditure and the public investments and their financing by the revenues of various years. The government can influence an internal economy development by right decisions for the state debt funding and by choosing appropriate coefficients for the revenue redistribution in time modified by the annual inflation and interest rate. One of the goals of this thesis is to create a suitable model for isolated economy with an optimal solution which represents an optimal choose of parameter values $\tilde{\alpha}$ used for the revenue reallocation.

The budget assignment described in the bachelor thesis [1] is defined by the following equation

$$\begin{aligned}\text{Budget}_k = \alpha_0 R_k \quad &+ \quad \sum_{j=1}^{-k_\alpha} \alpha_{-j} R_{k-j} \left(\frac{1+r_k}{1+\pi_k}\right)^{\frac{1}{2}} \left(\frac{1+r_{k-j}}{1+\pi_{k-j}}\right)^{-\frac{1}{2}} \prod_{m=1}^{j} \left(\frac{1+r_{k-m}}{1+\pi_{k-m}}\right) + \\ &+ \quad \sum_{j=1}^{k_\alpha} \alpha_j R_{k+j} \left(\frac{1+\pi_k}{1+r_k}\right)^{\frac{1}{2}} \left(\frac{1+\pi_{k+j}}{1+r_{k+j}}\right)^{-\frac{1}{2}} \prod_{m=1}^{j} \left(\frac{1+\pi_{k+m}}{1+r_{k+m}}\right) \ ,\end{aligned}$$

where the variable r_k defines the annual interest rate, the variable π_k captures the annual inflation rate and the coefficients α characterize the time redistribution of the government revenues.

1.8 State Debt D_k

Current state debt is affected by previous state debt and by a national interest and inflation rate. A fluctuation of the state debt level is a result of a recent debt change produced by the difference between the current revenues and the government consumption and the public investments $(I_{k+1} + C_{k+1} - R_{k+1})$. The difference equation (12) interprets the state debt development in time

$$D_{k+1} = (1 + \tilde{\delta}\Delta t) D_k + (I_{k+1} + C_{k+1} - R_{k+1}) + (\widehat{R}_{k+1} - R_{k+1}) \ , \tag{12}$$

where the parameter $\tilde{\delta} = 3.025\%$ *p.a.* characterizes the national annual real debt interest rate, which includes also the inflation rate [7]. The recent revenues are used for the debt financing and on the other hand the public investments and the consumption are used for deepening of the debt.

For a creation of investment opportunities the government should estimate the revenues for next year. If their estimation differs from the real revenues in the same year, the government has to consider the shortfall between the real and estimated revenues $(\widehat{R}_{k+1} - R_{k+1})$, which affects the current level of the debt.

Summary: Discrete Optimal Control Problem

$$\max_H \sum_{j=0}^{T} (1+\tilde{\theta}\Delta t)^{-j} \frac{1}{\Delta t} \left[-(D_{j+1}-D_j) + \tilde{\eta}(S_{j+1}-S_j)\right]$$

$$\begin{aligned}
N_{k+1} &= (1+\Delta t\tilde{\nu})N_k \\
S_{k+1} &= S_k + H_k \\
C_{k+1} &= \tilde{\beta}N_{k+1}S_{k+1} \\
P_{k+1} &= P_k + \tilde{\pi}_{IP}\Delta t \left[\frac{IP_k - \tilde{\lambda}_P IP_{k-1}}{E_k}\right] + \tilde{\pi}_I \Delta t \left[\frac{I_k - \tilde{\lambda}_P I_{k-1}}{E_k}\right] \\
E_{k+1} &= E_k \left[1 + \left(\frac{N_k - E_k}{N_k}\right) \left\{\tilde{\zeta}_{IP}\Delta t \frac{IP_k - \tilde{\lambda}_E IP_{k-1}}{E_k} + \tilde{\zeta}_I \Delta t \frac{I_k - \tilde{\lambda}_E I_{k-1}}{E_k}\right\}\right. \\
&\qquad\qquad\qquad\qquad\qquad \left.\left\{1 - \frac{E_k}{(1 - \frac{S_k}{\tilde{\mu}P_k}\frac{N_k}{N_k - E_k})N_k}\right\}\right] \\
\widehat{R}_{k+1} &= (1-\tilde{\omega})R_{k-1} + \tilde{\omega}R_k \\
G_{k+1} &= E_{k+1}P_{k+1} \\
R_{k+1} &= \tilde{\kappa}G_{k+1} + \sum_{j=k+1-k_I}^{k} \tilde{\imath}\Delta t I_j + \sum_{j=k+1-k_C}^{k} \tilde{\gamma}C_j \\
I_{k+1} &= -C_{k+1} + \sum_{j=-k_{-\alpha}}^{0} \tilde{\alpha}_j R_{(k+1)+j} + \sum_{j=1}^{k_\alpha} \tilde{\alpha}_j \widehat{R}_{(k+1)+j} \\
IP_{k+1} &= IP_k + \tilde{\rho}(R_{k+1} - R_k) \\
D_{k+1} &= (1+\tilde{\delta}\Delta t)D_k + (I_{k+1} + C_{k+1} - R_{k+1}) + (\widehat{R}_{k+1} - R_{k+1})
\end{aligned}$$

2 Nondimesionalization

To avoid unnecessary computational errors we create a dimensionless economy model with variables represented by relative growth of all selected quantities. Before simulation and modelling we have to define the fixed fiscal period $\Delta t = 1\ month$ characterized by a frequency of government decisions, the target time horizon $T = 10\ years\ \frac{1}{\Delta t} = 120$ and the initial time.

Population

The population n in the dimensionless economy model describes a relative fluctuation of the population since the initial time, i.e. the new variable is defined as a proportion of the number of people at time $k\Delta t$ and the number of people at the initial time:

$$n_k = \frac{N_k}{N_0}\ . \tag{13}$$

Therefore, we nondimensionalize the difference equation describing population dynamics using equation (13) as follows

$$\begin{aligned} N_{k+1} &= (1 + \tilde{\nu}\Delta t)N_k \\ n_{k+1}N_0 &= (1 + \tilde{\nu}\Delta t)n_k N_0 \\ n_{k+1} &= (1 + \nu)n_k\ , \end{aligned} \tag{14}$$

where the parameter $\nu = \tilde{\nu}\Delta t$ represents the rate of the population growth for one fiscal period and the initial condition for the equation is $n_0 = \frac{N_0}{N_0} = 1$.

Social Benefits

A relative level of social benefits s determined by social benefits change h and the initial social benefits level S_0 is affected by the government decisions and the type of its policy. Using equations (15) representing relative growth since the initial time

$$s_k = \frac{S_k}{S_0} \qquad h_k = \frac{H_k}{H_0} \tag{15}$$

we deduce a nondimensional form of the difference equation describing the social benefits development (see eq. (16)) with the initial condition $s_0 = \frac{S_0}{S_0} = 1$.

$$\begin{aligned} S_{k+1} &= S_k + H_k \\ s_{k+1}S_0 &= s_k S_0 + h_k H_0 \\ s_{k+1} &= s_k + h_k \end{aligned} \tag{16}$$

The change of the social benefits h is one of the control variables in this economy model. Therefore, we approximate the relative change of social benefits level as a proportion of social benefits variation and their initial level, i.e. we set $S_0 = H_0$ where a multiplication $h_k S_0$ represents the change of the amount of the social benefits per capita since the initial time.

Consumption

Consumption is determined by the social benefits per capita and the number of people in the country. We deduce a dimensionless difference equation for the consumption development by using relation $C_0 = \tilde{\beta} N_0 S_0$ as follows

$$\begin{aligned} C_{k+1} &= \tilde{\beta} N_{k+1} S_{k+1} \\ c_{k+1}C_0 &= \tilde{\beta} n_{k+1} N_0 s_{k+1} S_0 \\ c_{k+1} &= n_{k+1} s_{k+1} \ , \end{aligned} \tag{17}$$

where the initial condition is a multiplication of the social benefits per capita and the population at initial time $c_0 = n_0 s_0 = 1$.

Productivity

Productivity per employee is affected by the current productivity level, the current employment level and the added investments above the stabilization level, which signifies the amount of the investments needed for maintenance of the productivity per employee. The impacts of the private and public investments for improving the productivity in the nondimensional model are determined by the parameters $\pi_{ip} = \widetilde{\pi}_{IP} \frac{IP_0}{E_0 P_0} \Delta t$ and $\pi_i = \widetilde{\pi}_I \frac{I_0}{E_0 P_0} \Delta t$.

New dimensionless difference equation for the relative productivity growth is represented by equation (18)

$$\begin{aligned} P_{k+1} &= P_k + \tilde{\pi}_{IP}\Delta t \left[\frac{IP_k - \tilde{\lambda}_P IP_{k-1}}{E_k}\right] + \tilde{\pi}_I \Delta t \left[\frac{I_k - \tilde{\lambda}_P I_{k-1}}{E_k}\right] \\ p_{k+1}P_0 &= p_k P_0 + \tilde{\pi}_{IP}\frac{IP_0}{E_0}\Delta t \left[\frac{ip_k - \tilde{\lambda}_P ip_{k-1}}{e_k}\right] + \tilde{\pi}_I \frac{I_0}{E_0}\Delta t \left[\frac{i_k - \tilde{\lambda}_P i_{k-1}}{e_k}\right] \\ p_{k+1} &= p_k + \pi_{ip}\left[\frac{ip_k - \lambda_P ip_{k-1}}{e_k}\right] + \pi_i \left[\frac{i_k - \lambda_P i_{k-1}}{e_k}\right] , \end{aligned} \tag{18}$$

where the parameter $\lambda_P = \tilde{\lambda}_P$ describes the level of the investments necessary for the stabilization of the productivity modified by the inflation rate and the initial condition is $p_0 = \frac{P_0}{P_0} = 1$.

Employment

The relative change of the employment is a result of the initial relative employment in the country

$$\epsilon = \frac{E_0}{N_0}$$

and the proportion of the social benefits and the average wages at initial time

$$\sigma = \frac{S_0}{\tilde{\mu}P_0} .$$

The added value of the investments above the stabilization level affects the employment by the same way as affects the productivity, i.e. the private investments have greater impact on a creation of new working positions than the public investments. The parameters $\zeta_{ip} = \tilde{\zeta}_{IP}\frac{IP_0}{E_0}\Delta t$ and $\zeta_i = \tilde{\zeta}_I \frac{I_0}{E_0}\Delta t$ define the difference between these impacts.

The difference equation characterizing the dynamics of the relative employment changes with the initial condition $e_0 = \frac{E_0}{E_0} = 1$ (see eq. (19)) has to be constrained by zero $e_k > 0$ and by the population $e_k \le n_k \frac{N_0}{E_0}$ at each time $k\Delta t$.

$$E_{k+1} = E_k \left[1 + \left(\frac{N_k - E_k}{N_k}\right)\left\{\tilde{\zeta}_{IP}\Delta t \frac{IP_k - \tilde{\lambda}_E IP_{k-1}}{E_k} + \tilde{\zeta}_I \Delta t \frac{I_k - \tilde{\lambda}_E I_{k-1}}{E_k}\right\} \right.$$
$$\left. \left\{1 - \frac{E_k}{(1 - \frac{S_k}{\tilde{\mu}P_k}\frac{N_k}{N_k - E_k})N_k}\right\}\right]$$

$$e_{k+1}E_0 = e_kE_0\left[1+\frac{n_k-\frac{E_0}{N_0}e_k}{n_k}\left\{\tilde{\zeta}_{IP}\frac{IP_0}{E_0}\Delta t\frac{(ip_k-\tilde{\lambda}_E ip_{k-1})}{e_k}+\right.\right.$$
$$\left.\left.+\tilde{\zeta}_I\frac{I_0}{E_0}\Delta t\frac{(i_k-\tilde{\lambda}_E i_{k-1})}{e_k}\right\}\left\{1-\frac{\frac{E_0}{N_0}e_k}{[1-\frac{s_kS_0}{\tilde{\mu}p_kP_0}\frac{n_k}{n_k-\frac{E_0}{N_0}e_k}]n_k}\right\}\right]$$

$$e_{k+1} = e_k\left[1+\frac{n_k-\epsilon e_k}{n_k}\left\{\frac{\zeta_{ip}(ip_k-\lambda_E ip_{k-1})+\zeta_i(i_k-\lambda_E i_{k-1})}{e_k}\right\}\right.$$
$$\left.\left\{1-\frac{\epsilon e_k}{[1-\sigma\frac{s_k}{p_k}\frac{n_k}{n_k-\epsilon e_k}]n_k}\right\}\right] \qquad (19)$$

A differential equation for the employment development would be well defined for the continuous time where the natural employment level is a stable equilibrium and the investments at the stabilization level is unstable equilibrium as well as $e_k = 0$. If the employment possibly ever reaches the zero level, the employment maintains on this level. In contrary, because of the discrete time the difference equation (19) might give us the level of relative employment less than zero. Therefore we have to define the restrictions for the employment in the discrete dimensionless model.

Gross Domestic Product

Gross domestic product in the model is defined as multiplication of the productivity per employee and the number of people employed $G_k = E_kP_k$ at each time $k\Delta t$, i.e. the dimensionless variable is represented by multiplication of the relative growth of the employment and the relative growth of the productivity as well (see eq. (20)).

$$\begin{aligned} G_{k+1} &= E_{k+1}P_{k+1} \\ g_{k+1}G_0 &= e_{k+1}E_0p_{k+1}P_0 \\ g_{k+1} &= e_{k+1}p_{k+1} \end{aligned} \qquad (20)$$

The initial condition for the relative GDP is dictated by the initial conditions for the relative employment and for the relative productivity $g_0 = e_0p_0 = 1$.

Revenues

State revenues are consisted of the investments and consumption returns and the money from the taxation represented by the percentage of the gross domestic product. The explicit equation for the revenue development (21) contains the parameter $\kappa = \tilde{\kappa}\frac{G_0}{R_0}$ which characterizes the percentage of the revenues earned from the taxation, and the dimensionless parameters $\iota = \tilde{\iota}\frac{I_0}{R_0}\Delta t$ and $\gamma = \tilde{\gamma}\frac{C_0}{R_0}$ which describe the relative returns of the investments and the consumption.

$$\begin{aligned} R_{k+1} &= \tilde{\kappa}G_{k+1} + \sum_{j=k-k_I}^{k} \tilde{\iota}\Delta t I_j + \sum_{j=k-k_C}^{k} \tilde{\gamma}C_j \\ r_{k+1}R_0 &= \tilde{\kappa}g_{k+1}G_0 + \sum_{j=k-k_I}^{k} \tilde{\iota}\Delta t i_j I_0 + \sum_{j=k-k_C}^{k} \tilde{\gamma}c_j C_0 \\ r_{k+1} &= \kappa g_{k+1} + \sum_{j=k-k_I}^{k} \iota i_j + \sum_{j=k-k_C}^{k} \gamma c_j \end{aligned} \tag{21}$$

The initial condition for the revenue development equation is $r_0 = \frac{R_0}{R_0} = 1$. The time horizons k_I and k_C define the number of the fiscal periods needed for the whole return from the investments and the consumption which are realized at time $(k - k_I)\Delta t$ or in case of the consumption at time $(k - k_C)\Delta t$.

Investments

The level of the public investments is constrained by the level of the consumption and by the available budget characterized by the redistribution of the revenues in time. Therefore, we have to set a restriction (see eq. (22)) for the social benefits level since the investments should never be less than zero.

$$C_k \leq R_k \Rightarrow s_k \leq \frac{R_0}{N_0 S_0}\frac{r_k}{n_k} \tag{22}$$

An explicit nondimensional equation for the state investments could be deduced as follows

$$\begin{aligned} I_k + C_k &= \sum_{j=-k_{-\alpha}}^{0} \tilde{\alpha}_j R_{j+k} + \sum_{j=1}^{k_\alpha} \tilde{\alpha}_j \widehat{R}_{j+k} \\ i_k I_0 + c_k C_0 &= \sum_{j=-k_{-\alpha}}^{0} \tilde{\alpha}_j r_{j+k} R_0 + \sum_{j=1}^{k_\alpha} \tilde{\alpha}_j \widehat{r}_{j+k} R_0 \\ i_k + \xi c_k &= \sum_{j=-k_{-\alpha}}^{0} \alpha_j r_{j+k} + \sum_{j=1}^{k_\alpha} \alpha_j \widehat{r}_{j+k} \end{aligned} \tag{23}$$

where the fluctuation of the state investments is affected by the dimensionless parameter $\xi = \frac{C_0}{I_0}$ representing the impact of the consumption changes on the public investments and by the dimensionless parameters $\alpha_j = \tilde{\alpha}_j \frac{R_0}{I_0}$ defining the weight of the revenues in time redistribution. The time horizon $k_{-\alpha}$ speaks of the number of the fiscal periods used for the redistribution of some previous state revenues. In contrary, the time horizon k_α is defined as the number of the fiscal period included in the time redistribution of the future revenues and the length of the extinction of the debt made by the borrowing for the current investments and the consumption funding.

Private Investments

Private investments represent the investments of owners and shareholders into their companies including their will for improving their capital assets. The willingness for investing is correlated with the state economy development which is characterized by the changes of the government revenue level.

$$
\begin{aligned}
IP_{k+1} &= IP_k + \tilde{\rho}(R_{k+1} - R_k) \\
ip_{k+1} IP_0 &= ip_k IP_0 + \tilde{\rho}(r_{k+1} R_0 - r_k R_0) \\
ip_{k+1} &= ip_k + \rho(r_{k+1} - r_k)
\end{aligned}
\tag{24}
$$

The parameter $\rho = \tilde{\rho} \frac{R_0}{IP_0}$ in the dimensionless difference equation with the initial condition $ip_0 = \frac{IP_0}{IP_0} = 1$ (see eq. (24)) defines the impact of the economy changes on the private investment changes.

State debt

State debt is determined by the last year state debt, current debt change and revenue estimation error. Considering the impacts of the revenues, the consumption and the investments on the debt level we deduce the difference equation for the relative changes of the debt (see eq. (25)) where $\delta = \tilde{\delta} \Delta t$ characterizes the real interest debt rate for one fiscal period.

$$
\begin{aligned}
D_{k+1} &= (1 + \tilde{\delta} \Delta t) D_k + (I_{k+1} + C_{k+1} - R_{k+1}) + (\widehat{R}_{k+1} - R_{k+1}) \\
d_{k+1} D_0 &= (1 + \tilde{\delta} \Delta t) d_k D_0 + (i_{k+1} I_0 + c_{k+1} C_0 - r_{k+1} R_0) + (\widehat{r}_{k+1} - r_{k+1}) R_0 \\
d_{k+1} &= (1 + \delta) d_k + (\psi_i i_{k+1} + \psi_c c_{k+1} - \psi_r r_{k+1}) + \psi_r (\widehat{r}_{k+1} - r_{k+1})
\end{aligned}
\tag{25}
$$

The particular impacts of all quantities participated on the debt development are included in the dimensionless parameters $\psi_i = \frac{I_0}{D_0}$, $\psi_c = \frac{C_0}{D_0}$ and $\psi_r = \frac{R_0}{D_0}$. The initial condition $d_0 = 1$ characterizes the relative debt development in time considering the initial debt level $D_0 > 0$.

Objective function

The objective function in this model should maximize the populistic decisions of the government and help to choose right social benefits progress in the selected time horizon. We modify the objective function for nondimensional model as follows

$$\begin{aligned}
&\max_H \sum_{j=0}^{T} \ (1+\tilde{\theta}\Delta t)^{-j} \frac{1}{\Delta t} \left[-(D_{j+1} - D_j) + \tilde{\eta}(S_{j+1} - S_j)\right] \\
&\max_H \sum_{j=0}^{T} \ (1+\tilde{\theta}\Delta t)^{-j} \frac{1}{\Delta t} \left[-(d_{j+1} - d_j) + \tilde{\eta}\frac{S_0}{D_0}(s_{j+1} - s_j)\right] \\
&\max_h \sum_{j=0}^{T} \ (1+\theta)^{-j} \left[-\frac{d_{j+1} - d_j}{\Delta t} + \eta h_j\right] \ , \qquad (26)
\end{aligned}$$

where $\tilde{\theta}\Delta t = \theta$ represents the discount of the government decisions impact and the parameter $\eta = \tilde{\eta}\frac{S_0}{D_0}$ captures the preferences of the current government for the social benefits increase.

Summary: Dimensionless Optimal Control Problem

$$\max_h \sum_{j=0}^{T} (1+\theta)^{-j} \left[-\frac{d_{j+1} - d_j}{\Delta t} + \eta h_j \right]$$

$$\begin{aligned}
n_{k+1} &= (1+\nu) n_k \\
s_{k+1} &= s_k + h_k \\
c_{k+1} &= n_{k+1} s_{k+1} \\
p_{k+1} &= p_k + \pi_{ip} \left[\frac{ip_k - \lambda_P ip_{k-1}}{e_k} \right] + \pi_i \left[\frac{i_k - \lambda_P i_{k-1}}{e_k} \right] \\
e_{k+1} &= e_k \left[1 + \frac{n_k - \epsilon e_k}{n_k} \left\{ \frac{\zeta_{ip}(ip_k - \lambda_E ip_{k-1}) + \zeta_i (i_k - \lambda_E i_{k-1})}{e_k} \right\} \right. \\
&\qquad \left. \left\{ 1 - \frac{\epsilon e_k}{[1 - \sigma \frac{s_k}{p_k} \frac{n_k}{n_k - \epsilon e_k}] n_k} \right\} \right] && 0 < \epsilon e_k < n_k \quad \forall k \\
g_{k+1} &= e_{k+1} p_{k+1} \\
r_{k+1} &= \kappa g_{k+1} + \sum_{j=k-k_I}^{k} \iota i_j + \sum_{j=k-k_C}^{k} \gamma c_j \\
i_{k+1} &= -\xi c_{k+1} + \sum_{j=-k_{-\alpha}}^{0} \alpha_j r_{(k+1)+j} + \sum_{j=1}^{k_\alpha} \alpha_j \widehat{r}_{(k+1)+j} && 0 \le s_k \le \frac{R_0}{N_0 S_0} \frac{r_k}{n_k} \quad \forall k \\
ip_{k+1} &= ip_k + \rho (r_{k+1} - r_k) \\
d_{k+1} &= (1+\delta) d_k + (\psi_i i_{k+1} + \psi_c c_{k+1} - \psi_r r_{k+1}) + \psi_r (\widehat{r}_{k+1} - r_{k+1})
\end{aligned}$$

2.1 Interpretation of Parameters of the Model

Each parameter in the model has a different meaning and a different impact on the development of all quantities. Firstly we have to define the values of all constant parameters in the model by the nominal values of all selected quantities using the Eurostat financial and economic tables [3]. In this model we use the Slovak Republic data as an example for computation of the values of all constant parameters which are summarized in table 2 and for simulation of the optimal control problem.

Parameter	Value	Explanation
$\tilde{\nu}$	0.2 % p.a.	annual rate of population growth
$\widetilde{\pi}_{IP}$	10 % p.a.	private investments impact on productivity
$\widetilde{\pi}_{I}$	12 % p.a.	public investments impact on productivity
$\tilde{\lambda}_P$	10.6 %	level of investments necessary for stabilization of productivity
$\widetilde{\zeta}_{IP}$	0.013 % p.a.	private investments impact on employment
$\widetilde{\zeta}_{I}$	0.0035 % p.a.	public investments impact on employment
$\tilde{\lambda}_E$	86.33 %	level of investments necessary for stabilization of employment
$\tilde{\mu}$	29 %	level of productivity used for wages for employees
$\tilde{\kappa}$	29.5 %	level of GDP used for government expenditure funding
$\tilde{\iota}$	7 % p.a.	level of annual public investments return
$\tilde{\gamma}$	20 %	level of consumption return
$\tilde{\rho}$	10 %	impact of economy development on private investments
$\tilde{\delta}$	3.025 % p.a.	annual real debt interest rate
$\widetilde{\omega}$	2	parameter used for next year revenue estimation

Table 2: The values of all parameters of the model calculated from Slovak Republic data

In the nondimensional model we have to modify the values of particular parameters to represent impacts on relative changes of the selected quantities (see table 3). We use the dimensionless model with nondimensional parameters for avoiding the computation errors and for right simulation of the state economy behaviour.

Parameter	Value	Expression	Explanation
ν	0.00017	$\widetilde{\nu}\Delta t$	rate of population growth for fiscal period
π_{ip}	0.01698	$\widetilde{\pi}_{IP}\frac{IP_0}{G_0}\Delta t$	private investments impact on productivity
π_i	0.00926	$\widetilde{\pi}_I\frac{I_0}{G_0}\Delta t$	public investments impact on productivity
λ_P	0.10600	$\widetilde{\lambda}_P$	level of investments necessary for stabilization of productivity
ϵ	0.43636	$\frac{E_0}{N_0}$	initial percentage of employed people
σ	0.29374	$\frac{S_0}{\bar{\mu}P_0}$	initial attractivity to work
ζ_{ip}	0.00415	$\widetilde{\zeta}_{IP}\frac{IP_0}{E_0}\Delta t$	private investments impact on employment
ζ_i	0.00050	$\widetilde{\zeta}_I\frac{I_0}{E_0}\Delta t$	public investments impact on employment
λ_E	0.86330	$\widetilde{\lambda}_E$	level of investments necessary for stabilization of employment
κ	0.83113	$\widetilde{\kappa}\frac{G_0}{R_0}$	level of GDP used for government expenditure funding
ι	0.00127	$\widetilde{\iota}\frac{I_0}{R_0}\Delta t$	public investments return for fiscal period
γ	0.16500	$\widetilde{\gamma}\frac{C_0}{R_0}$	consumption return
ξ	3.79500	$\frac{C_0}{I_0}$	consumption impact on public investments
ρ	0.20909	$\widetilde{\rho}\frac{R_0}{IP_0}$	impact of economy development on private investments
δ	0.00252	$\widetilde{\delta}\Delta t$	real debt interest rate for fiscal period
ψ_i	0.17857	$\frac{I_0}{D_0}$	public investments impact on debt change
ψ_c	0.67768	$\frac{C_0}{D_0}$	consumption impact on debt change
ψ_r	0.82143	$\frac{R_0}{D_0}$	revenue impact on debt change

Table 3: The values of all nondimensional parameters of the model

Example 1. *The government of Slovak Republic has to consider an economy impact of their decisions of the social benefits level changes made each month, in the following 10 years, i.e. the government chooses the fixed fiscal period* $\Delta t = 1/12$*, the fixed time* *horizon* $T = 10/\Delta t = 120$ *and the social benefits progress. The redistribution of the revenues in the time is uniformly divided to three years, i.e. the revenues of each year used for counting of the available budget are multiplied by fraction* $\frac{1}{3}$. *The estimation of the next year revenue is defined by the simple rule* $\hat{R}_{k+1} = (1-\omega)R_{k-1} + \omega R_k$, *where the government fixes the parameter* $\omega = 2$. *The policy of the government is progressive considering the value of parameter* ω *which defines the future duration of the recent economic trend. The government believes that the employment react to the social benefits change quickly and thus it wants to rise the impact on the relative employment change by multiplying the employment growth by an expression* $e_k^2 E_0^{\frac{1}{4}}$.

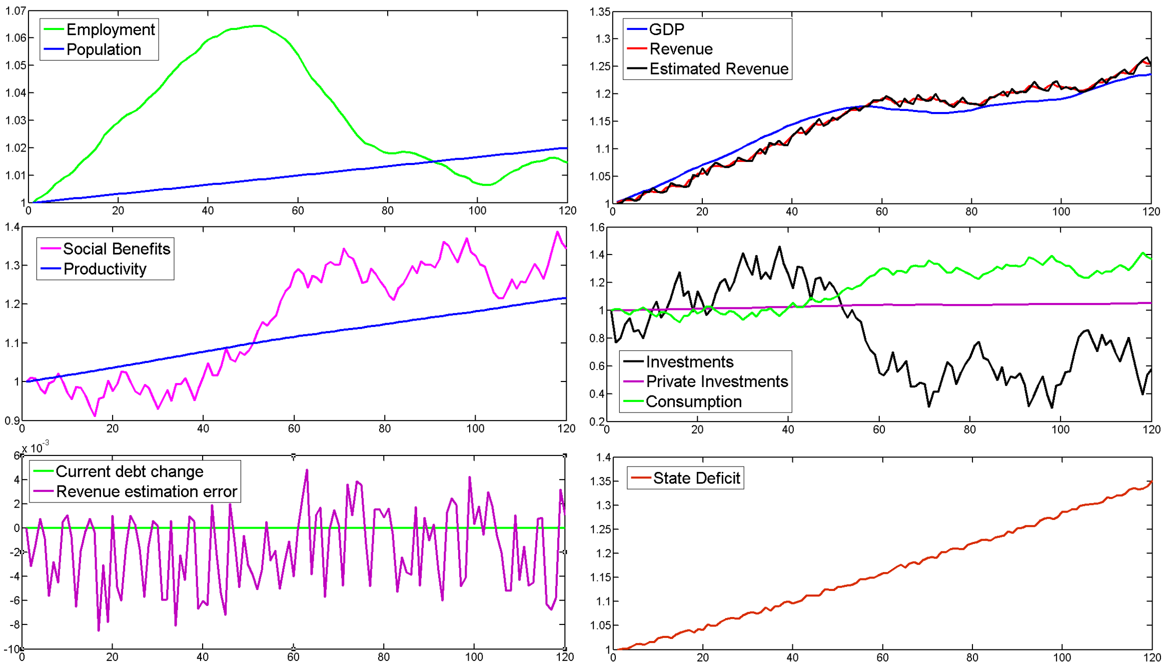

Figure 3: The simulation shows the development of all selected quantities over 10 years affected by the monthly government influences. The social benefit changes are randomly selected fixed variables characterizing the government decisions. The development of all quantities is determined by the system of the difference equation of the created model. The change of current debt describes the difference between current government expenditure and the state revenues. The revenue estimation error characterizes wrong prediction of the future revenues and it impact on the state debt development.

The simulation of all quantities behaviour over 10 years is displayed in figure 3 where all constant parameters of the model are approximated from Slovak Republic data (see *table 3). The population growth is not affected by the development of other quantities and it is dictated just by the previous level of the population and by the annual rate of the population growth. The consumption consists of all social benefits paid to people, thus it copies the social benefits level changes.*

The productivity is affected by the proportion of the investments above the stabilization level and the employment. Since the employment and the investments have similar progress the productivity is represented by the constant fluent growth. On the other hand, the employment reacts on the available investments used for a creation of new job opportunities and on the recent attractivity of being unemployed which is defined as proportion of the social benefits for the unemployed and the average wages. When the rate of the social benefits growth is greater than the rate of the productivity growth, the employment could decrease with the attractivity to work depreciation.

The level of the investments is constrained by the current state available budget, i.e. the recent revenues are split for funding of the consumption exactly defined by the population and the social benefits level and for the investments financing. The government has to restrict the investments at the expense of full consumption funding.

The increase of the state debt is mostly caused by a capitalization of the previous debt. The state debt is slightly affected by the current debt change based on the difference between the expenditure and the revenues and by the revenue estimation error caused by the difference between the estimated revenues and the real government revenues.

The rate of the population growth primary affects the difference equation representing the population dynamics. The rise of this parameter to two percent level $\tilde{\nu} = 2\%$ causes a significant population growth but just a slight employment increase. The

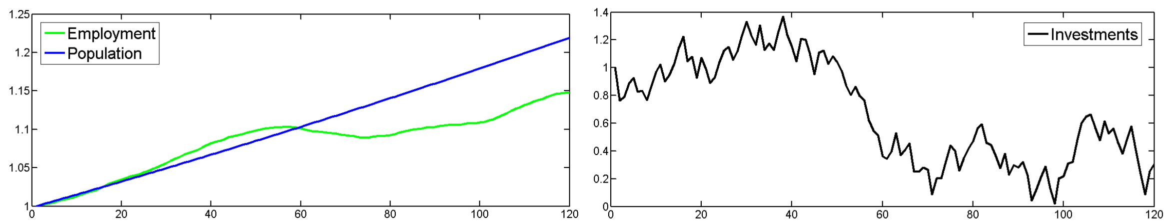

Figure 4: The simulation of the isolated economy development with two percent annual rate of population growth $\tilde{\nu} = 2\%$ captures the slight increase of the employment, the high investment drop and the significant accrual of the population.

investments depreciation in the last half of the target time horizon reacts to the noticeable increase of the population and consecutive increase of the consumption. The simulation with increased rate of the population growth is depicted in figure 4.

The parameters $\tilde{\pi}_{ip}$ and $\tilde{\pi}_i$ represent the influence of total private and public investments on the productivity changes. Figure 5 shows the difference between these parameters and their influence on another economical quantities development. The increase of the parameter $\tilde{\pi}_{ip} = 0.3$ helps to accelerate the productivity growth and to raise the public investments resultantly. On the other hand, the increase of the parameter $\tilde{\pi}_i = 0.3$ has the same but a bit lower impact on the productivity and investment growth. In both cases the increase of the parameter causes the rise of the employment, because the higher level of the productivity positively affects the attraction of being employed.

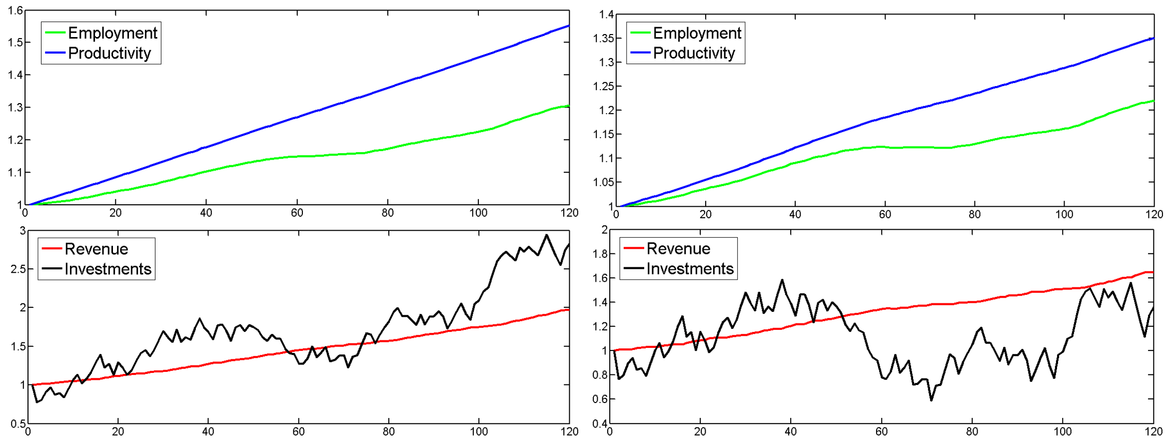

Figure 5: The comparison of the simulations of the isolated economy development with increased impact of total private $\tilde{\pi}_{ip} = 0.3$ and public $\tilde{\pi}_i = 0.3$ investments on the productivity changes captures the higher impact of the private investment (on the left) on the rise of the productivity, the investments, and the employment than the public investments (on the right).

The stabilization levels captured in the parameters $\tilde{\lambda}_P$ and $\tilde{\lambda}_E$ characterize the potential of the economy growth and define the levels of the investments for remaining the employment, the productivity and the whole economy at the same level. If this stabilization level of the investments necessary for maintenance of the productivity raises $\tilde{\lambda}_P = 0.3$, the economic growth is retarded, the government has lower revenues for reinvesting and the employment drops. In contrary, the increase of stabilization level of the necessary investments for maintenance of the employment $\tilde{\lambda}_E = 1$ causes approximately linear growth of the employment with small variations. In those circumstances, the economy development is dictated mostly by the productivity growth because of the low impact of the investments on the employment variation. Figure 6 depicts the simulations of selected economy factors in both cases.

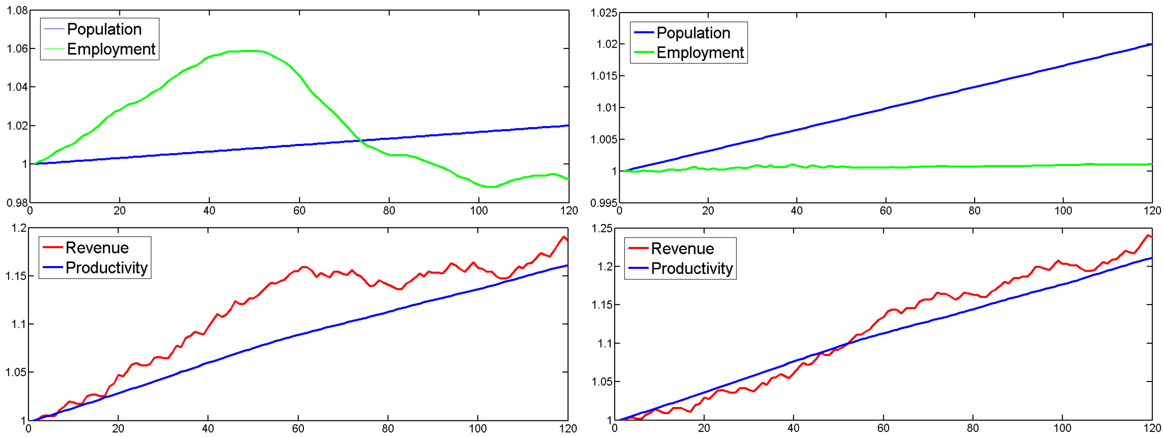

Figure 6: The figure compares the simulations of the selected economy factors with increased level of the investments necessary for the stabilization of the productivity and the employment. The productivity retardation is caused by the level of parameter $\tilde{\lambda}_P = 0.3$ (on the left) and consecutive low values of the revenues. The increase of parameter $\tilde{\lambda}_E = 1$ (on the right) represents the employment stagnation, the revenues are affected mostly by the level of the productivity.

The impacts of the private and public investments on the employment are characterized by the parameters $\tilde{\zeta}_{IP}$ and $\tilde{\zeta}_I$ which represent the inverted value of the annual investments used for new job opportunities for the unemployed. The rise of the parameter $\tilde{\zeta}_{IP} = 0.006$ causes the accrual of the sensitivity of the employment to the private investment changes. The increase of the parameter $\tilde{\zeta}_I = 0.0003$ highlights the dependence of the employment changes on the public investment changes. Figure 7 records the sensitivity of the employment to the public and private investments with the increased value of these parameters.

The natural employment level is directly proportional to the parameter $\tilde{\mu}$ which affects the attractivity of being unemployed. Figure 8 displays the increase and the stabilization of employment caused by the rise of the parameter $\tilde{\mu} = 0.35$ and by the

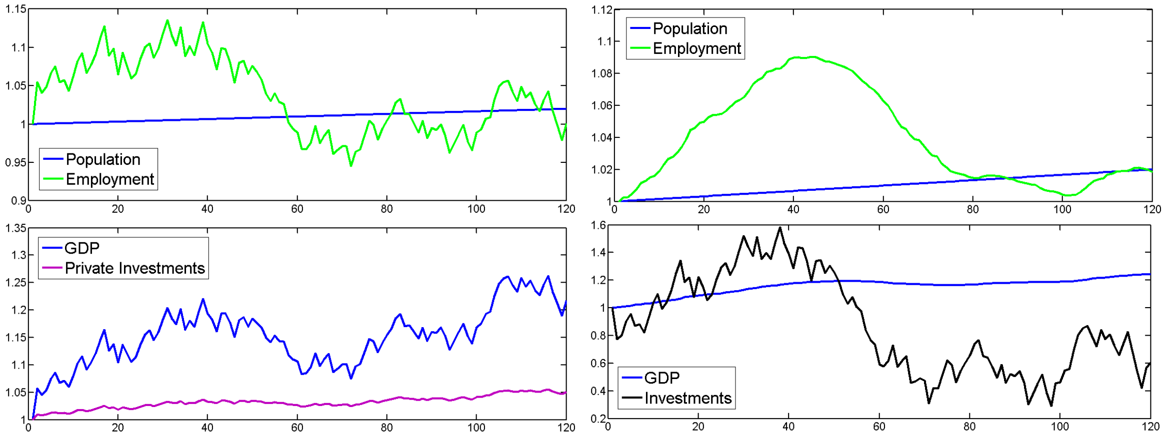

Figure 7: The parameter $\tilde{\zeta}_{IP} = 0.006$ rises the impact of the private investment changes on the employment (on the left) and the parameter $\tilde{\zeta}_I = 0.0003$ highlights the dependence of the employment changes on the public investment changes (on the right).

social benefits level. The stagnation of the employment in the last half of the target time horizon is affected by the decrease of attractivity to work. On the other hand, the lower value of parameter $\tilde{\mu} = 0.27$ signalizes the future accrual of attractivity of being unemployed. Therefore, the figure 8 shows moderate decrease of the employment in the first half and a significant drop of the employment with a rise of the social benefits levels in the second half of the target time horizon.

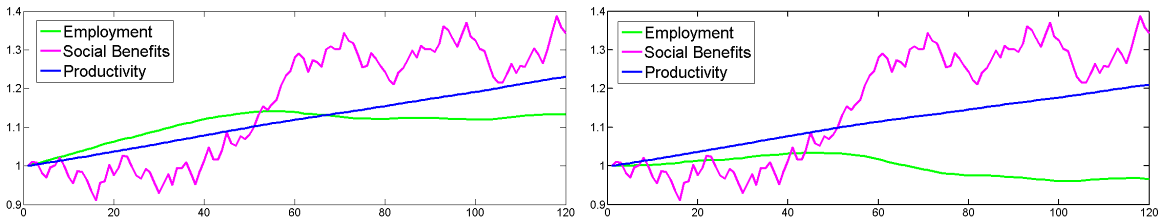

Figure 8: The figure compares the simulations of employment development with different initial values of the attractivity of being unemployed. The lower value of this quantity affected by greater value of parameter $\tilde{\mu} = 0.35$ (on the left) signifies the increase of natural employment level. In contrary, the greater value of the attractivity of being unemployed caused by value of the parameter $\tilde{\mu} = 0.27$ (on the right) decreases the employment level.

The increase of the parameters $\tilde{\kappa}$, $\tilde{\iota}$ and $\tilde{\gamma}$ visibly affects a rise of the level of the government revenues which causes the relevant increase of the public investments and subsequently the increase of the productivity and the employment (see fig. 9).

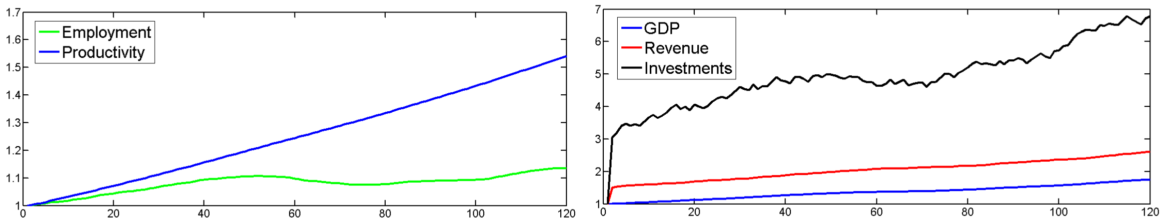

Figure 9: The simulation of economy development with increased values of parameters $\tilde{\kappa} = 0.4$, $\tilde{\iota} = 0.5$ and $\tilde{\gamma} = 0.3$ captures the significant increase of the government revenues, the public investments, the productivity, and the employment.

The parameter $\tilde{\rho}$ represents the impact of economy changes on the private investments development. An increase of this parameter $\tilde{\rho} = 0.7$ has a slight impact on the state economy, the level of the private investments rises but other changes of the economy factors are slightly observed just in the employment and productivity progress (see figure 10).

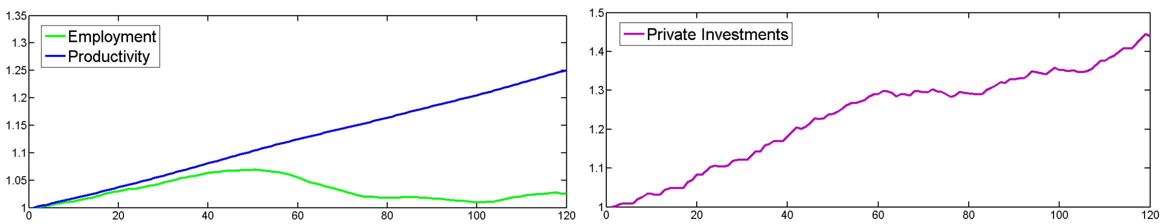

Figure 10: The simulation of economy development with increased value of parameter $\tilde{\rho} = 0.7$ shows the significant increase of the private investments, but a slight increase of the productivity and the employment.

The parameter $\tilde{\beta}$ defines the value of the total consumption affected by the level of the social benefits and the current population. The decrease of the consumption causes

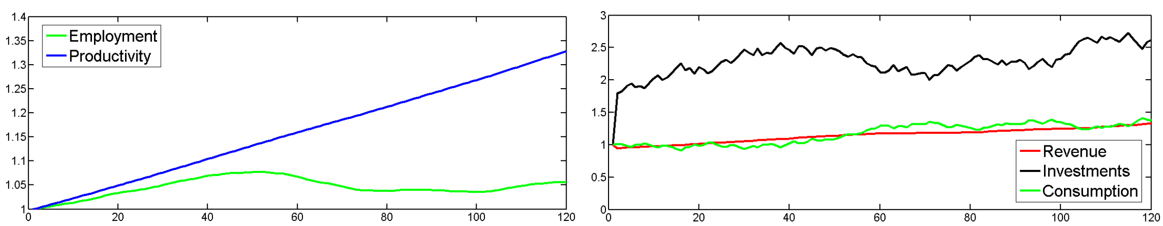

Figure 11: The simulation of economy development with decreased value of parameter $\tilde{\beta} = 1$ shows the increase of the public investments, the productivity and the employment.

a significant public investment rise and consecutive increase of the productivity and the employment. Figure 11 shows the behaviour of the selected quantities simulated with the parameter $\tilde{\beta} = 1$.

The initial employment level ϵ can affect the following employment development. Figure 12 shows two simulations of the employment development affected by the initial percentage of people employed. If the value of parameter $\epsilon = 0.45$ is greater than the real initial natural level of the employment defined by the initial attractivity of being unemployed, the employment decreases to new natural level of the employment. On the other hand, if the value of parameter $\epsilon = 0.3$ is lower, the employment significantly increases.

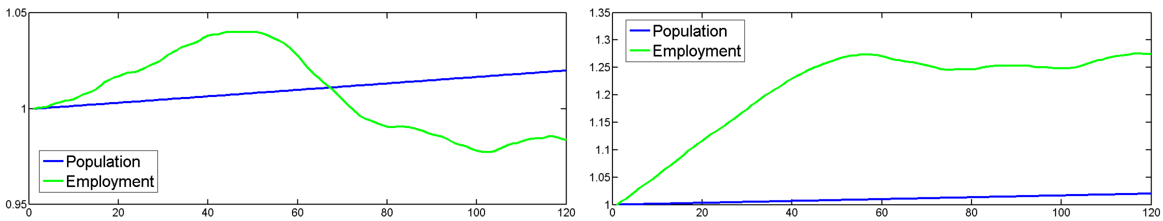

Figure 12: The simulations of employment development show a drop of the employment level caused by the parameter $\epsilon = 0.45$ (on the left) and a significant increase of the employment level affected by the parameter $\epsilon = 0.35$ (on the right).

The parameter ω defines the impact of the current and previous revenues on the estimation of the future revenues. The high value of this parameter highlights the government revenue development and predicts the same development of the revenues for the following fiscal periods. On the other hand, the low value of ω represents prediction of the revenue stagnation, i.e. the government expects the future and previous revenues

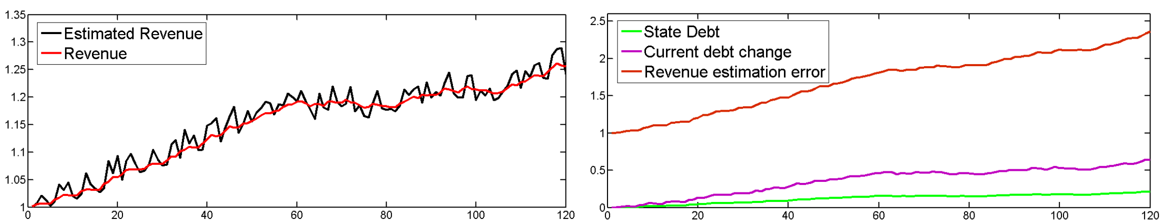

Figure 13: The simulation of economy development with parameter $\omega = 5$ shows significant errors in the future revenue estimation which affect rapid increase of the state debt.

at the same level. Figure 13 captures the debt development when the value of the parameter ω is equal to 5. The optimistic estimation of the future revenues causes deepening of the debt affected by the revenue estimation error and also by the wrong estimation of the available budget.

The real debt interest rate $\tilde{\delta}$ affects the growth of the state debt, i.e. the state debt is the only variable in this model, which reacts on the real debt interest rate changes. With the high value of the parameter $\tilde{\delta}$ the state debt uncontrollably increases and it is hard to repay it. On the other hand, the lower value of real debt interest rate raises the possibility of the debt funding.

2.2 The Constant Economy Growth

An idea of a constant economy growth represents the same living condition for all next generations, i.e. the selected quantities increase by a constant growth rate for each fiscal period. The future generations will have exactly same proportion of state debt to GDP, exactly same proportion of productivity level to social benefits level or exactly same percent of people employed.

We compute the constant growth rate α_* for all quantities using equations in the nondimensional optimal control problem. The constant rate of the population growth α_n is defined by the rate of population growth for one fiscal period (see eq. (27)).

$$n_{k+1} = (1+\nu)n_k \Rightarrow \frac{n_{k+1}}{n_k} = (1+\nu) = \alpha_n \tag{27}$$

The social benefits level is affected by the changes of the social benefits level and can be defined as relative growth of the previous social benefits $(1+\chi)s_k$ which is represented by the constant rate of the social benefits growth α_s.

$$s_{k+1} = s_k + h_k \Rightarrow s_{k+1} = s_k + \chi s_k \Rightarrow \frac{s_{k+1}}{s_k} = \alpha_s \tag{28}$$

The proportion of the consumption at time $(k+1)\Delta t$ and consumption at time $k\Delta t$ indicates the rate of the consumption growth α_c. The value of the constant rate of consumption growth is defined as a multiplication of the rate of the population and social benefits growth (see eq. (29)).

$$c_{k+1} = n_{k+1}s_{k+1} \Rightarrow \frac{c_{k+1}}{c_k} = \frac{n_{k+1}}{n_k}\frac{s_{k+1}}{s_k} = \alpha_n\alpha_s = \alpha_c \tag{29}$$

The constant rate of the productivity has to be deduced using the equation (18) where all addends must have the same constant growth rate. The equation (30) captures the comparison of growth rates of all addends which affect the productivity level.

$$\alpha_p = \frac{\alpha_{ip}}{\alpha_e} = \frac{\alpha_i}{\alpha_e} \Rightarrow \alpha_p \alpha_e = \alpha_{ip} = \alpha_i \tag{30}$$

We deduce the constant rate of the employment growth by using the equation (19) where all expressions added in this difference equation must increase by the same constant growth rate. The equation (31) specifies the relation between the constant rate of the public and private investments growth and the rate of the employment growth. The government affects the social benefits level and thus we may presuppose that the constant rate of the productivity growth is equal to the constant rate of the social benefits growth $\alpha_s = \alpha_p$.

$$\alpha_e = \frac{\alpha_{ip}}{\alpha_e} = \frac{\alpha_{ip}}{\alpha_i} \Rightarrow \alpha_e^2 = \alpha_{ip} = \alpha_i \tag{31}$$

The constant rate of the GDP growth is defined as a multiplication of rate of the employment and productivity growth (see eq. (32)).

$$g_{k+1} = e_{k+1} p_{k+1} \Rightarrow \frac{g_{k+1}}{g_k} = \frac{e_{k+1}}{e_k} \frac{p_{k+1}}{p_k} = \alpha_e \alpha_p = \alpha_g \tag{32}$$

We use the equation (21) for deducing the constant rate of the revenue growth where this rate is equal to rates of all individual addends in the difference equation (see eq. (33)).

$$\alpha_r = \alpha_g = \alpha_i = \alpha_c \tag{33}$$

The explicit equation (23) for the level of the public investments gives us the relations amongst the constant rates of the investments, the consumption, the revenues and the estimated revenues.

$$\alpha_i = \alpha_c = \alpha_r = \alpha_{\tilde{r}} \tag{34}$$

Using the equations (27) - (34) we define the equivalency of the constant growth rates of the public and private investments, the consumption, the revenues, the estimated revenues, and the gross domestic product. These quantities increase with the constant rate $(1+\nu)^2$ each fiscal period (see eq. (35)) and the population, the social benefits, the productivity, and the employment rise with the constant rate $(1+\nu)$ each fiscal period (see eq. (36)). Figure 14 displays the constant economy development over 120 months.

$$\alpha_i = \alpha_{ip} = \alpha_c = \alpha_r = \alpha_{\tilde{r}} = \alpha_g = (1+\nu)^2 \tag{35}$$

$$\alpha_n = \alpha_s = \alpha_p = \alpha_e = (1+\nu) \tag{36}$$

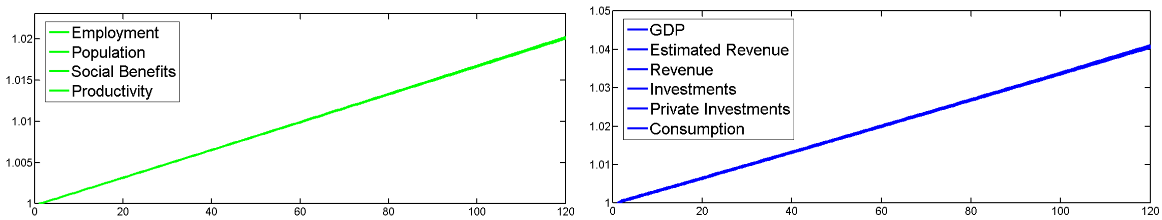

Figure 14: The simulation of the economy development captures the rise of the productivity, the social benefits, the employment, and the population by the constant growth rate $(1+\nu)$ (on the left) and the rise of the public and private investments, the consumption, the revenues, the estimated revenues and the GDP by the constant growth rate $(1+\nu)^2$ (on the right).

For the constant economy growth the government has to set various conditions like the suitable added value tax or the balanced level of the social benefits. We deduce the theoretical changes in the parameters of the model by using all difference and explicit equations of the optimal control problem for times $k = 1$ and $k = 0$.

The parameter ρ defines the impact of the economy growth on the private investment growth and by using the equation (24) for time $k = 1$ we deduce the value of the parameter $\rho = 1$ for the constant economy growth (see eq. (37)) which represents the amount of the private investments needed for maintenance of the constant growth and the duty of the government to produce opportunities for private investing. The values of the relative private investments and the relative revenues at time $k = 0$ are equal to one and the rates of constant growth of these quantities are exactly the same.

$$(\alpha_{ip} - 1) = (\alpha_{ip} - 1)ip_0 = ip_1 - ip_0 = \rho(r_1 - r_0) = \rho(\alpha_r r_0 - r_0) = \rho(\alpha_r - 1) \tag{37}$$

The modification of the parameter $\xi = 3.74$ represents the adaptation of the social benefits level or public investments to the initial condition for starting the constant economy growth. We compute the value of this parameter from explicit equation defining the level of the public investments (23) for time $k = 0$, where the relative values of the consumption, the public investments and the revenues at time zero are equal to one and the constant rate of the revenue growth is define by the parameter α_r.

$$\xi = \xi c_0 = -i_0 + \alpha_{-1} r_{-1} + \alpha_0 r_0 + \alpha_1 r_1 = \alpha_{-1}\alpha_r^{-1} + \alpha_0 + \alpha_1\alpha_r - 1 \tag{38}$$

The government can affect the value added tax in the country represented by the parameter γ. The value of this parameter $\gamma = 0.191$ is defined by the explicit equation for the revenue development 21 for time $k = 1$, where $r_0,\ g_0,\ i_0$ and c_0 equal to one and each of these quantities increase by the same constant growth rate.

$$\gamma = \gamma c_0 = r_1 - \kappa g_1 - \iota i_0 - \iota i_{-1} = \alpha_r - \kappa\alpha_g - \iota - \iota\alpha_i^{-1} \tag{39}$$

The parameters π_i and ζ_i represent the opposite impacts of the investments on economy. The first one defines the relative change of the productivity per employee and the second one the change of the employment. Both can be affected by the specific selection of investment projects, i.e. the investments for hiring new employees usually decrease the productivity per employee and on the other hand the investments increasing the productivity per employee usually do not raise the employment level. The change of the impact of the public investments on these quantities can be deduce by using the difference equations (19) and (18) for time $k = 0$ (see eqs. (40), (41)). The theoretical values of the parameters $\zeta_i = 0.02018$ and $\pi_i = -0.0012$ signalize that the public investments are used mostly for the employment maintenance, not for the productivity development.

$$\pi_i = \left[\alpha_p - 1 + \pi_{ip}\left(1 - \frac{\lambda_P}{\alpha_{ip}}\right)\right]\frac{\alpha_i}{\alpha_i - \lambda_P} \Rightarrow \pi_i = \frac{(\alpha_p - 1)\alpha_i}{\alpha_i - \lambda_P} - \pi_{ip} \tag{40}$$

$$\zeta_i = \frac{(\alpha_e - 1)(1 - \epsilon - \sigma)\alpha_i}{((1-\epsilon)^3 - \sigma(1-\epsilon))(\alpha_i - \lambda_E)} - \zeta_{ip} \tag{41}$$

It is hard to define the constant growth of the state debt because the debt consists of two components with different growth rates - the capitalization of the debt and the debt change caused by the difference between the government expenditures and revenues. Each of the components has a different impact on the state debt and the dependence between these components causes the decrease of the rate of the debt growth in time. Figure 15 shows the decrease of the debt growth rate and the simulating state debt development comparing with the state debt development with a constant rate of the debt growth $\alpha_d = 0.0121$ which is characterized by the initial rate of the debt growth $\frac{d_1}{d_0}$. The decrease of the state debt growth rate in time is affected by the changes of the components impacts.

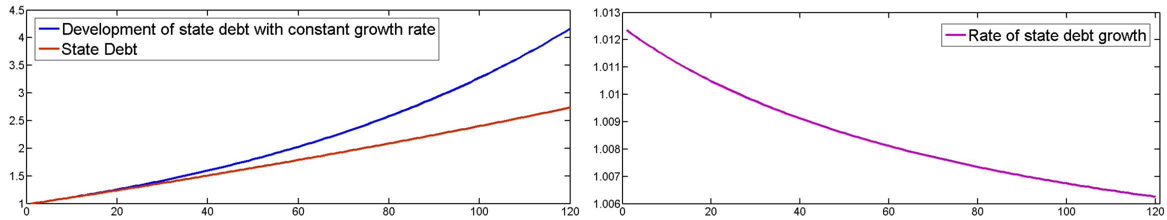

Figure 15: The comparison of the state debt development and the possible state debt development with a constant growth rate $\alpha_d = 0.0121$ (on the left) captures better sustainability of the real state debt because the rate of debt growth decreases with time (on the right).

3 Results - Influence of Government Preferences on Economy Policy

The policy of the government shows the difference between the government promises and the real government preferences which are affected by the unknown future depreciation of the government objective. We assume that the objective function of the optimal control problem represents maximization of the populistic decisions of the government, its desire for minimizing the state debt and raising the social benefits level.

The impact of the social benefits changes on the state debt development could be the best noticeable when the government splits the whole available budget to the consumption and the public investments, i.e. the sum of the parameters $\tilde{\alpha}$ used in the revenues time redistribution equals to one. Therefore, we maximize the values of the objective function with the constant social benefits level change as the control variable and with the vector $\tilde{\alpha} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T$. The optimizations of the populistic decisions of the government are captured in figure 16 where the constant social benefits level change depends on the value of parameter $\tilde{\eta}$ and on the fixed value of the parameter $\tilde{\theta}$. Small changes in the value of the parameter $\tilde{\eta}$ affect the optimal value of the constant social benefits level change, which does not directly influence the changes of the state debt, but it influences the level of the consumption and subsequently the level of the public investments.

The impact of the consumption changes on the state debt is captured in the parameter $\psi_c = 0.67768$ and is greater than the impact of the public investments changes $\psi_i = 0.17857$. The public investments changes react conversely on the government consumption changes because total government expenditure are funded by whole current available budget, i.e. the increase of the consumption causes the decrease of the public investments and vice versa. Therefore, the constant decrease of the social benefits level slightly affects the decrease of the state debt and conversely the constant increase of the social benefits level causes the moderate increase of the state debt.

Figure 16 shows a short interval of the values of the parameter $\tilde{\eta}$ where the constant social benefits level change significantly rises and subsequently stabilizes at the constant relative growth 0.04662. The initial value of $\tilde{\eta}$ where the constant social benefits growth

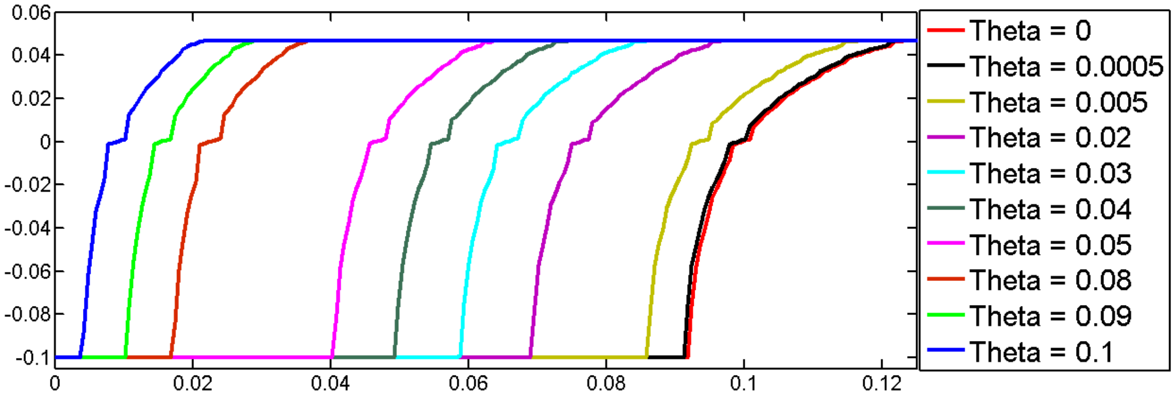

Figure 16: The behaviour of the optimal annual constant social benefits change affected by the fixed value of the parameter $\tilde{\theta}$ depends on the value of the $\tilde{\eta}$ and characterizes the difference between liberal and conservative type of government.

suddenly increases depends on the values of the parameter $\tilde{\theta}$. The increase of the parameter $\tilde{\theta}$ causes shortening and shifting of the interval to the left. The function (42) is an approximation of the dependence of the parameters $\tilde{\eta}$ on the parameter $\tilde{\theta}$ captured in figure 17.

$$\eta(\theta) = 0.17467e^{-7.1\theta} - 0.0829 \tag{42}$$

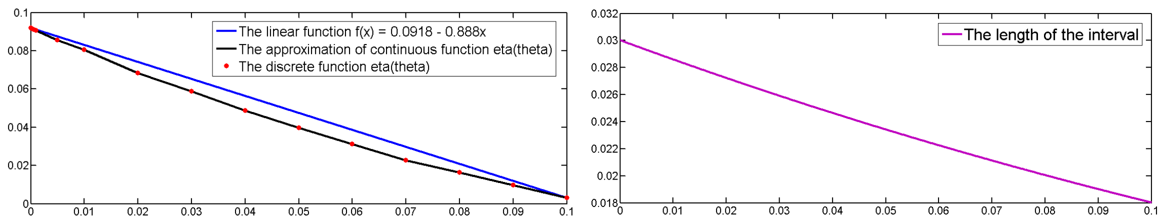

Figure 17: The figure displays the dependence of the parameter $\tilde{\eta}$ on the parameter $\tilde{\theta}$, its comparison with a linear approximation (on the left), and the shortening of the interval with increase of the value of $\tilde{\theta}$ (on the right).

Considering the values of the constant growth of the social benefits level and the parameter $\tilde{\eta}$ we can differ between solely conservative and solely liberal type of the government policy. The decrease of the state debt and the necessary decrease of the rate of the social benefits growth belong to the conservative government aspirations and in contrary the effort for the increase of the social benefits level regardless of the state debt development characterizes the liberal government. The interval of the parameter $\tilde{\eta}$ where the social benefits growth rate changes, defines the type of the government policy which tries to maximize the social benefits level and minimize the state debt at the same time.

Recently, no governments are solely conservative or solely liberal, they want to improve the whole economy considering the social benefits level and the state debt. Therefore, the government probably uses the values of the parameter $\tilde{\eta}$ from the short interval where the constant social benefits growth changes and the fixed value of the parameter $\tilde{\theta}$ representing its time preferences. Figure 18 shows a difference between two types of the government time preferences with the same level of importance of social benefits increase. The liberal government chooses the higher value of the constant social benefits growth and therefore its decision determines the government time preferences and shows that the increase of social benefits characterizes just a short term populistic government objective. In contrary, the lower value of the social benefits increase determines the government time preferences representing a long term government objective.

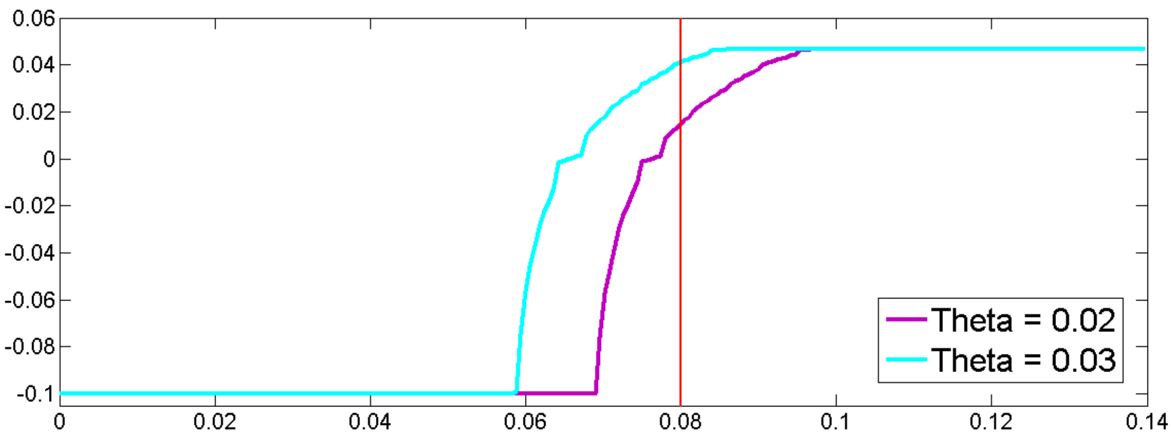

Figure 18: The difference in the time preferences affects the optimal government decisions of the social benefits level changes.

The changes of the constant parameters of the model affect the dependence of the constant social benefits growth on the value of the parameter $\tilde{\eta}$. Figure 19 captures the constant social benefits level change for fixed parameter $\tilde{\theta}$ as solution of the optimal control problem with modified parameters $\tilde{\lambda}_P = 0.05$, $\tilde{\lambda}_E = 0.6$, $\tilde{\iota} = 0.11$ and $\tilde{\rho} = 0.5$. The changes of the parameters affect the length of the interval of the parameter $\tilde{\eta}$ and the rate of the increase of the social benefits growth.

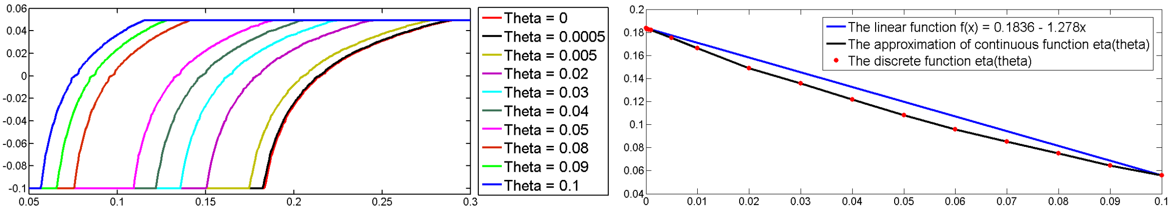

Figure 19: The figure shows the behaviour of optimal annual constant social benefits growth affected by the parameter $\tilde{\theta}$ and $\tilde{\eta}$ in the simulation with the modified parameters $\tilde{\lambda}_P = 0.05$, $\tilde{\lambda}_E = 0.6$, $\tilde{\iota} = 0.11$ and $\tilde{\rho} = 0.5$ (on the left) and the dependence of the parameter $\tilde{\eta}$ on the parameter $\tilde{\theta}$ and its comparison with a linear function (on the right).

Solely Conservative versus Solely Liberal Type of the Government Policy

The optimal solution of the optimal control problem for maximizing the populistic decisions of the solely conservative government defines the constant social benefits level change as the annual decrease equals to $-0.1S_0$ for $\tilde{\eta} = 0.063$ and $\tilde{\theta} = 0.02$, i.e. the level of the social benefits at the end of the target time horizon reaches the zero level of the social benefits. The radical drop of the social benefits causes the decrease of the consumption, the significant increase of the public investments and subsequently the noticeable increase of the productivity. The high level of the public investments at the end of the target time horizon is caused by the fluent decrease of the consumption to zero level and by the consumption impact on the public investments $\xi = 3.795$. The social benefits decrease causes just a moderate decrease of the state debt affected mostly by the estimation revenue error.

The optimal decisions for the liberal government rise the social benefits level by the social benefits increase $0.04662S_0$ each fiscal period. The raise of the social benefits determines the consumption increase and conversely the public investments depreciation. The decrease of the public investments to zero level at the end of the target time horizon retards the increase of the productivity and causes stagnation of the employment considering the increase of the social benefits. The economy development affected by these optimal decisions slightly raises the state debt.

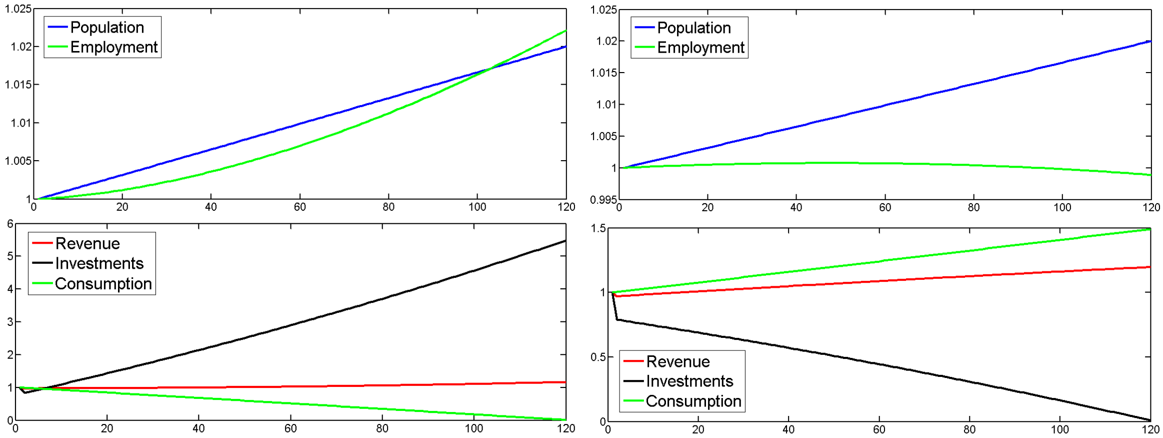

Figure 20: Figure captures the differences between the economy development with the constant annual decrease of the social benefits $h = -0.1$ (on the left) and the economy development with the constant annual increase of the social benefits $h = 0.04662$ (on the right).

Figure 20 captures the different economy development for strictly conservative and strictly liberal type of the government policy. The employment reacts on the changes of attractivity of being employed affected by the social benefits level in the country. The conservative optimal decisions increase the natural level of employment to 100% of the population at the time T and on the other hand the liberal government decrease the natural level of employment to 36.86% of the population at the end of the target time horizon.

4 Results - Influence of Restricted Budgeting on Economy Development

Economy statue of the Slovak Republic defines the optimal solution for social spending and for the government expenditure funding by the budget restricted by optimal weight of the revenue in time redistribution. The optimal value of the social benefits change is restricted by the unemployment in the country and by the public investments level rather than by the defined available budget.

The optimization of the social benefits changes and of the parameters $\tilde{\alpha}$ depends on the initial value of the vector $\tilde{\alpha}$ where the optimization process gives the local maximum of the objective function in the optimal solution. For the optimal solution representing the global maximum of the objective function we have to simulate the optimization problem with various input values of the vector $\tilde{\alpha}$ and we could approximate the optimal time redistribution for the economy model by a comparison of the values of the objective function.

4.1 Optimal Sum of Vector $\tilde{\alpha}$

We optimize the constant social benefits changes and the coefficients $\tilde{\alpha}$ defining the weight of the revenues participated on the budget creation for different input values of $\tilde{\alpha}$ (see table 4). Each optimal solution defined in the table represents the local maximum of the objective function, therefore we compare the values of the objective function and we approximate the optimal solution representing the global maximum of the objective function.

The input $\tilde{\alpha} = (1.130, -0.320, 0.370)^T$ characterizes the time redistribution theoretically used in Slovak Republic which is presented in the bachelor thesis [1]. In this case, the optimal value in table 4 define the optimal government strategy for a budget creation because greater changes in the budget assignment or greater changes in the constant social benefits growth could have disastrous impact on the isolated economy development. For reaching the global optimal solution for this kind of economy the government has to repeat the optimization after the stabilization of economy with restricted budget and with the constant nominal social benefits changes $0.0472S_0$.

Input values				Optimal values			
h	$\tilde{\alpha}_{-1}$	$\tilde{\alpha}_0$	$\tilde{\alpha}_1$	h	$\tilde{\alpha}_{-1}$	$\tilde{\alpha}_0$	$\tilde{\alpha}_1$
0	0.333	0.333	0.333	0.025	0.292	0.295	0.301
0	1.130	-0.320	0.370	0.047	1.063	-0.377	0.344
0	0.167	0.333	0.500	0.024	0.123	0.295	0.467
0	-0.167	0.333	0.833	0.024	-0.209	0.296	0.800
0	0.833	0.333	-0.167	0.024	0.791	0.296	-0.200
0	0.500	0.500	0.500	0.053	0.379	0.398	0.414
0	0.250	0.250	0.333	0.004	0.247	0.248	0.332
0	0.667	-0.333	0.667	0.024	0.624	-0.371	0.633

Table 4: The optimal value of vector $\tilde{\alpha}$ for different inputs represents local maximum of the objective function.

The optimal sum of $\tilde{\alpha}$ reaches the 82.94% level of the budget defined by the average revenues used in the time redistribution. Figure 21 shows the values of the objective function for different sums of $\tilde{\alpha}$ and corresponding constant social benefits level growth. We consider the sum $\sum \tilde{\alpha} = 0.829$ and social benefits change $h = 0.0137$ as the optimal

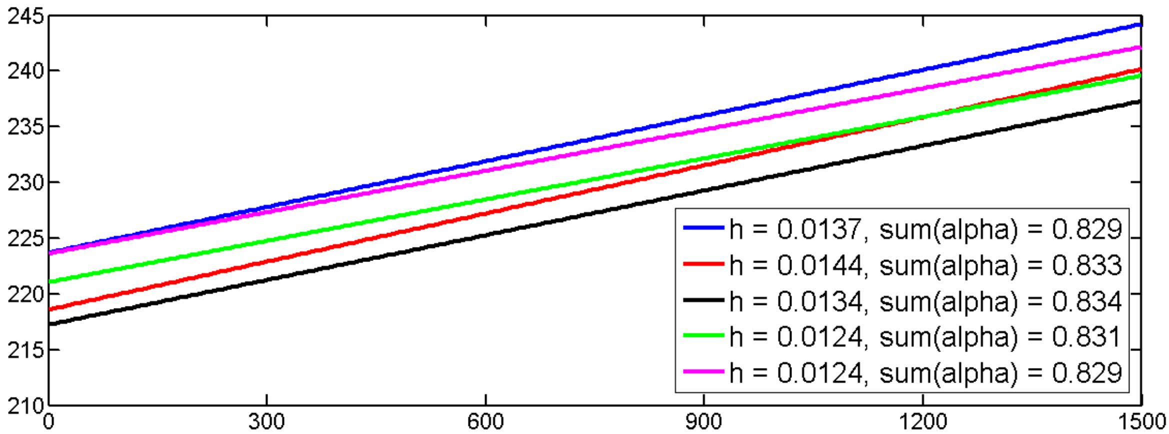

Figure 21: The values of the objective function increase with the value of the $\tilde{\eta}$. Figure captures the values of the objective function with the parameter $\tilde{\theta} = 0.02$ for various constant growth rates of the social benefits and various sums of $\tilde{\alpha}$.

solution of the optimal control problem with constant parameters characterizing the Slovak Republic economy.

The value of the optimal solution is strictly defined by the constant parameters and restrictions of the model and does not depend on the values of parameters $\tilde{\eta}$ and $\tilde{\theta}$. The parameter $\tilde{\eta}$ highlights the impact of the social benefits changes on the government objective. Considering the different dimensions of the state debt and the social benefits the parameter $\tilde{\eta}$ has to reach the values from an interval $(\frac{D_0}{S_0}, \infty)$ to prioritize the social benefits increase at the expense of the significant debt deepening. The usage of the $\tilde{\eta}$ values from this interval defines the optimal solution leading to the uncontrollable economy development which may cause the collapse of the economy of the state.

The dependence of the optimal solutions on the values of the parameter $\tilde{\theta}$ does not show in the case of the constant social benefits growth. We assume that the absolute value of the social benefits changes diminishes in time and the decrease rate of this absolute value depends on the value of the parameter $\tilde{\theta}$.

4.2 Optimal Values of Parameters $\tilde{\alpha}_{-1}$, $\tilde{\alpha}_0$ and $\tilde{\alpha}_1$

We define the optimal sum of the parameters $\tilde{\alpha}$, but the global maximum of the objective function is affected also by the difference amongst the weights of the revenues in

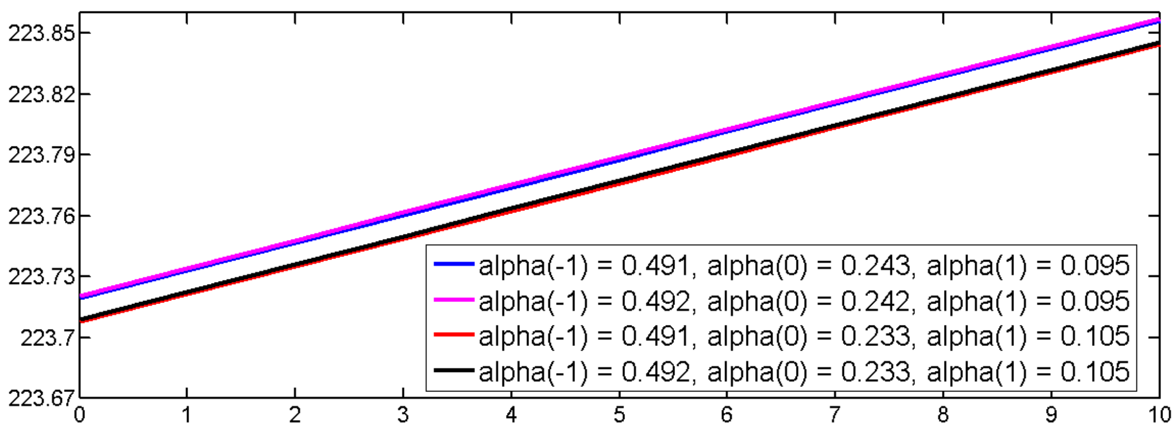

Figure 22: The dependence of the values of the objective function on the values of $\tilde{\eta}$ shows the difference amongst the slightly different vectors $\tilde{\alpha}$ restricted by the optimal sum of weights used in optimal time redistribution.

the time redistribution. Figure 22 shows the dependence of the values of the objective function on the values of $\tilde{\eta}$ for slightly different vectors $\tilde{\alpha} = (\tilde{\alpha}_{-1}, \tilde{\alpha}_0, \tilde{\alpha}_1)^T$ which sum maintains approximately about the same level.

Considering the method used for the revenue estimation with the parameter $\tilde{\omega} = 2$ (see eq. (10)) the optimal budget assignment is defined by the equation (43).

$$\text{Budget}_k = 0.492R_{k-1} + 0.242R_k + 0.095\hat{R}_{k+1} = 0.397R_{k-1} + 0.337R_k \tag{43}$$

The government should create the available budget considering the level of the previous and current revenues which have the similar weight on the budget assignment. We can assume that the government finances the expenditures in the first half of the fiscal period by the previous revenues and the expenditures in the second half by the recent revenues of the state. The usage of this method for the expenditure funding offers the possibility for the state debt funding by 17% of the average revenues each fiscal period.

4.3 Economy Development with Optimal Governing

The model describes the relatively stable isolated economy of the state which is characterized by the previous and current government decisions. The optimal type of the governing for this economy is defined by the optimal value of the vector $\tilde{\alpha} = (0.4917, 0.2422, 0.0955)^T$ and by the optimal nominal constant social benefits changes $H = 0.0137S_0$. If the government starts using the optimal method for the budget assignment and starts increasing the social benefits level, it maximizes its populistic objective at the expense of the economy development. Figure 23 captures the development of the isolated economy, if the government uses the optimal method for whole target time horizon.

The increase of the social benefits level causes the employment stagnation because the attractivity to work maintains approximately on the same level. The increase of the employment is defined just by the moderate increase of the private investments. The restrained rise of the productivity and the employment determines the moderate increase of the government revenues which defines the amount of the available budget. The government has to finance the consumption by the restricted budget and thus the government stops investing. On the other hand, this type of governing decreases the state debt and even the government can start lending money to another countries.

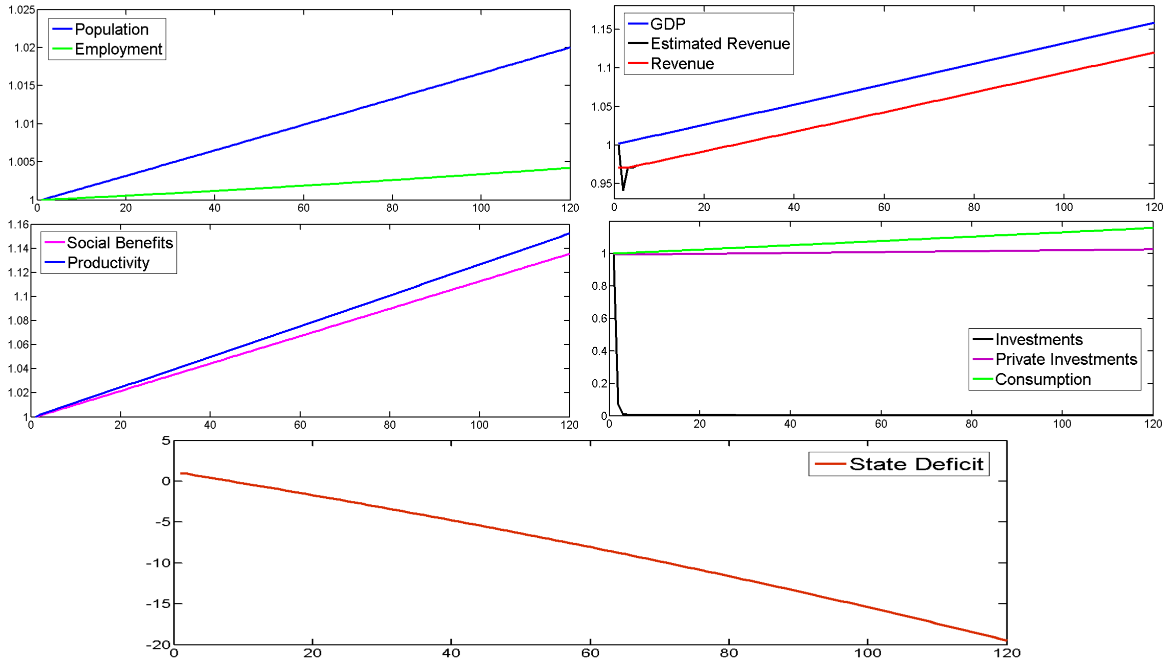

Figure 23: The economy development affected by the optimization of the budget assignment and of the social benefits increase shows the significant state debt and public investments depreciation caused by the restrictive policy of the government.

This kind of governing totally restricts the public investments important for improving the living condition for people and the whole economy stability, therefore the investments at zero level are not optimal for the long term optimal economy progress. The optimal solution defined by the optimal control problem characterizes an attitude of the government for the maximization of their populistic gain, but for the best economy development the government has to optimize the current statue of the economy and use the optimal solution representing the local maximum as the basis for its decisions. The government has to repeatedly optimize its influence on the state economy and wait for the stabilization of the economy affected by the previous optimal government decisions.

Conclusions and Discussion

The main aim of the thesis was the comparison of the government promises and the real government policy. We created the optimal control problem with the objective function which maximizes the populistic decisions of the government and with the system of difference equations which characterizes the development of isolated economy of the state. The parameters of the model and the initial values of all selected quantities correspond to Slovak Republic economy development. Subsequently we deduced the dimensionless optimal control problem for better interpretation and for avoiding computation errors. We optimized the state governing and the gain from populistic decisions considering the constant social benefits change for each fiscal period and the expenditure funding by the whole or restricted budget which assignment is based on the idea of the weight average of the state revenues of several years.

The results of the research point out the impact of the government time preferences on its governing and highlight the difference between the government promises and real policy of the government affected by the future depreciation of the government objective. The increased social benefits growth rate characterizes the short term government objective considering the debt deepening and the liberal type of its policy. On the other hand, lower rate of social benefits growth belongs to long term objectives of conservative government which focuses principally on the state debt financing.

The optimization defines the optimal method for restriction for the budget assignment and the constant optimal increase of the social benefits level. The optimal solution shows the possibility of the increase of the social benefits and the debt funding at the same time. The levels of the budget restriction is affected by the low sum of the weights of the state revenues used in the time redistribution and is primary based on the level of the previous revenues even in the period of economical stability.

The analysis of the results offers many suggestions for improvements of model: (i) The created model characterizes the isolated economy development and thus it offers the possibility for including the foreign influences on the state economy. (ii) The optimal governing for the model causes the total debt funding and even offers the possibility for the state money lending. Therefore, we may add some new restrictions for the minimal level of the debt, the minimal level of the public investments, the maximal level of the employment, etc. (iii) The time preferences of the government affect the non-constant social benefits changes in time, thus the optimization could find the optimal social benefits changes which absolute value diminishes in the future. (iv) The estimation of future state revenues in this model is defined by the simple rule which can be improved to more exact definition of future revenues.

Resumé

Cieľom tejto práce je porovnanie vládnych prísľubov a reálnych rozhodnutí vlády, ktoré sú ovplyvnené nám neznámymi okolnosťami. S týmto zámerom sme vytvorili úlohu optimálneho riadenia, ktorá by mala maximalizovať populistické rozhodnutia vlády, nakoľko v mohých prípadoch majú rozhodnutia vlády skôr tento charakter, ktorý nie je zďaleka optimálnym pre ekonomický vývoj. Systém diferenčných rovníc v úlohe optimálneho riadenia reprezentuje vývoj jednotlivých ekonomických faktorov a ich dopad na celú uzavretú ekonomiku. Počiatočné podmienky pre systém rovníc a odhad konštantných parametrov, ktoré ovplyvňujú vývoj, sme odhadli z dát Slovenskej republiky.

Vzhľadom k veľkému počtu rovníc zodpovedajúcich premenným, ktoré majú rozdielne dimenzie, sme upravili celú úlohu optimálneho riadenia na ekvivalentnú nedimenzovanú formu a následne sme na nej robili optimalizáciu. Proces optimalizácie sa zameriaval na stanovenie optimálnej zmeny sociálnych dávok v priebehu jedného fiškálneho obdobia a na stanovenie optimálneho štáteho rozpočtu, ktorý je ovplyvnený váženým priemerom príjmov z viacerých rokov.

Výsledky práce poukázali na dopad časových preferencií na vládne rozhodovanie a zvýraznili rozdiely medzi vládnymi sľubmi a skutočnou politikou. Povtrdil sa predpoklad, že ľavicové vlády stanovujúce vyšší nárast sociálnych dávok sa zameriavajú na rozhodnutia ohľadne zlepšenia životných podmienok, ktoré ovplyvňujú blízku budúcnosť. Naopak pravicové vlády sa zameriavajú na stabilizovanie ekonomiky štátu a stanovujú nižší rast (prípadne pokles) sociálnych dávok a vytvárajú tým dlhodobý cieľ splatenia dlhu.

Optimálne riešenie úlohy poukázalo na možnosť navyšovania sociálnych dávok a možnosť znižovania štátneho dlhu zároveň. Splácanie štátneho dlhu umožňuje reštriktívna tvorba štátneho rozpočtu, ktorá je určená súčtom váh jednotlivých štátnych príjmov podieľajúcich sa na tvorbe rozpočtu. Okrem optimálneho súčtu veľkosť štátneho rozpočtu ovplyvňujú aj hodnoty jednotlivých váh. Poukázali sme na fakt, že vláda by aj v prípade ekonomickej stability mala stanovovať štátny rozpočet predovšetkým na základe príjmov z minulého fiškálneho obdobia.

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Appendix: Matlab Scripts

Initial Conditions, Nominal Values of the Quantities at Initial Time and Values of the Parameters

```
% THE FIXED PARAMETERS FOR SIMULATION
    % the fixed fiscal period
    dt = 1/12;
    % the target time horizon
    nn = 10/dt;
    % the vector of the revenues redistribution parameters
    tildealpha=[1/3,1/3,1/3];
% THE INITIAL NOMINAL VALUES OF ALL SELECTED QUANTITIES
    % the population at time zero
        N0 = 5.5*10^6;
    % the public investments per one fiscal period at time zero
        I0 = 5*10^9*dt;
    % the social benefits level per one fiscal period at time zero
        S0 = 2300*dt;
    % the consumption per one fiscal period at time zero
        tildebeta = 1.5;
        C0 = tildebeta*N0*S0;
    % the productivity per employee per one fiscal period at time zero
        P0 = 27000*dt;
    % the employment at time zero
        E0 = 2.4*10^6;
    % the GDP per one fiscal period at time zero
        G0 = E0*P0;
    % the government revenues per one fiscal period at time zero
        R0 = 23.7*10^9*dt;
    % the estimated government revenues for the first fiscal period
        ER0 = R0;
    % the private investments per one fiscal period at time zero
        IP0 = 11*10^9*dt;
    % the state debt at time zero
        D0 = 28*10^9*dt;
% THE PARAMETER VALUES IN THE MODEL
     % the annual rate of population growth
        tildeni = 0.002;
     % the public investments impact on the productivity
        tildepii = 0.12;
     % the private investments impact on the productivity
        tildepiip = 0.1;
     % the level of investments necessary for stabilization of the productivity
        tildelambdap = 0.106;
```

```
    % the level of investments necessary for stabilization of the employment
      tildelambdae = 0.9566;
    % the public investments impact on the employment
      tildezetai = 0.00003477;
    % the private investments impact on the employment
      tildezetaip = 0.0001304;
    % the level of productivity used for wages for employees
      tildemi = 0.29;
    % the level of the GDP used for the government expenditure funding
      tildekappa = 0.295;
    % the level of the public investments return
      tildeiota = 0.07;
    % the level of the consumption return
      tildegamma = 0.2;
    % the impact of the economy development on the private investments
      tilderho = 0.1;
    % the real annual debt interest rate
      tildedelta = 0.03025;
    % the parameter used for next year revenue estimation
      tildeomega = 2;
% THE DIMENSIONLESS PARAMETER VALUES IN THE NONDIMENSINAL MODEL
    % the annual rate of population growth
      ni = tildeni*dt;
    % the public investments impact on the productivity
      pii = tildepii*dt*I0/G0;
    % the private investments impact on the productivity
      piip = tildepiip*dt*IP0/G0;
    % the level of investments necessary for stabilization of the productivity
      lambdap = tildelambdap;
    % the level of investments necessary for stabilization of the employment
      lambdae = tildelambdae;
    % the public investments impact on the employment
      zetai = tildezetai*dt*I0/E0;
    % the private investments impact on the employment
      zetaip = tildezetaip*dt*IP0/E0;
    % the percentage of the employment in the country at time zero
      epsilon = E0/N0;
    % the attractivity of being employed at time zero
      sigma = S0/(tildemi*P0);
    % the level of the GDP used for the government expenditure funding
      kappa = tildekappa*G0/R0;
    % the level of the public investments return
      iota = tildeiota*dt*I0/R0;
    % the level of the consumption return
      gamma = tildegamma*C0/R0;
    % the impact of the economy development on the private investments
      rho = tilderho*R0/IP0;
    % the real annual debt interest rate
```

```
        delta = tildedelta*dt;
    % the parameter used for next year revenue estimation
        omega = tildeomega;
    % the vector of the revenues redistribution in time
        alpha = tildealpha*R0/I0;
    % the impact of the public investments on the state debt
        psii = I0/D0;
    % the impact of the consumption on the state debt
        psic = C0/D0;
    % the impact of the revenues on the state debt
        psir = R0/D0;
    % the impact of the consumption on the public investments
        xi = C0/I0;
% THE INITIAL CONDITIONS FOR THE DIMENSIONLESS EQUATIONS IN THE MODEL
        n(1) = N0/N0;
        s(1) = S0/S0;
        c(1) = n(1)*s(1);
        p(1) = P0/P0;
        e(1) = E0/E0;
        r(1) = R0/R0;
        ip(1) = IP0/IP0;
        i(1) = I0/I0;
        g(1) = e(1)*p(1);
        er(1) = ER0/ER0;
        d(1) = D0/D0;
```

Simulation of the Constant Economy Growth

```
% THE CONSTANT RATE OF GROWTH OF ALL SELECTED QUANTITIES
    % the constant rate of growth of the employment, the productivity, the social
         benefits level and the population
        growth1=(1+ni);
    % the constant rate of growth of the GDP, the private investments, the public
         investments, the consumption, the revenues and the estimated revenues
        growth2=(1+ni)^2;
% THE INITIALIZATION OF THE SIMULATION
    % the previous level of the public and private investments considering the
         constant growth rate
        previp=ip(1)/growth2;
        previ=i(1)/growth2;
    % the change of the dimensionless parameters for initialization of constant
         growth of the state economy
        rho=1;
        xi=alpha*[1/growth2;1;growth2]-1;
        gamma=(growth2-kappa*growth2-iota*(1/growth2+1));
        pii=(growth2-growth1)/(growth2-lambdap)-piip;
        help=(1-epsilon)*(1-epsilon/(1-sigma/(1-epsilon)));
```

```
        zetai=growth2*(growth1-1)/help/(growth2-lambdae)-zetaip;
      % the initialization of the iterations
        j = 1;
% THE SIMULATION OF THE ECONOMIC MODEL
while j < nn
    % the iterations
        j = j +1;
    % the difference equation for the population dynamics
        n(j)=(1+ni)*n(j-1);
    % the difference equation for the social benefits development
        s(j)=(1+ni)*s(j-1);
    % the equation for the consumption level
        c(j)=n(j)*s(j);
    % the difference equation for the productivity development
        if j==2
            p(j)=p(j-1)+(piip*(ip(j-1)-lambdap*previp)+pii*(i(j-1)-lambdap*previ))/e(j
                -1);
        else
            p(j)=p(j-1)+(piip*(ip(j-1)-lambdap*ip(j-2))+pii*(i(j-1)-lambdap*i(j-2)))/e
                (j-1);
        end
    % the difference equation for the employment dynamics
        if j==2
            q=(n(j-1)-e(j-1)*epsilon)/e(j-1)/n(j-1)*(zetaip*(ip(j-1)-lambdae*previp)+
                zetai*(i(j-1)-lambdae*previ));
            q=q*(1-e(j-1)/(1-(sigma*s(j-1)/p(j-1)*(n(j-1)/(n(j-1)-e(j-1)*epsilon))))/n
                (j-1)*epsilon);
            e(j)=e(j-1)*(1+q);
        else
            q=(n(j-1)-e(j-1)*epsilon)/e(j-1)/n(j-1)*(zetaip*(ip(j-1)-lambdae*ip(j-2))+
                zetai*(i(j-1)-lambdae*i(j-2)));
            q=q*(1-e(j-1)/(1-(sigma*s(j-1)/p(j-1)*(n(j-1)/(n(j-1)-e(j-1)*epsilon))))/n
                (j-1)*epsilon);
            e(j)=e(j-1)*(1+q);
        end
    % the equation for the GDP level
        g(j)=e(j)*p(j);
    % the difference equation for the revenues development
        if j ==2
            r(j)=kappa*g(j)+iota*(i(j-1)+previ)+gamma*c(j-1);
        else
            r(j)=kappa*g(j)+iota*(i(j-1)+i(j-2))+gamma*c(j-1);
        end
    % the equation for the estimated revenues level for next fiscal period
        er(j)=(1-omega)*r(j-1)+omega*r(j);
    % the equation for the public investments level
        i(j)=-c(j)*xi+alpha*[r(j-1);r(j);er(j)];
    % the difference equation for the private investments development
```

```
        ip(j)=ip(j-1)+rho*(r(j)-r(j-1));
    % the difference equation for the state debt development
        d(j)=(1+delta)*d(j-1)+psii*i(j)+psic*c(j)-psir*r(j)+psir*(er(j-1)-r(j));
end;
% THE GRAPH OF THE EMPLOYMENT, THE POPULATION, THE SOCIAL BENEFITS LEVEL AND THE
    PRODUCTIVITY PROGRESS
    subplot(3,1,1);
    plot(e,'g');
    hold on
    plot(n,'g');
    plot(s,'g');
    plot(p,'g');
    legend('Employment','Population','Social Benefits','Productivity',4);
    hold off
% THE GRAPH OF THE GDP, THE REVENUE, THE ESTIMATED REVENUE, THE PUBLIC INVESTMENTS,
    THE PRIVATE INVESTMENTS AND THE CONSUMPTION PROGRESS
    subplot(3,1,2);
    plot(g,'b');
    hold on
    plot(r,'b');
    plot(er,'b');
    plot(i,'b');
    plot(ip,'b');
    plot(c,'b');
    legend('GDP','Revenue','Estimated Revenue','Investments','Private Investments','
        Consumption',6);
    hold off
% THE GRAPH OF THE STATE DEBT PROGRESS
    subplot(3,1,3);
    plot(d,'r');
    legend('State Deficit',1);
```

Optimal Control Problem

```
% THE OPTIMAL CONTROL PROBLEM
    % the input is defined as [constant h,vector alpha]'
    function value = model(input)
% THE DEFINITION OF THE GLOBAL VARIABLES
        global nn
        global dt
        global restriction
        global tildetetha
        global tildeeta
% THE PARAMETERS FOR THE OPTIMALIZATION
    % the vector of the revenues redistribution parameters
        tildealpha = input(2:4)';
    % the nondimensional vector alpha
```

```
        alpha = tildealpha*R0/I0;
    % the vector of constant social benefits growth for each fiscal period
        h=input(1)*ones(nn+1,1);
% THE INITIALIZATION OF THE SIMULATION
    % the previous level of the public investments, the private investmetns and
    % and the revenues
        previp = 0.99*ip(1);
        previ = 0.99*i(1);
        prevr = 0.99*r(1);
    % the initialization of the iterations
        j = 1;
    % the variable defining the restrictions of the model
        restriction = 1;
    % the restriction for the maximal change of the social benefits level
    % per fical period
        ss0=0.5;
% THE SIMULATION OF THE ECONOMIC MODEL
while (j < nn+1)&&(restriction==1)
    % the verification of the restrictions
        if (h(j)*dt>=-s(j))&&(h(j)*dt<=s(j))&&(h(j)<ss0)&&(s(j)>=0)&&(restriction==1)
                restriction = 1;
            else restriction = -1;
        end
    % the iterations
        j = j +1;
    % the difference equation for the population dynamics
        n(j)=(1+ni)*n(j-1);
    % the difference equation for the social benefits development
        s(j)=s(j-1)+h(j-1)*dt;
    % the equation for the consumption level
        c(j)=n(j)*s(j);
    % the difference equation for the productivity development
        if j==2
            p(j)=p(j-1)+(piip*(ip(j-1)-lambdap*previp)+pii*(i(j-1)-lambdap*previ))/e(j
                -1);
        else
            p(j)=p(j-1)+(piip*(ip(j-1)-lambdap*ip(j-2))+pii*(i(j-1)-lambdap*i(j-2)))/e
                (j-1);
        end
    % the difference equation for the employment dynamics
        if j==2
            q=(n(j-1)-e(j-1)*epsilon)/e(j-1)/n(j-1)*(zetaip*(ip(j-1)-lambdae*previp)+
                zetai*(i(j-1)-lambdae*previ));
            q=q*(1-e(j-1)/(1-(sigma*s(j-1)/p(j-1)*(n(j-1)/(n(j-1)-e(j-1)*epsilon))))/n
                (j-1)*epsilon);
            e(j)=e(j-1)*(1+q);
        else
            q=(n(j-1)-e(j-1)*epsilon)/e(j-1)/n(j-1)*(zetaip*(ip(j-1)-lambdae*ip(j-2))+
```

```
                zetai*(i(j-1)-lambdae*i(j-2)));
            q=q*(1-e(j-1)/(1-(sigma*s(j-1)/p(j-1)*(n(j-1)/(n(j-1)-e(j-1)*epsilon))))/n
                (j-1)*epsilon);
            e(j)=e(j-1)*(1+q);
        end
    % the equation for the GDP level
        g(j)=e(j)*p(j);
    % the difference equation for the revenues development
        if j ==2
            r(j)=kappa*g(j)+iota*(i(j-1)+previ)+gamma*c(j-1);
        else
            r(j)=kappa*g(j)+iota*(i(j-1)+i(j-2))+gamma*c(j-1);
        end
    % the equation for the estimated revenues level for next fiscal period
        er(j)=(1-omega)*r(j-1)+omega*r(j);
    % the equation for the public investments level
        i(j)=-c(j)*xi+alpha*[r(j-1);r(j);er(j)];
    % the difference equation for the private investments development
        ip(j)=ip(j-1)+rho*(r(j)-r(j-1));
    % the difference equation for the state debt development
        d(j)=(1+delta)*d(j-1)+psii*i(j)+psic*c(j)-psir*r(j)+psir*(er(j-1)-r(j));
    % the verification of the restrictions
        if (phi*r(j)>=c(j)*psic)&&(e(j)>0)&&(p(j)>0)&&(i(j)>=0)&&(E0*e(j)<N0*n(j))&&(
            restriction==1)
                restriction = 1;
            else restriction = -1;
        end
end;
% THE VECTOR DEFINING THE VALUE OF THE OBJECTIVE FUCNTION FOR EACH TIME
    eta = tildeeta*S0/D0;
    theta = tildetheta*dt;
    objfunction = ones(nn,1);
    objfunction(1) = (-d(2)+d(1)+eta*(s(2)-s(1)))/dt;
    if  restriction==1
        for j=2:nn
            objfunction(j) = ((1+theta)^(1-j))/dt*(-d(j+1)+d(j)+eta*(s(j+1)-s(j)));
        end
    end
    if (restriction==-1) value = 10000;
    else value = -sum(objfunction);
    end;
```