COMENIUS UNIVERSITY IN BRATISLAVA
FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS

CALIBRATION OF A MODEL FOR OPTION PRICES WITH FEEDBACK EFFECT

DIPLOMA THESIS

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Aim: The aim of the thesis is to study a model for option prices taking so called feedback effect into account, which has been suggested by Sircar and Papanicolaou: 1.propose a procedure for calibration of the parameters, using an approximation of the solution based on asymptotic methods derived in their paper, 2. perform this procedure using real market data and assess the results, 3. in more detail, study the possibility of pricing a portfolio of options, which mathematically leads to a system of partial differential equations and is only briefly outlined in the original paper by Sircar and Papanicolaou.

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Abstract

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Our thesis is dedicated to exploring a possible improvement of the Black-Scholes model for option pricing, through incorporation of so-called feedback effect. Our objective is to introduce an upgraded model, derived in the paper by Sircar and Papanicolaou, collect a sample of market data, perform, on this sample, the calibration of parameters using an approximation of the solution, based on asymptotic methods derived in the mentioned paper, and asses the results with implications in real-life stock market.

Keywords: Black-Scholes model, option pricing, feedback effect, asset price volatility

Abstrakt v štátnom jazyku

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V našej práci sa venujeme skúmaniu možného vylepšenia Black-Scholesovho modelu na oceňovanie opcií, prostredníctvom zohľadnenia takzvaného feedback efektu. Našim cieľom je predstaviť tento vylepšený model odvodený v práci Sircara a Papanicolaoua, zozbierať vzorku trhových dát, zrealizovať na tejto vzorke kalibráciu parametrov, využívajúc aproximáciu riešenia odvodenú v spomenutom článku a vyhodnotiť výsledky s dôsledkami pre skutočný trh s cennými papiermi.

Keywords: Black-Scholes model, oceňovanie opcií, feedback effect, volatilita ceny aktíva

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Introduction

The Black-Scholes model, improvements of which we are going to examine in this thesis, is a commonly used tool in finance. This model enables us to calculate the "fair" price of a derivative, an option for example, which depends on the so-called underlying asset, such as a stock. Its main advantage is, that it requires only a few basic premises about the behaviour of the asset price, on which the value of the derivative security is based and about the market on which the asset is traded.

One of these assumptions for example is the one, that the market in the underlying asset is perfectly elastic. That means, no matter in how large quantities the asset is traded, the equilibrium price will not be affected. If we decide to relax this assumption, we proceed to the problem discussed in this thesis, that is, the impact which trading in the underlying can have on its price, whence then naturally follows also the change in the corresponding derivative. This effect is called the *feedback effect*.

Modelling of this phenomenon, as described in the paper [7], usually begins with an economy of two types of traders. The first type are the so-called *reference traders*, who are the majority on the market and invest in the asset with expectations of gain. Second, much smaller group are the *program traders*. These trade the asset in order to insure against the risk from holding or writing an option, using a strategy, based on the Black-Scholes model, such as *delta hedging*.

In the first chapter, we will recall a few basic definitions of some financial terms that will be used in this thesis. We will unify, what we will understand by stock, option or the Black-Scholes model.

The second chapter will be dedicated to the derivation of a possible improvement or extention of the classical Black-Scholes model. We will examine how the presence of program traders on the market affects the asset price process. After obtaining this new price process, we will derive the Black-Scholes model anew, using the adjusted asset price volatility. Thus we obtain the extended Black-Scholes model incorporating the feedback effect, caused by the program traders and their hedging strategies.

The third chapter will contain the description of the approach to the calculation of the new derivative price and the explanation, how the algorithm is programmed in the software Scilab. We need to calculate the approximation of the derivative price which is the result of the newly derived model. This approach which we took over from the paper [7] consists in computation of a first order correction to the original Black-Scholes derivative price. The terms of higher order are omitted. This chapter will also include the description how parameters will be calibrated.

In the last fourth chapter we will analyze and interpret the results which we will obtain for two sets of market data. One of these will be call options for one chosen stock with the same expiration date, but different strike prices, and the other for another stock, with both, different strike prices and dates of expiration.

1 Recall of Basic Concepts

Before we derive the new model, we need to recall and unify what we understand by some basic financial terminology and introduce notation used throughout this thesis. We give some definitions from the website [3].

An *asset* is a resource with economic value that an individual, corporation or country owns or controls with the expectation that it will provide future benefit. Our notation for the asset price at time t will be X_t.

A *stock* is a type of security that signifies ownership in a corporation and represents a claim on part of the corporation's assets and earnings.

A *derivative* is a security with a price that is dependent upon or derived from one or more underlying assets. The derivative itself is a contract between two or more parties based upon the asset or assets. Its value is determined by fluctuations in the underlying asset. The most common underlying assets include stocks, bonds, commodities, currencies, interest rates and market indices. The derivative price at time t for the asset price X_t will be denoted by $V(X_t, t)$.

An *option* is a financial derivative that represents a contract sold by one party (option writer) to another party (option holder). The contract offers the buyer the right, but not the obligation, to buy (call) or sell (put) a security or other financial asset at an agreed-upon price (the strike/excercise price) during a certain period of time (American) or on a specific date (exercise date/maturity) (European). Call options give the option to buy at certain price, so the buyer would want the stock to go up. Put options give the option to sell at a certain price, so the buyer would want the stock to go down. Strike price will be denoted by K and maturity by T. In this thesis we will perform calculations for European call options with notation V^{EC}.

The *Black-Scholes model* is a model of price variation over time of financial instruments such as stocks that can, among other things, be used to determine the price of a European call option. The model assumes that the price of heavily traded assets follows a geometric Brownian motion with constant drift and volatility. When applied to a stock option, the model incorporates the constant price variation of the stock, the time value of money, the option's strike price and the time to the option's expiry.

The option price is a solution to the *Black-Scholes partial diferential equation* when following is satisfied on the market.

- There is a constant riskless interest rate r,
- no transaction costs,
- one can sell and buy an arbitrary amount of stocks or bonds,
- short selling is allowed, and
- options are of European type.

We consider an economy, where a certain asset is continually traded and its equilibrium price process is described by a Geometric Brownian Motion $\{X_t, t \geq 0\}$. Itô process for the price of the asset is given by

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \tag{1}$$

where $\{W_t, t \geq 0\}$ is the Wiener process on a probability space (Ω, F, P), for constants μ, the drift, and σ, the volatility.

There are two other securities in this economy. A riskless bond with price process

$$\beta_t = \beta_0 e^{rt},$$

where r is the constant spot interest rate, and a derivative security with price process $\{P_t, t \geq 0\}$, whose payoff at a certain time of maturity $T > 0$ is dependent on the price X_T of the underlying asset at time T and it holds

$$P_T = h(X_T),$$

for some function $h(.)$. We assume, the asset pays no dividends during the time interval $0 \leq t \leq T$.

Price of the derivative is then given by

$$P_t = V(X_t, t)$$

for some function $V(x,t)$, which is sufficiently smooth to satisfy the Black-Scholes partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + r\left(x\frac{\partial V}{\partial x} - V\right) = 0, \tag{2}$$

which is a result of construction of a self-financing, derivative replicating, strategy, using the underlying asset and the bond.

2 Derivation of the Model

In this chapter, we present the derivation of the Black-Scholes model incorporating the feedback effect. We follow the derivation in the paper [7], with some contributions of our own.

2.1 Feedback Effect

In recent years, increases in market volatility of asset prices have been observed and, as some, like M. Miller [6], believe, the reason of this icrease can be the popularity of portfolio insurance strategies for derivatives. As the Black-Scholes model is used so widely, as several sources claim, like for example [2] and [8], it is assumed to be likely to influence the market itself to some extent. Specifically, that its possible utilization to create hedging strategies could be the cause of different quantities of an asset traded on the market, and without the assumption of elasticity, this could be the origin of the mentioned increase of volatilities. Changes in the asset price then also cause a change in the corresponding derivative price and we call this the feedback effect.

The hedging strategy in question is the following. Let us say, an investor wants to insure himself against the risk from writing a derivative. At time t, he needs to hold the amount $V_x(X_t, t)$ of the underlying asset, continually trading to maintain it, and invest the amount $V(X_t, t) - X_t V_x(X_t, t)$ in the bond at time t. The price of these transactions is exactly the price of the derivative $V(X_t, t)$. Analogically, in case of holding the derivative, for a riskless investment, he needs to hold the amount $-V_x(X_t, t)$ of the underlying asset.

2.2 Extended Black-Scholes Model

We assume, that the hedging strategy is unknown and we derive equations for it using the modified underlying asset diffusion process. The previously described market can be characterized by two groups trading the asset.

The first group are the reference traders, that is, investors, who trade in the asset in such a way that, were they the only ones in the economy, the equillibrium asset price would be exactly the solution of the Itô process (1). Also, this price would be independent of the distribution of wealth among the traders in this group. That is why we can consider one aggregate reference trader, who represents the whole groups actions in the market. In order to derive the model for the asset price incorporating feedback effect, we describe the reference trader using two attributes:

1. an aggregate stochastic income modelled by an Itô process $\{Y_t, t \geq 0\}$ satisfying

$$dY_t = \mu(Y_t, t)dt + \eta(Y_t, t)dW_t, \tag{3}$$

where $\{W_t, t \geq 0\}$ is the Wiener process, and μ and η are exogenously given functions satisfying all conditions for the existence and uniqueness of the solution to (3). These functions will not appear in the pricing equations that we will derive, therefore, the income process, which is not directly observable, need not be known for our model.

2. a demand function $\widetilde{D}(X_t, Y_t, t)$, arguments of which are the income and equillibrium price process.

The second group of traders on the market are the program traders, whose characteristics are the dynamic hedging strategies, which they follow to insure their portfolios. Hedging against the risk from writing or holding a derivative is the only reason why they trade in the asset. Their aggregate demand function is given by a function $\phi(X_t, t)$, which indicates the amount of the asset that the program traders want to hold at time t given the price X_t. ϕ is naturally independent of the income Y_t of the reference traders, which is unknown to the program traders. We assume that the program traders have written ξ identical derivatives, which they want to hedge. For simplicity, we introduce

$$\phi(X_t, t) = \xi\Phi(X_t, t),$$

where Φ is the demand after the asset per derivative security being hedged. The function Φ also need not be given.

2.2.1 Asset Price under Feedback

Now we want to find out, how is the price process X_t determined by the market equillibrium and the income process Y_t. Let S_0 be the constant supply of the asset and

$$\widetilde{D}(x, y, t) = S_0 D(x, y, t),$$

so that D is the demand of reference traders relative to the supply.

We define the relative demand of the representative reference trader and the program traders as

$$G(x, y, t) = D(x, y, t) + \rho\Phi(x, t), \tag{4}$$

where $\rho = \frac{\xi}{S_0}$ is the ratio of the volume of the derivatives being hedged to the total supply of the asset. The normalization by the total supply is included in the definition of the function D and $\rho\Phi$ is the proportion of the total supply of the asset that is being traded by the program traders.

When we set at each time point *demand* $\equiv$ *supply* $= 1$ to enforce the market equillibrium, we get

$$G(X_t, Y_t, t) = 1, \tag{5}$$

which determines the relationship between the trajectory X_t and the trajectory (3). We suppose that $G(x, y, t)$ is strictly monotonous in first two arguments and has continuous first derivatives in x and y, so that we can invert (5) to obtain

$$X_t = \psi(Y_t, t), \tag{6}$$

for some smooth function $\psi(y, t)$. Now we know that the process X_t is driven by the same Wiener process as Y_t.

Using the Itô's lemma on (6), with (3) we get

$$dX_t = \left[\mu(Y_t, t)\frac{\partial\psi}{\partial y} + \frac{\partial\psi}{\partial t} + \frac{1}{2}\eta^2(Y_t, t)\frac{\partial^2\psi}{\partial y^2}\right] dt + \eta(Y_t, t)\frac{\partial\psi}{\partial y}(Y_t, t)dW_t, \tag{7}$$

and after differentiating the constraint $G(\psi(y,t),y,t)=1$, we have

$$\frac{\partial \psi}{\partial y} = -\frac{\frac{\partial G}{\partial y}}{\frac{\partial G}{\partial x}}, \tag{8}$$

where $\frac{\partial G}{\partial x} \neq 0$ due to strict monotonicity of G in x.

From (7) we can see that the asset price process under feedback effect satisfies the stochastic differential equation

$$dX_t = \alpha(X_t, Y_t, t)dt + \eta(Y_t, t)\nu(X_t, Y_t, t)dW_t, \tag{9}$$

where

$$\alpha(X_t, Y_t, t) = \mu(Y_t, t)\frac{\partial \psi}{\partial y} + \frac{\partial \psi}{\partial t} + \frac{1}{2}\eta^2(Y_t, t)\frac{\partial^2 \psi}{\partial y^2} \tag{10}$$

and

$$\nu(X_t, Y_t, t) = \frac{\partial \psi}{\partial y}(Y_t, t). \tag{11}$$

When we insert (4) into (8) we get

$$\frac{\partial \psi}{\partial y} = -\frac{D_y(X_t, Y_t, t) + \rho\phi_y(X_t, t)}{D_x(X_t, Y_t, t) + \rho\phi_x(X_t, t)}. \tag{12}$$

Subsequently inserting (12) into (11) and (10) the modified asset price volatility takes the form

$$\nu(X_t, Y_t, t) = -\frac{D_y(X_t, Y_t, t) + \rho\phi_y(X_t, t)}{D_x(X_t, Y_t, t) + \rho\phi_x(X_t, t)} \tag{13}$$

and the adjusted drift is

$$\alpha(X_t, Y_t, t) = -\left[\mu\frac{G_y}{G_x} + \frac{G_t}{G_x} + \frac{1}{2}\eta^2\left(\frac{G_{yy}}{G_x} - 2\frac{G_{xy}G_y}{G_x^2} + \frac{G_y^2 G_{xx}}{G_x^3}\right)\right].$$

2.2.2 Modified Black-Scholes under Feedback Effect

When we already have the new asset price process, in which the feedback effect is accounted for and a new volatility, we will examine how this changed volatility affects the derivation of the Black-Scholes partial differential equation for the price P_t. We will follow the derivation procedure of Black and Scholes also performed in lecture notes [?]. The only change is that the price of the underlying asset is not driven by (1) but by the process (9), which is dependent on the other Itô process Y_t. In the course of the derivation, we will get the amount of the asset, the program traders should buy or sell to cover the risk arising from holding or writing the derivative. We should get an expression for Φ in terms of the derivative price $V(X_t, t)$.

Firstly, we construct a self-financing replicating strategy (a_t, b_t) in the underlying asset and the riskless bond. In time T it holds

$$a_T X_T + b_T \beta_T = P_T,$$

and for $0 \leq t \leq T$

$$a_t X_t + b_t \beta_t = a_0 X_0 + b_0 \beta_0 + \int_0^t a_s dX_s + \int_0^t b_s d\beta_s.$$

Since a_t is exactly the amount of the asset, which the traders must hold at time t, it is the demand for the asset per derivative security being hedged, that is

$$a_t = \Phi(X_t, t).$$

To rule out arbitrage opportunities, we must set

$$a_t X_t + b_t \beta_t = P_t \tag{14}$$

for $0 \leq t \leq T$ and the self-financing property of the strategy (a_t, b_t) can be, according to chapter 5 in [5], expressed by the following equation

$$\Delta P_t = a_t \Delta X_t + b_t \Delta \beta_t,$$

which expresses that the strategy starts with the value P_0 at time 0 and then only the proportion of held assets and bonds is changed, no further resources are neither added, nor generated. In a continuous case, the equation has a form

$$dP_t = a_t dX_t + b_t d\beta_t.$$

After we replace dX_t using (9) and insert $d\beta_t = r\beta_t dt$ we have one expression for dP_t

$$dP_t = \left[a_t \alpha(X_t, Y_t, t) + b_t r \beta_t\right] dt + a_t \nu(X_t, Y_t, t) \eta(Y_t, t) dW_t. \tag{15}$$

When we now use the Itô's lemma, for $P_t = V(X_t, t)$, we get the equality

$$dP_t = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial x}dX_t + \frac{1}{2}\eta^2\nu^2\frac{\partial^2 V}{\partial x^2}dt,$$

and again after inserting (9) we obtain another expression for dP_t

$$dP_t = \left[\frac{\partial V}{\partial t} + \alpha(X_t, Y_t, t)\frac{\partial V}{\partial x} + \frac{1}{2}\eta^2\nu^2\frac{\partial^2 V}{\partial x^2}\right]dt + \frac{\partial V}{\partial x}\nu(X_t, Y_t, t)\eta(Y_t, t)dW_t. \quad (16)$$

Comparing the coefficients of dW_t in (15) and (16) we obtain the expression for Φ, which we were looking for

$$a_t = \Phi(X_t, t) = \frac{\partial V}{\partial x} \quad (17)$$

and from (14) again we get

$$b_t = \frac{P_t - a_t X_t}{\beta_t}. \quad (18)$$

When we equate the coefficients of dt in (15) and (16), and insert (17) and (18) we have

$$\frac{\partial V}{\partial t} + \alpha\frac{\partial V}{\partial x} + \frac{1}{2}\eta^2\nu^2\frac{\partial^2 V}{\partial x^2} = \alpha\Phi + r(V - x\Phi). \quad (19)$$

Next we put to use the fact, that the volatility ν comes from the feedback from the hedging strategies. From (13) we see that the adjusted volatility for X_t is a function of Φ and its derivative, so we can write

$$\nu(X_t, Y_t, t) = H\left(\frac{\partial\Phi}{\partial x}(X_t, t), \Phi(X_t, t), X_t, Y_t, t\right), \quad (20)$$

for some function H. Then from (17) and (19) we can see that the function $V(X_t, t)$ must satisfy the nonlinear partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\eta^2 H^2\left(\frac{\partial^2 V}{\partial x^2}, \frac{\partial V}{\partial x}, x, y, t\right)\frac{\partial^2 V}{\partial x^2} + r\left(x\frac{\partial V}{\partial x} - V\right) = 0, \quad (21)$$

for $x, y > 0$ and $0 \leq t \leq T$ with following

$$V(x, T) = h(x),$$
$$\Phi(x, T) = h'(x),$$
$$V(0, t) = 0,$$
$$\Phi(0, t) = 0.$$

The dependence of functions H and η on the variable y can be removed by inverting (4), so that we obtain a relationship

$$y = \hat{\psi}(x, \rho\Phi(x,t), t).$$

We can rewrite the equation (21) in terms of the relative demand function (4). Since from the equalities (13) and (20)

$$H = -\frac{G_y}{G_x},$$

we get the Black-Scholes partial differential equation with the feedback effect

$$\frac{\partial V}{\partial t} + \frac{1}{2}\eta^2 \left(\frac{D_y}{D_x + \rho V_x x} \right) \frac{\partial^2 V}{\partial x^2} + r\left(x\frac{\partial V}{\partial x} - V \right) = 0. \tag{22}$$

The last step to the new model is ensuring the consistency with the Black-Scholes model when $\rho \to 0$. Thereby we get an important constraint for the demand function.

2.2.3 Consistency and Reduction to Black-Scholes Model

Now we finish our model so that it reduces to the Black-Scholes model in case program traders are not present. We start again with the demand function $D(x, y, t)$ and the income process Y_t, the only modification in Section 2.2.1 is that it holds $\rho\Phi = 0$, so the new volatility has a form

$$\nu_0(x, y, t) = -\frac{D_y(x, y, t)}{D_x(x, y, t)}.$$

The derivation procedure is the same as in Section 2.2.2 and the equation (19) now has the following form

$$\frac{\partial V}{\partial t} + \frac{1}{2}\eta^2(y,t)\nu_0^2(x,y,t)\frac{\partial^2 V}{\partial x^2} + r(x\frac{\partial V}{\partial x} - V) = 0. \tag{23}$$

We will call the equation (23) the limit case of the partial differential equation (22) when we omit the program traders. Lastly, we need to determine some conditions on the demand function of reference traders D, so that we get the original Black-Scholes partial differential equation (2).

2.2.4 Conditions for the Demand Function

Let us suppose that D does not depend explicitly on t. Further, we know that reference traders have the rational characteristics $D_x < 0$ that is, the demand of reference traders decreases with increasing asset price and $D_y > 0$, which means that their demand increases with their income.

Let the income process Y_t be a Geometric Brownian Motion which satisfies

$$dY_t = \mu_1 Y_t dt + \eta_1 Y_t dW_t$$

for constants μ_1 and η_1. Then (23) reduces to (2) if and only if coefficients by the second derivative member are the same, that is, if the diffusion coefficient satisfies

$$\frac{1}{2}\eta_1^2 y^2 \left[\frac{D_y(x,y)}{D_x(x,y)}\right]^2 = \frac{1}{2}\sigma^2 x^2.$$

Hence, D must satisfy the condition

$$\frac{D_y}{D_x} = -\frac{\gamma x}{y}, \tag{24}$$

where $\gamma = \frac{\sigma}{\eta_1}$. We take the negative square root because the left-hand side is negative under the assumption of rationality.

We can easily check that when we take the function D equal to

$$D(x,y) = \frac{y^\gamma}{x},$$

the ratio of its derivatives

$$\mathrm{D}_x(x,y) = y^\gamma\left(-\frac{1}{x^2}\right),$$
$$\mathrm{D}_y(x,y) = \frac{1}{x}\gamma y^{\gamma-1}$$

equals

$$\frac{D_y}{D_x} = \frac{\frac{1}{x}\gamma y^{\gamma-1}}{-y^\gamma \frac{1}{x^2}} = -\frac{\gamma x}{y}.$$

Moreover, if we choose D as

$$D(x,y) = U\left(\frac{y^\gamma}{x}\right),$$

for some differentiable function U, the ratio of derivatives will be the same

$$\frac{D_y}{D_x} = \frac{U'\left(\frac{y^\gamma}{x}\right)\frac{1}{x}\gamma y^{\gamma-1}}{U'\left(\frac{y^\gamma}{x}\right)\left(-y^\gamma\frac{1}{x^2}\right)} = -\frac{\gamma x}{y},$$

that means the general solution to the partial differential equation (24) is

$$D(x,y) = U\left(\frac{y^\gamma}{x}\right) \tag{25}$$

for an arbitrary differentiable function $U(.)$. The diffusion coefficient can be rewritten as follows

$$\frac{1}{2}\eta_1^2 y^2 \left[\frac{D_y(x,y)}{D_x(x,y)}\right]^2 = \frac{1}{2}\eta_1^2 y^2 \left[\frac{U'\left(\frac{y^\gamma}{x}\right)\frac{1}{x}\gamma y^{\gamma-1}}{U'\left(\frac{y^\gamma}{x}\right)\left(-y^\gamma \frac{1}{x^2}\right)}\right]^2.$$

We can use the modified market clearing equation, which we get from inserting (4) and (25) into (5)

$$U\left(\frac{Y_t^\gamma}{X_t}\right) = 1 - \rho\Phi(X_t, t)$$

to eliminate y. Let us introduce a function $Z(.)$, which is the inverse function of $U(.)$ and its existence is guaranteed thanks to the strict monotonicity of U. Substituting

$$\frac{y^\gamma}{x} = Z(1-\rho\Phi)$$

and using

$$\eta_1\gamma = \sigma$$

the diffusion coefficient becomes

$$\frac{1}{2}\eta_1^2 y^2 \left[\frac{U'\left(y^\gamma x\right)\frac{1}{x}\gamma y^{\gamma-1}}{U'\left(\frac{y^\gamma}{x}\right)\left(-y^\gamma \frac{1}{x^2}\right)}\right]^2 = \frac{1}{2}\sigma^2 x^2 \left[\frac{Z(1-\rho\Phi)U'(Z(1-\rho\Phi))}{Z(1-\rho\Phi)U'(Z(1-\rho\Phi)) - \rho x \Phi_x}\right]^2.$$

As we mentioned above, some conditions of rationality, namely $D_x < 0$ and $D_y > 0$ for $x, y > 0$, must hold. As one can see

$$D_x = U'\left(\frac{y^\gamma}{x}\right)\left(-y^\gamma \frac{1}{x^2}\right) < 0,$$

and

$$D_y = U'\left(\frac{y^\gamma}{x}\right)\frac{1}{x}\gamma y^{\gamma-1} > 0$$

hold if and only if $U'(.) > 0$, therefore, the function U is increasing. The paper [7] features the derivation for an arbitrary increasing function, though in our thesis we will study a model which arises from taking U linear, $U(z) = \beta z$, $\beta > 0$.

Now that we have the linear demand function U, we get following relations

$$U(\frac{y^\gamma}{x}) = \beta\frac{y^\gamma}{x} = 1 - \rho\Phi$$
$$\frac{y^\gamma}{x} = Z(1-\rho\Phi) = \frac{1}{\beta}(1-\rho\Phi),$$

and the derivative of U equals

$$U'(z) = \beta.$$

The diffusion coefficient takes the form

$$\frac{1}{2}\sigma^2 x^2 \left[\frac{Z(1-\rho\Phi)U'(Z(1-\rho\Phi))}{Z(1-\rho\Phi)U'(Z(1-\rho\Phi)) - \rho x\Phi_x}\right]^2 = \frac{1}{2}\sigma^2 x^2 \left[\frac{1-\rho\Phi}{1-\rho\Phi-\rho x\Phi_x}\right]^2.$$

One can notice that Φ can be replaced by the derivative of the option price function (17), so that the coefficient is expressed in terms of ρ and V

$$\frac{1}{2}\left[\frac{1-\rho\frac{\partial V}{\partial x}}{1-\rho\frac{\partial V}{\partial x} - \rho x\frac{\partial^2 V}{\partial x^2}}\right]^2 \sigma^2 x^2,$$

and then the pricing equation takes the form

$$\frac{\partial V}{\partial t} + \frac{1}{2}\left[\frac{1-\rho\frac{\partial V}{\partial x}}{1-\rho\frac{\partial V}{\partial x} - \rho x\frac{\partial^2 V}{\partial x^2}}\right]^2 \sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + r\left(x\frac{\partial V}{\partial x} - V\right) = 0, \qquad (26)$$

which does not depend on β, is consistent with and reduces to the Black-Scholes equation (2) in the absence of program trading.

As we already mentioned before, this equation is not dependent on the parameters of the income process Y_t, but only on the function U and σ, the observable market volatility of the underlying asset.

If we set $\rho = 0$ in (26) we immediately obtain (2). Since ρ is a fraction of the asset market held by program traders, it is likely to be a small number in practice. Thus as long as $\Phi = V_x$ and V_{xx} remain bounded by a reasonable constant, the expression

$$\frac{1-\rho\frac{\partial V}{\partial x}}{1-\rho\frac{\partial V}{\partial x} - \rho x\frac{\partial^2 V}{\partial x^2}},$$

will approximately be equal to 1, which means that the whole diffusion coefficient will not differ much from the one in the Black-Scholes partial differential equation, therefore, we can study (26) as a small perturbation of (2).

2.2.5 European Options Pricing

We focus on the problem of the feedback caused by insuring against the risk from writing one European call option, which gives the owner the right, but not the obligation to buy the underlying asset at the strike price K at the expiration time T. The terminal payoff functions is

$$h(x) = (x - K)^+. \tag{27}$$

For this kind of a derivative security was originally derived a pricing formula by Black and Scholes, known as the Black-Scholes formula

$$V^{EC}(x,t) = xN(d_1) - Ke^{-r(T-t)}N(d_2), \tag{28}$$

where

$$\begin{aligned} d_1 &= \frac{ln\left(\frac{x}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned} \tag{29}$$

and

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{s^2}{2}} ds. \tag{30}$$

The terminal condition (27) in the model (26) causes that the denominator of the diffusion coefficient

$$\frac{1}{2}\left[\frac{1 - \rho\frac{\partial V}{\partial x}}{1 - \rho\frac{\partial V}{\partial x} - \rho x\frac{\partial^2 V}{\partial x^2}}\right]^2 \sigma^2 x^2$$

might become equal to zero. The reason is that for the second derivative with respect to x holds

$$h''(x) = \delta(x - K),$$

where δ is the Dirac delta function

$$\delta(z) = \begin{cases} \infty & \text{for} \quad z = 0, \\ 0 & \text{for} \quad z \in \mathbb{R} \setminus \{0\} \end{cases}$$

and therefore, at $t = T$, no matter how small ρ is, the denominator is negative in some neighborhood of K. Since we expect the terminal data to smooth as we run the equation backwards in time from T, the denominator will go through zero with V_x and V_{xx} becoming smaller, which causes the equation to become meaningless.

To avoid this situation which arises only due to the breaking point in the option's payoff function, we introduce a second consistency condition with the Black-Scholes model, as the maturity approaches. That means, we will ignore the feedback effect as $t \to T$, because of the oversensitivity of the hedging strategies to price changes around $x = K$, which is reflected by the fact that $V_x^{EC}(x,t) \sim \mathcal{H}(x-K)$ and $V_{xx}^{EC}(x,t) \sim \delta(x-K)$, as $t \to T$, where $\mathcal{H}(z)$ is the Heaviside function

$$\mathcal{H}(z) = \begin{cases} 1 & \text{pre} \quad z \geq 0, \\ 0 & \text{pre} \quad z < 0, z \in \mathbb{R} \end{cases}$$

and $\delta(z)$ is its derivative with respect to z, the Dirac delta function. In practice, hectic program trading close to expiration is dampened by the transaction costs, which can be considered a natural smoothing.

Technically, it means that in some small interval $T - \epsilon \leq t \leq T$ we set the feedback price V equal to Black-Scholes price V^{EC}. It can be shown that ϵ can be calculated and expressed in terms of ρ and σ to obtain sufficient smoothing of the data for the right setting of the nonlinear partial differential equation. Thus specified smoothing parameter ϵ then completes our feedback incorporating pricing model.

2.2.6 The Smoothing Parameter

Derivation of an equation, specifying the smoothing parameter is only briefly outlined in the paper [7], but we present the full course of derivation. We define the smoothing parameter ϵ as the minimum value of $\epsilon > 0$ such that

$$\min_{x>0} F_{BS}(x, T-\epsilon) = 0,$$

where $F_{BS}(x,t)$ is the denominator of the diffusion coeficient in equation (26), for Black-Scholes price

$$F_{BS}(x,t) = 1 - \rho \frac{\partial V^{EC}}{\partial x}(x,t) - \rho x \frac{\partial^2 V^{EC}}{\partial x^2}(x,t).$$

The problem we are solving is the following

$$\min_{x>0} \left[1 - \rho \frac{\partial V^{EC}}{\partial x}(x, T-\epsilon) - \rho x \frac{\partial^2 V^{EC}}{\partial x^2}(x, T-\epsilon) \right] = 0. \tag{31}$$

The two partial derivatives can be substituted by the greeks Δ^{EC} and Γ^{EC}

$$\min_{x>0} \left[1 - \rho \Delta^{EC}(x, T-\epsilon) - \rho x \Gamma^{EC}(x, T-\epsilon) \right] = 0. \tag{32}$$

Formulas for their calculation are listed in book [5] (chapter 3)

$$\begin{aligned} \Delta^{EC} &= N(d_1) \\ \Gamma^{EC} &= \frac{e^{-\frac{1}{2}d_1^2}}{\sigma x \sqrt{2\pi(T-t)}}. \end{aligned} \tag{33}$$

When we insert (33) in the equation (32), we obtain

$$\min_{x>0} \left[1 - \rho N(d_1) - \rho \frac{e^{-\frac{1}{2}d_1^2}}{\sigma \sqrt{2\pi\epsilon}} \right] = 0$$

After replacing of the cummulative distribution function of normal distribution and d_1 by (30) and (29) we get

$$\min_{x>0} \left[1 - \rho \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{ln\frac{x}{K} + \left(r + \frac{\sigma^2}{2}\right)\epsilon}{\sigma\sqrt{\epsilon}}} e^{-\frac{s^2}{2}} ds - \rho \frac{e^{-\frac{1}{2}\left(\frac{ln\frac{x}{K} + \left(r + \frac{\sigma^2}{2}\right)\epsilon}{\sigma\sqrt{\epsilon}}\right)^2}}{\sigma\sqrt{2\pi\epsilon}} \right] = 0 \tag{34}$$

Now we need to find the point of minimum, that is set the first derivative with respect to x equal to 0 and express x from equation (35).

$$\left[1 - \rho \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{ln\frac{x}{K} + \left(r + \frac{\sigma^2}{2}\right)\epsilon}{\sigma\sqrt{\epsilon}}} e^{-\frac{s^2}{2}} ds - \rho \frac{e^{-\frac{1}{2}\left(\frac{ln\frac{x}{K} + \left(r + \frac{\sigma^2}{2}\right)\epsilon}{\sigma\sqrt{\epsilon}}\right)^2}}{\sigma\sqrt{2\pi\epsilon}} \right]' = 0 \tag{35}$$

The integral in the equation (35) is differentiated using the following theorem (for reference, check [4], page 16).

Theorem 2.1. *Let $f : [c,d] \to \mathbb{R}$ be a continuous function, φ, ψ are differentiable on interval I, and let $\varphi(I) \subset [c,d]$,$\psi(I) \subset [c,d]$. Then function $G : I \to \mathbb{R}$ defined as*

$$G(x) = \int\limits_{\varphi(x)}^{\psi(x)} f(t)dt$$

is differentiable on I and it holds

$$G'(x) = f(\psi(x))\psi'(x) - f(\varphi(x))\varphi'(x).$$

The equation (35) is differentiated and further simplified as folows.

$$-\rho\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{ln\frac{x}{K}+\left(r+\frac{\sigma^2}{2}\right)\epsilon}{\sigma\sqrt{\epsilon}}\right)^2}\frac{1}{\sigma\sqrt{\epsilon}}\frac{K}{x}-\rho\frac{1}{\sigma\sqrt{2\pi\epsilon}}e^{-\frac{1}{2}\left(\frac{ln\frac{x}{K}+\left(r+\frac{\sigma^2}{2}\right)\epsilon}{\sigma\sqrt{\epsilon}}\right)^2}(-1)\frac{ln\frac{x}{K}+\left(r+\frac{\sigma^2}{2}\right)\epsilon}{\sigma\sqrt{\epsilon}}\frac{1}{\sigma\sqrt{\epsilon}}\frac{K}{x}=$$

$$=-\rho\frac{1}{\sigma\sqrt{2\pi\epsilon}}\frac{K}{x}e^{-\frac{1}{2}\left(\frac{ln\frac{x}{K}+\left(r+\frac{\sigma^2}{2}\right)\epsilon}{\sigma\sqrt{\epsilon}}\right)^2}+\rho\frac{1}{\sigma\sqrt{2\pi\epsilon}}\frac{K}{x}\frac{1}{\sigma^2\epsilon}\left(ln\frac{x}{K}+\left(r+\frac{\sigma^2}{2}\right)\epsilon\right)e^{-\frac{1}{2}\left(\frac{ln\frac{x}{K}+\left(r+\frac{\sigma^2}{2}\right)\epsilon}{\sigma\sqrt{\epsilon}}\right)^2}=$$

$$=\rho\frac{K}{x\sigma\sqrt{2\pi\epsilon}}e^{-\frac{1}{2}\left(\frac{ln\frac{x}{K}+\left(r+\frac{\sigma^2}{2}\right)\epsilon}{\sigma\sqrt{\epsilon}}\right)^2}\left(-1+\frac{1}{\sigma^2\epsilon}\left(ln\frac{x}{K}+\left(r+\frac{\sigma^2}{2}\right)\epsilon\right)\right)=0,$$

This is equivalent to

$$\frac{ln\frac{x}{K}+\left(r+\frac{\sigma^2}{2}\right)\epsilon}{\sigma^2\epsilon}-1=0$$

$$ln\frac{x}{K}+\left(r+\frac{\sigma^2}{2}\right)\epsilon=\sigma^2\epsilon$$

$$ln\frac{x}{K}=\epsilon\left(\frac{\sigma^2}{2}-r\right)$$

The point of minimum is

$$x=Ke^{\epsilon\left(\frac{\sigma^2}{2}-r\right)}$$

and now we can insert it in the equation (34), which gives us

$$1-\rho\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\frac{ln\frac{Ke^{\epsilon\left(\frac{\sigma^2}{2}-r\right)}}{K}+\left(r+\frac{\sigma^2}{2}\right)\epsilon}{\sigma\sqrt{\epsilon}}}e^{-\frac{s^2}{2}}ds-\rho\frac{e^{-\frac{1}{2}\left(\frac{ln\frac{Ke^{\epsilon\left(\frac{\sigma^2}{2}-r\right)}}{K}+\left(r+\frac{\sigma^2}{2}\right)\epsilon}{\sigma\sqrt{\epsilon}}\right)^2}}{\sigma\sqrt{2\pi\epsilon}}=0$$

$$1 - \rho \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\sigma\sqrt{\epsilon}} e^{-\frac{s^2}{2}} ds - \rho \frac{e^{-\frac{1}{2}\epsilon\sigma^2}}{\sigma\sqrt{2\pi\epsilon}} = 0$$

Thus we derived the equation satisfied by the smoothing parameter ϵ

$$\frac{1}{\rho} = N(\sigma\sqrt{\epsilon}) + \rho \frac{e^{-\frac{1}{2}\epsilon\sigma^2}}{\sigma\sqrt{2\pi\epsilon}}.$$

The solution of this equation is depicted in Figure 1

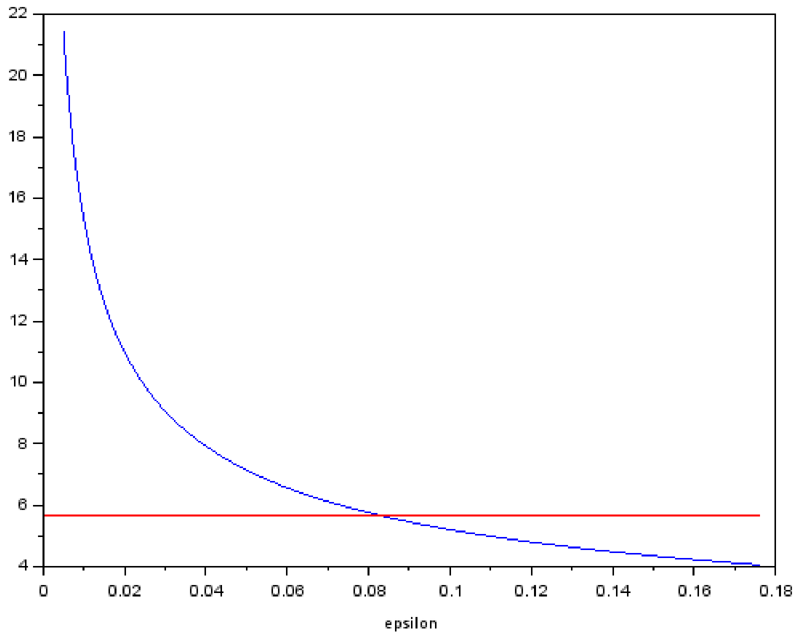

Figure 1: Blue line represents the function $N(\sigma\sqrt{\epsilon}) + \frac{e^{-\frac{1}{2}\sigma^2\epsilon}}{\sigma\sqrt{2\pi\epsilon}}$ and the red line is the constant function $\frac{1}{\rho}$, for parameter values $\sigma = 0,26903$, $\rho = 0,17630$. The point of their intersection is the ϵ we are looking for.

2.2.7 The Full Model

When we summarize everything derived in this chapter, we get the final feedback incorporating pricing model for a European call option

$$\frac{\partial V}{\partial t} + \frac{1}{2}\left[\frac{1-\rho\frac{\partial V}{\partial x}}{1-\rho\frac{\partial V}{\partial x}-\rho\frac{\partial^2 V}{\partial x^2}}\right]^2 \sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + r\left(x\frac{\partial V}{\partial x} - V\right) = 0, \quad t < T - \epsilon \tag{36}$$

$$V(x, T-\epsilon) = V^{EC}(x, T-\epsilon)$$

$$V(0,t) = 0$$

$$\lim_{x\to\infty} |V(x,t) - (x - Ke^{r(T-t)})| = 0,$$

where $V(x,t) = V^{EC}(x,t)$ for $T - \epsilon \le t \le T$.

3 Calculation and Programming

This chapter is dedicated to describing the way the results are calculated and then how the algorithm is programmed in the software Scilab. The results are valid as ρ tends to zero, so that it can be considered that (36) is a small perturbation to the classical Black-Scholes equation (2). We are looking for the price of the option $V(x,t)$ when the underlying stock price $x > 0$ at time $t < T$.

3.1 Regular Perturbation Series Solution

Firstly, we explain how the feedback price is computed. Once again, the idea of calculation and derivation of used formulas, are taken over from the paper [7]. For a European option we calculate the first-order correction to the Black-Scholes pricing formula under the feedback effect when $\rho << 1$. We construct a regular perturbation series

$$V(x,t) = V^{EC}(x,t) + \rho\bar{V}(x,t) + O(\rho^2) \tag{37}$$

and we label the left-hand side of the Black-Scholes partial differential equation

$$L_{BS}V := V_t + \frac{1}{2}\sigma^2 x^2 V_{xx} + r(xV_x - V). \tag{38}$$

If we insert (37) into (26), considering a small ρ, we will obtain for $\bar{V}$ the expression

$$L_{BS}\bar{V} = -\sigma^2 x^3 \left[V_{xx}^{EC}\right]^2. \tag{39}$$

Once again we employ the formula (33) for Γ^{EC} from the lecture notes [?] and (39) becomes the problem for the first-order correction $\bar{V}$

$$\begin{gathered}
\bar{V}_t + \tfrac{1}{2}\sigma^2 x^2 \bar{V}_{xx} + r(x\bar{V}_x - \bar{V}) = -\tfrac{xe^{-d_1^2}}{2\pi(T-t)}, \qquad t < T - \epsilon \\
\bar{V}(x, T-\epsilon) = 0 \\
\bar{V}(0,t) = 0 \\
\lim_{x\to\infty} |\bar{V}(x,t)| = 0.
\end{gathered}$$

Now we do the transformation of the problem for $\bar{V}$ to an inhomogenous heat equation

$$\begin{aligned}
x &= Ke^y \\
t &= T - \frac{2\tau}{\sigma^2}
\end{aligned} \tag{40}$$

$$\bar{V}(x,t) = Ke^{-\frac{1}{2}(k-1)y-\frac{1}{4}(k+1)^2\tau}u(y,\tau), \tag{41}$$

where $k = \frac{2r}{\sigma^2}$ and we obtain the following problem for $u(y,\tau)$ for $-\infty < y < \infty$ and $\tau > \frac{\epsilon\sigma^2}{2}$

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial y^2} = \frac{1}{2\pi\tau}e^{-\frac{y^2}{2\tau}-\frac{1}{4}(k+1)^2\tau-\frac{y}{2}(k+1)} \tag{42}$$

$$u(y, \tfrac{\epsilon}{2}\sigma^2) = 0$$

$$e^{-\frac{1}{2}(k-1)y}u(y,\tau) \to 0 \text{ as } y \to -\infty$$

and u is bounded for $y \to \infty$.

From the theory of partial differential equations, we know that if we denote the right-hand side of (42) as follows

$$f(y,\tau) = \frac{1}{2\pi\tau}e^{-\frac{y^2}{2\tau}-\frac{1}{4}(k+1)^2\tau-\frac{y}{2}(k+1)}, \tag{43}$$

the solution to the inhomogenous heat equation is

$$u(y,\tau) = \int_{\frac{\epsilon}{2}\sigma^2}^{\tau}\int_{-\infty}^{\infty} B(\xi,s;y,\tau)f(\xi,s)d\xi ds,$$

where

$$B(\xi,s;y,\tau) = \frac{1}{\sqrt{4\pi(\tau-s)}}e^{-\frac{(\xi-y)^2}{4(\tau-s)}}.$$

When we put the last two expressions together with (43), we obtain

$$u(y,\tau) = \int_{\frac{\epsilon}{2}\sigma^2}^{\tau}\int_{-\infty}^{\infty} \frac{1}{2\pi s\sqrt{4\pi(\tau-s)}}e^{-\frac{(\xi-y)^2}{4(\tau-s)}-\frac{\xi^2}{2s}-\frac{1}{4}(k+1)^2 s-\frac{\xi}{2}(k+1)}d\xi ds,$$

which can be rewritten into

$$u(y,\tau) = \int_{\frac{\epsilon}{2}\sigma^2}^{\tau} \frac{e^{-\frac{1}{4}(k+1)^2 s} - \frac{y^2}{4(\tau-s)}}{2\pi s\sqrt{4\pi(\tau-s)}} \int_{-\infty}^{\infty} e^{-\alpha\xi^2-\beta\xi}d\xi ds, \tag{44}$$

where

$$\alpha = \frac{1}{2s} + \frac{1}{4(\tau-s)} \tag{45}$$

and

$$\beta = -\frac{y}{2(\tau-s)} + \frac{1}{2}(k+1). \tag{46}$$

The inner integral in (44) can be evaluated as

$$\int_{-\infty}^{\infty} e^{-\alpha\xi^2-\beta\xi}d\xi = \sqrt{\frac{\pi}{\alpha}}e^{\frac{\beta^2}{4\alpha}}, \tag{47}$$

where we can see that if α and β satisfy (45) and (46), it holds

$$\tfrac{\beta^2}{4\alpha} = \frac{s[y-(k+1)(\tau-s)]^2}{4(2\tau-s)(\tau-s)}.$$

After inserting (45), (46), and (47) into (44), the solution takes the form

$$u(y,\tau) = \frac{1}{2\pi}\int_{\frac{\epsilon}{2}\sigma^2}^{\tau} \frac{e^{-\frac{1}{4}(k+1)^2 s-\frac{y^2}{4(\tau-s)}+\frac{s[y-(k+1)(\tau-s)]^2}{4(2\tau-s)(\tau-s)}}}{\sqrt{2\tau s-s^2}}ds.$$

In order to eliminate the singularity for $s=0$, when the denominator equals zero and the integrand becomes large close to the lower limit, which could cause some problems to the quadrature methods, we make the following transformation

$$v = \sqrt{\frac{s}{2\tau}}$$

and obtain the solution in the form

$$u(y,\tau) = \int_{\sigma\sqrt{\frac{\epsilon}{4\tau}}}^{\sqrt{\frac{1}{2}}} M(y,\tau,v)dv, \tag{48}$$

where

$$M(y,\tau,v) = \frac{1}{\pi\sqrt{1-v^2}}e^{-\frac{1}{2}(k+1)^2\tau v^2-\frac{y^2}{4\tau(1-2v^2)}+\frac{v^2[y-\tau(k+1)(1-2v^2)]^2}{4\tau(1-v^2)(1-2v^2)}}. \tag{49}$$

Clearly, $M>0$ in the interval of integration, therefore, the first-order correction $\bar{V}$ given by (41) and (48) is positive in $x>0$, $t<T$. The perturbation of the classical Black-Scholes model consequently has the effect of increasing the no-arbitrage price of the European option, due to the presence of the program traders. As the Black-Scholes formula (28) is increasing in the parameter σ, it confirms the initial guess that program traders cause the market volatility to increase. Moreover, from the construction of the perturbation series (37) we see that it is linearly increasing in the parameter ρ.

3.2 Program Code in Scilab

Now we describe step by step, how the above described algorithm is programmed in Scilab.

Firstly, we need to define the function $M(y,\tau,v)$ according to (49), which will subsequently be integrated to obtain the solution to the heat equation $u(y,\tau)$.

Next we define some partial functions, which will be used. The cumulative distribution function for the normal distribution *normcdf*(x), the function for the price of a European call option *Call*(S,K,r,σ,τ), and the function *Epsilon*(ρ,σ), which calculates the value of ϵ, the smoothing parameter, where the other parameters, ρ and σ, are input arguments. The computation of ϵ consists in finding the zero point of the expression

$$N(\sigma\sqrt{\epsilon}) + \frac{e^{-\frac{1}{2}\sigma^2\epsilon}}{\sigma\sqrt{2\pi\epsilon}} - \frac{1}{\rho},$$

for which the in Scilab incorporated function $fsolve(x_0, function)$ is used, where x_0 is the initial value of function argument.

These partial functions are then used as building blocks of the function calculating the Black-Scholes price under feedback, $BSUnderFeedback(x,T,K(i),r,\sigma,\rho)$. Its input arguments are:

x-the asset price,

T-expiration time,

$K(i)$-ith component of the vector of strike prices K,

r-constant spot interest rate,

σ-asset volatility, and

ρ-ratio of the volume of options being hedged to the total supply of the asset.

After the initial transformations of variables (40), arising from the transformation of the pricing problem to the heat equation, it calculates the classical Black-Scholes price using the previously defined function *Call*(), then it computes the smoothing parameter using the function *Epsilon*() and determines the boundaries a and b for the following integral, which needs to be calculated. As next it computes the integral with respect to v, of the function $M(y, \tau, v)$ on the interval $[a, b]$ to obtain the function $u(y, \tau)$ according to (48), transforms it to the first-order correction $\bar{V}$ following (41) and finally, calculates the feedback price according to the relation (37).

Thus, we already have a function, which returns the feedback price for any maturity, strike price or asset price we choose, however, we still do not know how to select the two last parameters, which are σ and ρ. It would be reasonable, for the new price under feedback effect to be closest possible to actual trading prices on the market. In order to optimize the price, we define one more function $distance(\sigma, \rho)$, which calculates the distance between the real stock-market price and the price under feedback, which then will be minimized with respect to its two parameters σ and ρ.

We get the vector of the real option prices V_{real} from the website [9]. The function $distance()$ then returns the sum of squared differences between the stock-market prices and correspondent calculated feedback prices. We would now like to find the minimum distance for some optimal σ and ρ.

There is a built-in function in Scilab, which finds the minimum of a function with respect to a chosen variable, however, we encountered some numerical problems for the value range of our parameters. For that reason, we use a different approach. We take the unit vectors e_1 and e_2 as a set of directions. First, we find the minimum in direction of the first vector, e_1 (σ-direction). From there we move along the second direction, e_2 (ρ-direction), and look for its minimum, then again the first direction, and so on, cycling as many times as necessary, until the function stops decreasing. Thus, starting with an initial guess, always optimizing one parameter at a time, we obtain the optimal parameter values and the minimum distance between the real and feedback price.

4 Results and Their Evaluation

For the purposes of this thesis we chose options for two stocks from different industry branches.

4.1 Results For Amazon

The first are the call options for Amazon.com Inc with expiration date April 15, 2016. We took 17 most traded options (all of those with trading volumes above 100) with this expiration date, but different strike prices. The trading option prices, strike prices and asset price in USD, maturity, and constant spot interest rate from [1] are

V_{real}=(24.08; 21.12; 18.70; 15.90; 13.85; 11.32; 9.80; 8.31; 6.90; 5.60; 4.69; 3.83; 3.01; 2.39; 1.87; 1.20; 0.49)
K=(540; 545; 550; 555; 560; 565; 570; 575; 580; 585; 590; 595; 600; 605; 610; 620; 640)
x=559.44
T=18/250
r=0.0025.

For this data, we now want to compare the two models, the original Black-Scholes and our, derived in Section 2, upgraded Black-Scholes model incorporating feedback effect. We want to determine, whether consideration of program traders that leads to the new volatility, gives a possibility of improvement in accuracy of option price assesment and whether this improvement is worth its cost, which is the consequential non-linearity of the model, and therefore, related calculations are made slightly more difficult.

What we need to do, to be able to make a sensible comparison, is to estimate, for both models, optimal parameters, namely, the volatility σ and the ratio of the volume of derivatives being hedged to the total supply of the asset, ρ. These two parameters are optimized, such that the new-calculated option prices copy the market price V_{real}

as accurately as possible, that is, for both mentioned models, minimize the distance between the actual market price and the calculated one. To attain this, we created the function $distance(\sigma, \rho)$, as decribed in Section 3.2, through which we can monitor, how the distance changes with these two parameters.

At first, we would like a general idea about what this function looks like. If it is simple with only one point of local minimum, or if there are more of them. Having this preliminary picture will help us choose a way of optimization, which leads to the right result.

One way to obtain a first outline, is to select a rectangle area $(\sigma_1, \sigma_n) \times (\rho_1, \rho_m)$, divide it into an equidistant grid $(\sigma_i, \rho_j), i = 1, ..., n,\ j = 1, ..., m$, and calculate the function values $distance(\sigma_i, \rho_j)$, for each i, j. In Table 1, there are listed values of the function for the area

$$(\sigma_1, \sigma_n) \times (\rho_1, \rho_m) = (0.175, 0.215) \times (0.02, 0.14),$$

which we thought could be a good initial guess for the optimal values of parameters σ and ρ.

Table 1: Values for the function $distance(\sigma, \rho)$ on an equidistant grid, for the area $(0.175, 0.215) \times (0.02, 0.14)$. Highlighted, there are several possible points of local minima.

$\rho \backslash \sigma$	0.175	0.185	0.195	0.205	0.215
0.02	29.33	15.97	**10.24**	12.44	22.81
0.035	15.22	14.27	22.90	41.26	69.46
0.05	16.65	24.33	43.98	75.52	118.91
0.065	17.96	28.30	53.75	93.92	148.51
0.08	18.11	21.44	44.37	85.97	145.49
0.095	34.85	15.53	22.64	54.27	108.87
0.11	112.66	44.89	**14.75**	18.35	52.64
0.125	339.23	178.14	75.04	21.41	**11.09**
0.14	888.58	545.17	304.19	143.32	48.42

One can easily notice that the graph of the function is rather complex. In σ-direction, its values mostly decrease at first and start increasing later. In ρ-direction, on the other hand, our monitored function $distance(\sigma, \rho)$ seems to have more than one different local minima in this area selected for observation, because it can be divided into several sections, where the monotonicity of the function changes from increasing to decreasing and back again. For instance, if we take a closer look at layer $\sigma = 0.185$, the function values decrease at first, then they increase on the interval from $\rho = 0.035$ to $\rho = 0.065$, decrease for a bit, and then start increasing again from the point $\rho = 0.095$ on.

Another, more visual and straight-forward, way to observe behaviour of the examined 3D function, is to display its partial graph. By partial graph we now mean the graph of this function, displayed in a reduced dimension, that is in this case in 2D. This reduction is achieved by fixating one of the function's variables, in a certain value, and drawing the graph of a function of only one variable.

In Figure 2, we can see the situation, if we take the variable ρ fixated at $\rho = 0.02$ and display function values for σ from the interval $(0.175, 0.215)$.

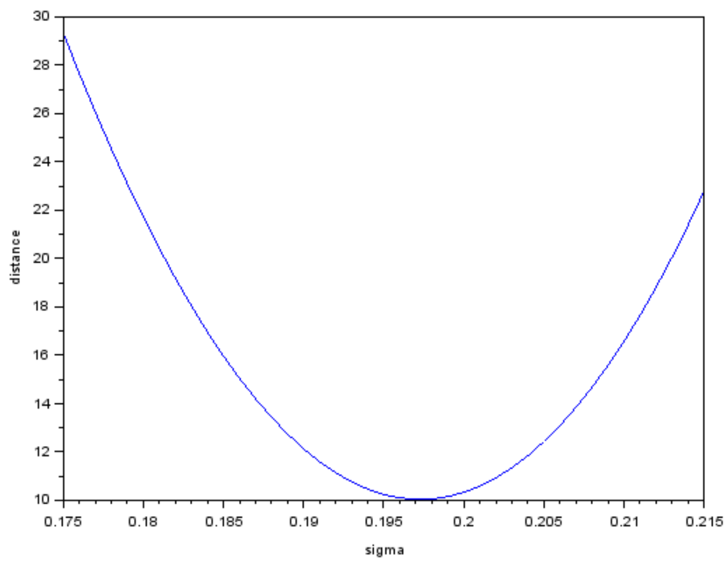

Figure 2: Graph of the function $distance(\sigma, \rho)$ for $\rho = 0.02$ and σ from the interval $(0.175, 0.215)$.

We can see that our assumption about the behaviour of our function in σ-direction was correct. More precisely, we now know that on this particular interval, it is a convex function in the variable σ, which reaches its minimum somewhere in the vicinity of the point $\sigma = 0.197$.

To see if this assumption is true not only for this layer, but also for different values of parameter ρ, we drew some more graphs. In Figure 3, there are sketched three curves for three different values of ρ.

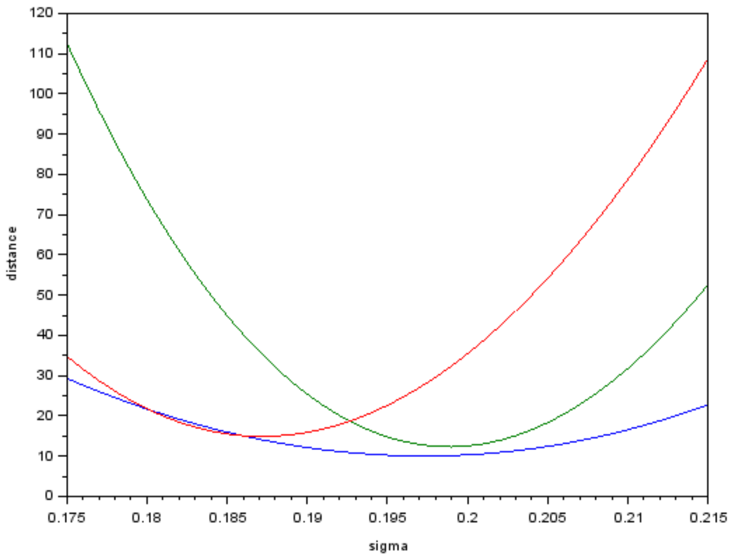

Figure 3: Graphs of the function $distance(\sigma, \rho)$ for $\rho = 0.02$ (blue line), $\rho = 0.11$ (green line), $\rho = 0.095$ (red line), and σ from the interval $(0.175, 0.215)$.

The Figure 4 shows us the second situation, when σ is taken as fixated at the value $\sigma = 0.2$ again, and ρ from the interval $(0.019, 0.145)$ is displayed.

What we can deduce from this displayed part of the distance function is, that it is neither convex, nor any other easily explorable type of function, in the variable ρ, but has two local minima. First of them close to $\rho = 0.02$ and another around the point $\rho = 0.11$. Like in the previous case when parameter ρ was fixed, we displayed also few other chosen layers of the distance function.

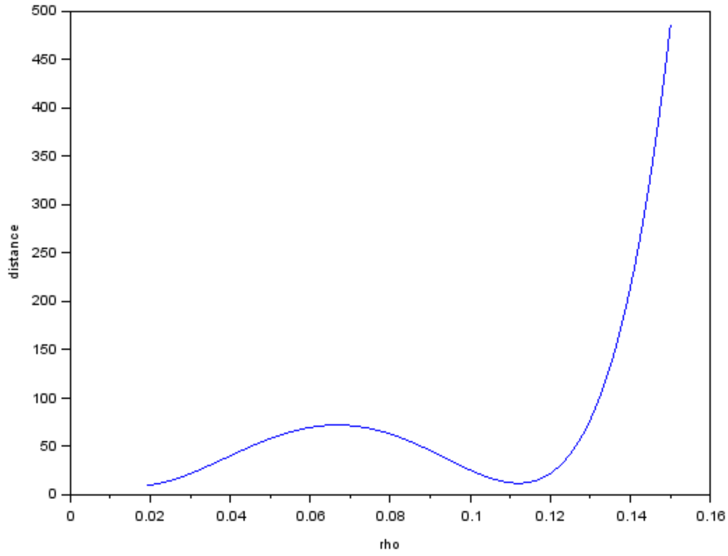

Figure 4: Graph of the function $distance(\sigma, \rho)$ for $\sigma = 0.2$ and $\rho \in (0.019, 0.145)$.

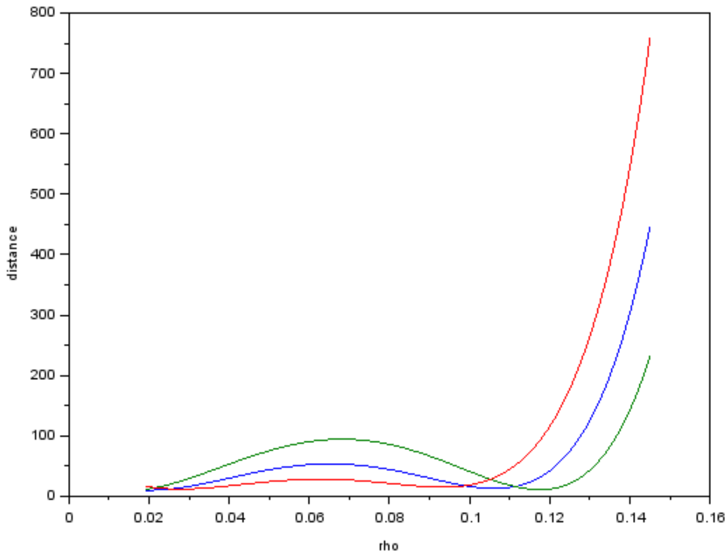

Figure 5: Graphs of the function $distance(\sigma, \rho)$ for $\sigma = 0.195$ (blue line), $\sigma = 0.205$ (green line), $\sigma = 0.185$ (red line), and ρ from the interval $(0.019, 0.145)$.

All of the curves in Figure 5 seem to point to the same mentioned characteristic behaviour. This means that again we were right in our anticipation about the function having more than one local minima.

Now that we have a brief idea about what the optimized function looks like, we can proceed to the calculation. Let us begin with the easier part, that is, solving the problem for the Black-Scholes model. This means to compute the minimal distance between the Black-Scholes price and the actual trading price on the market V_{real}, for a particular optimal σ. We are taking the second parameter ρ equal to zero, since the original model does not take the program traders into consideration. Therefore, this is optimization through only one variable, the volatility σ.

We need to slightly modify our formula for finding a minimal distance, programmed as described in Section 3.2. Instead of summing squared differences between market price and feedback price, the latter is substituted by a call option price computed using the Black-Scholes formula.

After this change, the formula has the following form

$$distance(\sigma) = \sum_{i=1}^{n} (V_{real}(i) - Call(x, K(i), r, \sigma, T))^2 \rightarrow min, \tag{50}$$

where

n is the number of options included in the calculation,

$V_{real}(i)$ is the ith component of the trading option prices vector,

$K(i)$ is the ith component of the strike prices vector, and

$Call()$ is the computed price of the ith call option for a particular set of input parameters.

After minimization through all $\sigma \in \mathbb{R}^{++}$, the obtained resulting cummulated distance between the prices is

$$distance(0.234, 0) = 7.3894316, \tag{51}$$

for the optimal volatility

$$\sigma = 0.234. \tag{52}$$

Now that we have the optimal distance of Black-Scholes price from the market price, we shall move on to the feedback price. There we have two options, since the function

$distance(\sigma, \rho)$ has two minima in variable ρ.

Since we expect ρ to be very small, we assume that the first minimum, close to $\rho = 0.02$, is the one we are looking for. In order to find it, we use the method of sequential optimization, described in Section 3, that is, find the minimal value when ρ is fixated, then find the minimum with σ fixated, and so on in cycles, till the values stop decreasing altogether.

The formula for the minimized distance now remains in the same form, as we originally programmed it (description in Section 3.2), summing the squared differences between market and feedback price, as follows

$$distance(\sigma, \rho) = \sum_{i=1}^{n} (V_{real}(i) - BSUnderFeedback(x, T, K(i), r, \sigma, \rho))^2 \rightarrow min, \quad (53)$$

where
n is the number of options included in the calculation,
V_{real} is the ith component of the trading option prices vector,
$K(i)$ is the ith component of the strike prices vector, and
$BSUnderFeedback()$ is the computed feedback price of the ith option for a particular set of input parameters.

Compared to the Black-Scholes formula for call option price, which we used in the previous part of our calculations, where we computed the shortest distance for the original Black-Scholes model, the function $BSUnderFeedback()$ is indirectly dependent also on the smoothing parameter ϵ. This follows from the fact that it calculates the option price only for a time period excluding a very short time interval before maturity T, determined by ϵ. In Section 3.2, we introduced the function $Epsilon()$, which calculates the value of ϵ, and one of its input arguments is a starting value, ϵ_0, which yet remains to be asssigned a value. Throughout Section 3 Asymptotic Results for small ρ, in the article [7], $\epsilon = 0.003$ is used, so it could be a good candidate for the starting point ϵ_0. However, we decided to try and find a time period before maturity $T - \epsilon$, during which the feedback effect is ignored, which would be as short, as possible, therefore, we chose to set ϵ_0 to a really small value $\epsilon_0 = 0.000001$, and see where it goes.

Thus, we applied the method of sequential optimization, ϵ_0 set to 0.000001, aiming to find the first local minimum. Starting with $\rho = 0.035$, returning $\sigma = 0.181$, and so on, σ increased and ρ decreased, till we reached the point

$$(\sigma, \rho) = (0.233, 0.00039).$$

From this point, calculations couldn't be continued due to numerical problems. The function computing the smoothing parameter ϵ couldn't return any result for a value of the input parameter ρ, smaller than 0.00039.

For these parameters, $(\sigma, \rho) = (0.233, 0.00039)$, the distance between calculated and trading price would be

$$distance(0.233, 0.00039) = 7.4140867. \tag{54}$$

The next logical step is to determine, whether this could be the point of minimum we are looking for. In order to do that, we compared the result with the one we gained for the Black-Scholes model. We know that the best attainable distance value for Black-Scholes is (51), which is smaller than the current result for the first minimum for the feedback incorporating model (54). It means that the Black-Scholes price is closer to the trading price V_{real}, than the new-calculated price from the feedback incorporating model. This does not make much sense, since we wanted to achieve an improvement to the original Black-Scholes model. In other words, we wanted to construct a model, which would be able to return a price even more precisely describing the situation on the market, which is equivalent to the difference between the feedback price and the real trading price V_{real} being smaller than the difference between the Black-Scholes price and V_{real}.

Due to this setback we decided to change ϵ_0 to 0.003, as suggested in the article [7], and try finding the first minimum again, hoping that with the new starting value for ϵ, we will not encounter any more numerical issues. We started with $\sigma = 0.2$, which we presumed, based on the result for previous ϵ_0, could be quite close to an optimal value of the parameter σ. The first iteration returned the best value for $\rho = 0.019$, this led to another value for σ, and so the algorithm continued until a point was reached, in surrounding of which, no better values could be found.

This try, however, seems to be even less successful, since the final, shortest distance for parameters $\sigma = 0.1987$ and $\rho = 0.01803$ was

$$distance(0.1987, 0.01803) = 9.7264517.$$

It is clear that this point

$$(\sigma, \rho) = (0.1987, 0.01803)$$

cannot be the point of the minimum we are looking for, because we still have not reached a distance smaller than Black-Scholes (51). For that reason, the first minimum is not the one we wanted to find, after all.

This leaves us with the second option for minimum of the observed distance function, that is the one around the value $\rho = 0.11$ (see Figure 4). We kept the parameter $\epsilon_0 = 0.003$, since it does not seem to cause numerical problems. As a starting value for optimization, we chose $\rho = 0.09$, which should lead us to the second minimum.

This value of ρ led to the best value of $\sigma = 0.1839$, which then returned a new value for ρ, and so it continued till the algorithm reached the point

$$(\sigma, \rho) = (0.26903, 0.1763), \tag{55}$$

in which our examined distance function reaches value

$$distance(0.26903, 0.1763) = 5.9121576. \tag{56}$$

This point, indeed, satisfies the characteristics of a minimum we are searching for, since the price difference is significantly smaller, than for the Black-Scholes price, and the values of parameters σ and ρ are in line with our expectations or requirements. Just to be sure, we calculated the second minimum also for $\epsilon_0 = 0.000001$, but we gained the same result as for $\epsilon_0 = 0.003$.

To illustrate the course of calculation, in Figure 6 there is visible the developement of distance function with increasing number of performed iterations.

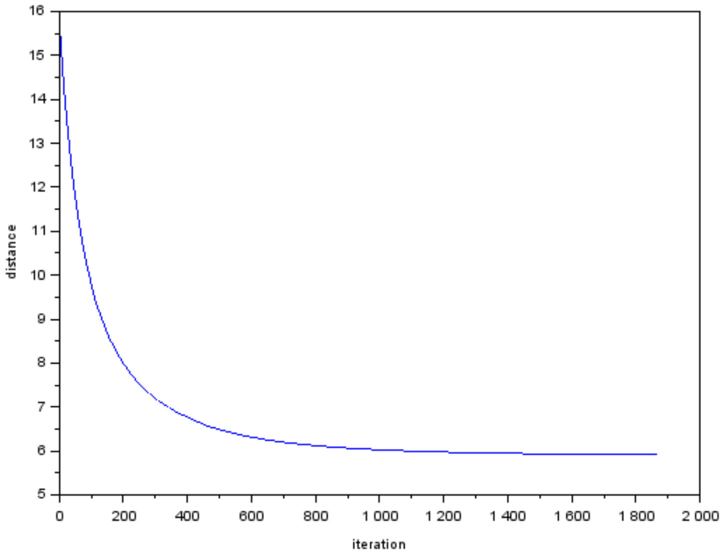

Figure 6: Graph of the developement of distance function with increasing number of performed iterations.

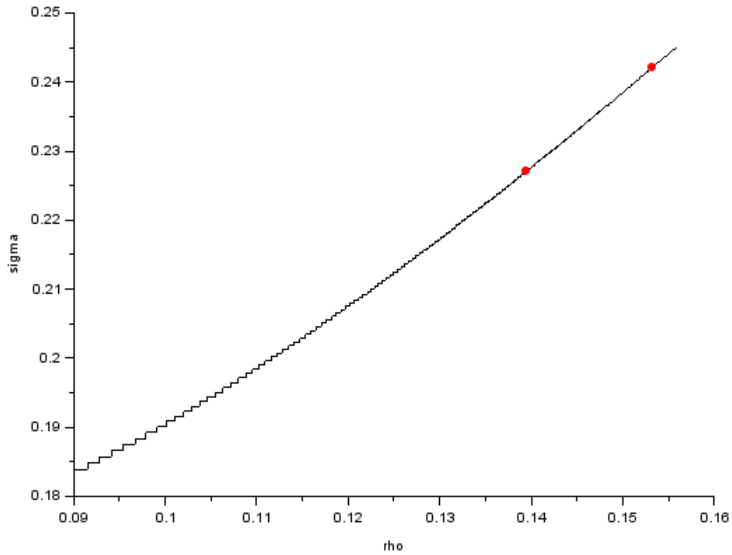

Figure 7: Graph of the developement of parameters σ and ρ, for the first 504 iterations, with highlighted iterations number 200 and 400.

The first returned distance value for the starting point $\rho = 0.09$ was 15.686146, followed by a rapid decrease until the 400-th iteration, where the process slowed down and little by little the algorithm converged to the resulting optimal distance 5.9121576 in iteration number 1866.

The Figure 7, for a change, shows the progress of the parameters σ and ρ. Starting $\rho = 0.09$ led to $\sigma = 0.1839$, then again to $\rho = 0.0915$, followed by $\sigma = 0.1848$. Thus, both parameters increased almost linearly, until the final point (0.26903, 0.1763).

To sum up, in Table 2 we list all the resulting distances and optimal parameters for both values of ϵ_0.

Table 2: Values of parameters σ and ρ, and the function $distance(\sigma, \rho)$ for different values of ϵ_0.

ϵ_0	σ	ρ	*distance*
		Black-Scholes	
	0.234	0	7.3894316
		First minimum	
0.000001	0.233	0.00039	7.4140867
0.003	0.1987	0.01803	9.7264517
		Second minimum	
0.000001	0.26903	0.1763	5.9121576
0.003	0.26903	0.1763	5.9121576

4.2 Evaluation of Results For Amazon

Now that we have the minimum distance values for the two models we are comparing, we should be able to draw some conclusions and see what effects on real-life market could it have, if the original Black-Scholes model is replaced by the newly-derived, feedback incorporating Black-Scholes model.

The numbers 7.3894316 and 5.9121576, as results of distance minimization, however, still do not tell us very much, except, that the prices returned by the new model could, indeed, be copying the market price better. So let us have a closer look at what the two numbers describe.

Let us again begin with the Black-Scholes model and decompose its optimal value of the distance function. We already know that the formula used for its calculation is the following

$$distance(\sigma) = \sum_{i=1}^{n} (V_{real}(i) - Call(x, K(i), r, \sigma, T))^2.$$

Putting the resulting optimal distance and the used formula together we obtain an equation

$$7.3894316 = \sum_{i=1}^{n} (V_{real}(i) - Call(x, K(i), r, \sigma, T))^2, \tag{57}$$

from which we want to deduce some kind of an average difference of Black-Scholes price from the one traded on the market.

In order to make our further reasoning and involved calculations correct, let us assume that the partial differences between prices are constant in absolute value, for each i

$$|V_{real}(i) - Call(i)| = c_{BS}. \tag{58}$$

After we replace all the partial differences between prices in (57) by the absolute value (58) we will receive a simplified equation, independent of i

$$7.3894316 = n.c_{BS}^2. \tag{59}$$

If we divide both sides of the equation (59) by the length of the vector of option prices $n = 17$, what we obtain, is

$$0.434672 = c_{BS}^2. \tag{60}$$

Now what remains to be done is the square root of both sides of (60), which results in the constant absolute difference of trading price from Black-Scholes price

$$c_{BS} = 0.659297. \tag{61}$$

The same procedure we shall now repeat for the feedback model. The formula, used for its calculation, is the following

$$distance(\sigma, \rho) = \sum_{i=1}^{n} (V_{real}(i) - BSUnderFeedback(x, T, K(i), r, \sigma, \rho))^2.$$

Again the resulting optimal distance and the used formula are put into one equation

$$5.9121576 = \sum_{i=1}^{n} (V_{real}(i) - BSUnderFeedback(x, T, K(i), r, \sigma, \rho))^2 \tag{62}$$

and all the partial differences between prices in (62) are substituted by a constant absolute difference for feedback model

$$|V_{real}(i) - BSUnderFeedback(i)| = c_{FB}.$$

The simplified equation

$$5.9121576 = n.c_{FB}^2$$

is divided by the length of the vector of option prices $n = 17$, and we obtain a square of the average distance

$$0.347774 = c_{FB}^2. \tag{63}$$

The last step is the square root of both sides of (63), from which we obtain the constant absolute difference between trading price and feedback price

$$c_{FB} = 0.5897236. \tag{64}$$

These numbers, (61) and (64), hold an information, which is much more useful for us, than the cummulated distances (51) and (56). The conclusion is, that on average, original Black-Scholes price differs from the real market price in about 0.66 currency units, while the feedback model deviates from market price approximately 0.59 currency units. All prices so far in this thesis were listed in US dollars, so accordingly, these differences are 66 US cents for Black-Scholes and 59 cents for feedback model.

If we elaborate this idea even further, one can notice that the difference between these two deviations, rounded to two decimal places, equals

$$c_{BS} - c_{FB} = 0.07,$$

in other words, the upgraded model accounting for feedback effect is on average by 7 US cents more accurate in option pricing, than the original Black-Scholes model.

Whether this difference is significant or profitable, cannot be generally decided, but must be left for the trader to consider. For comparison, we list the calculated prices for both models and corresponding market prices in the Table 3.

Table 3: Comparison of Black-Scholes price, feedback price, and the market price for Amazon.

Black-Scholes price	Feedback price	Market price
25.68	25.75	24.08
22.30	22.25	21.12
19.18	19.03	18.7
16.34	16.13	15.9
13.79	13.55	13.85
11.52	11.29	11.32
9.52	9.34	9.8
7.79	7.69	8.31
6.31	6.30	6.9
5.06	5.15	5.6
4.01	4.21	4.69
3.14	3.43	3.83
2.44	2.79	3.01
1.87	2.27	2.39
1.42	1.84	1.87
0.79	1.19	1.2
0.22	0.46	0.49

4.3 Results For Disney

The next step, as a refinement of the previous strategy, when we were considering only options for one particular expiration date, is to choose derivatives with different times of maturity. We decided to perform this set of calculations for most traded call options written on the stock of The Walt Disney Company, for expiration dates April 21, 2017; May 12, 2017; May 19, 2017; June 16, 2017; and July 21, 2017.

Our assumption is that taking more different times of expiration into consideration will ensure higher quality of information. A single maturity date does not remotely describe the situation on the market, thence, to achieve an imitation of real life trading conditions, we need to attempt to capture the element of diversity.

Apart from the quality of information, stemming from the diversity of expiration dates, another positive consequence of taking most traded options for a variety of maturity dates is that in the majority of cases, the trading volumes are considerably higher than some of the previously taken call options. These both facts, we reckon, could lead to a better fit on the market and therefore to a smaller difference of calculated price from real market price.

Thus, we collected data for 15 call options with the highest trading volume. The trading option prices and strike prices in USD, as well, as corresponding expiration dates, stock price, and constant spot interest rate (source [1]) are

V_{real}=(0.9; 0.23; 3.55;0.08; 0.48; 1.7; 2.66; 1.6; 1.78; 0.48; 0.94; 0.32; 2.52; 0.5; 1.25)
K=(113; 115; 110; 116; 114; 112; 111; 115; 115; 120; 120; 125; 115; 125; 120)
T=(April 21, 2017; April 21, 2017; April 21, 2017; April 21, 2017; April 21, 2017; April 21, 2017; April 21, 2017; May 12, 2017; May 19, 2017; May 19, 2017; June 16, 2017; June 16, 2017; June 16, 2017; July 21, 2017; July 21, 2017)
x=113.20
r=0.0025.

As we already have all the data necessary for the calculation of both the Black-Scholes and the feedback price, our goal is again to compute option prices for both models and compare the cummulated distances from the market prices.

At this point, just like the last time, prior to finding the minimum, we shall have a look at the basic characteristics of the distance function $distance(\sigma, \rho)$ on a particular, sensibly chosen, rectangular area $(\sigma_1, \sigma_n) \times (\rho_1, \rho_m)$, divided into an equidistant grid (σ_i, ρ_j), $i = 1, ..., n$, $j = 1, ..., m$.

The question now arises, how do we best choose the ranges of σ and ρ. After some preliminary efforts, we discerned that for combinations of σ from interval $(0.1, 0.19)$ and ρ from $(0.02, 0.045)$, the distance function returns reasonable values, without any numerical problems.

In Table 4, there are listed calculated values of the distance function $distance(\sigma_i, \rho_j)$, for each i, j, for the area

$$(\sigma_1, \sigma_n) \times (\rho_1, \rho_m) = (0.1, 0.19) \times (0.02, 0.045).$$

Table 4: Values for the function $distance(\sigma, \rho)$ on an equidistant grid, for the area $(0.1, 0.19) \times (0.02, 0.045)$. Highlighted, there is again a possible point of local minimum.

$\rho\backslash\sigma$	0.1	0.115	0.13	0.145	0.16	0.175	0.19
0.02	3.12	1.58	0.75	0.74	1.65	3.54	6.49
0.025	2.77	1.28	0.61	0.86	2.09	4.39	7.80
0.03	2.55	1.05	0.49	0.94	2.49	5.17	9.03
0.035	2.50	0.89	0.37	0.99	2.80	5.83	10.12
0.04	2.74	0.87	**0.28**	0.99	3.01	6.36	11.04
0.045	3.47	1.08	0.29	0.98	3.14	6.74	11.77

When we analyze the figures in the table, what we notice is that all of the function values are significantly smaller from those in the previous run. The general behaviour of the function, however, seems to stay the same. In σ-direction we can observe the same feature as for the data in previous computation, that is, values decrease at first and increase with higher σ.

In case of the second parameter, ρ, we see that for lower values of σ_i from the grid, the observed function first decreases in ρ and later increases. Only one local minimum is visible here, because we focused on the second minimum, which meant that we included only higher values of ρ, leading to it.

Again, to get even better view of what the function looks like, we drew some partial graphs for chosen values of both parameters σ and ρ. In Figure 8 and Figure 9, layers from Table 4 are depicted, namely those for $\rho = 0.04$ and $\sigma = 0.13$, which are the coordinates of the lowest value in the table, $distance(0.13, 0.04) = 0.28$.

As we can see in Figure 8, the distance function has the same properties in variable σ as for data for the single maturity date in previous calculation, that is, in variable σ, it is a convex function with only one minimum.

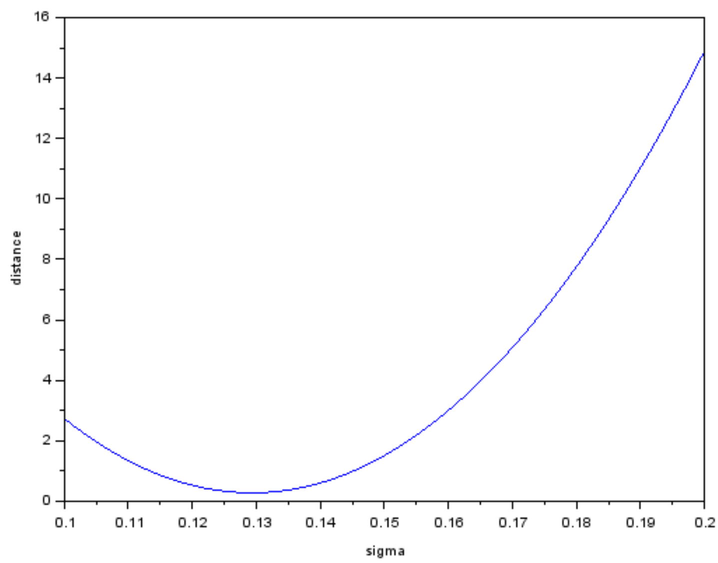

Figure 8: Graph of the function $distance(\sigma, \rho)$ for $\rho = 0.04$ and σ from the interval $(0.1, 0.2)$.

In Figure 9, only neighbourhood of the second minimum is displayed, therefore, we must extend the interval to include lower values of ρ to see, whether the attribute of two local minima is preserved.

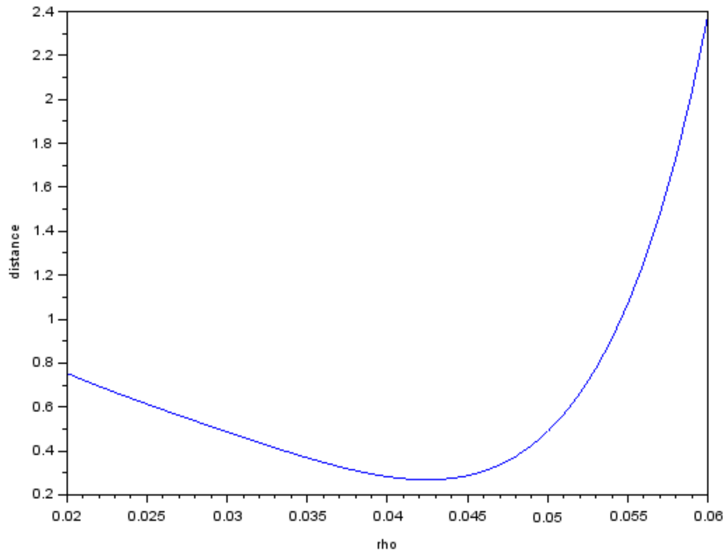

Figure 9: Graph of the function $distance(\sigma, \rho)$ for $\sigma = 0.13$ and ρ from the interval $(0.02, 0.06)$.

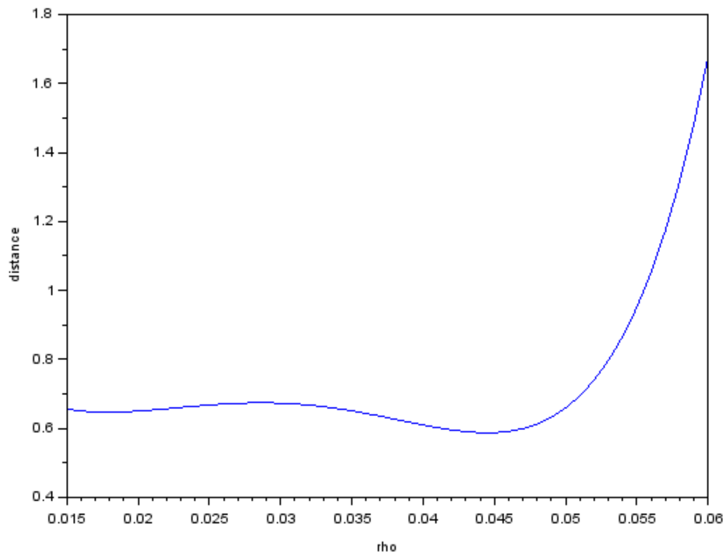

Figure 10: Graph of the function $distance(\sigma, \rho)$ for $\sigma = 0.14$ and ρ from the interval $(0.015, 0.06)$.

The Figure 10 shows us that truly, with lower values of ρ, the function values decrease toward $\rho = 0.015$, ergo, the characteristics of the observed function did not change for this particular range of parameter values.

Just like before, we sketched also several other contour lines, to see how the behaviour of the distance function changes with various values of parameters. These are displayed in Figures 11 and 12.

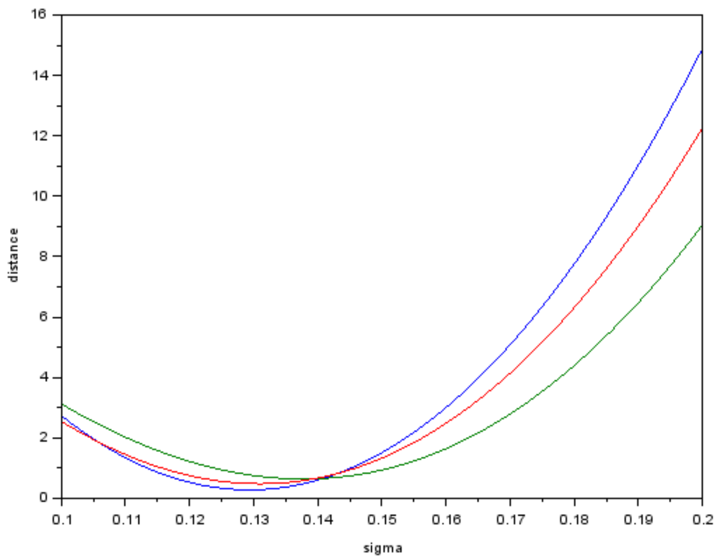

Figure 11: Graphs of the function $distance(\sigma, \rho)$ for $\rho = 0.04$ (blue line), $\rho = 0.02$ (green line), $\rho = 0.03$ (red line), and σ from the interval $(0.1, 0.2)$.

We needed to adjust some of the intervals to be displayed to avoid some numerical issues, and eventually, one can see in Figure 11 that the typical characeristics are preserved in σ-direction for all chosen fixed values of ρ, although, the same cannot be said about the other case, when we examined the behavior of the function for several fixed σ, that is in ρ-direction. We can see that the displayed parts of the function are very different and in case of $\sigma = 0.16$, no possibility of a local minima is visible. It seems that for this set of data, the distance function has many undefined areas, caused by numerical problems, and does not seem to have a typical behaviour for an arbitrary combination of paramaters.

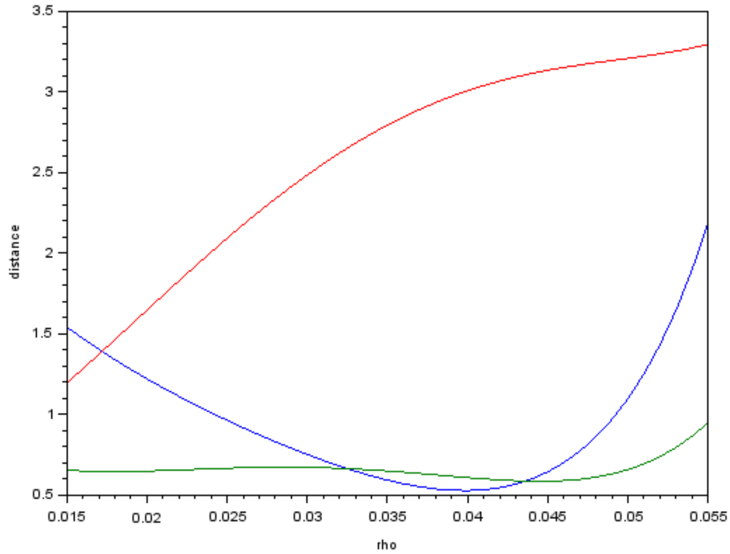

Figure 12: Graphs of the function $distance(\sigma, \rho)$ for $\sigma = 0.12$ (blue line), $\sigma = 0.14$ (green line), $\sigma = 0.16$ (red line), and ρ from the interval $(0.015, 0.055)$.

Let us start then with the calculation of prices. The first one to optimize will again be the Black-Scholes model. The procedure is exactly the same as before, except that the expiration date T in the formula (50) now also depends on the index i

$$distance(\sigma) = \sum_{i=1}^{n} (V_{real}(i) - Call(x, K(i), r, \sigma, T(i)))^2 \rightarrow min.$$

The results of this optimization are the minimal value of cummulated squared differences

$$distance(0.165, 0) = 0.4532516 \tag{65}$$

and the optimal σ, in which the minimum is achieved

$$\sigma = 0.165. \tag{66}$$

Now we can move on to the feedback incorporating model. This model, as we recall, depends also on the variable ϵ, the smoothing parameter, which needs to be assigneed a starting value ϵ_0. In the previous calculation for a single expiration date, we tried two different starting values, one of which, $\epsilon_0 = 0.000001$, tended to cause numerical problems. Therefore, we continue with the latter, more reliable one, $\epsilon_0 = 0.003$.

We chose to start with the value of parameter ρ equal to $\rho = 0.02$. For this input, our algorithm returned $\sigma = 0.1377$, then again, this value led to $\rho = 0.0442$, and alltogether in 7 iterations it reached the optimal point

$$(\sigma, \rho) = (0.1297, 0.0424). \tag{67}$$

The optimal distance in the point $(\sigma, \rho) = (0.1297, 0.0424)$ is

$$distance(\sigma, \rho) = 0.2682257. \tag{68}$$

Depicted in Figure 13 is the developement of parameters for the 7 iterations.

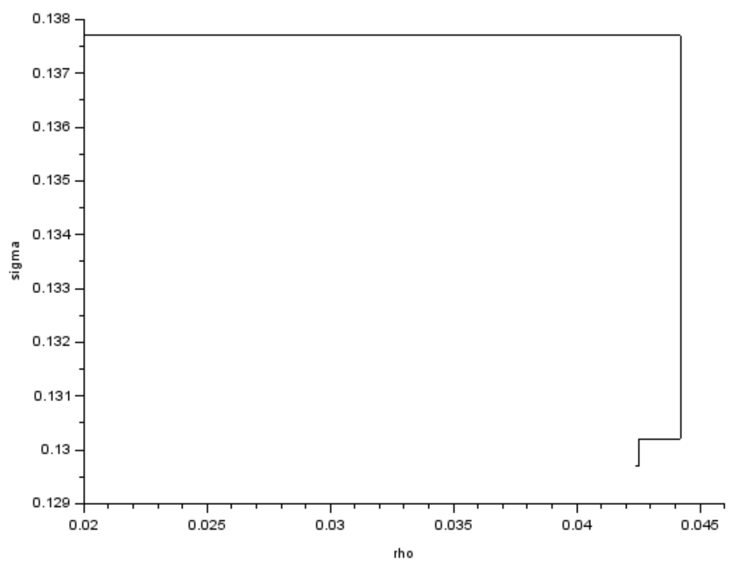

Figure 13: Graph of the developement of parameters σ and ρ.

The decrease of the distance function with the number of iterations is displayed in Figure 14. What one can observe is that the distance decreases rapidly in first 3 iterations and in the remaining 4 descends only little by little as it reaches the optimal value. The number of necessary iterations to reach the minimum is really low. This is probably caused by the shape of the distance function for this data.

Both results, (65) and (68), seem to be reasonable, considering the condition that the resulting distance for the feedback incorporating model needs to be smaller than the one returned by the Black-Scholes model. We presume that the second minimum of the distance function, for the feedback incorporating model, is also its global minimum. The first minimum does not seem a reasonable option after looking at Figure 10 and Figure 12.

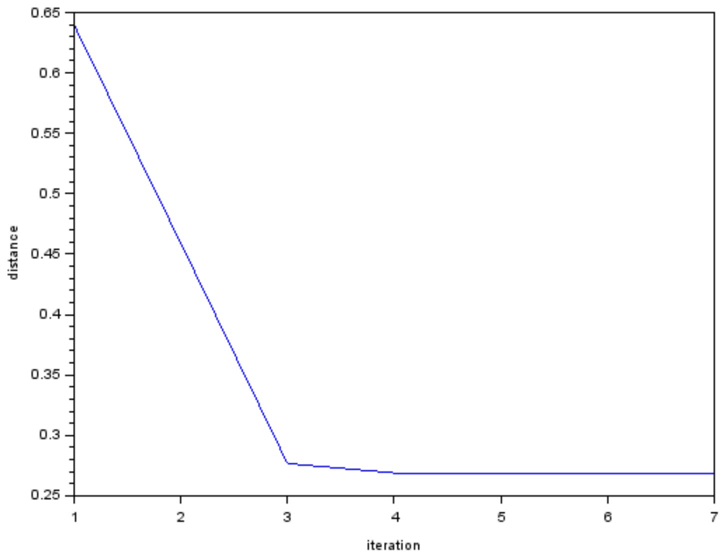

Figure 14: Graph of the developement of distance values with increasing number of iterations.

4.4 Evaluation of Results For Disney

Let us now proceed to the evaluation of the obtained optimal cummulative distances of the new-calculated option prices from corresponding market prices.

In Table 5, there is again a summary of results, new ones alongside those obtained from previous set of data.

Table 5: Summary of results from both calculations, for $\epsilon_0 = 0.003$.

	Amazon			Disney		
	σ	ρ	*distance*	σ	ρ	*distance*
Black-Scholes	0.234	0	7.3894316	0.165	0	0.4532516
Second minimum	0.26903	0.1763	5.9121576	0.1297	0.0424	0.2682257

It is quite easy to notice that all the new figures, from the multiple expiration dates case, are much smaller than the previous results, when a single time of maturity was taken into consideration. We cannot, however, make any sensible comparison of the pairs of distances, for single and multiple expiration dates, until we perform the decomposition to the average difference of calculated and real price. This is because there were different numbers of options included in each calculation. Nevertheless, we can at least note that in the second run, for both cases, the Black-Scholes model and the second local minimum of the feedback incorporating model, the minimal distances are obtained for lower values of parameters.

Let us now compute the average deviation of the prices. Starting with the Black-Scholes model, we will once again follow the same course of decomposition of the computed optimal distance value. The slightly changed formula compared to the previous has the form

$$distance(\sigma) = \sum_{i=1}^{m} (V_{real}(i) - Call(x, K(i), r, \sigma, T(i)))^2.$$

After we put the resulting optimal distance and the used formula together we obtain an equation

$$0.4532516 = \sum_{i=1}^{m} (V_{real}(i) - Call(x, K(i), r, \sigma, T(i)))^2, \tag{69}$$

and from this equation we want to deduce the same kind of an average difference of Black-Scholes price from the one traded on the market, as before.

We shall once more continue with the assumption that all the partial differences between prices are equal to a constant in absolute value

$$|V_{real}(i) - Call(i)| = k_{BS}. \tag{70}$$

Now all the partial differences between prices in (69) are replaced by the absolute value (70) and we will receive the simplified equation, independent of i

$$0.4532516 = m.k_{BS}^2. \tag{71}$$

As a next step, we divide the equation (71) by the number of options taken, $m = 15$, and we obtain

$$0.0302168 = k_{BS}^2. \tag{72}$$

What remains to be done is the square root of both sides of (72), which gives us the desired result, the constant absolute difference of trading price from Black-Scholes price

$$k_{BS} = 0.1738297. \tag{73}$$

The same procedure is repeated for the feedback model. The modified formula, which is used for its calculation, looks as follows

$$distance(\sigma, \rho) = \sum_{i=1}^{m} (V_{real}(i) - BSUnderFeedback(x, T(i), K(i), r, \sigma, \rho))^2.$$

The general distance is replaced by the optimal value

$$0.2682257 = \sum_{i=1}^{m} (V_{real}(i) - BSUnderFeedback(x, T(i), K(i), r, \sigma, \rho))^2 \tag{74}$$

and all the partial differences between prices in (74) are substituted by a constant absolute difference for feedback model

$$|V_{real}(i) - BSUnderFeedback(i)| = k_{FB}.$$

The simplified equation

$$0.2682257 = m.k_{FB}^2$$

is divided by the length of the vector of option prices $m = 15$, and we obtain a square of the average distance

$$0.0178817 = k_{FB}^2. \tag{75}$$

The last step is the square root of both sides of (75), from which we obtain the constant absolute difference between trading price and feedback price

$$k_{FB} = 0.1337225. \tag{76}$$

From these two numbers, (73) and (76), we can now see that on average, original Black-Scholes price differs from the real market price in about 0.17 dollars, while the deviation of the feedback model from market price is approximately 0.13 dollars. In other words, the difference is 17 US cents for Black-Scholes and 13 US cents for derived feedback model.

What immediately follows is that when we round the two deviations to two decimal numbers, so that their meaning as ammounts in US dollars is emphasized, and calculate their difference

$$k_{BS} - k_{FB} = 0.04,$$

we find that after we take into consideration the diversity of the market, the upgraded model accounting for feedback effect is on average by 4 US cents more accurate in option pricing, than the original Black-Scholes model.

Now that we have all the necessary information, we can compare all the price differences for both models and both selections of data. For better overview, we list all of these in Table 6

Table 6: Overview of price differences for both models and both selections of data.

	Amazon	Disney	Difference between data sets
Black-Scholes	66	17	66>17
Feedback	59	13	59>13
Difference between models	7	4	

We can see that when Black-Scholes is compared to the feedback model, for both data sets, the latter gives better results, that is, the average difference of the price it calculates, from the actual market price, is smaller.

Also when we compare the improvement brought about by the diversity of expiration times, it is clearly visible that the diverse data gives us average differences significantly smaller from those obtained with consideration of only one expiration date.

Again, we list the calculated prices for both models and corresponding market prices in the Table 7.

Table 7: Comparison of Black-Scholes price, feedback price, and the market price for Disney.

Black-Scholes price	Feedback price	Market price
1.1588319	1.0193029	0.9
0.3974500	0.2704965	0.23
3.3425127	3.2651916	3.55
0.2043303	0.1063277	0.08
0.7074382	0.5690013	0.48
1.759402	1.6247202	1.7
2.4967069	2.3827161	2.66
1.352619	1.5336545	1.6
1.5932196	1.786473	1.78
0.4063568	0.3586658	0.48
0.9086260	0.8781581	0.94
0.2847972	0.1917536	0.32
2.3518271	2.5154753	2.52
0.6423357	0.5037300	0.5
1.5059591	1.4514878	1.25

Conclusion

The aim of this thesis was to explore a possible improvement of the Black-Scholes model, which is widely known and used for option pricing. This examined upgrade consists in incorporating of the feedback effect in the original model. The feedback effect, as we explained, arises if we decide to relax one of the assumptions required for the use of Black-Scholes model, namely, the perfect elasticity of the underlying asset. The non elasticity of the underlying stock means that its equilibrium price can be affected by the quantity in which it is traded on the market. The additional volume of the asset traded has its origin in a small fraction of traders, called the program traders, who trade the asset solely to ensure their portfolios and not for gain. This excessive amount traded can cause a change in the price of the stock which would naturally lead to a change in corresponding derivative price.

After the first chapter, purpose of which was to recall and unify basic definitions and terms to be used throughout the thesis, in the second chapter we derived in detail the new, upgraded Black-Scholes model according to the original work of Sircar and Papanicolaou [7], with some additions of our own. We explained more closely how the presence of program traders on the market affects the asset price process. The new price process and its adjusted volatility were then used in the derivation process which is the same as the original of Black and Scholes. Thus, we obtained the extended Black-Scholes model incorporating the feedback effect, caused by the program traders and their hedging strategies.

Third chapter was dedicated to the description of how the theory from previous chapter was transformed and programmed in Scilab. We explained that we would consider the approximation of the new derivative price to be a sum of the Black-Scholes price, calculated using the Black-Scholes formula, and a first order correction, outlined how the parameters would be calibrated through minimizaation of the distance function, and then we shortly introduced our programmed Scilab functions and how they worked.

In the last chapter, we recounted the whole process of calculation for two sets of market data, how we obtained the correct results, the issues we encountered, and how we solved them. We found that the optimized distance function is not simple, but has more than one local minimum. After we explored several options for calculation and found the global minimum, we compared the results and drew conclusions for the real-life market.

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