COMENIUS UNIVERSITY IN BRATISLAVA
FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS

TWO-PERIOD MODEL FOR CONSUMPTION - INVESTMENT DECISION WITH A PROSPECT THEORY HOUSEHOLD

MASTER THESIS

2017

Bc. Barbora MAREČÁKOVÁ

COMENIUS UNIVERSITY IN BRATISLAVA
FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS

Two-period model for consumption - investment decision with a prospect theory household.

MASTER THESIS

Study programme:	Mathematical Economic, Finance and Modelling
Branch of study:	9.1.9. Applied Mathematics (1114)
Department:	Department of Applied Mathematics and Statistics
Supervisor:	RNDr. Jaroslava Hloušková, PhD.

Bratislava 2017

Bc. Barbora MAREČÁKOVÁ

UNIVERZITA KOMENSKÉHO V BRATISLAVE
FAKULTA MATEMATIKY, FYZIKY A INFORMATIKY

Dvojperiódový model spotreby - investičné rozhodovanie domácnosti pomocou prospektovej teórie

DIPLOMOVÁ PRÁCA

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Katedra:	Katedra aplikovanej matematiky a štatistiky
Supervisor:	RNDr. Jaroslava Hloušková, PhD.

Bratislava 2017

Bc. Barbora MAREČÁKOVÁ

75343393

Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

THESIS ASSIGNMENT

Name and Surname: Bc. Barbora Marečáková
Study programme: Mathematical Economics, Finance and Modelling (Single degree study, master II. deg., full time form)
Field of Study: Applied Mathematics
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Title: Two-period model for consumption - investment decision with a prospect theory household.

Aim: (i) To derive the solution and its properties of a two-period consumption-investment problem of a household with prospect theory type of preferences when the second period income is random.
(ii) Compare results to the case when the second period income is deterministic but the risky asset return is random and/or when both the second period income and the return of the risky asset are random.
(iii) If possible, generalize the set-up for a general (continuous) distribution of the second period income.

Supervisor: RNDr. Jaroslava Hloušková, PhD.
Department: FMFI.KAMŠ - Department of Applied Mathematics and Statistics
Head of department: prof. RNDr. Daniel Ševčovič, CSc.

Assigned: 22.06.2016

Approved: 22.06.2016 prof. RNDr. Daniel Ševčovič, CSc.
Guarantor of Study Programme

Student

Supervisor

Univerzita Komenského v Bratislave
Fakulta matematiky, fyziky a informatiky

75343393

ZADANIE ZÁVEREČNEJ PRÁCE

Meno a priezvisko študenta: Bc. Barbora Marečáková
Študijný program: ekonomicko-finančná matematika a modelovanie (Jednoodborové štúdium, magisterský II. st., denná forma)
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Názov: Two-period model for consumption - investment decision with a prospect theory household.
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Vedúci: RNDr. Jaroslava Hloušková, PhD.
Katedra: FMFI.KAMŠ - Katedra aplikovanej matematiky a štatistiky
Vedúci katedry: prof. RNDr. Daniel Ševčovič, CSc.

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prof. RNDr. Daniel Ševčovič, CSc.
garant študijného programu

....
študent

....
vedúci práce

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Abstract

MAREČÁKOVÁ, Barbora: Two-period model for consumption - investment decision with a prospect theory household [Master Thesis], Comenius University in Bratislava, Faculty of Mathematics, Physics and Informatics, Department of Applied Mathematics and Statistics; Supervisor: RNDr. Jaroslava Hloušková, PhD., Bratislava, 2017, 70 pages.

This study examines the choices of a sufficiently loss averse household described by the two-period consumption model with an uncertain income in the second period. Household is maximizing its utility by optimization of the first period consumption. The utility function is reference based with quadratic loss averse form. The optimal solution primarily depends on the magnitude of the first period reference level of consumption. There are eight different optimal values of the first period consumption according to the parameter set-up. If the optimal solution is in the feasibility interval, then the changes of the risk aversion parameter influence the optimal values. If the reference level of the first period consumption is low, then the avoidance of losses in the first period depends on the relation between risk free rate and discount factor. If the first period reference level of consumption is larger than in the previous cases, the solution again depends on the risk free rate and discount factor, which defines the willingness to avoid the domain of losses in the first period. A self-improving household, which is characterized by sufficiently large first period reference level, experiences loss in the first period and in the bad state of nature in the second period.

Keywords: prospect theory, loss aversion, utility function

Abstrakt

MAREČÁKOVÁ, Barbora: Dvojperiódový model spotreby - investičné rozhodovanie domácnosti pomocou prospektovej teórie [Diplomová práca], Univerzita Komenského v Bratislave, Fakulta matematiky, fyziky a informatiky, Katedra aplikovanej matematiky a štatistiky; školiteľ: RNDr. Jaroslava Hloušková, PhD., Bratislava, 2017, 70 strán.

Práca skúma správanie sa dostatočne stratovo averznej domácnosti popísanej dvojperiódovým modelom spotreby s neurčitým príjmom v druhej perióde. Domácnosť maximalizuje svoju užitočnosť pomocou optimalizácie spotreby v prvej perióde. Funkcia užitočnosti závisí od veľkosti referenčných hladín a má kvadratickú formu averzie voči stratám. Optimálne riešenie primárne závisí od veľkosti referenčnej hladiny spotreby v prvej perióde. Je uvedených osem optimálnych hodnôt spotreby v prvej perióde na základe usporiadania vzťahov medzi parametrami. Ak je optimálne riešenie vo vnútri oblasti prípustných riešení, tak zmeny parametra averzie voči stratám ovplyvňujú optimálne hodnoty. Ak je referenčná hladina spotreby v prvej perióde nízka, tak sa domácnosť snaží vyhýbať stratám v prvej perióde na základe vzťahu medzi bezrizikovou mierou a diskontným faktorom. Ak je referenčná hladina spotreby v prvej perióde väčšia ako v predošlom prípade, tak riešenia znova závisia od bezrizikovej úrokovej miery a diskontného faktora, ktoré definujú snahu domácnosti vyhnúť sa stratám v prvej perióde. Domácnosť s vysokými ašpiráciami, ktorá je charakteristická dostatočne vysokou hodnotou referenčnej hladiny v prvej perióde, je v strate počas prvej periódy a aj v zlom prirodzenom stave druhej periódy.

Kľúčové slová: prospektová teória, averzia voči starte, úžitková funkcia

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Introduction

How do people make decisions? What are the important factors which influence their consumption? These questions and many more are asked by the economist since the introduction of prospect theory by Kahnemann and Tversky in 1979 in [11].

In this thesis we explore the behaviour of a household during two periods. Every household has some expectations about standard of living, which is represented by consumption reference levels in both periods. In the first period the household has labour income, which can be consumed immediately or saved for the next period. The amount of consumed income depends on the willingness of the household to live in the loss compared to it's reference level. Second period is characterized by the uncertainty of exogenous income.

We study the optimal consumption with aim to maximize the happiness of the household. Happiness is given by the utility function, which has similar properties as the one in the prospect theory. Our motivation is to describe the behaviour of the household which has large loss aversion parameter. Thus the utility function is concave in the domain of losses and linear in the domain of gains.

First chapter explains theories of decision making and contribution of our work. Following one focuses on the detailed explanation of our problem set-up. The last chapter contains economical interpretation of the results along with the sensitivity analysis. Results are presented according to the parameter set-up of the first period reference level.

1 Theoretical Background of Decision Making

1.1 Expected Utility Theory

The term *expected utility* was first used by Daniel Bernoulli in 1738 in [3]. Shortly, it states, that if an individual is making a decision under uncertainty then he maximizes his expected utility. The function of the expected utility is the sum of products of probability (p_i) and utility over all possible outcomes (x_i) for m losses and n gains.

$$U(\mathbf{x},\, \mathbf{p}) = \sum_{i=-m}^{n} p_i \cdot x_i$$

John von Neumann and Oskar Morgenstern proved the hypothesis in study [13], that under four axioms any individual has the utility function. Those axioms are defined as:

1. Completeness - an individual has well defined preferences between any two alternatives, i.e. with two alternatives A and B - he either prefers A to B, or prefers B to A or is indifferent between them.

2. Transitivity - if for every alternatives A, B and C holds that A is preferred to B and B is preferred to C, then A is preferred to C. It means that the individual decides consistently.

3. Independence - alternatives A and B maintain the same order of preference independently of the third irrelevant alternative C.

4. Continuity - for three alternatives where A is preferred to B and B is preferred to C exists a mixed combination of A and C, which is indifferent to B.

This theory is an abstraction and simplification of rationally acting agents. The main limits were discussed in Kahneman and Tversky analysis [11].

1.2 Prospect Theory

Prospect theory was introduced by Daniel Kahneman and Amos Tversky in the journal *Econometrica* in 1979 [11]. The study - *Prospect Theory: An Analysis of Decision under Risk* demonstrates that in laboratory settings people under risk aren't rational players according to the predictions of the expected utility theory. Expected utility theory was the normative model of rational choice and descriptive model of economic behaviour before prospect theory was introduced.

Irrational behaviour occurs especially when people are making decisions under risk. They tend not to decide by maximizing the sum of probabilities times utility rather decide by comparing the losses and gains with the reference levels. Furthermore, sensitivity to losses is greater than to gains. These specifications are presented by the results of the experiment in study [11].

The experiment consisting of a dozen hypothetical choice questions was performed on students at the Israeli university, University of Stockholm and University of Michigan. For example respondents had to choose between two options:

PROBLEM 1

A: gain 2500 with probability 0.33
gain 2400 with probability 0.66
gain 0 with probability 0.01

B: gain 2400 for sure

The stated example demonstrates *certainty effect*, firstly described in 1953 in Allais [1]. Following the expected utility theory respondents should choose option **A**. Expected utility of **A** exceeds the expected utility of **B**. However, 82% of them chose option **B**. More experiments with similar questions showed that people prefer outcomes that are considered certain, relative to outcomes which are merely probable.

Answers to the pair of the questions below violated utility theory from a different angle.

PROBLEM 2

A: gain 6000 with probability 0.45

B: gain 3000 with probability 0.90

PROBLEM 3

A: gain 6000 with probability 0.001

B: gain 3000 with probability 0.002

In the mentioned experiment the majority of respondents chose **B** in Problem 2 (86%) but **A** in Problem 3 (73%). The answers describe *possibility effect*, when the probabilities of winning are high, the person chooses the more probable option, however, in Problem 3 the probabilities are low and the person chooses higher gain.

Reflection effect means that the preferences between two options of losses are the mirror image of the preferences in the domain of gains. The percentage of the preferences from the respondents in the experiment is shown in the following pair of the questions.

PROBLEM 4 - gains

A: 4000 with probability 0.80 - chosen by 20% respondents,

B: 3000 for sure - chosen by 80%.

PROBLEM 5 - losses

A: -4000 with probability 0.80 - chosen by 92% respondents,

B: -3000 for sure - chosen by 8%.

Isolation effect occurs when people are choosing between two alternatives and their focus is aimed only on the differences between them, more described in Tversky [18]. Inconsistent preferences appear because identification of the features can be done in different ways followed by different preferences.

Kahneman and Tversky [11] described the process of choice by two phases. The first phase is editing. It is characterized by several operations:

- Coding - defining the outcomes as gains or losses according to the neutral reference point. Reference point can be either current asset position or expectations.
- Combination - simplifying the probabilities with the same outcomes. For example option A contains two parts of gaining 200 with 25% probability, it is simplified to gain 200 with 50% probability.
- Segregation - another type of simplification. If there is probability 40% of loss 400 and 60% of loss 100, it can be simplified to loss of 300 with 40% probability and sure loss of 100.
- Cancellation - ignoring the same output in both options. Let option A be gaining 200 with probability 20%, 100 with probability 50% and losing 50 with probability 30%. Option B is gaining 200 with probability 20%, 150 with probability 50% and losing 100 with probability 30%. It can be cancelled to choice between A, gain 100 by 50%, lose 50 by 30%, and B, gain 150 by 50%, lose 100 by 30%.
- Rounding - numbers in choices are rounded.
- Detection of dominance - dominated options are rejected.

Note that different order of editing results in different final options.

The second part is evaluation. Every respondent assigns decision weights to the outputs. Decision weights are associated with probability, however, they are rarely equal. Then to the outputs are assigned values according to the reference point - gains and losses. Following the work of Kahneman and Tversky, the basic equation of theory can be defined such that:

Definition 1.1. *If $(x, p; y, q)$ is a regular prospect (option), where x and y are the amounts of gains and losses respectively, p and q are the assigned probabilities and either $p + q < 1$ or $x \geq 0 \geq y$ or $x \leq 0 \leq y$ then the value of regular prospect is defined as*

$$V(x, p; y, q) = \pi(p)v(x) + \pi(q)v(y), \tag{1}$$

where π is the function of decision weights and v is the function of values according to the reference point and $v(0) = 0, \pi(0) = 0$ and $\pi(1) = 1$

1.2.1 The Utility Function

According to the adaptation level theory firstly introduced by Helson in [9], our perception is attuned to evaluate changes rather than the absolute magnitudes. Prospect theory value (utility) function depends only on one variable - change, which is assumed to be satisfactory approximation. However, more realistic value function has two variables - initial wealth and change in the wealth.

The important feature of the function is concavity in the domain of gains ($v''(x) < 0$ for $x > 0$) and convexity in the domain of losses ($v''(x) > 0$ for $x < 0$). People evaluate the difference between gaining 100 and 200 to be greater than the difference between 1100 and 1200. Similar evaluation is in the domain of losses, unless the loss is intolerable. The assumption is based on the choices of the respondents in the experiment. Note that in special cases this feature may change. Especially, when someone's decision is accompanied by the changes in lifestyle.

Another feature reflects the difference between perception of losses and gains of the same amount. The experiment shows that if $x > y \geq 0$, then the option to gain y or lose $-y$ with 50% probability is preferred to gain x or lose $-x$ with the same probabilities. It leads to the conclusion that $v'(x) < v'(-x)$, which means that the value function is steeper in the domain of losses than in the domain of gains.

Kahneman and Tversky introduced the hypothetical utility function displayed in Figure 1.

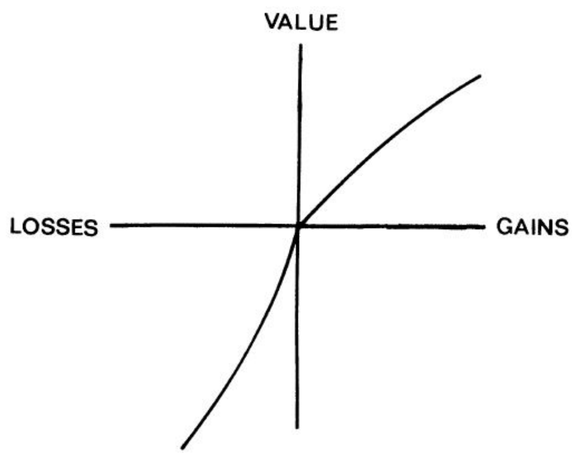

Figure 1: Hypothetical utility function according to the Kahneman and Tversky [11].

1.2.2 The Weight Function

In prospect theory decision weights measure the impact of events on the desirability of options. Function of weights is the same as the probabilities when one decides by the expected utility theory. Weight function depends only on stated probabilities in the prospect theory approximation, however, it might be influenced by other factors as well.

The weight function π is an increasing function of probability p, with $\pi(0) = 0$ and $\pi(1) = 1$. First equation describes ignorance of impossible events. Second equation means that the weights associated with probabilities are normalized.

Based on the answers in the experiment Kahneman and Tversky assume that the weight function behaves differently with small p than large p. It overvalues low probabilities and undervalues high probabilities. It is based on the fact that people like lotteries and also insurance.

The figure of hypothetical weight function was presented in the article [11] according to the features described above, see figure 2.

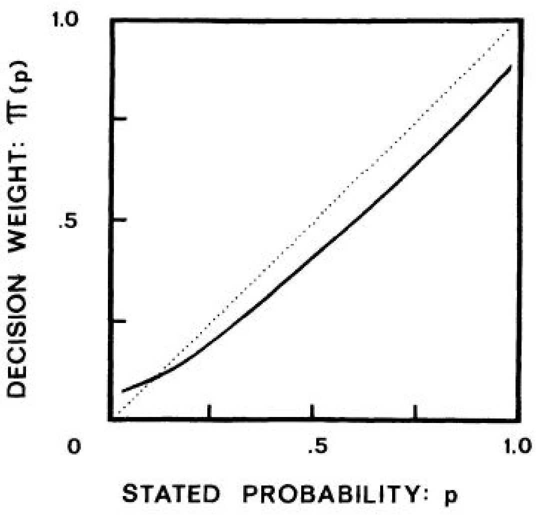

Figure 2: Hypothetical weight function according to the Kahneman and Tversky [11].

1.2.3 Cumulative Prospect Theory

Cumulative prospect theory resolves some limitations of the theory explained above. It applies to uncertain and risky options with any number of outcomes. It means that for a gamble with m losses, n gains and one zero output the value function is defined as

$$U(\mathbf{x},\mathbf{p}) = \sum_{i=-m}^{n} \pi(p_i)v(x_i).$$

It also allows different weight function for losses and for gains. The focus of the study aims on the explanation of the curvature of the value function and weight functions. In the study [19] of cumulative prospect theory Kahneman and Tversky discussed the experimental evidence of both - risk-averse respondents for losses, risk seeking for gains and risk-seeking respondents for losses, risk averse for gains.

1.2.4 Disadvantages

The prospect theory was introduced by the experiment in the laboratory conditions. One may question whether the accuracy of this theory holds even in the real world. Barberis [2] describes situations when the theory holds with sufficiently large financial incentives. However, results of experiments provided in the field of financial trading are not that clear - List [12] found prospect theory insufficient. On the other hand, study [14] of experienced professional golfers (instead of traders) shows, that they follow the theory in gambles on their performance, prospect theory was sufficient. Also decision weights may be sensitive to the formulation of the questions.

Another obstacle is in defining gains and losses. For a portfolio investor it might be a change in overall wealth, a change in the value of stock market holdings or difference of the real value and reference level. Different definitions give different utility.

People tend to employ heuristics procedures. These procedures were mostly described above as editing phase. As mentioned before, different order of operations gives different value.

1.3 Other Theories

Most other theories differ from the prospect theory in the assumptions for utility function. The simplest difference is to have marginal utility of gains and losses fixed. It means, that the loss aversion is linear. Studies following this assumption are, for example, [5], [7] and [17].

Another assumption is that people are more averse to large losses than the small ones and that the gains have constant marginal utility. That is quadratic form of loss aversion. Function is concave in the domain of losses and linear in the domain of gains. It was precisely studied in Fortin and Hlouskova [4].

1.4 Our Model

Our work focuses on the model where the loss aversion is the main feature of the utility function. We expect that the penalty in the domain of losses is increasing with larger losses. In the domain of gains the marginal utility is fixed, which is the simplest form of aversion. Gains and losses are defined relative to a given reference point. Our utility function has quadratic form of loss aversion.

The utility function is displayed in the figure 3.

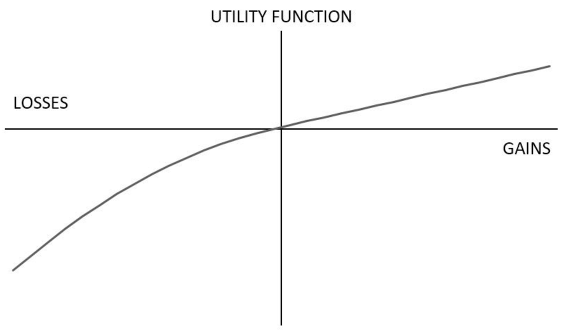

Figure 3: Hypothetical utility function with quadratic form of loss aversion.

Quadratic form of loss aversion is mostly used in the fields with downside risk and portfolio management [10] and in statistical decision theory [8], [20]. Large losses are punished more severely than small losses, it is referred as quadratic shortfall in the studies [16], [17].

Another important assumption is that the future income is uncertain. Thus the consumption is influenced by exogenous income with two possible amounts. The consumer has to make the choice between saving income for later and immediate consumption. The effect of uncertain future income (continuous case) on savings decision is described by Sandmo in article [15]. The role of accumulated savings is being a buffer for future consumption.

2 Problem Set-up

A household is living for two periods - productive period and retirement. During the productive period it receives an income that is non-stochastic and exogenous - Y_1. It might be labor income or endowment income. Household allocates the income to the current consumption and savings. Consumption in the first period is C_1 and risk-free investment is notated as S. The following equation describes the *first period*:

$$Y_1 = C_1 + S. \tag{2}$$

The second period is characterized by stochastic income Y_{2i}. The exogenous income might be for example government pension. It is defined according to the state of nature, which can be good or bad, as follows:

$$Y_{2i} = \begin{cases} Y_{2g} & \text{with probability } p \\ Y_{2b} & \text{with probability } 1-p \end{cases} \tag{3}$$

with $p \in (0, 1)$. Where lower index b stands for the bad state of nature and g stands for the good state of nature. We assume that $Y_{2g} > Y_{2b} \geq 0$. C_2 is the second period consumption and r_f is a net of the dollar return of the risk-free asset. The *second period* is modeled as follows:

$$C_{2i} = Y_{2i} + (1 + r_f)\ S, \tag{4}$$

$$= Y_{2i} + (1 + r_f)(Y_1 - C_1), \tag{5}$$

where $i \in \{b, g\}$. Consumption can not be negative, but we don't assume any liquidity constraints for consuming Y_2 in the first period. This means that the household is allowed to borrow money.

According to the prospect theory, every household sets it's reference levels of consumptions for both periods. Note that $\bar{C}_1$ is the first period reference level of consumption and the second period consumption reference level is noted as $\bar{C}_2$. Consumptions in both periods are greater than some minimum consumption C_{1L} and C_{2L}, so we assume $C_1 > C_{1L} \geq 0$, $C_{2g} > C_{2b} > C_{2L} \geq 0$ and reference levels $\bar{C}_1$, $\bar{C}_2$ are such that $0 \leq C_{1L} \leq \bar{C}_1$, $0 \leq C_{2L} \leq \bar{C}_2$.

Household's preferences are described by the following reference based utility function

$$U(C_1) = V(C_1 - \bar{C}_1) + \delta V(C_2 - \bar{C}_2), \tag{6}$$

where $\delta \in (0,1)$ is a discount factor and the quadratic loss averse value function $V(\cdot)$ is such that

$$V(C_i - \bar{C}_i) = \begin{cases} \gamma_1(C_i - \bar{C}_i) & \text{if } C_i \geq \bar{C}_i \\ \\ \gamma_1(C_i - \bar{C}_i) - \lambda\frac{\gamma_2}{2}(\bar{C}_i - C_i)^2 & \text{if } C_i < \bar{C}_i \quad \text{and} \quad \lambda > 0 \end{cases}$$

for $i = 1, 2$. Parameter $\lambda > 0$ is the loss aversion parameter and $\gamma_1,\ \gamma_2 > 0$ are coefficients. The function refers to relative gains when reference level is below actual consumption and similarly to relative losses when reference level is above actual consumption. As discussed in the previous chapter the function has following features:

- values are steeper in the domain of losses than gains,
- below the reference point function is concave.

The common features with the prospect theory are reference based utility and loss aversion principle. The same amount of loss has higher absolute utility value than the amount of gain, i.e. household is more penalized by the loss than awarded by the same amount of gain. We also assume that household is risk averse in the domain of losses and risk neutral in the domain of gains. The marginal utility is constant in the domain of gains.

In general, the household maximizes the following quadratic loss aversion based expected utility

$$\begin{aligned} \mathrm{Max}_{\mathrm{C}_1}: &\quad \mathbb{E}(U(C_1)) = V(C_1 - \bar{C}_1) + \delta\,\mathbb{E}V(C_2 - \bar{C}_2) \\ \text{such that}: &\quad C_1 \geq C_{1L} \text{ and } \mathrm{C_{2b}} \geq \mathrm{C_{2L}} \end{aligned}$$

Based on this and (5) the maximization problem can be reformulated as follows

$$\left.\begin{aligned} \mathrm{Max}_{\mathrm{C}_1}: &\quad \mathbb{E}(U(C_1)) = V(C_1 - \bar{C}_1) + \delta\,\mathbb{E}V\left((1 + r_f)(Y_1 - C_1) + Y_{2i} - \bar{C}_2\right) \\ \text{such that}: &\quad C_{1L} \leq C_1 \leq Y_1 + \tfrac{Y_{2b} - C_{2L}}{1 + r_f} \end{aligned}\right\} \tag{7}$$

where we assume that

$$C_{1L} + \frac{C_{2L}}{1+r_f} \leq Y_1 + \frac{Y_{2b}}{1+r_f} \text{ or } C_{1L} \leq Y_1 + \frac{Y_{2b} - C_{2L}}{1+r_f} \tag{8}$$

We use following notation to simplify the expressions:

$\gamma \equiv \frac{\gamma_1}{\gamma_2}$

$\Omega \equiv (1+r_f)(Y_1 - \bar{C}_1) + Y_{2b} - \bar{C}_2$

$k \equiv \frac{\gamma[1-\delta(1+r_f)]}{\delta(1-p)(1+r_f)^2}$

Solving (7) breaks down to solving the following eight maximization problems assuming (8) holds and that $C_{1L} \leq \bar{C}_1$ and $C_{2L} \leq \bar{C}_2$.

(P1): $C_1 \geq \bar{C}_1$ and $\bar{C}_2 \leq C_{2b} < C_{2g}$

$$\left.\begin{aligned} \text{Max}_{C_1} : \frac{1}{\gamma_2}\mathbb{E}(U(C_1)) = \; & \gamma(C_1 - \bar{C}_1) + \delta p\, \gamma \left[(1+r_f)(Y_1 - C_1) + Y_{2g} - \bar{C}_2\right] \\ & + \delta(1-p)\, \gamma \left[(1+r_f)(Y_1 - C_1) + Y_{2b} - \bar{C}_2\right] \\ \text{such that}: \; & \bar{C}_1 \leq C_1 \leq Y_1 + \tfrac{Y_{2b} - \bar{C}_2}{1+r_f} \end{aligned}\right\} \tag{P1}$$

Note that (P1) has a feasible solution only if

$$\bar{C}_1 \leq Y_1 + \frac{Y_{2b} - \bar{C}_2}{1+r_f} \tag{A-P1}$$

(P2): $C_1 \geq \bar{C}_1$ and $C_{2L} \leq C_{2b} < \bar{C}_2 \leq C_{2g}$

$$\left.\begin{aligned} \text{Max}_{C_1} : \; & \tfrac{1}{\gamma_2}\mathbb{E}(U(C_1)) = \gamma(C_1 - \bar{C}_1) + \delta p\, \gamma \left[(1+r_f)(Y_1 - C_1) + Y_{2g} - \bar{C}_2\right] \\ & + \delta(1-p) \left[\gamma \left((1+r_f)(Y_1 - C_1) + Y_{2b} - \bar{C}_2\right)\right] \\ & + \delta(1-p) \left[-\lambda \tfrac{1}{2} \left(\bar{C}_2 - (1+r_f)(Y_1 - C_1) - Y_{2b}\right)^2\right] \\ \text{such that}: \; & \max\left\{\bar{C}_1, Y_1 + \tfrac{Y_{2b} - \bar{C}_2}{1+r_f}\right\} \leq C_1 \leq \min\left\{Y_1 + \tfrac{Y_{2b} - C_{2L}}{1+r_f}, Y_1 + \tfrac{Y_{2g} - \bar{C}_2}{1+r_f}\right\} \end{aligned}\right\} \tag{P2}$$

Note that (P2) has a feasible solution only if

$$\bar{C}_1 \leq \min\left\{Y_1 + \frac{Y_{2b} - C_{2L}}{1+r_f}, Y_1 + \frac{Y_{2g} - \bar{C}_2}{1+r_f}\right\} \tag{A-P2}$$

Note in addition that if (A-P1) holds then (A-P2) boils down to $\bar{C}_1 \leq Y_1 + \frac{Y_{2g} - \bar{C}_2}{1+r_f}$.

(P3): $C_1 \geq \bar{C}_1$ and $C_{2L} \leq C_{2g} < \bar{C}_2 \leq C_{2b}$ - the problem has no feasible solutions since $C_{2b} < C_{2g}$.

(P4): $C_1 \geq \bar{C}_1$ and $C_{2L} \leq C_{2b} < C_{2g} < \bar{C}_2$

$$\left.\begin{array}{rl} \text{Max}_{\text{C}_1}: & \frac{1}{\gamma_2}\mathbb{E}(U(C_1)) = \gamma(C_1 - \bar{C}_1) \\ & +\delta p\left[\gamma\left((1+r_f)(Y_1 - C_1) + Y_{2g} - \bar{C}_2\right)\right] \\ & +\delta p\left[-\lambda\frac{1}{2}\left(\bar{C}_2 - (1+r_f)(Y_1 - C_1) - Y_{2g}\right)^2\right] \\ & +\delta(1-p)\left[\gamma\left((1+r_f)(Y_1 - C_1) + Y_{2b} - \bar{C}_2\right)\right] \\ & +\delta(1-p)\left[-\lambda\frac{1}{2}\left(\bar{C}_2 - (1+r_f)(Y_1 - C_1) - Y_{2b}\right)^2\right] \\ \text{such that}: & \max\left\{\bar{C}_1, Y_1 + \frac{Y_{2g} - \bar{C}_2}{1+r_f}\right\} \leq C_1 \leq Y_1 + \frac{Y_{2b} - C_{2L}}{1+r_f} \end{array}\right\} \quad \text{(P4)}$$

Note that (P4) has a feasible solution only if

$$\bar{C}_1 \leq Y_1 + \frac{Y_{2b} - C_{2L}}{1+r_f} \quad \text{and} \quad \bar{C}_2 \geq C_{2L} + Y_{2g} - Y_{2b} \quad \text{(A-P4)}$$

Note in addition that if (A-P1) holds then (A-P4) boils down to

$$C_{2L} + Y_{2g} - Y_{2b} \leq \bar{C}_2 \leq (1+r_f)(Y_1 - \bar{C}_1) + Y_{2b}$$

(P5): $C_{1L} \leq C_1 < \bar{C}_1$ and $\bar{C}_2 \leq C_{2b} < C_{2g}$

$$\left.\begin{array}{rl} \text{Max}_{\text{C}_1}: \frac{1}{\gamma_2}\mathbb{E}(U(C_1)) = & \gamma(C_1 - \bar{C}_1) - \lambda\frac{1}{2}(\bar{C}_1 - C_1)^2 \\ & +\delta p\,\gamma\left[(1+r_f)(Y_1 - C_1) + Y_{2g} - \bar{C}_2\right] \\ & +\delta(1-p)\gamma\left[(1+r_f)(Y_1 - C_1) + Y_{2b} - \bar{C}_2\right] \\ \text{such that}: & C_{1L} \leq C_1 \leq \min\left\{\bar{C}_1, Y_1 + \frac{Y_{2b} - \bar{C}_2}{1+r_f}\right\} \end{array}\right\} \quad \text{(P5)}$$

Note that (P5) has a feasible solution only if

$$C_{1L} \leq Y_1 + \frac{Y_{2b} - \bar{C}_2}{1+r_f} \quad \text{(A-P5)}$$

Note in addition that if (A-P1) holds then (A-P5) holds as well.

(P6): $C_{1L} \leq C_1 < \bar{C}_1$ and $C_{2L} \leq C_{2b} < \bar{C}_2 \leq C_{2g}$

$$\left.\begin{array}{rl} \text{Max}_{\text{C}_1}: & \frac{1}{\gamma_2}\mathbb{E}(U(C_1)) = \gamma(C_1 - \bar{C}_1) - \lambda\frac{1}{2}(\bar{C}_1 - C_1)^2 \\ & +\delta p\,\gamma\left[(1+r_f)(Y_1 - C_1) + Y_{2g} - \bar{C}_2\right] \\ & +\delta(1-p)\left[\gamma\left((1+r_f)(Y_1 - C_1) + Y_{2b} - \bar{C}_2\right)\right] \\ & +\delta(1-p)\left[-\lambda\frac{1}{2}\left(\bar{C}_2 - (1+r_f)(Y_1 - C_1) - Y_{2b}\right)^2\right] \\ \text{such that}: & \max\left\{C_{1L}, Y_1 + \frac{Y_{2b} - \bar{C}_2}{1+r_f}\right\} \leq C_1 \leq \min\left\{\bar{C}_1, Y_1 + \frac{Y_{2b} - C_{2L}}{1+r_f}, Y_1 + \frac{Y_{2g} - \bar{C}_2}{1+r_f}\right\} \end{array}\right\} \quad \text{(P6)}$$

Note that (P6) has a feasible solution only if

$$C_{1L} \leq Y_1 + \frac{1}{1+r_f} \min \left\{ Y_{2b} - C_{2L}, Y_{2g} - \bar{C}_2 \right\} \quad \text{and} \quad \bar{C}_1 \geq Y_1 + \frac{Y_{2b} - \bar{C}_2}{1+r_f} \tag{A-P6}$$

Note in addition that if (A-P5) holds then (A-P6) boils down to $\bar{C}_1 \geq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$. If even stronger assumption (A-P1) holds then (P6) has no feasible solution if the strong inequality in (A-P1) takes place. However, if $\bar{C}_1 = Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ then the only feasible (and thus optimal) solution of (P6) is $C_1^* = \bar{C}_1 = Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$.

Note finally that as the set of feasible solutions of (P6) is the sub-set of the set of feasible solutions of (P2) then for any C_1 feasible for (P6) will $\mathbb{E}(U(C_1))^{P2} \geq \mathbb{E}(U(C_1))^{P6}$ as $\frac{1}{\gamma_2}\mathbb{E}(U(C_1))^{P6} = \frac{1}{\gamma_2}\mathbb{E}(U(C_1))^{P2} - \lambda\frac{1}{2}(\bar{C}_1 - C_1)^2$.

(P7): $C_{1L} \leq C_1 < \bar{C}_1$ and $C_{2L} \leq C_{2g} < \bar{C}_2 \leq C_{2b}$ - the problem has no feasible solutions since $C_{2b} < C_{2g}$.

(P8): $C_{1L} \leq C_1 < \bar{C}_1$ and $C_{2L} \leq C_{2b} < C_{2g} \leq \bar{C}_2$

$$\left.\begin{aligned}
\text{Max}_{C_1}: \quad & \tfrac{1}{\gamma_2}\mathbb{E}(U(C_1)) = \gamma(C_1 - \bar{C}_1) - \lambda\tfrac{1}{2}(\bar{C}_1 - C_1)^2 \\
& +\delta p \left[\gamma\left((1+r_f)(Y_1 - C_1) + Y_{2g} - \bar{C}_2\right)\right] \\
& +\delta p \left[-\lambda\tfrac{1}{2}\left(\bar{C}_2 - (1+r_f)(Y_1 - C_1) - Y_{2g}\right)^2\right] \\
& +\delta(1-p) \left[\gamma\left((1+r_f)(Y_1 - C_1) + Y_{2b} - \bar{C}_2\right)\right] \\
& +\delta(1-p) \left[-\lambda\tfrac{1}{2}\left(\bar{C}_2 - (1+r_f)(Y_1 - C_1) - Y_{2b}\right)^2\right] \\
\text{such that}: \quad & \max\left\{C_{1L}, Y_1 + \tfrac{Y_{2g}-\bar{C}_2}{1+r_f}\right\} \leq C_1 \leq \min\left\{\bar{C}_1, Y_1 + \tfrac{Y_{2b}-C_{2L}}{1+r_f}\right\}
\end{aligned}\right\} \tag{P8}$$

Note that (P8) has a feasible solution only if

$$\bar{C}_1 \geq Y_1 + \frac{Y_{2g} - \bar{C}_2}{1+r_f} \quad \text{and} \quad \bar{C}_2 \geq C_{2L} + Y_{2g} - Y_{2b} \tag{A-P8}$$

Note in addition that if the first inequality in (A-P8) holds then also (A-P5) holds and that the second inequality in (A-P8) coincides with the second inequality in (A-P4). Note finally that if (A-P1) holds then (A-P8) boils down to the same constrains as in (P4), namely

$$C_{2L} + Y_{2g} - Y_{2b} \leq \bar{C}_2 \leq (1+r_f)(Y_1 - \bar{C}_1) + Y_{2b}$$

3 Results and Economic Interpretation

Formal analysis of the problem leads to solving the eight problems mentioned in the previous chapter. Each problem has specific solutions for different parameter set-up, see Appendix A. The solution can be either inner point or the border point of the feasibility interval. Firstly note that the objective function is concave for all problems. We determine the stationary points. Considering the feasibility conditions - if the stationary point is in the feasibility interval then it is also its point of maxima. If it is not in the interval then the point of maxima is one of the end points of the interval of feasible solutions. For each problem we define the points of its maxima and detailed parameter conditions under which it is maximum.

Ten different parameter set-ups of $\bar{C}_1$, mentioned in Appendix B, lead to the solutions of the maximization problem (7). We obtain solution for each of the set-up by comparing the values of objective functions in the points of maxima. Overall we present eight optimal values for C_1.

Note that we focus only on (sufficiently) loss averse households, i.e. on largest possible λ. Their objective functions have high "penalty" in the domain of losses. Penalty depends on loss aversion parameter λ and the amount of loss, i.e. it is the amount of happiness decrease. In the domain of gains the household experiences reward, depending on the amount of gain. In the following subsections we present the solutions for optimal first period consumption.

3.1 $C_1^* = Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} + \frac{k}{\lambda}$

Firstly let's assume parameter set-up, when household consumption reference level is sufficiently low, namely $\bar{C}_1 \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$. The household with low reference levels can be viewed as the ones driven by the self-enhancement motive (the need to feel good and maintain self-esteem, as described in [6]). In addition we assume that the discount factor is lower than discounting by risk free rate. The optimal values for consumptions in the first and second period are stated in the following proposition.

Proposition 1. *Let* $\bar{C}_1 \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ *and* $\delta < \frac{1}{1+r_f}$*. Then the following holds for*

$\lambda \geq \frac{(1+r_f)k}{\min\left\{\bar{C}_2 - C_{2L}, Y_{2g} - Y_{2b}\right\}}$

$$C_1^* = \frac{k}{\lambda} + Y_1 + \frac{Y_{2b} - \bar{C}_2}{1+r_f} > \bar{C}_1 \tag{9}$$

$$C_{2g}^* = \bar{C}_2 + Y_{2g} - Y_{2b} - \frac{(1+r_f)k}{\lambda} \geq \bar{C}_2 \tag{10}$$

$$C_{2L} \leq C_{2b}^* = \bar{C}_2 - \frac{(1+r_f)k}{\lambda} < \bar{C}_2 \tag{11}$$

$$\frac{1}{\gamma_2}\mathbb{E}(U(C_1^*)) = \left[\frac{\Omega}{1+r_f} + \delta p\left(Y_{2g} - Y_{2b}\right) + \frac{\left(1 - \delta(1+r_f)\right)k}{2\lambda}\right]\gamma \tag{12}$$

Proof. Proof follows directly from Lemma C.1. ■

The optimal consumption in the first period is strictly higher than its reference level, so household tries to avoid losses. In the second period the optimal consumption in the good state of nature is higher than the corresponding reference level, however, in the bad state of nature the reference level exceeds the optimal consumption. Thus, the household's optimal solution according to the relationship between reference levels and actual consumption is reached in (P2), see in the previous section where problems (P1)-(P8) are defined. For increasing loss-aversion parameter λ, the consumption in the bad state of nature, C_{2b}, is increasing as well and thus the gap between the second period reference level and C_{2b} is decreasing. Thus, the loss averse household wants to have as small losses as possible.

Table 1 captures sensitivity of the optimal solutions for consumptions, relative consumption, amount of savings and "happiness" with respect to changes in loss-averse parameter and reference levels. Relative consumption is the distance between the optimal level of consumption and the corresponding reference consumption, $|C_i^* - \bar{C}_i|$, $i = 1, 2$; savings are the difference between first period income and actual consumption in the first period, $S^* = Y_1 - C_1^*$, and happiness is the value of the objective function (expected utility) at its optimal level of the first period consumption (indirect utility).

If savings are negative, i.e. $Y_1 - C_1^* < 0$, then the household is a borrower, otherwise a lender. Specifically for this case if the second period reference level is sufficiently large the household is a lender, i.e. $Y_{2b} - \frac{k(1+r_f)}{\lambda} < \bar{C}_2$.

Table 1 on sensitivity analysis shows, that the more loss averse is the household, the lower consumption is in the first period and higher are the second period optimal consumptions. The household is lowering its losses in the bad state of nature in the

	dC_1^*	dC_{2g}^*	dC_{2b}^*	$d(C_1^* - \bar{C}_1)$	$d(C_{2g}^* - \bar{C}_2)$	$d(\bar{C}_2 - C_{2b}^*)$	dS^*	$d(\mathbb{E}(U(C_1^*)))$
$d\lambda$	< 0	> 0	> 0	< 0	> 0	< 0	> 0	< 0
$d\bar{C}_1$	$= 0$	$= 0$	$= 0$	< 0	$= 0$	$= 0$	$= 0$	< 0
$d\bar{C}_2$	< 0	> 0	> 0	< 0	$= 0$	$= 0$	> 0	< 0

Table 1: Sensitivity analysis for $Y_1 + \frac{Y_{2b} - \bar{C}_2}{1+r_f} + \frac{k}{\lambda}$

second period, when the penalization, λ increases. When the optimal consumption in the first period decreases, the savings have to increase. Household is saving to minimize the losses in the bad state of nature in the second period. The indirect utility is also decreasing when the loss averse parameter increases, since the penalty in the losses is larger and the reward in gains is linear.

Household is always in the domain of gains in the first period, where the objective function is linear. Thus, changes in the first period reference level don't influence the optimal values of the consumptions. However, the happiness decreases because the relative gain also decreases.

Second period reference level influences more values. Both second period optimal consumptions increase by increasing the reference level. On the other hand, the optimal consumption in the first period decreases. It simulates the situation when households are saving for retirement because they want to enjoy earned value. Savings increases but happiness decreases because in the bad state of nature the penalty increases.

3.2 $C_1^* = \bar{C}_1 + \frac{\gamma}{\lambda}[1 - \delta(1 + r_f)]$

This case differs from the previous one in the condition for relation between the discount factor and risk-free rate. Household can be described as driven by self-enhancement motive, as its first period reference level is relatively low, i.e. $\bar{C}_1 \leq Y_1 + \frac{Y_{2b} - \bar{C}_2}{1+r_f}$. Optimal values are stated in the following proposition.

Proposition 2. *Let $\bar{C}_1 \leq Y_1 + \frac{Y_{2b} - \bar{C}_2}{1+r_f}$ and $\delta > \frac{1}{1+r_f}$. Then the following holds for*

$\lambda > \lambda_L^{P5} = \frac{\gamma[\delta(1+r_f)-1]}{\bar{C}_1 - C_{1L}}$

$$\begin{aligned} C_{1L} < C_1^* &= \bar{C}_1 + \frac{\gamma}{\lambda}[1-\delta(1+r_f)] < \bar{C}_1 && (13) \\ C_{2g}^* &= Y_{2g} + (1+r_f)(Y_1 - \bar{C}_1) + \frac{\gamma}{\lambda}\left(\delta - \frac{1}{1+r_f}\right) > \bar{C}_2 && (14) \\ C_{2b}^* &= Y_{2b} + (1+r_f)(Y_1 - \bar{C}_1) + \frac{\gamma}{\lambda}\left(\delta - \frac{1}{1+r_f}\right) \geq \bar{C}_2 && (15) \\ \frac{1}{\gamma_2}\mathbb{E}(U(C_1^*)) &= \delta\left[\Omega + p(Y_{2g} - Y_{2b})\right]\gamma + \frac{\gamma^2}{2\lambda}[1-\delta(1+r_f)]^2 && (16) \end{aligned}$$

Proof. Proof follows directly from Lemma C.2. ■

Optimal consumption in the first period is lower than the corresponding reference level. The loss is decreasing by larger loss aversion. However, second period optimal consumption levels for both periods are larger than the corresponding reference level. It means that household is in the domain of losses in the first period, so it can be in the domain of gains in the second period. From the previous optimal solution stated in Proposition 1 the change in sensitivity appears due to the change in the relationship of discount factor and risk-free rate. The solution is part of the set of feasible solutions of problem (P5) based on the condition on $\bar{C}_1$. In general, we can say, that the household is living in losses to have gains in the future. Savings can be either positive or negative, depending on the reference level of consumption in the second period. If $S > 0$ then $\bar{C}_1 \leq Y_1 + \frac{\gamma[\delta(1+r_f)-1]}{\lambda}$, which holds for $\bar{C}_2 \geq Y_{2b} - \frac{\gamma(1+r_f)[\delta(1+r_f)-1]}{\lambda}$. Household is a lender when the second period reference level is sufficiently large.

Table 2 is the sensitivity analysis with the same focus as in the previous case.

	dC_1^*	dC_{2g}^*	dC_{2b}^*	$d(\bar{C}_1 - C_1^*)$	$d(C_{2g}^* - \bar{C}_2)$	$d(C_{2b}^* - \bar{C}_2)$	dS^*	$d(\mathbb{E}(U(C_1^*)))$
$d\lambda$	> 0	< 0	< 0	< 0	< 0	< 0	< 0	< 0
$d\bar{C}_1$	> 0	< 0	< 0	$= 0$	< 0	< 0	< 0	< 0
$d\bar{C}_2$	$= 0$	$= 0$	$= 0$	$= 0$	< 0	< 0	$= 0$	< 0

Table 2: Sensitivity analysis for $\bar{C}_1 + \frac{\gamma}{\lambda}[1-\delta(1+r_f)]$

Household is avoiding the losses $(\bar{C}_1 - C_1^*)$ in the first period with increasing loss-aversion parameter. On the other hand, the second period optimal consumption is decreasing when λ increases. This case might represent the household, which needs to decrease savings in order to decrease the penalty in the first period, so the optimal consumption in the second period is lower. Change in happiness is also negative, since the first period penalty gets larger and second period reward gets smaller.

Change of the first period reference level $\bar{C}_1$ has impact similar to change of loss-aversion parameter. Household experiences higher optimal value of consumption in the first period and smaller gains in the second period with increasing $\bar{C}_1$. It means that savings decrease and also optimal values of the second period consumptions decrease. But there is no influence on relative consumption in the first period - loss remains the same.

Second period optimal consumptions in both states of the nature are higher than the reference level. If there is an increase of the second period reference level, it only has impact on relative gains in the second period and happiness. The gains get smaller so the happiness also decreases.

3.3 $C_1^* = \bar{C}_1$

Let $\bar{C}_1 \geq Y_1 + \frac{Y_{2b} - \bar{C}_2}{1+r_f}$. The household is more driven by the self-improvement motive, with high aspirations, than in the previous cases, due to the $\bar{C}_1$ being above the threshold. In general, the more thresholds are below the value of the first period reference level, the more household is self-improvement and it is comparing itself to household with higher economic status - upward comparison.

The following proposition states the conditions when the optimal value of consumption equals $\bar{C}_1$.

Proposition 3. *Let* $Y_1 + \frac{Y_{2b} - \bar{C}_2}{1+r_f} \leq \bar{C}_1 \leq \min\left\{Y_1 + \frac{Y_{2g} - \bar{C}_2}{1+r_f}, Y_1 + \frac{Y_{2b} - C_{2L}}{1+r_f}\right\}$, *then for*

(i) $\delta < \min\left\{\frac{1}{(1+r_f)}, \frac{1}{(1-p)(1+r_f)^2}\right\}$ *and* $\lambda \geq \frac{(1+r_f)k}{-\Omega}$ *and* $Y_1 + \frac{Y_{2b} - \bar{C}_2}{1+r_f} < C_{1L}$ *or*

(ii) $\frac{1}{(1+r_f)} < \delta \leq \frac{1}{(1-p)(1+r_f)^2}$ *and* $\lambda \geq \frac{2\gamma\left(\frac{1}{1+r_f} - \delta\right)}{(-\Omega)\left[\delta(1-p) - \frac{1}{(1+r_f)^2}\right]} = \lambda^{P2-P5}$ *or*

(iii) $\delta = \frac{1}{(1+r_f)}$ *and* $p \geq \frac{r_f}{1+r_f}$

holds that

$$C_{1L} \leq C_1^* = \bar{C}_1 \tag{17}$$

$$C_{2g}^* = Y_{2g} + (1+r_f)(Y_1 - \bar{C}_1) \geq \bar{C}_2 \tag{18}$$

$$C_{2L} \leq C_{2b}^* = Y_{2b} + (1+r_f)(Y_1 - \bar{C}_1) < \bar{C}_2 \tag{19}$$

$$\frac{1}{\gamma_2}\mathbb{E}(U(C_1^*)) = \delta\left[(1+r_f)(Y_1 - \bar{C}_1) + \mathbb{E}(Y_2) - \bar{C}_2\right]\gamma - \delta(1-p)\frac{\lambda}{2}(-\Omega)^2 \tag{20}$$

Proof. Proof for (i) follows directly from Lemma C.4, (ii) follows directly from Lemma C.5 and finally (iii) follows from Lemma C.4, C.5 and also C.6. ■

Another important assumption is $\bar{C}_1 \leq Y_1 + \frac{\min\{Y_{2b}-C_{2L}, Y_{2b}-C_{2L}\}}{1+r_f}$. It means that the first period consumption reference level is limited. Household has higher aspirations than in the previous cases $\left(Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} \leq \bar{C}_1\right)$, but still limited.

There are no gains or losses for the first period, since the optimal consumption coincides with its reference level. In the second period the losses occur in the bad state of nature. Difference between consumptions in different states of nature is driven by the difference between incomes in those states. Due to the relationships between optimal consumptions and reference levels, the solution is reached in problem (P2).

Savings are negative if the household has the first period reference level larger than the first period income. It is also feasible to have smaller reference level and the household is a lender with positive savings.

Table 3 shows the sensitivity of the optimal values, when there is a change in $\lambda, \bar{C}_1$ or $\bar{C}_2$.

	dC_1^*	dC_{2g}^*	dC_{2b}^*	$d(C_1^* - \bar{C}_1)$	$d(C_{2g}^* - \bar{C}_2)$	$d(\bar{C}_2 - C_{2b}^*)$	dS^*	$d(\mathbb{E}(U(C_1^*)))$
$d\lambda$	$=0$	$=0$	$=0$	$=0$	$=0$	$=0$	$=0$	<0
$d\bar{C}_1$	>0	<0	<0	$=0$	<0	>0	<0	<0
$d\bar{C}_2$	$=0$	$=0$	$=0$	$=0$	<0	>0	$=0$	<0

Table 3: Sensitivity analysis for $\bar{C}_1$

The loss-aversion parameter has no influence on the optimal values except happiness. The happiness decreases with increasing loss aversion because of the losses in the bad state of nature.

Change in the reference level for the first period means the change of C_1^*, since reference level is the optimal value. By increasing it the values of second period consumptions are decreasing. Also the savings and happiness are decreasing. Household consume more in the first period, so the gains in the second period are lower in the good state of nature and losses in the bad state of nature are higher, even though the relative consumption is lower.

Second period reference level has impact only on the relative consumptions and happiness. In the good state of nature the gains are smaller and in the bad state of nature the losses increase. Overall the happiness has to decrease.

3.4 $C_1^* = Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$

The parameter set-up for this optimal consumption is suggesting that the household can be described as self-improving and with high aspirations, as in the previous case with optimal values defined in 3. Usually it is comparing with higher-status households. Specifically the conditions are stated in the following proposition.

Proposition 4. *Let* $C_{1L} \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < \bar{C}_1 \leq \min\{Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}, Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}\}$ *and* $\delta \geq \frac{1}{(1-p)(1+r_f)^2}$*, then for*

(i) $\delta < \frac{1}{(1+r_f)}$ *and* $\lambda \geq \lambda^{P2-P5} = \frac{2\gamma\left(\frac{1}{1+r_f}-\delta\right)}{(-\Omega)\left[\delta(1-p)-\frac{1}{(1+r_f)^2}\right]}$ *or*

(ii) $\delta > \frac{1}{(1+r_f)}$ *and* $\lambda \geq \lambda_L^{P5}$ *or*

(iii) $\delta = \frac{1}{(1+r_f)}$ *and* $\lambda > 0$

holds that

$$C_{1L} \leq C_1^* = Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < \bar{C}_1 \quad (21)$$

$$C_{2g}^* = Y_{2g} - Y_{2b} + \bar{C}_2 > \bar{C}_2 \quad (22)$$

$$C_{2b}^* = \bar{C}_2 \quad (23)$$

$$\frac{1}{\gamma_2}\mathbb{E}(U(C_1^*)) = \frac{\gamma}{1+r_f}\left[(1+r_f)(Y_1-\bar{C}_1) + \mathbb{E}(Y_2) - \bar{C}_2\right] - \frac{\lambda}{2}\left(\frac{\Omega}{1+r_f}\right)^2 \quad (24)$$

Proof. Proof for (i) follows directly from Lemma C.4, (ii) follows directly from Lemma C.5 and finally (iii) follows directly from Lemma C.6. ■

The first period optimal consumption is below the reference level, so the household is in the domain of losses. In the second period household experiences only gains, that in the bad state of nature are zero. This parameter set-up means that the maximum is reached in the problem (P5).

Savings are equal to $\frac{\bar{C}_2 - Y_{2b}}{1+r_f}$, which means that for sufficiently large consumption reference level in the second period, household is a lender.

Table 4 summarizes the sensitivity analysis.

	dC_1^*	dC_{2g}^*	dC_{2b}^*	$d(\bar{C}_1 - C_1^*)$	$d(C_{2g}^* - \bar{C}_2)$	$d(C_{2b}^* - \bar{C}_2)$	dS^*	$d(\mathbb{E}(U(C_1^*)))$
$d\lambda$	$= 0$	$= 0$	$= 0$	$= 0$	$= 0$	$= 0$	$= 0$	< 0
$d\bar{C}_1$	$= 0$	$= 0$	$= 0$	> 0	$= 0$	$= 0$	$= 0$	< 0
$d\bar{C}_2$	< 0	> 0	> 0	> 0	$= 0$	$= 0$	> 0	< 0

Table 4: Sensitivity analysis for $Y_1 + \frac{Y_{2b} - \bar{C}_2}{1+r_f}$

Loss-aversion parameter has no impact on optimal values of consumption in neither of the periods. The only change is in the happiness which decreases. It happens due to the higher penalty in the first period.

If we change the first period reference level, the only change is in the size of the losses in the first period, which increases. Following that, the happiness should decrease which correspondents with the sensitivity analysis.

Second period reference level has big influence on the analyzed optimal values. If there is an increase of $\bar{C}_2$, then second period optimal consumption in the bad state of nature has to increase, since they are equal. Also the optimal consumption in the good state of nature has to increase by the same amount. It means that the household is consuming more in the second period than before. Following this, the first period consumption has to decrease, which causes higher losses. Finally, the happiness decreases, because of the increase of losses.

3.5 $C_1^* = \bar{C}_1^{P6}$

In this case, the parameter set-up can have six different forms but all of them have common inequality $\bar{C}_1 > Y_1 + \frac{Y_{2b} - \bar{C}_2}{1+r_f}$. It means that the household is also more self-improving and with high aspirations, since the first period consumption reference level has the lower limit. However, it is also limited from becoming too large by either $\bar{C}_1 < \min\left\{Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}, \bar{C}_1^{T1}\right\}$ or $\bar{C}_1 < \min\left\{Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}, \bar{C}_1^{T}\right\}$. Following proposition summarizes the conditions under which $C_1^* = \bar{C}_1^{P6}$.

Proposition 5. *For each of the following parameter set-ups*

(1-a) $C_{1L} \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < \bar{C}_1 < \min\left\{Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}, \bar{C}_1^{T1}\right\}$ *and* $\lambda > \max\{0, \lambda_U^{P5}, \lambda_2^{P6}\}$

(1-b) $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < C_{1L} \leq \max\left\{Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}, \bar{C}_1^{T2}\right\} < \bar{C}_1 \leq \min\left\{Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}, \bar{C}_1^{T1}\right\}$ *and* $\lambda > \max\{0, \lambda_1^{P6}, \lambda_2^{P6}\}$

(2-a) $C_{1L} \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq \bar{C}_1 < \min\left\{Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}, \bar{C}_1^{T}\right\}$ *and* $\lambda > \max\{0, \lambda_{U2}^{P4}, \lambda_U^{P5}, \lambda_3^{P6}\}$

(2-b) $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < C_{1L} \leq \max\left\{Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}, \bar{C}_1^{T2}\right\} \leq \bar{C}_1 < \min\left\{Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}, \bar{C}_1^{T}\right\}$ *and* $\lambda > \max\{0, \lambda_{U2}^{P4}, \lambda_U^{P5}, \lambda_3^{P6}, \lambda_1^{P6}\}$

(3-a) $C_{1L} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} < \bar{C}_1 < \bar{C}_1^{T1}$ *and* $\lambda > \max\{0, \lambda_U^{P5}, \lambda_1^{P6}, \lambda_2^{P6}\}$

(3-b) $C_{1L} \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < \bar{C}_1 < \bar{C}_1^{T}$ *and* $\lambda > \max\{0, \lambda_U^{P5}, \lambda_1^{P6}, \lambda_3^{P6}\}$

where

$$
\begin{aligned}
\lambda_{U2}^{P4} &= \frac{(1-p)(1+r_f)k}{(1+r_f)(\bar{C}_1 - Y_1) + \bar{C}_2 - \mathbb{E}(Y_2)}, \\
\lambda_U^{P5} &= \frac{(1+r_f)[1-\delta(1+r_f)]\gamma}{\Omega}, \\
\lambda_1^{P6} &= \frac{[1-\delta(1+r_f)]\gamma}{\bar{C}_1^{T2} - \bar{C}_1}, \\
\lambda_2^{P6} &= \frac{[1-\delta(1+r_f)]\gamma}{\bar{C}_1^{T1} - \bar{C}_1}, \\
\lambda_3^{P6} &= \frac{[1-\delta(1+r_f)]\gamma}{\bar{C}_1^{T} - \bar{C}_1}, \\
\bar{C}_1^{T} &= Y_1 + \frac{Y_{2g} - \bar{C}_2}{1+r_f} + \delta(1-p)(1+r_f)(Y_{2g} - Y_{2b}), \\
\bar{C}_1^{T1} &= Y_1 + \frac{Y_{2b} - C_{2L}}{1+r_f} + \delta(1-p)(1+r_f)(\bar{C}_2 - C_{2L}), \\
\bar{C}_1^{T2} &= C_{1L} + \delta(1-p)(1+r_f)[(1+r_f)(C_{1L} - Y_1) + \bar{C}_2 - Y_{2b}]
\end{aligned}
$$

the optimal solution is $C_1^* = \bar{C}_1^{P6}$ *and the following holds*

$$\begin{aligned} C_{1L} \leq C_1^* &= \bar{C}_1^{P6} < \bar{C}_1 && (25)\\ C_{2g}^* &= Y_{2g} + (1+r_f)(Y_1 - C_1^*) \geq \bar{C}_2 && (26)\\ C_{2L} \leq C_{2b}^* &= Y_{2b} + (1+r_f)(Y_1 - C_1^*) < \bar{C}_2 && (27)\\ & \frac{\lambda}{\gamma_2}\left[1+\delta(1-p)(1+r_f)^2\right]\mathbb{E}(U(\bar{C}_1^{P6})) = \frac{1}{2}\left[1-\delta(1+r_f)\right]^2\gamma^2 \\ & +\lambda\delta\left[(1+(1-p)(1+r_f))\,\Omega + p\,(Y_{2g}-Y_{2b})\left(1+\delta(1-p)(1+r_f)^2\right)\right]\gamma \\ & -\frac{1}{2}\lambda^2\Omega^2\delta(1-p) && (28) \end{aligned}$$

Note that $\bar{C}_1^{P6} = \bar{C}_1 + \frac{1}{1+\delta(1-p)(1+r_f)^2}\left[\delta(1-p)(1+r_f)\Omega + \frac{\gamma}{\lambda}(1-\delta(1+r_f))\right]$.

Proof. Proofs for (1-a) and (1-b) follow directly from Lemma C.7, (2-a) and (2-b) follow directly from Lemma C.8 and finally (3-a) and (3-b) follow directly from Lemma C.9. ■

This solution is the inner solution, since it is not on the border of the feasibility interval. The optimal consumption is below the reference level in the first period. Second period relationships are $C_{2b}^* < \bar{C}_2 \leq C_{2g}^*$. The solution is reached in the problem (P6). It means that the household is in the domain of losses during the first period. In the second period, if the nature is in the good state, it is in the domain of gains otherwise in domain of losses. It is the example where household is more likely to compare itself to the neighbours with higher economic status.

Household is a borrower, if the first period income does not cover the optimal consumption, which happens for sufficiently large first period consumption reference level:

$$Y_1 - \frac{1}{1+\delta(1-p)(1+r_f)^2}\left[\delta(1-p)(1+r_f)\Omega + \frac{\gamma}{\lambda}(1-\delta(1+r_f))\right] < \bar{C}_1.$$

Note that if the household is a borrower, it has the reference level of consumption higher than the income in the first period.

The sensitivity analysis results are presented in Table 5.

Changes in loss aversion parameter depend on the relation between $\frac{1}{1+r_f}$ and δ. If $\delta < \frac{1}{1+r_f}$ than increase in loss-aversion causes decrease of the optimal consumption in the first period. Household has the tendency to spend less in the first period for consumption, because δ is smaller. This has impact on relative consumption in the first period and savings. The first period losses increase and savings increases as well.

	dC_1^*	dC_{2g}^*	dC_{2b}^*	$d(\bar{C}_1 - C_1^*)$	$d(C_{2g}^* - \bar{C}_2)$	$d(\bar{C}_2 - C_{2b}^*)$	dS^*	$d(\mathbb{E}(U(C_1^*)))$
(1) $d\lambda$	< 0	> 0	> 0	> 0	> 0	< 0	> 0	< 0
(2) $d\lambda$	$= 0$	$= 0$	$= 0$	$= 0$	$= 0$	$= 0$	$= 0$	< 0
(3) $d\lambda$	> 0	< 0	< 0	< 0	< 0	> 0	< 0	< 0
$d\bar{C}_1$	> 0	< 0	< 0	> 0	< 0	> 0	< 0	< 0
$d\bar{C}_2$	< 0	> 0	> 0	> 0	< 0	> 0	> 0	< 0

(1) $d\lambda$ holds for $\delta < \frac{1}{1+r_f}$, (2) $d\lambda$ holds for $\delta = \frac{1}{1+r_f}$, (3) $d\lambda$ holds for $\delta > \frac{1}{1+r_f}$

Table 5: Sensitivity results for $\bar{C}_1^{P6}$

On the other hand, if $\delta > \frac{1}{1+r_f}$ then optimal consumption in the first period increases, household wants to decrease its losses. It means that the savings decreases. In both cases the happiness decreases with increasing loss-aversion, since the penalty is higher than reward in the domain of gains, which are only in the good state of nature in the second period.

If the household increases the first period reference level, then the optimal value increases in the first period as well. Again, the household wants to avoid losses. Second period consumptions get smaller. Losses in the first period and in the second period in the bad state of nature become larger and the overall happiness decreases. Since the household consumes more in the first period also the savings decreases.

Increase of second period reference level means that the optimal consumption in the first period decreases, so second period consumptions increase. Household is not able to lower the losses in the first period when the award from gains is almost still. This means that the savings increases and happiness decreases.

3.6 $C_1^* = Y_1 + \frac{Y_{2b} - C_{2L}}{1+r_f}$

Specific parameter set-up is notated in the following proposition.

Proposition 6. *For each of the following parameter set-ups*

(1-a) $C_{1L} \leq Y_1 + \frac{Y_{2b} - \bar{C}_2}{1+r_f} \leq Y_1 + \frac{Y_{2b} - C_{2L}}{1+r_f} \leq \bar{C}_1^{T1} < \bar{C}_1 \leq Y_1 + \frac{Y_{2g} - \bar{C}_2}{1+r_f}$ *and* $\lambda > \max\{0, \lambda_U^{P5}, \lambda_2^{P6}\}$

(1-b) $Y_1 + \frac{Y_{2b} - \bar{C}_2}{1+r_f} < C_{1L} \leq Y_1 + \frac{Y_{2b} - C_{2L}}{1+r_f} \leq \bar{C}_1^{T1} < \bar{C}_1 \leq Y_1 + \frac{Y_{2g} - \bar{C}_2}{1+r_f}$ *and* $\lambda >$

$\max\{0, \lambda_1^{P6}, \lambda_2^{P6}\}$

(2-a) $C_{1L} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq \max\left\{Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}, \bar{C}_1^{T1}\right\} < \bar{C}_1$ *and* $\lambda > \max\{0, \lambda_U^{P5}, \lambda_1^{P6}, \lambda_2^{P6}\}$

(2-b) $C_{1L} \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} \leq \bar{C}_1^{T3} < \bar{C}_1$ *and* $\lambda > \max\{0, \lambda_U^{P5}, \lambda_1^{P6}, \lambda_3^{P6}, \lambda_1^{P8}\}$

(2-c) $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq C_{1L} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} \leq \bar{C}_1^{T3} < \bar{C}_1$ *and* $\lambda > \max\{0, \lambda_1^{P8}, \lambda_2^{P8}\}$

where

$$
\begin{aligned}
\bar{C}_1^{T3} &= Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} + \delta(1+r_f)[\bar{C}_2 - C_{2L} - p(Y_{2g} - Y_{2b})] \\
\lambda_1^{P8} &= \frac{[1-\delta(1+r_f)]\gamma}{\bar{C}_1^{T3} - \bar{C}_1} \\
\lambda_2^{P8} &= \frac{[1-\delta(1+r_f)]\gamma}{\bar{C}_1^{T4} - \bar{C}_1}
\end{aligned}
$$

the optimal solution is $C_1^* = Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ *and the following holds*

$$
\begin{aligned}
C_{1L} \leq C_1^* &= Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < \bar{C}_1 && (29) \\
C_{2g}^* &= Y_{2g} - Y_{2b} + C_{2L} \geq \bar{C}_2 \quad \textit{in cases } (1-a),\ (1-b),\ (2-a) && (30) \\
C_{2g}^* &= Y_{2g} - Y_{2b} + C_{2L} \leq \bar{C}_2 \quad \textit{in cases } (2-b), (2-c) && (31) \\
C_{2b}^* &= C_{2L} < \bar{C}_2 && (32) \\
\frac{1}{\gamma_2}\mathbb{E}(U(C_1^*)) &= -\gamma\left(\bar{C}_1 - Y_1 - \frac{Y_{2b}-C_{2L}}{1+r_f}\right) - \frac{\lambda}{2}\left(\bar{C}_1 - Y_1 - \frac{Y_{2b}-C_{2L}}{1+r_f}\right)^2 \\
&\quad + \delta p\,\gamma(Y_{2g} - Y_{2b} + C_{2L} - \bar{C}_2) \\
&\quad - \delta(1-p)\gamma(\bar{C}_2 - C_{2L}) - \frac{\lambda}{2}(\bar{C}_2 - C_{2L})^2 && (33)
\end{aligned}
$$

Proof. Proof for (1-a) and (1-b) follow directly from Lemma C.7, (2-a), (2-b) and (2-c) follow directly from Lemma C.9. ■

Relationships between reference values of consumption and optimal consumptions are same as in the previous case, which correspondents with problem (P6) for cases (1-a), (1-b), (2-a). Household is in the domain of losses in the first period and second period in the bad state of nature. Only in the second period and good state of nature it is in the domain of gains. Also C_{2b} equals minimal value, so the household is consuming most of the income in the first period. The good state consumption is close to the difference between the incomes in the second period. For cases (2-b) and (2-c) household is in the second period in the domain of losses independently of the state of the nature. It means that it has stronger self-enhancement motives and higher aspirations than cases before. The maximum is reached in the problem (P8).

Note that savings are given by $\frac{C_{2L}-Y_{2b}}{1+r_f}$. Household is a borrower when the second period income in the bad state of the nature is large enough to cover the minimal consumption in the second period.

In the sensitivity analysis shows that the analyzed parameters influence just a few optimal values, see in Table 6 for cases (1-a), (1-b) and (2-a) and in Table 7 for cases (2-b) and (2-c).

	dC_1^*	dC_{2g}^*	dC_{2b}^*	$d(\bar{C}_1 - C_1^*)$	$d(C_{2g}^* - \bar{C}_2)$	$d(\bar{C}_2 - C_{2b}^*)$	dS^*	$d(\mathbb{E}(U(C_1^*)))$
$d\lambda$	$=0$	$=0$	$=0$	$=0$	$=0$	$=0$	$=0$	<0
$d\bar{C}_1$	$=0$	$=0$	$=0$	>0	$=0$	$=0$	$=0$	<0
$d\bar{C}_2$	$=0$	$=0$	$=0$	$=0$	<0	>0	$=0$	<0

Table 6: Sensitivity analysis for $Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ for cases (1-a), (1-b), (2-a)

	dC_1^*	dC_{2g}^*	dC_{2b}^*	$d(\bar{C}_1 - C_1^*)$	$d(\bar{C}_2 - C_{2g}^*)$	$d(\bar{C}_2 - C_{2b}^*)$	dS^*	$d(\mathbb{E}(U(C_1^*)))$
$d\lambda$	$=0$	$=0$	$=0$	$=0$	$=0$	$=0$	$=0$	<0
$d\bar{C}_1$	$=0$	$=0$	$=0$	>0	$=0$	$=0$	$=0$	<0
$d\bar{C}_2$	$=0$	$=0$	$=0$	$=0$	>0	>0	$=0$	<0

Table 7: Sensitivity analysis for $Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ for cases (2-b), (2-c)

The loss-aversion parameter has impact only on the happiness. If the household becomes more loss-averse, then the losses are more penalized. In this case the household is in the domain of losses in the first period and in the second in the bad state of nature for cases (1-a), (1-b), (2-a). There is no domain of gains for the household in the cases when $C_{2g}^* \leq \bar{C}_2$. Since the optimal values of consumptions are not affected, then the happiness has to decrease. The solution is on the border of the interval of feasibility, so λ has no impact on the optimal consumptions.

Changes in reference levels in both periods have only impact on the relative reference consumption. They change the amount of losses and gains. This also influence happiness. By increase in the reference levels the losses get larger (or gains smaller), so the happiness decreases.

3.7 $C_1^* = C_{1L}$

The parameter set-up is such that household's optimal consumption during the first period is the smallest possible. It means that the household is having as large savings as possible to use them in the second period.

Note that $\max\left\{Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}, Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}\right\} < \bar{C}_1$ - reference level is larger than in the previous cases, so household has self-improvement motivates. In the following proposition are specific parameter set-ups.

Proposition 7. *For each of the following parameter set-ups*

(1) $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < C_{1L} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < \bar{C}_1 < \min\left\{Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}, \bar{C}_1^{T2}\right\}$ *and* $\lambda > \max\{0, \lambda_1^{P6}, \lambda_2^{P6}\}$

(2) $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} \leq C_{1L} \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq \bar{C}_1 < \min\left\{Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}, \bar{C}_1^{T2}\right\}$ *and* $\lambda > \max\{0, \lambda_{U2}^{P4}, \lambda_U^{P5}, \lambda_1^{P6}, \lambda_3^{P6}\}$

(3) $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq C_{1L} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < \bar{C}_1 < \bar{C}_1^{T4}$ *and* $\lambda > \max\{0, \lambda_1^{P8}, \lambda_2^{P8}\}$

the optimal solution is $C_1^* = C_{1L}$ *and the following holds*

$$C_1^* = C_{1L} \leq \bar{C}_1 \tag{34}$$

$$C_{2g}^* = Y_{2g} + (1+r_f)(Y_1 - C_{1L}) \geq \bar{C}_2 \quad \textit{for (1) and (2)} \tag{35}$$

$$C_{2g}^* = Y_{2g} + (1+r_f)(Y_1 - C_{1L}) \leq \bar{C}_2 \quad \textit{for (3)} \tag{36}$$

$$C_{2L} \leq C_{2b}^* = Y_{2b} + (1+r_f)(Y_1 - C_{1L}) < \bar{C}_2 \tag{37}$$

$$\begin{aligned}\frac{1}{\gamma_2}\mathbb{E}(U(C_{1L})) &= \delta\left[(1+r_f)(Y_1 - C_{1L}) + \mathbb{E}(Y_2) - \bar{C}_2\right]\gamma - (\bar{C}_1 - C_{1L})\gamma \\ &\quad -\frac{\lambda}{2}\left[(\bar{C}_1 - C_{1L})^2 + \delta(1-p)\left((1+r_f)(Y_1 - C_{1L}) + Y_{2b} - \bar{C}_2\right)^2\right]\end{aligned} \tag{38}$$

Proof. Proof for (1) follows directly from Lemma C.7, (2) follows directly from Lemma C.8 and (3) from Lemma C.9. ■

Note that the maximum can be reached in the problem (P6) and (P8). The household is penalized by losses in the first period and in the bad state of nature in the second period and also in the good state of nature in the case (3). Also note that the first period consumption is at the lower border, so household has the largest savings. Second period consumption levels differ by the difference in income levels in the bad and in the good state of nature.

In this optimal solution, savings are positive if income in the first period is sufficiently large to cover the minimal consumption in the first period and the household is a lender.

Sensitivity Table 8 and 9 is similar to the previous one, when the solution is on the border of the feasibility interval.

	dC_1^*	dC_{2g}^*	dC_{2b}^*	$d(\bar{C}_1 - C_1^*)$	$d(C_{2g}^* - \bar{C}_2)$	$d(\bar{C}_2 - C_{2b}^*)$	dS^*	$d(\mathbb{E}(U(C_1^*)))$
$d\lambda$	$= 0$	$= 0$	$= 0$	$= 0$	$= 0$	$= 0$	$= 0$	< 0
$d\bar{C}_1$	$= 0$	$= 0$	$= 0$	> 0	$= 0$	$= 0$	$= 0$	< 0
$d\bar{C}_2$	$= 0$	$= 0$	$= 0$	$= 0$	< 0	> 0	$= 0$	< 0

Table 8: Sensitivity analysis for C_{1L} for cases (1) and (2)

	dC_1^*	dC_{2g}^*	dC_{2b}^*	$d(\bar{C}_1 - C_1^*)$	$d(\bar{C}_2 - C_{2g}^*)$	$d(\bar{C}_2 - C_{2b}^*)$	dS^*	$d(\mathbb{E}(U(C_1^*)))$
$d\lambda$	$= 0$	$= 0$	$= 0$	$= 0$	$= 0$	$= 0$	$= 0$	< 0
$d\bar{C}_1$	$= 0$	$= 0$	$= 0$	> 0	$= 0$	$= 0$	$= 0$	< 0
$d\bar{C}_2$	$= 0$	$= 0$	$= 0$	$= 0$	> 0	> 0	$= 0$	< 0

Table 9: Sensitivity analysis for C_{1L} for case (3)

The loss-aversion parameter affects only happiness, due to the character of the solution. If the loss-aversion increases then the penalty in the domain of losses increases as well and the happiness has to decrease.

Changes in the reference levels in both periods have impact only on the size of the losses or gains and happiness. If any of them increases its value, then the happiness has to decrease. Households is experiencing larger losses and smaller gains in the cases (1) and (2), which means larger penalty and less reward. In the case (3) there are no gains, so by increasing the reference levels household just increases the self-improving motive and has larger losses.

3.8 $C_1^* = \bar{C}_1^{P8}$

The last solution that we present is inside the interval of feasibility. The conditions under which it is the optimal value of consumption in the first period are presented in the following proposition.

Proposition 8. *For each of the following parameter set-ups*

(1-a) $C_{1L} \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq \bar{C}_1^T < \bar{C}_1 \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ *and* $\lambda > \max\{0, \lambda_{U2}^{P4}, \lambda_U^{P5}, \lambda_3^{P6}\}$

(1-b) $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < C_{1L} \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq \bar{C}_1^T < \bar{C}_1 \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ *and* $\lambda > \max\{0, \lambda_{U2}^{P4}, \lambda_U^{P5}, \lambda_1^{P6}, \lambda_3^{P6}\}$

(1-c) $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} < C_{1L} \leq \bar{C}_1 \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ *and* $\lambda > \max\{0, \lambda_{U2}^{P4}, \lambda_3^{P6}\}$

(2-a) $C_{1L} \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} \leq \max\left\{Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}, \bar{C}_1^T\right\} < \bar{C}_1 < \bar{C}_1^{T3}$ *and* $\lambda > \max\{0, \lambda_U^{P5}, \lambda_1^{P6}, \lambda_3^{P6}, \lambda_1^{P8}\}$

(2-b) $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq C_{1L} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} \leq \max\left\{Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}, \bar{C}_1^T\right\} < \bar{C}_1 < \bar{C}_1^{T3}$ *and* $\lambda > \max\{0, \lambda_1^{P8}, \lambda_2^{P8}\}$

the optimal solution is $C_1^* = \bar{C}_1^{P8}$ *and the following holds*

$$
\begin{aligned}
C_{1L} \leq C_1^* &= \bar{C}_1^{P8} < \bar{C}_1 && (39)\\
C_{2L} < C_{2g}^* &= Y_{2g} + (1+r_f)(Y_1 - C_1^*) \leq \bar{C}_2 && (40)\\
C_{2L} \leq C_{2b}^* &= Y_{2b} + (1+r_f)(Y_1 - C_1^*) < \bar{C}_2 && (41)\\
\frac{1}{\gamma_2}\mathbb{E}(U(C_1^*)) &= \gamma(C_1^* - \bar{C}_1) - \lambda\frac{1}{2}(\bar{C}_1 - C_1^*)^2 \\
&\quad +\delta p \left[\gamma\left((1+r_f)(Y_1 - C_1^*) + Y_{2g} - \bar{C}_2\right)\right. \\
&\quad \left. -\lambda\frac{1}{2}\left(\bar{C}_2 - (1+r_f)(Y_1 - C_1^*) - Y_{2g}\right)^2\right] \\
&\quad +\delta(1-p)\left[\gamma\left((1+r_f)(Y_1 - C_1^*) + Y_{2b} - \bar{C}_2\right)\right. \\
&\quad \left. -\lambda\frac{1}{2}\left(\bar{C}_2 - (1+r_f)(Y_1 - C_1^*) - Y_{2b}\right)^2\right] && (42)
\end{aligned}
$$

where $\bar{C}_1^{P8} = \frac{1}{\lambda}\frac{[1-\delta(1+r_f)]\gamma}{1+\delta(1+r_f)^2} + \frac{\bar{C}_1 - \delta(1+r_f)[\bar{C}_2 - (1+r_f)Y_1 - \mathbb{E}(Y_2)]}{1+\delta(1+r_f)^2}$.

Proof. Proof for (1-a), (1-b) and (1-c) follow directly from Lemma C.8, (2-a) and (2-b) follow directly from Lemma C.9. ■

In this case, optimal consumptions in both periods and both states are below the reference levels. The maximum is reached in the problem (P8). The household is experiencing losses in all scenarios. Also household is driven by self-improvement motives. This motive is stronger than in the cases before, since both reference levels in the second period are larger than the optimal value.

Household is a lender, if the first period income covers the optimal consumption, which happens for sufficiently small first period consumption:

$$Y_1 + \delta(1+r_f)^2 \left[2Y_1 + \frac{\mathbb{E}(Y_2) - \bar{C}_2}{1+r_f}\right] - \frac{\gamma}{\lambda}[1 - \delta(1+r_f)] > \bar{C}_1.$$

Sensitivity analysis shows, that all the parameters given by the household have impact on the optimal values, see Table 10 below.

	dC_1^*	dC_{2g}^*	dC_{2b}^*	$d(\bar{C}_1 - C_1^*)$	$d(\bar{C}_2 - C_{2g}^*)$	$d(\bar{C}_2 - C_{2b}^*)$	dS^*	$d(\mathbb{E}(U(C_1^*)))$
(1) $d\lambda$	< 0	> 0	> 0	> 0	< 0	< 0	> 0	< 0
(2) $d\lambda$	$= 0$	$= 0$	$= 0$	$= 0$	$= 0$	$= 0$	$= 0$	< 0
(3) $d\lambda$	> 0	< 0	< 0	< 0	> 0	> 0	< 0	< 0
$d\bar{C}_1$	> 0	< 0	< 0	> 0	> 0	> 0	< 0	< 0
$d\bar{C}_2$	< 0	> 0	> 0	< 0	> 0	> 0	> 0	< 0

(1) $d\lambda$ holds for $\delta < \frac{1}{1+r_f}$, (2) $d\lambda$ holds for $\delta = \frac{1}{1+r_f}$, (3) $d\lambda$ holds for $\delta > \frac{1}{1+r_f}$

Table 10: Sensitivity results for $\bar{C}_1^{P8}$

We need to consider three cases - (1) where $\delta < \frac{1}{1+r_f}$, (2) where $\delta = \frac{1}{1+r_f}$ and (3) where $\delta > \frac{1}{1+r_f}$.

In the case (1) by increasing loss-aversion parameter, the optimal consumption in the first period drops, however, the second period consumptions increase. It means that the household is saving in the first period to have larger consumption in the second period. The first period loss increases but the second period losses decrease. In the case (3) loss-aversion has exactly opposite influence, household spends more in the first period and less in the second period. Savings are decreasing. In both cases happiness decreases, because the penalty is higher and changes in the consumptions are not sufficient. In the case (2) optimal consumption in the first period does not change and it neither does in the second period. Household does not change savings and the only change is in the penalties, which causes less happiness.

If the first period consumption reference level increases, then the first period optimal consumption increases. This causes the decrease in the second period consumption, since household is spending more in the first period than in the second one. Also happiness decreases, because losses increase.

Increase in the second period consumption reference level causes decrease in the optimal first period consumption. Household has higher savings and spends more in the second period to lower the losses. However, the happiness is still decreasing.

Conclusion

Our work describes the main features of prospect theory - utility function and weight function, as well as the motivation, which led Kahneman and Tversky to publish their study [11]. Prospect theory takes into consideration that people tend to be risk averse in the domain of losses and risk seeking in the domain of gains. We are using two period consumption-investment model for a household to explore the optimal consumption while household maximizes the overall happiness. We assume that the household is highly loss averse. We also take into consideration that the second period income is uncertain with two possible scenarios.

Our utility function is quadratic loss averse, which means it is concave in the domain of losses, linear in the domain of gains and is based on the differences of the reference levels and corresponding consumptions. We found the close form solutions for every parameter set-up. Results presented in the last chapter depend on the relationships between parameters, especially, the position of the first period reference level of consumption, $\bar{C}_1$. If the household has low first period reference level of consumption, we assume that it has self-enhancement motives. Propositions 1 and 2 holds when $\bar{C}_1$ is low. The optimal values are sensitive to the risk averse parameter λ. Sensitivity of the solutions depends on the relation between δ and r_f. The optimal values are sensitive to the second period reference level for $\delta < \frac{1}{1+r_f}$, otherwise to the first period reference level. Also this relation influences whether in the first period the household is in the domain of gains or losses, respectively. The optimal value stated in the proposition 3 holds for larger first period reference level than in the previous cases. It means that the household is becoming less self-enhanced. The optimal consumption is equal to the first period reference level. The solution holds for δ lower than in the next case stated in the proposition 4. In this case, second period reference level of consumption has the influence on the optimal values. The household is also becoming less self-enhanced than in the first two cases. The other four solutions stated in the propositions 5, 6, 7 and 8 hold for self-improving households. It means that the reference level is relatively large in contrast to the previous conditions. Household experiences losses in all four cases in the first period and in the bad state of nature in the second period. In the good state of the nature it varies. Optimal values, that are inner points of the feasibility interval, depend on the risk-averse parameter.

The study of loss averse household may continue by exploring modifications of the model. Interesting might be comparing the results after introducing risky asset with uncertainty as the possible investment option in the first period. Another modification might be in changing the uncertainty of the second period income and assuming a continuous distribution. Also the comparison with other studies following quadratic averse function with regard to the two period model might bring interesting conclusions.

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Appendix

A Solutions of the Problems

(P1). Note that

$$C_1^{P1,*} = \begin{cases} \bar{C}_1, & \text{for } \delta > \frac{1}{1+r_f} \\ \in \left[\bar{C}_1, Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}\right], & \text{for } \delta = \frac{1}{1+r_f} \\ Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}, & \text{for } \delta < \frac{1}{1+r_f} \end{cases}$$

and

$$\begin{aligned} \frac{1}{\gamma_2}\mathbb{E}(U(C_1^{P1,*})) &= \frac{1}{\gamma_2}\mathbb{E}(U(\bar{C}_1)) = \gamma\delta\left[(1+r_f)(Y_1-\bar{C}_1)+\mathbb{E}(Y_2)-\bar{C}_2\right] \quad (43) \\ &= \gamma\delta[\Omega + p\,(Y_{2g}-Y_{2b})] \;\; \text{for } \delta > \frac{1}{1+r_f} \\ \frac{1}{\gamma_2}\mathbb{E}(U(C_1^{P1,*})) &= \frac{1}{\gamma_2}\mathbb{E}(U(C_1)) = \frac{\gamma}{1+r_f}\left[(1+r_f)(Y_1-\bar{C}_1)+\mathbb{E}(Y_2)-\bar{C}_2\right] \quad (44) \\ & \text{for } \delta = \frac{1}{1+r_f} \;\; \text{and any } C_1 \in \left[\bar{C}_1, Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}\right] \\ \frac{1}{\gamma_2}\mathbb{E}(U(C_1^{P1,*})) &= \frac{1}{\gamma_2}\mathbb{E}\left(U\left(Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}\right)\right) = \gamma\frac{\Omega}{1+r_f} + \delta p\,\gamma(Y_{2g}-Y_{2b}) \quad (45) \\ & \text{for } \delta < \frac{1}{1+r_f} \end{aligned}$$

(P2). It can be shown that the objective function of (P2) is concave and thus its stationary point is the point of its maxima. The stationary point is

$$\bar{C}_1^{P2} = \frac{k}{\lambda} + Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} \qquad (46)$$

where

$$k \equiv \frac{\gamma[1-\delta(1+r_f)]}{\delta(1-p)(1+r_f)^2}$$

Note that k is non-negative for $\delta \le \frac{1}{1+r_f}$. Let's assume that $\bar{C}_1 \le Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$. Then $\bar{C}_1^{P2}$ is feasible for (P2), and thus its maximum ($C_1^{P2,*} = \bar{C}_1^{P2}$), when $\delta \le \frac{1}{1+r_f}$, i.e., $\bar{C}_1^{P2} \ge Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} \ge \bar{C}_1$, and $\lambda \ge \frac{(1+r_f)k}{\min\left\{\bar{C}_2-C_{2L}, Y_{2g}-Y_{2b}\right\}} \equiv \lambda^{P2}$, i.e., $\bar{C}_1^{P2} \le \min\left\{Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}, Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}\right\}$. Thus, the following holds

(P2)-(i) If $\delta \le \frac{1}{1+r_f}$ and $\lambda \ge \lambda^{P2}$ then $C_1^{P2,*} = \bar{C}_1^{P2}$

(P2)-(ii) If $\delta > \frac{1}{1+r_f}$ then $\bar{C}_1^{P2} < Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ and thus $C_1^{P2,*} = Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$

(P2)-(iii) If $0 < \lambda < \lambda^{P2}$ then $\bar{C}_1^{P2} > \min\left\{Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}, Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}\right\}$ and thus $C_1^{P2,*} = \min\left\{Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}, Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}\right\}$

Let $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < \bar{C}_1 \leq \min\left\{Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}, Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}\right\}$. Then

(P2)-(iv) $C_1^{P2,*} = \bar{C}_1^{P2}$ if $\lambda^{P2} \leq \lambda \leq \frac{(1+r_f)k}{-\Omega}$. Note that $\lambda^{P2} \leq \frac{(1+r_f)k}{-\Omega}$ only when $(1+r_f)(\bar{C}_1 - Y_1) \leq Y_{2b} \leq (1+r_f)(\bar{C}_1 - Y_1) + \bar{C}_2 \leq Y_{2g}$ and when $C_{2L} \leq Y_{2b} + (1+r_f)(Y_1 - \bar{C}_1) \leq \bar{C}_2$.

(P2)-(v) If $0 < \lambda < \lambda^{P2}$ then $\bar{C}_1^{P2} > \min\left\{Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}, Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}\right\}$ and thus $C_1^{P2,*} = \min\left\{Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}, Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}\right\}$

(P2)-(vi) If $\lambda > \frac{(1+r_f)k}{-\Omega}$ then $\bar{C}_1^{P2} < \bar{C}_1$ and thus $C_1^{P2,*} = \bar{C}_1$.

Note in addition that

$$\frac{1}{\gamma_2}\mathbb{E}(U(\bar{C}_1^{P2})) = \left[\frac{\Omega}{1+r_f} + \delta p\,(Y_{2g} - Y_{2b}) + \frac{(1-\delta(1+r_f))\,k}{2\lambda}\right]\gamma \tag{47}$$

$$\begin{aligned}
\frac{1}{\gamma_2}\mathbb{E}\left(U\left(Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}\right)\right) &= \left[\frac{\Omega}{1+r_f} + \delta p\,(Y_{2g} - Y_{2b})\right]\gamma \\
\frac{1}{\gamma_2}\mathbb{E}\left(U\left(Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}\right)\right) &= \left[\frac{\Omega}{1+r_f} + \delta p\,(Y_{2g} - Y_{2b})\right]\gamma + \left(\frac{1}{1+r_f} - \delta\right)\gamma\,(\bar{C}_2 - C_{2L}) \\
&\quad - \frac{\delta(1-p)\lambda\left(\bar{C}_2 - C_{2L}\right)^2}{2} \\
\frac{1}{\gamma_2}\mathbb{E}\left(U\left(Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}\right)\right) &= \left(Y_1 - \bar{C}_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}\right)\gamma \\
&\quad -\delta(1-p)\left[\gamma + \frac{\lambda}{2}(Y_{2g} - Y_{2b})\right](Y_{2g} - Y_{2b}) \\
\frac{1}{\gamma_2}\mathbb{E}(U(\bar{C}_1)) &= \delta\left[(1+r_f)(Y_1 - \bar{C}_1) + \mathbb{E}(Y_2) - \bar{C}_2\right]\gamma - \delta(1-p)\frac{\lambda}{2}(-\Omega)^2
\end{aligned} \tag{48}$$

(P4). It can be shown that the objective function of (P4) is concave and thus its stationary point is the point of its maxima. The stationary point is

$$\bar{C}_1^{P4} = \frac{(1-p)k}{\lambda} + Y_1 + \frac{\mathbb{E}(Y_2) - \bar{C}_2}{1+r_f} \tag{49}$$

Before proceeding further, let's introduce the following notation:

$$\lambda_L^{P4} = \frac{(1-p)(1+r_f)k}{\bar{C}_2 - C_{2L} - p(Y_{2g} - Y_{2b})} \tag{50}$$

$$\lambda_U^{P4} = \frac{(1+r_f)k}{Y_{2g} - Y_{2b}} \tag{51}$$

$$\lambda_{U2}^{P4} = \frac{(1-p)(1+r_f)k}{(1+r_f)(\bar{C}_1 - Y_1) + \bar{C}_2 - \mathbb{E}(Y_2)} \tag{52}$$

Let $\bar{C}_1 \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ and $\bar{C}_2 \geq C_{2L} + Y_{2g} - Y_{2b}$. Then

(P4)-(i) $C_1^{P4,*} = \bar{C}_1^{P4}$ if $\delta < \frac{1}{1+r_f}$ and $\lambda_L^{P4} \leq \lambda \leq \lambda_U^{P4}$

(P4)-(ii) If $0 < \lambda < \lambda_L^{P4}$ and $\delta < \frac{1}{1+r_f}$ then $\bar{C}_1^{P4} > Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ and thus $C_1^{P4,*} = Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$

(P4)-(iii) If $\lambda \geq \lambda_U^{P4}$ then $\bar{C}_1^{P4} \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ and thus $C_1^{P4,*} = Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$

Let $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} < \bar{C}_1 \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$, i.e., $\bar{C}_2 \geq C_{2L} + Y_{2g} - Y_{2b}$. Then

(P4)-(iv) $C_1^{P4,*} = \bar{C}_1^{P4}$ if $\lambda_L^{P4} \leq \lambda \leq \lambda_{U2}^{P4}$

(P4)-(v) If $0 < \lambda < \lambda_L^{P4}$ then $\bar{C}_1^{P4} > Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ and thus $C_1^{P4,*} = Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$

(P4)-(vi) If $\lambda > \lambda_{U2}^{P4}$ then $\bar{C}_1^{P4} < \bar{C}_1$ and thus $C_1^{P4,*} = \bar{C}_1$

Note in addition that

$$\begin{aligned}
\frac{1}{\gamma_2}\mathbb{E}(U(\bar{C}_1^{P4})) &= \left(Y_1 - \bar{C}_1 + \frac{\mathbb{E}(Y_2) - \bar{C}_2}{1+r_f}\right)\gamma \\
&\quad + \frac{(1-p)k}{\lambda}[1-\delta(1+r_f)]\gamma \\
&\quad - \frac{\delta\lambda}{2}\left[p(1-p)(Y_{2g}-Y_{2b})^2 + \left(\frac{(1-p)(1+r_f)k}{\lambda}\right)^2\right] \\
\frac{1}{\gamma_2}\mathbb{E}\left(U\left(Y_1 + \frac{Y_{2b} - C_{2L}}{1+r_f}\right)\right) &= \left(Y_1 - \bar{C}_1 + \frac{Y_{2b} - C_{2L}}{1+r_f}\right)\gamma \\
&\quad + \delta[p\,(Y_{2g} - Y_{2b}) - (\bar{C}_2 - C_{2L})]\gamma \\
&\quad - \delta\frac{\lambda}{2}\left[p\,(\bar{C}_2 - C_{2L} - (Y_{2g} - Y_{2b}))^2\right] \\
&\quad - \delta\frac{\lambda}{2}\left[(1-p)(\bar{C}_2 - C_{2L})^2\right] \\
\frac{1}{\gamma_2}\mathbb{E}\left(U\left(Y_1 + \frac{Y_{2g} - \bar{C}_2}{1+r_f}\right)\right) &= \left(Y_1 - \bar{C}_1 + \frac{Y_{2g} - \bar{C}_2}{1+r_f}\right)\gamma \\
&\quad - \delta(1-p)\left[\gamma + \frac{\lambda}{2}(Y_{2g} - Y_{2b})\right](Y_{2g} - Y_{2b}) \\
&= \frac{\Omega}{1+r_f}\gamma + (Y_{2g} - Y_{2b})\left[\frac{1}{1+r_f} - \delta(1-p)\right]\gamma \\
&\quad - \delta(1-p)\frac{\lambda}{2}(Y_{2g} - Y_{2b})^2 \qquad (53)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\gamma_2}\mathbb{E}(U(\bar{C}_1)) &= \delta\left[(1+r_f)(Y_1 - \bar{C}_1) + \mathbb{E}(Y_2) - \bar{C}_2\right]\gamma \\
&\quad - \delta\frac{\lambda}{2}\left[\left((1+r_f)(Y_1 - \bar{C}_1) - \bar{C}_2\right)^2 + \mathbb{E}(Y_2^2)\right] \\
&\quad + \delta\lambda\left[(1+r_f)(Y_1 - \bar{C}_1) - \bar{C}_2\right]\mathbb{E}(Y_2)
\end{aligned}$$

(P5). It can be shown that the objective function of (P5) is concave and thus its stationary point is the point of its maxima. The stationary point is

$$\bar{C}_1^{P5} = \bar{C}_1 + \frac{\gamma}{\lambda}[1 - \delta(1+r_f)] \qquad (54)$$

Before proceeding further, let's introduce the following notation:

$$\begin{aligned}
\lambda_L^{P5} &= \frac{\gamma[\delta(1+r_f) - 1]}{\bar{C}_1 - C_{1L}} \qquad (55) \\
\lambda_U^{P5} &= \frac{\gamma(1+r_f)[\delta(1+r_f) - 1]}{(1+r_f)(\bar{C}_1 - Y_1) + \bar{C}_2 - Y_{2b}} \qquad (56)
\end{aligned}$$

The following can be observed for $C_{1L} \leq Y_1 + \frac{Y_{2b} - \bar{C}_2}{1+r_f}$

(P5)-(i) For $\delta < \frac{1}{1+r_f}$ is the objective function of (P5) increasing in its set of feasible solutions and thus $C_1^{P5,*} = \min\left\{\bar{C}_1, Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}\right\}$

(P5)-(ii) If $\delta \geq \frac{1}{1+r_f}$ and $\bar{C}_1 \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ then $C_1^{P5,*} = \bar{C}_1^{P5}$ if $\lambda \geq \lambda_L^{P5}$. If $0 < \lambda < \lambda_L^{P5}$ then $C_1^{P5,*} = C_{1L}$.

(P5)-(iii) If $\delta \geq \frac{1}{1+r_f}$ and $\bar{C}_1 > Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ then $C_1^{P5,*} = \bar{C}_1^{P5}$ if $\lambda_L^{P5} \leq \lambda \leq \lambda_U^{P5}$. If $\lambda < \lambda_L^{P5}$ then $C_1^{P5,*} = C_{1L}$ and if $\lambda > \lambda_U^{P5}$ then $C_1^{P5,*} = Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$.

Note in addition that

$$\begin{aligned}
\frac{1}{\gamma_2}\mathbb{E}(U(\bar{C}_1^{P5})) &= \delta\left[(1+r_f)(Y_1-\bar{C}_1)+\mathbb{E}(Y_2)-\bar{C}_2\right]\gamma \\
&\quad +\frac{\gamma^2}{2\lambda}[1-\delta(1+r_f)]^2 \\
&= \delta\left[\Omega+p(Y_{2g}-Y_{2b})\right]\gamma+\frac{\gamma^2}{2\lambda}[1-\delta(1+r_f)]^2 \qquad (57)\\
\frac{1}{\gamma_2}\mathbb{E}(U(\bar{C}_1)) &= \delta\left[(1+r_f)(Y_1-\bar{C}_1)+\mathbb{E}(Y_2)-\bar{C}_2\right]\gamma \\
&= \delta\left[\Omega+p\left(Y_{2g}-Y_{2b}\right)\right]\gamma \qquad (58)\\
\frac{1}{\gamma_2}\mathbb{E}\left(U\left(Y_1+\frac{Y_{2b}-\bar{C}_2}{1+r_f}\right)\right) &= \frac{\Omega\gamma}{1+r_f}-\frac{\lambda}{2}\left(\frac{\Omega}{1+r_f}\right)^2+\delta p\left(Y_{2g}-Y_{2b}\right)\gamma \qquad (59)\\
\frac{1}{\gamma_2}\mathbb{E}(U(C_{1L})) &= -(\bar{C}_1-C_{1L})\gamma-\frac{\lambda}{2}(\bar{C}_1-C_{1L})^2 \\
&\quad +\delta[(1+r_f)(Y_1-C_{1L})+\mathbb{E}(Y_2)-\bar{C}_2]\gamma \qquad (60)
\end{aligned}$$

(P6). It can be shown that the objective function of (P6) is concave and thus its stationary point is the point of its maxima. The stationary point is

$$\bar{C}_1^{P6} = \bar{C}_1 + \frac{1}{1+\delta(1-p)(1+r_f)^2}\left[\delta(1-p)(1+r_f)\Omega+\frac{\gamma}{\lambda}(1-\delta(1+r_f))\right] \qquad (61)$$

Before proceeding further, let's introduce the following notation:

$$\bar{C}_1^T = Y_1 + \frac{Y_{2g} - \bar{C}_2}{1 + r_f} + \delta(1-p)(1+r_f)(Y_{2g} - Y_{2b}) \tag{62}$$

$$\bar{C}_1^{T1} = Y_1 + \frac{Y_{2b} - C_{2L}}{1 + r_f} + \delta(1-p)(1+r_f)(\bar{C}_2 - C_{2L}) \tag{63}$$

$$\bar{C}_1^{T2} = C_{1L} + \delta(1-p)(1+r_f)[(1+r_f)(C_{1L} - Y_1) + \bar{C}_2 - Y_{2b}] \tag{64}$$

$$\lambda_2^{P6} = \frac{[1 - \delta(1+r_f)]\gamma}{\bar{C}_1^{T1} - \bar{C}_1} \tag{65}$$

$$\lambda_1^{P6} = \frac{[1 - \delta(1+r_f)]\gamma}{\bar{C}_1^{T2} - \bar{C}_1} \tag{66}$$

$$\lambda_3^{P6} = \frac{[1 - \delta(1+r_f)]\gamma}{\bar{C}_1^{T} - \bar{C}_1} \tag{67}$$

Note that $\bar{C}_1^{T2} \leq \bar{C}_1^{T1}$ which follows from (8). Note in addition that $\bar{C}_1^{T2} \leq \bar{C}_1^{T}$ if $C_{1L} \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ and $\bar{C}_1^{T2} \geq \bar{C}_1^{T}$ if $C_{1L} \geq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$.

The following can be observed for $C_{1L} \leq Y_1 + \frac{1}{1+r_f} \min\left\{Y_{2b} - C_{2L}, Y_{2g} - \bar{C}_2\right\}$ and $\bar{C}_1 \geq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$

(P6)-(i) Let $C_{1L} \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$[1] and $\bar{C}_1 \leq \min\left\{Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}, Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}\right\}$. Then $\bar{C}_1^{P6}$ is feasible, i.e., $C_1^{P6,*} = \bar{C}_1^{P6}$, if $\lambda \geq \max\{0, \lambda_{L1}^{P6}, \lambda_{L2}^{P6}\}$ where $\lambda_{L1}^{P6} \equiv \frac{[\delta(1+r_f)-1]\gamma}{\delta(1-p)(1+r_f)\Omega}$ and $\lambda_U^{P5} = \frac{(1+r_f)[1-\delta(1+r_f)]\gamma}{\Omega}$

(P6)-(ii) Let $C_{1L} > Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ and $\bar{C}_1 \leq \min\left\{Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}, Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}\right\}$. Then $C_1^{P6,*} = \bar{C}_1^{P6}$ if following holds: (i) $\bar{C}_2 < \frac{C_{1L}-\bar{C}_1}{\delta(1-p)(1+r_f)} + (1+r_f)(Y_1 - C_{1L}) + Y_{2b}$[2] and $\lambda \geq \max\{0, \lambda_{L1}^{P6}, \lambda_1^{P6}\}$ where $\lambda_1^{P6} \equiv \frac{[\delta(1+r_f)-1]\gamma}{\bar{C}_1 - C_{1L} + \delta(1-p)(1+r_f)[(1+r_f)(Y_1-C_{1L})+Y_{2b}-\bar{C}_2]}$ or (ii) $\bar{C}_2 > \frac{C_{1L}-\bar{C}_1}{\delta(1-p)(1+r_f)} + (1+r_f)(Y_1 - C_{1L}) + Y_{2b}$ and $\max\{0, \lambda_{L1}^{P6}\} \leq \lambda \leq \lambda_1^{P6}$.

(P6)-(iii) Let $C_{1L} \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$, $Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ and $\bar{C}_1 \geq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$. Thus, the set of feasible solutions is: $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} \leq C_1 \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$. Then $C_1^{P6,*} = \bar{C}_1^{P6}$ if (i) $\bar{C}_1 > \bar{C}_1^{T1}$ and $\max\{0, \lambda_U^{P5}\} \leq \lambda \leq \lambda_2^{P6}$, or (ii) $\bar{C}_1 < \bar{C}_1^{T1}$ and $\lambda \geq \max\{0, \lambda_U^{P5}, \lambda_2^{P6}\}$. Note that if $\lambda = \lambda_2^{P6}$ then $C_1^{P6,*} = Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ and if $\lambda = \lambda_U^{P5}$ then $C_1^{P6,*} = Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$. Thus, in case (i), $C_1^{P6,*} = Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ if $\lambda \geq \max\{0, \lambda_U^{P5}, \lambda_2^{P6}\}$.

(P6)-(iv) Let $C_{1L} > Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$, $Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ and $\bar{C}_1 \geq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$. Thus,

[1] I.e., $(1+r_f)(Y_1 - \bar{C}_1) + Y_{2b} \leq \bar{C}_2 \leq (1+r_f)(Y_1 - C_{1L}) + Y_{2b}$.

[2] Note that for Ω to be negative we need in this case $\delta > \frac{1}{(1+r_f)(1-p)}$.

the set of feasible solutions is: $C_{1L} \leq C_1 \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$. Then the following holds for $\lambda \geq \max\{0, \lambda_1^{P6}, \lambda_2^{P6}\}$:

(P6)-(iv)-(1) $C_1^{P6,*} = C_{1L}$ when $\bar{C}_1 < \bar{C}_1^{T2}$

(P6)-(iv)-(2) $C_1^{P6,*} = \bar{C}_1^{P6}$ when $\bar{C}_1^{T2} < \bar{C}_1 < \bar{C}_1^{T1}$

(P6)-(iv)-(3) $C_1^{P6,*} = Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ when $\bar{C}_1 > \bar{C}_1^{T1}$

(P6)-(v) Let $C_{1L} \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ and $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq \bar{C}_1 \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ and thus the set of feasible solutions is: $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} \leq C_1 \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$. Then $C_1^{P6,*} = \bar{C}_1^{P6}$ if the following holds: (i) $\bar{C}_1 < \bar{C}_1^T$ and $\lambda \geq \max\{0, \lambda_U^{P5}, \lambda_3^{P6}\}$ or (ii) $\bar{C}_1 > \bar{C}_1^T$ and $\max\{0, \lambda_U^{P5}\} \leq \lambda \leq \lambda_3^{P6}$. In case (ii) $\delta > \frac{1}{1+r_f}$ is necessary for feasibility of λ. Note that if $\lambda = \lambda_3^{P6}$ then $C_1^{P6,*} = Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ and thus in case (ii) if $\lambda \geq \max\{0, \lambda_U^{P5}, \lambda_3^{P6}\}$ then $C_1^{P6,*} = Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$. Note finally that if $\bar{C}_1 = \bar{C}_1^T$, $\delta \geq \frac{1}{1+r_f}$ and $\lambda \geq \max\{0, \lambda_U^{P5}\}$ then $C_1^{P6,*} = \bar{C}_1^{P6}$ and if $\bar{C}_1 = \bar{C}_1^T$, $\delta < \frac{1}{1+r_f}$ and $\lambda \geq \max\{0, \lambda_U^{P5}\}$ then $C_1^{P6,*} = Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$.

(P6)-(vi) Let $C_{1L} > Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ and $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq \bar{C}_1 \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ and thus the set of feasible solutions is: $C_{1L} \leq C_1 \leq \frac{Y_{2g}-\bar{C}_2}{1+r_f}$. Then the following holds (based also on the results of (P6)-(v)) for $\lambda > \max\{0, \lambda_1^{P6}, \lambda_3^{P6}\}$:

(P6)-(vi)-(1) $C_1^{P6,*} = C_{1L}$ when $\bar{C}_1 < \bar{C}_1^{T2}$

(P6)-(vi)-(2) $C_1^{P6,*} = \bar{C}_1^{P6}$ when $\bar{C}_1^{T2} < \bar{C}_1 < \bar{C}_1^T$

(P6)-(vi)-(3) $C_1^{P6,*} = Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ when $\bar{C}_1 > \bar{C}_1^T$

Note in addition that

$$\frac{\lambda}{\gamma_2}\left[1+\delta(1-p)(1+r_f)^2\right]\mathbb{E}(U(\bar{C}_1^{P6})) = \frac{1}{2}\left[1-\delta(1+r_f)\right]^2\gamma^2$$
$$+\lambda\delta\left[(1+(1-p)(1+r_f))\,\Omega + p\,(Y_{2g}-Y_{2b})\left(1+\delta(1-p)(1+r_f)^2\right)\right]\gamma$$
$$-\frac{1}{2}\lambda^2\Omega^2\delta(1-p)$$

$$
\begin{aligned}
\frac{1}{\gamma_2}\mathbb{E}\left(U\left(Y_1+\frac{Y_{2b}-\bar{C}_2}{1+r_f}\right)\right) &= \frac{\Omega}{1+r_f}\gamma-\frac{\lambda}{2}\left(\frac{\Omega}{1+r_f}\right)^2+\delta p\,(Y_{2g}-Y_{2b})\gamma \\
\frac{1}{\gamma_2}\mathbb{E}(U(C_{1L})) &= \delta\left[(1+r_f)(Y_1-C_{1L})+\mathbb{E}(Y_2)-\bar{C}_2\right]\gamma \\
&\quad -(\bar{C}_1-C_{1L})\gamma-\frac{\lambda}{2}(\bar{C}_1-C_{1L})^2 \\
&\quad -\frac{\lambda}{2}\delta(1-p)\left((1+r_f)(Y_1-C_{1L})+Y_{2b}-\bar{C}_2\right)^2 \\
\frac{1}{\gamma_2}\mathbb{E}(U(\bar{C}_1)) &= \delta\left[(1+r_f)(Y_1-\bar{C}_1)+\mathbb{E}(Y_2)-\bar{C}_2\right]\gamma-\frac{\lambda}{2}\delta(1-p)\Omega^2 \\
\frac{1}{\gamma_2}\mathbb{E}\left(U\left(Y_1+\frac{Y_{2b}-C_{2L}}{1+r_f}\right)\right) &= \left(Y_1-\bar{C}_1+\frac{Y_{2b}-C_{2L}}{1+r_f}\right)\gamma+\delta p\,(Y_{2g}-Y_{2b})\gamma \\
&\quad -\frac{\lambda}{2}\left(Y_1-\bar{C}_1+\frac{Y_{2b}-C_{2L}}{1+r_f}\right)^2-\delta(\bar{C}_2-C_{2L})\gamma \\
&\quad -\frac{\lambda}{2}\delta(1-p)(\bar{C}_2-C_{2L})^2 \\
\frac{1}{\gamma_2}\mathbb{E}\left(U\left(Y_1+\frac{Y_{2g}-\bar{C}_2}{1+r_f}\right)\right) &= \left(Y_1-\bar{C}_1+\frac{Y_{2g}-\bar{C}_2}{1+r_f}\right)\gamma-\frac{\lambda}{2}\left(Y_1-\bar{C}_1+\frac{Y_{2g}-\bar{C}_2}{1+r_f}\right)^2 \\
&\quad -\delta(1-p)(Y_{2g}-Y_{2b})\gamma-\frac{\lambda}{2}\delta(1-p)(Y_{2g}-Y_{2b})^2
\end{aligned}
$$

(P8). It can be show that the objective function of (P6) is concave and thus its stationary point is the point of its maxima. The stationary point is

$$
\bar{C}_1^{P8}=\frac{1}{\lambda}\frac{[1-\delta(1+r_f)]\gamma}{1+\delta(1+r_f)^2}+\frac{\bar{C}_1-\delta(1+r_f)[\bar{C}_2-(1+r_f)Y_1-\mathbb{E}(Y_2)]}{1+\delta(1+r_f)^2} \tag{68}
$$

Before proceeding further, let's introduce the following notation:

$$
\bar{C}_1^{T3} = Y_1+\frac{Y_{2b}-C_{2L}}{1+r_f}+\delta(1+r_f)[\bar{C}_2-C_{2L}-p(Y_{2g}-Y_{2b})] \tag{69}
$$

$$
\bar{C}_1^{T4} = C_{1L}+\delta(1+r_f)^2\left(C_{1L}-Y_1+\frac{\bar{C}_2-\mathbb{E}(Y)}{1+r_f}\right) \tag{70}
$$

$$
\lambda_1^{P8} = \frac{[1-\delta(1+r_f)]\gamma}{\bar{C}_1^{T3}-\bar{C}_1} \tag{71}
$$

$$
\lambda_2^{P8} = \frac{[1-\delta(1+r_f)]\gamma}{\bar{C}_1^{T4}-\bar{C}_1} \tag{72}
$$

Note that $\bar{C}_1^{T}\leq\bar{C}_1^{T3}$ if $\bar{C}_2\geq C_{2L}+Y_{2g}-Y_{2b}$ and $\bar{C}_1^{T}\geq\bar{C}_1^{T3}$ if $\bar{C}_2\leq C_{2L}+Y_{2g}-Y_{2b}$. Note in addition that $\bar{C}_1^{T4}\leq\bar{C}_1^{T3}$.

The following can be observed for $\bar{C}_1\geq Y_1+\frac{Y_{2g}-\bar{C}_2}{1+r_f}$ and $\bar{C}_2\geq C_{2L}+Y_{2g}-Y_{2b}$

(P8)-(i) Let $Y_1+\frac{Y_{2g}-\bar{C}_2}{1+r_f}\leq\bar{C}_1\leq Y_1+\frac{Y_{2b}-C_{2L}}{1+r_f}$ and thus the set of feasible solutions is: $Y_1+\frac{Y_{2g}-\bar{C}_2}{1+r_f}\leq C_1\leq\bar{C}_1$. For $\lambda\geq\lambda_{U2}^{P4}$ is $\bar{C}_1^{P8}\leq\bar{C}_1$ and $\bar{C}_1^{P8}\geq Y_1+\frac{Y_{2g}-\bar{C}_2}{1+r_f}$ for

$\bar{C}_1 > \bar{C}_1^T$ and $\lambda \geq \lambda_3^{P6}$ or for $\bar{C}_1 < \bar{C}_1^T$ and $\lambda \leq \lambda_3^{P6}$. Thus, $C_1^{P8,*} = \bar{C}_1^{P8}$ if $\bar{C}_1 > \bar{C}_1^T$ and $\lambda \geq \max\{0, \lambda_{U2}^{P4}, \lambda_3^{P6}\}$ and $C_1^{P8,*} = Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ if $\bar{C}_1 < \bar{C}_1^T$ and $\lambda \geq \lambda_3^{P6}$.

(P8)-(ii) Let $C_{1L} \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < \bar{C}_1$ and thus the set of feasible solutions is: $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq C_1 \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$. Then the following holds for $\lambda > \max\{0, \lambda_3^{P6}, \lambda_1^{P8}\}$:

(P8)-(ii)-(1) $C_1^{P8,*} = Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ when $\bar{C}_1 < \bar{C}_1^T \leq \bar{C}_1^{T3}$

(P8)-(ii)-(2) $C_1^{P8,*} = \bar{C}_1^{P8}$ when $\bar{C}_1^T < \bar{C}_1 < \bar{C}_1^{T3}$

(P8)-(ii)-(3) $C_1^{P8,*} = Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ when $\bar{C}_1^T \leq \bar{C}_1^{T3} < \bar{C}_1$

(P8)-(iii) Let $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq C_{1L} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < \bar{C}_1$ and thus the set of feasible solutions is: $C_{1L} \leq C_1 \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$. Then the following holds for $\lambda > \max\{0, \lambda_1^{P8}, \lambda_2^{P8}\}$:

(P8)-(iii)-(1) $C_1^{P8,*} = C_{1L}$ when $\bar{C}_1 < \bar{C}_1^{T4} \leq \bar{C}_1^{T3}$

(P8)-(iii)-(2) $C_1^{P8,*} = \bar{C}_1^{P8}$ when $\bar{C}_1^{T4} < \bar{C}_1 < \bar{C}_1^{T3}$

(P8)-(iii)-(3) $C_1^{P8,*} = Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ when $\bar{C}_1^{T4} \leq \bar{C}_1^{T3} < \bar{C}_1$

B Parameter Set-ups

The following observations can be made

(1) If $\bar{C}_1 \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$, then maximum can be reached either in (P1), (P2) or (P5). If $\bar{C}_2 \geq C_{2L} + Y_{2g} - Y_{2b}$, then maximum can be reached also in (P4).

(2) If $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < \bar{C}_1 \leq Y_1 + \frac{1}{1+r_f} \min\left\{Y_{2b} - C_{2L}, Y_{2g} - \bar{C}_2\right\}$ and $C_{1L} \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$, then maximum can be reached in (P2), (P5) or (P6). If $\bar{C}_2 \geq C_{2L} + Y_{2g} - Y_{2b}$, then maximum can be reached also in (P4).

(3) If $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < \bar{C}_1 \leq Y_1 + \frac{1}{1+r_f} \min\left\{Y_{2b} - C_{2L}, Y_{2g} - \bar{C}_2\right\}$ and $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < C_{1L} \leq Y_1 + \frac{1}{1+r_f} \min\left\{Y_{2b} - C_{2L}, Y_{2g} - \bar{C}_2\right\}$, then maximum can be reached either in (P2) or (P6). If $\bar{C}_2 \geq C_{2L} + Y_{2g} - Y_{2b}$, maximum can be reached also in (P4).

(4) If $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < \bar{C}_1 \leq Y_1 + \frac{1}{1+r_f} \min\left\{Y_{2b} - C_{2L}, Y_{2g} - \bar{C}_2\right\}$, $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} < C_{1L} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ and $\bar{C}_2 \geq C_{2L} + Y_{2g} - Y_{2b}$, then maximum can be reached in (P2) or (P4).

(5) If $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq \bar{C}_1 \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$, $\bar{C}_2 \geq C_{2L} + Y_{2g} - Y_{2b}$ and $C_{1L} \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$, then maximum can be reached in (P4) or (P5) or (P6) or (P8).

(6) If $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq \bar{C}_1 \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$, $\bar{C}_2 \geq C_{2L} + Y_{2g} - Y_{2b}$ and $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < C_{1L} \leq Y_1 + \frac{1}{1+r_f} \min\left\{Y_{2b} - C_{2L}, Y_{2g} - \bar{C}_2\right\}$, then maximum can be reached in (P4) or (P6) or (P8).

(7) If $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq \bar{C}_1 \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$, $\bar{C}_2 \geq C_{2L} + Y_{2g} - Y_{2b}$ and $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} < C_{1L} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$, then maximum can be reached in (P4) or (P8).

(8) If $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} \leq \bar{C}_1$, $\bar{C}_2 \geq C_{2L} + Y_{2g} - Y_{2b}$ and $C_{1L} \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$, then maximum can be reached in (P5) or (P6) or (P8).

(9) If $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} \leq \bar{C}_1$, $\bar{C}_2 \geq C_{2L} + Y_{2g} - Y_{2b}$ and $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < C_{1L} \leq Y_1 + \frac{1}{1+r_f} \min\left\{Y_{2b} - C_{2L}, Y_{2g} - \bar{C}_2\right\}$, then maximum can be reached in (P6) or (P8).

(10) If $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} \leq \bar{C}_1$, $\bar{C}_2 \geq C_{2L} + Y_{2g} - Y_{2b}$ and $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} < C_{1L} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$, then maximum can be reached only in (P8).

C Optimal Solutions by Specific Parameter Set-ups

Lemma C.1. *Let* $\bar{C}_1 \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$, $\delta < \frac{1}{1+r_f}$ *and* $\lambda \geq \frac{(1+r_f)k}{\min\left\{\bar{C}_2 - C_{2L}, Y_{2g}-Y_{2b}\right\}}$. *Then the following holds*

$$C_1^* = \frac{k}{\lambda} + Y_1 + \frac{Y_{2b} - \bar{C}_2}{1+r_f} > \bar{C}_1, \tag{73}$$

$$C_{2g}^* = \bar{C}_2 + Y_{2g} - Y_{2b} - \frac{(1+r_f)k}{\lambda} \geq \bar{C}_2, \tag{74}$$

$$C_{2L} \leq C_{2b}^* = \bar{C}_2 - \frac{(1+r_f)k}{\lambda} < \bar{C}_2, \tag{75}$$

$$\frac{1}{\gamma_2}\mathbb{E}(U(C_1^*)) = \left[\frac{\Omega}{1+r_f} + \delta p \left(Y_{2g} - Y_{2b}\right) + \frac{\left(1-\delta(1+r_f)\right)k}{2\lambda}\right]\gamma, \tag{76}$$

where $k = \frac{[1-\delta(1+r_f)]\gamma}{\delta(1-p)(1+r_f)^2}$ *and* $\Omega = (1+r_f)(Y_1 - \bar{C}_1) + Y_{2b} - \bar{C}_2$.

Proof. If $\bar{C}_1 \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$, then maximum can be reached either in (P1), (P2) or (P5). If $\bar{C}_2 \geq C_{2L} + Y_{2g} - Y_{2b}$, then maximum can be reached also in (P4). This follows from the set of feasible solutions. Thus, we compare the values of objective functions of problems (P1), (P2), (P4) and (P5) at their point of maxima.

Let $\lambda \geq \frac{(1+r_f)k}{\min\left\{\bar{C}_2 - C_{2L}, Y_{2g}-Y_{2b}\right\}}$. For $\delta < \frac{1}{1+r_f}$ is $C_1^{P1,*} = Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ and for $\delta < \frac{1}{1+r_f}$ and $\lambda \geq \lambda^{P2}$ is $C_1^{P2,*} = \bar{C}_1^{P2}$. Thus, comparing maximum values of (P1) and (P2), namely (46) and (47) implies that $\mathbb{E}(U(C_1^{P2,*})) > \mathbb{E}(U(C_1^{P1,*}))$.

For $\delta < \frac{1}{1+r_f}$ and $\bar{C}_1 \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ is $C_1^{P5,*} = \bar{C}_1$. Equations (47) and (58) imply that $\mathbb{E}(U(C_1^{P2,*})) > \mathbb{E}(U(C_1^{P5,*}))$ for $\delta < \frac{1}{1+r_f}$ and $\lambda > 0$.

If in addition $\bar{C}_2 \geq C_{2L} + Y_{2g} - Y_{2b}$, then the potential candidate for maximum can occur also in (P4) namely, $C_1^{P4,*} = Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ when $\lambda \geq \frac{(1+r_f)k}{Y_{2g}-Y_{2b}} = \lambda_U^{P4}$. Note that $\lambda^{P2} \geq \lambda_U^{P4}$ and thus for $\lambda > \lambda^{P2}$ is still $C_1^{P4,*} = Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$. It can be shown, after some derivations that

$$\begin{aligned}
\frac{1}{\gamma_2}\mathbb{E}(U(\bar{C}_1^{P2})) &= \frac{\Omega\gamma}{1+r_f} + \delta p \left(Y_{2g} - Y_{2b}\right)\gamma + \frac{\left(1-\delta(1+r_f)\right)k\gamma}{2\lambda} \\
&> \left(Y_1 - \bar{C}_1 + \frac{Y_{2g} - \bar{C}_2}{1+r_f}\right)\gamma - \delta(1-p)(Y_{2g} - Y_{2b})\left[\gamma + \frac{\lambda}{2}(Y_{2g} - Y_{2b})\right] \\
&= \frac{1}{\gamma_2}\mathbb{E}\left(U\left(Y_1 + \frac{Y_{2g} - \bar{C}_2}{1+r_f}\right)\right)
\end{aligned}$$

when $\left[(Y_{2g} - Y_{2b})\lambda - \frac{(1-\delta(1+r_f))\gamma}{\delta(1-p)(1+r_f)}\right]^2 > 0$ which holds for $\lambda > \lambda^{P2}$. Note that $\Omega = (1+r_f)(Y_1 - \bar{C}_1) + Y_{2b} - \bar{C}_2$. Thus, the maximum is reached in (P2). ■

Lemma C.2. *Let $\bar{C}_1 \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ and $\delta > \frac{1}{1+r_f}$ and $\lambda > \lambda_L^{P5}$. Then the following holds*

$$\begin{aligned} C_{1L} < C_1^* &= \bar{C}_1 + \frac{\gamma}{\lambda}[1-\delta(1+r_f)] < \bar{C}_1 && (77)\\ C_{2g}^* &= Y_{2g} + (1+r_f)(Y_1-\bar{C}_1) + \frac{\gamma}{\lambda}\left(\delta - \frac{1}{1+r_f}\right) > \bar{C}_2 && (78)\\ C_{2b}^* &= Y_{2b} + (1+r_f)(Y_1-\bar{C}_1) + \frac{\gamma}{\lambda}\left(\delta - \frac{1}{1+r_f}\right) \geq \bar{C}_2 && (79)\\ \frac{1}{\gamma_2}\mathbb{E}(U(C_1^*)) &= \delta\left[\Omega + p(Y_{2g}-Y_{2b})\right]\gamma + \frac{\gamma^2}{2\lambda}[1-\delta(1+r_f)]^2 && (80) \end{aligned}$$

Proof. If $\bar{C}_1 \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$, then maximum can be reached either in (P1), (P2) or (P5). If $\bar{C}_2 \geq C_{2L} + Y_{2g} - Y_{2b}$, then maximum can be reached also in (P4). This follows from the set of feasible solutions. Thus, we again compare the values of objective functions of problems (P1), (P2), (P4) and (P5) at their point of maxima.

For $\delta > \frac{1}{1+r_f}$ is $C_1^{P1,*} = \bar{C}_1$, see (43), and $C_1^{P2,*} = Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$. As $C_1 = Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ is feasible also for (P1) and $C_1^{P1,*} = \bar{C}_1$, then $\mathbb{E}(U(C_1^{P1,*})) > \mathbb{E}(U(C_1^{P2,*}))$.

(P5)-(ii) implies that for $\delta \geq \frac{1}{1+r_f}$ and $\lambda > \lambda_L^{P5}$ is $C_1^{P5,*} = \bar{C}_1^{P5}$ as given by (54). Comparing its utility value as given by (57) with the utility of (P1) at its maximum $C_1^{P1,*} = \bar{C}_1$ as given by (43) we see that $\mathbb{E}(U(C_1^{P5,*})) > \mathbb{E}(U(C_1^{P1,*}))$.

If $\bar{C}_2 \geq C_{2L} + Y_{2g} - Y_{2b}$, then also problem (P4) can be considered, namely case (P4)-(iii) as for $\delta > \frac{1}{1+r_f}$ is λ_U^{P4} negative. When comparing the utility of (P4) at its maximum, namely at $C_1^{P4,*} = Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$, see (53), to the utility of (P5) at its maximum, namely at $C_1^{P5,*} = \bar{C}_1^{P5}$, see (57), then it can be shown that for $\delta > \frac{1}{1+r_f}$ is $\mathbb{E}(U(C_1^{P5,*})) > \mathbb{E}(U(C_1^{P4,*}))$. ■

Lemma C.3. *Let $\bar{C}_1 \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ and $\delta = \frac{1}{1+r_f}$. Then*

$$\begin{aligned} C_1^* &\in \left[\bar{C}_1, Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}\right] && (81)\\ C_{2g}^* &= Y_{2g} + (1+r_f)(Y_1 - C_1^*) \geq \bar{C}_2 && (82)\\ C_{2b}^* &= Y_{2b} + (1+r_f)(Y_1 - C_1^*) \geq \bar{C}_2 && (83)\\ \frac{1}{\gamma_2}\mathbb{E}(U(C_1^*)) &= \left[Y_1 - \bar{C}_1 + \frac{\mathbb{E}(Y_2)-\bar{C}_2}{1+r_f}\right]\gamma && (84) \end{aligned}$$

Lemma C.4. *Let $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < \bar{C}_1 \leq \min\left\{Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}, Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}\right\}$ and $\delta < \frac{1}{1+r_f}$. Then the following holds*

(i) *For* $\lambda \geq \frac{(1+r_f)k}{-\Omega}$ *and* $\delta < \frac{1}{(1-p)(1+r_f)^2}$ *or* $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < C_{1L} \leq \bar{C}_1$ *is*

$$C_{1L} \leq C_1^* = \bar{C}_1 \tag{85}$$

$$C_{2g}^* = Y_{2g} + (1+r_f)(Y_1 - \bar{C}_1) \geq \bar{C}_2 \tag{86}$$

$$C_{2L} \leq C_{2b}^* = Y_{2b} + (1+r_f)(Y_1 - \bar{C}_1) < \bar{C}_2 \tag{87}$$

$$\frac{1}{\gamma_2}\mathbb{E}(U(C_1^*)) = \delta\left[(1+r_f)(Y_1 - \bar{C}_1) + \mathbb{E}(Y_2) - \bar{C}_2\right]\gamma - \delta(1-p)\frac{\lambda}{2}(-\Omega)^2 \tag{88}$$

(ii) *For* $\lambda \geq \frac{2\gamma\left(\frac{1}{1+r_f}-\delta\right)}{(-\Omega)\left[\delta(1-p)-\frac{1}{(1+r_f)^2}\right]} = \lambda^{P2-P5}$*,* $C_{1L} \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ *and* $\frac{1}{(1-p)(1+r_f)^2} \leq \delta < \frac{1}{1+r_f}$ *is*

$$C_{1L} \leq C_1^* = Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < \bar{C}_1 \tag{89}$$

$$C_{2g}^* = Y_{2g} - Y_{2b} + \bar{C}_2 \geq \bar{C}_2 \tag{90}$$

$$C_{2L} \leq C_{2b}^* = \bar{C}_2 \tag{91}$$

$$\frac{1}{\gamma_2}\mathbb{E}(U(C_1^*)) = \delta\left[(1+r_f)(Y_1 - \bar{C}_1) + \mathbb{E}(Y_2) - \bar{C}_2\right]\gamma - \delta(1-p)\frac{\lambda}{2}(-\Omega)^2 \tag{92}$$

Proof. Note that for $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} \leq \bar{C}_1 \leq \min\left\{Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}, Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}\right\}$ the maximum can be reached either in (P2), (P4), (P5) or (P6). Thus, we compare the values of objective functions of these problems at their points of maxima and show that the value of objective function of (P2) at its maxima exceeds, under certain conditions, those of (P4), (P5) or (P6), case (i) and under very specific conditions the value of objective function of (P5) at its maxima exceeds those of (P2), (P4) or (P6), case (ii).

As, under stated conditions, is the set of feasible solutions of (P6) the subset of the set of feasible solutions of (P2) and the objective function of (P2) exceeds the objective function of (P6), then the maximum of (P6) can not be the maximum of the stated problem.

Based on (P2)-(vi) is for $\lambda \geq \frac{(1+r_f)k}{-\Omega}$ the maximum of (P2) reached at $C_1^{P2,*} = \bar{C}_1$ and based on (P4)-(iii) is the maximum of (P4) reached at $C_1^{P4,*} = Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ for $\lambda \geq \frac{(1+r_f)k}{Y_{2g}-Y_{2b}}$. Note that $\frac{(1+r_f)k}{-\Omega} \geq \frac{(1+r_f)k}{Y_{2g}-Y_{2b}}$ as $\bar{C}_1 \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$. By comparing values of objective functions of (P2) and (P4) at these points, see (48) and (53), we obtain after some derivations that (P2) at its maximum exceeds (P4) at its maximum for $\lambda \geq \frac{2(1+r_f)k}{Y_{2g}-Y_{2b}-\Omega}$. Note again that $\frac{(1+r_f)k}{-\Omega} \geq \frac{2(1+r_f)k}{Y_{2g}-Y_{2b}-\Omega}$ as $\bar{C}_1 \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$.

Based on (P5)-(i) is the maximum of (P5) reached at $C_1^{P5,*} = Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ as $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < \bar{C}_1$. After some derivations we obtain that for $\delta < \frac{1}{(1-p)(1+r_f)^2}$, $\lambda \geq \lambda^{P2-P5}$ and $C_{1L} \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$[3], objective function of (P2) at its maximum exceeds objective function of (P5) at its maximum. Note in addition that for $\delta < \frac{1}{(1-p)(1+r_f)^2}$ is $\frac{(1+r_f)k}{-\Omega} \geq \lambda^{P2-P5}$.[4] In addition, it can be seen that for $\frac{1}{(1-p)(1+r_f)^2} \leq \delta < \frac{1}{1+r_f}$ and $\lambda \geq \lambda^{P2-P5}$ (P5) at its maximum exceeds (P2) at its maximum. ■

Lemma C.5. *Let* $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < \bar{C}_1 \leq \min\left\{Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}, Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}\right\}$ *and* $\delta > \frac{1}{1+r_f}$. *Then the following holds*

(i) *For* $C_{1L} \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$, $\delta > \max\left\{\frac{1}{1+r_f}, \frac{1}{(1-p)(1+r_f)^2}\right\}$ *and* $\lambda \geq \lambda_L^{P5}$ *is*

$$C_{1L} \leq C_1^* = Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < \bar{C}_1 \tag{93}$$

$$C_{2g}^* = Y_{2g} - Y_{2b} + \bar{C}_2 > \bar{C}_2 \tag{94}$$

$$C_{2b}^* = \bar{C}_2 \tag{95}$$

$$\frac{1}{\gamma_2}\mathbb{E}(U(C_1^*)) = \frac{\Omega\gamma}{1+r_f} - \frac{\lambda}{2}\left(\frac{\Omega}{1+r_f}\right)^2 + \delta p\,(Y_{2g} - Y_{2b})\gamma \tag{96}$$

(ii) *For* $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < C_{1L} \leq \bar{C}_1$ *or* $C_{1L} \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$, $\frac{1}{1+r_f} < \delta \leq \frac{1}{(1-p)(1+r_f)^2}$ *and*

$$\lambda \geq \frac{2\gamma\left(\frac{1}{1+r_f} - \delta\right)}{(-\Omega)\left[\delta(1-p) - \frac{1}{(1+r_f)^2}\right]} = \lambda^{P2-P5} \text{ is}$$

$$C_1^* = \bar{C}_1 \tag{97}$$

$$C_{2g}^* = Y_{2g} + (1+r_f)(Y_1 - \bar{C}_1) \geq \bar{C}_2 \tag{98}$$

$$C_{2L} \leq C_{2b}^* = Y_{2b} + (1+r_f)(Y_1 - \bar{C}_1) < \bar{C}_2 \tag{99}$$

$$\frac{1}{\gamma_2}\mathbb{E}(U(C_1^*)) = \delta\left[(1+r_f)(Y_1 - \bar{C}_1) + \mathbb{E}(Y_2) - \bar{C}_2\right]\gamma - \delta(1-p)\frac{\lambda}{2}(-\Omega)^2 \tag{100}$$

Proof. Similarly, as in the proof of Proposition C.4, the relevant problems to investigate (under conditions stated in this proposition) are (P2), (P4), (P5) or (P6). Using the same lines of arguments as in Proposition C.4, problems (P4) and (P6) become not the relevant ones and we show by simple comparisons of values of objective functions at their maxima that maximum is reached in (P5) when $C_{1L} \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ and

[3] Condition $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < C_{1L} \leq \bar{C}_1$ implies that there the set of feasible solutions for (P5) is empty.

[4] Note that λ^{P2-P5} can be written as $\lambda^{P2-P5} = \frac{2(1+r_f)k\delta(1-p)}{-\Omega\left[\delta(1-p) - \frac{1}{(1+r_f)^2}\right]}$.

$\lambda \geq \frac{\gamma[\delta(1+r_f)-1]}{\bar{C}_1 - C_{1L}}$, case (i), and in (P2) when $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < C_{1L} \leq \bar{C}_1$ and $\lambda > 0$, case (ii).

Based on (P2)-(vi) is the maximum of (P2) reached at $C_1^{P2,*} = \bar{C}_1$ for any for $\lambda > 0$ (as $\delta > \frac{1}{1+r_f}$) and based on (P5)-(iii) is the maximum of (P5) reached at $C_1^{P5,*} = Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ for $\lambda \geq \frac{\gamma(1+r_f)[\delta(1+r_f)-1]}{-\Omega}$ and $C_{1L} \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$. By comparing values of objective functions of (P2) and (P5) at these points, see (48) and (59), we get

$$\begin{aligned} &\frac{1}{\gamma_2}\mathbb{E}\left(U\left(Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}\right)\right) - \frac{1}{\gamma_2}\mathbb{E}(U(\bar{C}_1)) \\ &= \gamma(-\Omega)\left(\delta - \frac{1}{1+r_f}\right) + \frac{\lambda}{2}\Omega^2\left[\delta(1-p) - \frac{1}{(1+r_f)^2}\right] > 0 \end{aligned} \tag{101}$$

and thus (P5) at its maximum exceeds (P2) at its maximum for big λs only if $\delta > \frac{1}{1+r_f}$ if either $\delta \geq \frac{1}{(1-p)(1+r_f)^2}$. On the other hand, if $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < C_{1L} \leq \bar{C}_1$, i.e., the set of feasible solutions of (P5) is empty, or if $\delta < \frac{1}{(1-p)(1+r_f)^2}$ and $\lambda \geq \lambda^{P2-P5}$ then it follows from (101) that (P2) at its maximum exceeds (P5) at its maximum. ■

Lemma C.6. *Let* $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < \bar{C}_1 \leq \min\left\{Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}, Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}\right\}$, $\delta = \frac{1}{1+r_f}$ *and* $\lambda > 0$. *Then the following holds*

(i) *For* $C_{1L} \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ *and* $p < \frac{r_f}{1+r_f}$ *is*

$$C_{1L} \leq C_1^* = Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < \bar{C}_1 \tag{102}$$
$$C_{2g}^* = Y_{2g} - Y_{2b} + \bar{C}_2 > \bar{C}_2 \tag{103}$$
$$C_{2b}^* = \bar{C}_2 \tag{104}$$
$$\frac{1}{\gamma_2}\mathbb{E}(U(C_1^*)) = \frac{\gamma}{1+r_f}\left[(1+r_f)(Y_1-\bar{C}_1) + \mathbb{E}(Y_2) - \bar{C}_2\right] - \frac{\lambda}{2}\left(\frac{\Omega}{1+r_f}\right)^2 \tag{105}$$

(ii) *For* $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < C_{1L} \leq \bar{C}_1$ *or* $C_{1L} \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ *and* $p \geq \frac{r_f}{1+r_f}$ *is*

$$C_1^* = \bar{C}_1 \tag{106}$$
$$C_{2g}^* = Y_{2g} + (1+r_f)(Y_1-\bar{C}_1) \geq \bar{C}_2 \tag{107}$$
$$C_{2L} \leq C_{2b}^* = Y_{2b} + (1+r_f)(Y_1-\bar{C}_1) < \bar{C}_2 \tag{108}$$
$$\frac{1}{\gamma_2}\mathbb{E}(U(C_1^*)) = \frac{\gamma}{1+r_f}\left[(1+r_f)(Y_1-\bar{C}_1) + \mathbb{E}(Y_2) - \bar{C}_2\right] - \frac{\lambda(1-p)}{2(1+r_f)}(-\Omega)^2 \tag{109}$$

Proof. The statements of the proposition follow directly from (101), (105) and (109).

■

Lemma C.7. *Let* $Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < \bar{C}_1 \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$. *Then the following holds*

(i) *For* $C_{1L} \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ *and* $\lambda > \max\left\{0, \lambda_U^{P5}, \lambda_2^{P6}\right\}$ *is the solution of (7) given as follows*

$$C_1^* = \begin{cases} \bar{C}_1^{P6}, & for \;\; \bar{C}_1 < \bar{C}_1^{T1} \\ Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}, & for \;\; \bar{C}_1 > \bar{C}_1^{T1} \end{cases} \tag{110}$$

where

$$\begin{aligned} C_{1L} \leq C_1^* &= \bar{C}_1^{P6} < \bar{C}_1 && (111)\\ C_{2g}^* &= Y_{2g} + (1+r_f)(Y_1 - C_1^*) \geq \bar{C}_2 && (112)\\ C_{2L} \leq C_{2b}^* &= Y_{2b} + (1+r_f)(Y_1 - C_1^*) < \bar{C}_2 && (113)\\ & \frac{\lambda}{\gamma_2}\left[1+\delta(1-p)(1+r_f)^2\right]\mathbb{E}(U(\bar{C}_1^{P6})) = \frac{1}{2}\left[1-\delta(1+r_f)\right]^2\gamma^2 \\ & +\lambda\delta\left(1+(1-p)(1+r_f)\right)\Omega\;\gamma - \frac{1}{2}\lambda^2\Omega^2\delta(1-p) \\ & +\lambda\delta p\left(Y_{2g}-Y_{2b}\right)\left(1+\delta(1-p)(1+r_f)^2\right)\gamma && (114) \end{aligned}$$

when $\bar{C}_1 < \bar{C}_1^{T1}$ *and*

$$\begin{aligned} C_{1L} \leq C_1^* &= Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < \bar{C}_1 && (115)\\ C_{2g}^* &= Y_{2g} - Y_{2b} + C_{2L} \geq \bar{C}_2 && (116)\\ C_{2b}^* &= C_{2L} < \bar{C}_2 && (117)\\ \frac{1}{\gamma_2}\mathbb{E}(U(C_1^*)) &= -\gamma\left(\bar{C}_1 - Y_1 - \frac{Y_{2b}-C_{2L}}{1+r_f}\right) - \frac{\lambda}{2}\left(\bar{C}_1 - Y_1 - \frac{Y_{2b}-C_{2L}}{1+r_f}\right)^2 \\ & +\delta p\,\gamma(Y_{2g} - Y_{2b} + C_{2L} - \bar{C}_2) \\ & -\delta(1-p)\gamma(\bar{C}_2 - C_{2L}) - \frac{\lambda}{2}(\bar{C}_2 - C_{2L})^2 && (118) \end{aligned}$$

when $\bar{C}_1 > \bar{C}_1^{T1}$.

(ii) *For* $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < C_{1L} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ *and* $\lambda > \max\left\{0, \lambda_1^{P6}, \lambda_2^{P6}\right\}$ *is the solution of (7) given as follows*

$$C_1^* = \begin{cases} C_{1L}, & for \;\; \bar{C}_1 < \bar{C}_1^{T2} \\ \bar{C}_1^{P6}, & for \;\; \bar{C}_1^{T2} < \bar{C}_1 < \bar{C}_1^{T1} \\ Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}, & for \;\; \bar{C}_1 > \bar{C}_1^{T1} \end{cases} \tag{119}$$

where

$$
\begin{aligned}
C_1^* &= C_{1L} \leq \bar{C}_1 && (120)\\
C_{2g}^* &= Y_{2g} + (1+r_f)(Y_1 - C_{1L}) \geq \bar{C}_2 && (121)\\
C_{2L} \leq C_{2b}^* &= Y_{2b} + (1+r_f)(Y_1 - C_{1L}) < \bar{C}_2 && (122)\\
\frac{1}{\gamma_2}\mathbb{E}(U(C_{1L})) &= \delta\left[(1+r_f)(Y_1 - C_{1L}) + \mathbb{E}(Y_2) - \bar{C}_2\right]\gamma - (\bar{C}_1 - C_{1L})\gamma\\
&\quad -\frac{\lambda}{2}\left[(\bar{C}_1 - C_{1L})^2 + \delta(1-p)\left((1+r_f)(Y_1 - C_{1L}) + Y_{2b} - \bar{C}_2\right)^2\right] && (123)
\end{aligned}
$$

if $C_1^* = C_{1L}$. *If* $C_1^* = \bar{C}_1^{P6}$ *then results are stated by (111)-(114) and if* $C_1^* = Y_1 + \frac{Y_{2b} - C_{2L}}{1+r_f}$ *then results are stated by (115)-(118).*

Proof. Case (i). For $C_{1L} \leq Y_1 + \frac{Y_{2b} - \bar{C}_2}{1+r_f}$ the problems with non-empty sets of feasible solutions are (P5) and (P6). (P5)-(i) and (P5)-(iii) imply that for $\lambda \geq \lambda_U^{P5}$ is the maximum of (P5) reached at $Y_1 + \frac{Y_{2b} - \bar{C}_2}{1+r_f}$. (P6)-(iii) implies that for $\lambda > \max\left\{0, \lambda_U^{P5}, \lambda_2^{P6}\right\}$ is the maximum of (P6) reached at $\bar{C}_1^{P6}$, see (61), when $\bar{C}_1 < \bar{C}_1^{T1}$ and at $Y_1 + \frac{Y_{2b} - C_{2L}}{1+r_f}$ when $\bar{C}_1 > \bar{C}_1^{T1}$. As the objective functions of (P5) and (P6) coincide at $C_1 = Y_1 + \frac{Y_{2b} - \bar{C}_2}{1+r_f}$ which is part of the set of feasible solutions of both problems then based on what was said above is $C_1 = \bar{C}_1^{P6}$ the point where (7) reaches its maximum.

For $Y_1 + \frac{Y_{2b} - \bar{C}_2}{1+r_f} < C_{1L} \leq Y_1 + \frac{Y_{2b} - C_{2L}}{1+r_f}$, case (ii), the only problem with non-empty set of feasible solutions is (P6) and the statements thus follow from (P6)-(iv). ■

Lemma C.8. *Let* $Y_1 + \frac{Y_{2g} - \bar{C}_2}{1+r_f} \leq \bar{C}_1 \leq Y_1 + \frac{Y_{2b} - C_{2L}}{1+r_f}$. *Then the following holds*

(i) *For* $C_{1L} \leq Y_1 + \frac{Y_{2b} - \bar{C}_2}{1+r_f}$, $\bar{C}_1 < \bar{C}_1^T$ *and* $\lambda > \max\left\{0, \lambda_{U2}^{P4}, \lambda_U^{P5}, \lambda_3^{P6}\right\}$ *is the solution of (7) given as follows*

$$
C_1^* = \begin{cases} \bar{C}_1^{P6}, & for \quad \bar{C}_1 < \bar{C}_1^T \\ \bar{C}_1^{P8}, & for \quad \bar{C}_1 > \bar{C}_1^T \end{cases} \qquad (124)
$$

where results are stated by (111)-(114) if $C_1^* = \bar{C}_1^{P6}$ *and*

$$\begin{aligned}
C_{1L} \le C_1^* &= \bar{C}_1^{P8} < \bar{C}_1 && (125)\\
C_{2L} < C_{2g}^* &= Y_{2g} + (1+r_f)(Y_1 - C_1^*) \le \bar{C}_2 && (126)\\
C_{2L} \le C_{2b}^* &= Y_{2b} + (1+r_f)(Y_1 - C_1^*) < \bar{C}_2 && (127)\\
\frac{1}{\gamma_2}\mathbb{E}(U(C_1^*)) &= \gamma(C_1^* - \bar{C}_1) - \lambda\frac{1}{2}(\bar{C}_1 - C_1^*)^2 \\
&\quad +\delta p\, \gamma\left((1+r_f)(Y_1 - C_1^*) + Y_{2g} - \bar{C}_2\right) \\
&\quad -\delta p\frac{\lambda}{2}\left(\bar{C}_2 - (1+r_f)(Y_1 - C_1^*) - Y_{2g}\right)^2 \\
&\quad +\delta(1-p)\gamma\left((1+r_f)(Y_1 - C_1^*) + Y_{2b} - \bar{C}_2\right) \\
&\quad -\delta(1-p)\frac{\lambda}{2}\left(\bar{C}_2 - (1+r_f)(Y_1 - C_1^*) - Y_{2b}\right)^2 && (128)
\end{aligned}$$

otherwise,

(ii) *for* $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < C_{1L} \le Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ *and* $\lambda > \max\left\{0, \lambda_{U2}^{P4}, \lambda_U^{P5}, \lambda_1^{P6}\lambda_3^{P6}\right\}$ *is:*

(ii)-(1) $C_1^* = C_{1L}$, *see (120)-(123), for* $\bar{C}_1 < \bar{C}_1^{T2}$

(ii)-(2) $C_1^* = \bar{C}_1^{P6}$, *see (111)-(114), for* $\bar{C}_1^{T2} < \bar{C}_1 < \bar{C}_1^T$

(ii)-(3) $C_1^* = \bar{C}_1^{P8}$, *see (125)-(128), for* $\bar{C}_1 > \bar{C}_1^T$

(iii) *For* $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} < C_{1L} \le \bar{C}_1$ *and* $\lambda > \max\left\{0, \lambda_{U2}^{P4}, \lambda_3^{P6}\right\}$ *is* $C_1^* = \bar{C}_1^{P8}$, *see (125)-(128).*

Proof. Case (i): Let $C_{1L} \le Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} \le Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \le \bar{C}_1 \le Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$. For this parameter set-up the maximum can be reached either in (P4) or (P5) or (P6) or (P8) where

(P4): $\bar{C}_1 \le C_1 \le Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ is the set of feasible solutions and
$C_1^{P4,*} = \bar{C}_1$ for $\lambda > \lambda_{U2}^{P4}$, see (P4)-(vi)

(P5): $C_{1L} \le C_1 \le Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ is the set of feasible solutions and
$C_1^{P5,*} = Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ for $\lambda > \lambda_U^{P5}$, see (P5)-(i) and (P5)-(iii)

(P6): $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} \le C_1 \le Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ is the set of feasible solutions and
$C_1^{P6,*} = \bar{C}_1^{P6}$, see (61), for $\bar{C}_1 < \bar{C}_1^T$ and $\lambda > \max\{0, \lambda_U^{P5}, \lambda_3^{P6}\}$, see (P6)-(v) or
$C_1^{P6,*} = Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ for $\bar{C}_1 > \bar{C}_1^T$ and $\lambda > \lambda_3^{P6}$, see (P6)-(v)

(P8): $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq C_1 \leq \bar{C}_1$ is the set of feasible solutions and
$C_1^{P8,*} = \bar{C}_1^{P8}$, see (68), for $\bar{C}_1 > \bar{C}_1^T$ and $\lambda > \max\{0, \lambda_{U2}^{P4}, \lambda_3^{P6}\}$, see (P8)-(i) or
$C_1^{P8,*} = Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ for $\bar{C}_1 < \bar{C}_1^T$ and $\lambda > \lambda_3^{P6}$, see (P8)-(i)

As $C_1 = \bar{C}_1$ is part of the set of feasible solutions for both (P4) and (P8), values of utility functions coincide at this point and it is the point where (P4) reaches its maximum (not (P8)) then the objective function of (P8) at its maximum exceeds the objective function of (P4) at its maximum.

Similarly, as $C_1 = Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ is part of the set of feasible solutions for both (P5) and (P6), values of utility functions coincide at this point and it is the point where (P5) reaches its maximum (not (P6)) and thus the objective function of (P6) at its maximum exceeds the objective function of (P5) at its maximum.

Let $\bar{C}_1 < \bar{C}_1^T$. Then (P6)-(v) implies that (P6) has an interior solution. As $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ is the feasible solution of both (P6) and (P8), values of objective functions coincide at this point, and it is actually the point where (P8) reaches its maxima, see (P8)-(i), then the objective function of (P6) at its maximum exceeds the objective function of (P8) at its maximum.

On the other hand, let $\bar{C}_1 > \bar{C}_1^T$. Then (P8)-(i) implies that (P8) has an interior solution. As $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ is the feasible solution of both (P6) and (P8), values of objective functions coincide at this point, and it is actually the point where (P6) reaches its maxima then the objective function of (P8) at its maximum exceeds the objective function of (P6) at its maximum.

In summary, for $\bar{C}_1 < \bar{C}_1^T$ is (P6)>(P8)>(P4) and (P6)>(P5).[5] On the other hand, for $\bar{C}_1 > \bar{C}_1^T$ is (P8)>(P6)>(P5) and (P8)>(P4). This finished the proof.

Case (ii): Let $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < C_{1L} \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq \bar{C}_1 \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$. For this parameter set-up the maximum can be reached either in (P4), or (P6) or (P8) where (P4) and (P8) are defined as in the proof of case (i) and (P6) is given for $\lambda > \max\{0, \lambda_1^{P6}, \lambda_3^{P6}\}$ by

(P6): $C_{1L} \leq C_1 \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ is the set of feasible solutions and following from (P6)-

[5] To easy the notation, by writing (P6)>(P8), for instance, we mean: "the value of the utility function of problem (P6) at its maximum exceeds the value of the utility function of problem (P8) at its maximum.

(vi)
$C_1^{P6,*} = C_{1L}$ for $\bar{C}_1 < \bar{C}_1^{T2}$
$C_1^{P6,*} = \bar{C}_1^{P6}$ for $\bar{C}_1^{T2} < \bar{C}_1 < \bar{C}_1^{T}$
$C_1^{P6,*} = Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ for $\bar{C}_1 > \bar{C}_1^{T}$

The statements follow from the similar lines of arguments as in the proof of case (i) and from comparing (P8) and (P6).

Case (iii): let $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} < C_{1L} \leq \bar{C}_1 \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$. For this parameter set-up the maximum can be reached either in (P4) or (P8) and the fact that the objective function of (P8) at its maximum exceeds the objective function of (P4) at its maximum was already argued (and shown) in the proof of case (i). ■

Lemma C.9. *Let* $\max\left\{Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}, Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}\right\} < \bar{C}_1$. *Then the following holds*

(i) *For* $C_{1L} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} < \bar{C}_1$ *and* $\lambda > \max\{0, \lambda_U^{P5}, \lambda_1^{P6}, \lambda_2^{P6}\}$ *is the solution of (7) given as follows*

$$C_1^* = \begin{cases} \bar{C}_1^{P6}, & for \;\; \bar{C}_1 < \bar{C}_1^{T1} \\ Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}, & for \;\; \bar{C}_1 > \bar{C}_1^{T1} \end{cases} \tag{129}$$

where results are stated by (111)-(114) if $C_1^* = \bar{C}_1^{P6}$ *and results are given by (115)-(118) if* $C_1^* = Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$.

(ii) *For* $C_{1L} \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < \bar{C}_1$, $\bar{C}_1^{T} < \bar{C}_1 < \bar{C}_1^{T3}$ *and* $\lambda > \max\{0, \lambda_U^{P5}, \lambda_1^{P6}, \lambda_3^{P6}, \lambda_1^{P8}\}$ *or*
for $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq C_{1L} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < \bar{C}_1$, $\bar{C}_1^{T4} < \bar{C}_1 < \bar{C}_1^{T3}$ *and* $\lambda > \max\{0, \lambda_1^{P8}, \lambda_2^{P8}\}$
is the solution of (7) given by (125)-(128), as stated in Proposition C.8-(i).

(iii) *For* $C_{1L} \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < \bar{C}_1 < \bar{C}_1^{T}$ *and* $\lambda > \max\{0, \lambda_U^{P5}, \lambda_1^{P6}, \lambda_3^{P6}\}$ *is the solution of (7) given by (111)-(114), as stated in Proposition C.7-(i).*

(iv) *The following holds for* $C_{1L} \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < \bar{C}_1$, $\bar{C}_1 > \bar{C}_1^{T3}$ *and* $\lambda > \max\{0, \lambda_U^{P5}, \lambda_1^{P6}, \lambda_3^{P6}, \lambda_1^{P8}\}$

or for $Y_1+\frac{Y_{2g}-\bar{C}_2}{1+r_f} \le C_{1L} \le Y_1+\frac{Y_{2b}-C_{2L}}{1+r_f} < \bar{C}_1$, $\bar{C}_1 > \bar{C}_1^{T3}$ *and* $\lambda > \max\{0, \lambda_1^{P8}, \lambda_2^{P8}\}$

$$\begin{aligned} C_{1L} \le C_1^* &= Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < \bar{C}_1 && (130)\\ C_{2g}^* &= Y_{2g} - Y_{2b} + C_{2L} \le \bar{C}_2 && (131)\\ C_{2b}^* &= C_{2L} < \bar{C}_2 && (132)\\ \frac{1}{\gamma_2}\mathbb{E}(U(C_1^*)) &= -\gamma\left(\bar{C}_1 - Y_1 - \frac{Y_{2b}-C_{2L}}{1+r_f}\right) - \frac{\lambda}{2}\left(\bar{C}_1 - Y_1 - \frac{Y_{2b}-C_{2L}}{1+r_f}\right)^2 \\ &\quad -\delta p\,\gamma(\bar{C}_2 - C_{2L} - Y_{2g} + Y_{2b}) - \frac{\lambda}{2}(\bar{C}_2 - C_{2L} - Y_{2g} + Y_{2b})^2 \\ &\quad -\delta(1-p)\gamma(\bar{C}_2 - C_{2L}) - \frac{\lambda}{2}(\bar{C}_2 - C_{2L})^2 && (133) \end{aligned}$$

(v) *For* $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \le C_{1L} \le Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < \bar{C}_1 < \bar{C}_1^{T4}$ *and* $\lambda > \max\{0, \lambda_1^{P8}, \lambda_2^{P8}\}$ *is the solution of (7) given as follows*

$$\begin{aligned} C_1^* &= C_{1L} \le \bar{C}_1 && (134)\\ C_{2L} \le C_{2g}^* &= Y_{2g} + (1+r_f)(Y_1 - C_{1L}) \le \bar{C}_2 && (135)\\ C_{2L} \le C_{2b}^* &= Y_{2b} + (1+r_f)(Y_1 - C_{1L}) < \bar{C}_2 && (136)\\ \frac{1}{\gamma_2}\mathbb{E}(U(C_{1L})) &= && \\ & && (137) \end{aligned}$$

Proof. Case (i): Let $C_{1L} \le Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} < \bar{C}_1$ and $\lambda > \max\{0, \lambda_U^{P5}, \lambda_1^{P6}, \lambda_2^{P6}\}$. Then the following two sub-cases can be considered.

Sub-case (i)-(1): Let $C_{1L} \le Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} \le Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} < \bar{C}_1$. For this parameter set-up the maximum can be reached either in (P5) or (P6) where

(P5): $C_{1L} \le C_1 \le Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ is the set of feasible solutions and $C_1^{P5,*} = Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ for $\lambda > \lambda_U^{P5}$, see (P5)-(i) and (P5)-(iii)

(P6): $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} \le C_1 \le Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ is the set of feasible solutions and for $\lambda > \max\{0, \lambda_U^{P5}, \lambda_2^{P6}\}$ is $C_1^{P6,*} = \bar{C}_1^{P6}$, see (61), for $\bar{C}_1 < \bar{C}_1^{T1}$ and $C_1^{P6,*} = Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ for $\bar{C}_1 > \bar{C}_1^{T1}$ (see (P6)-(iii)).

Sub-case (i)-(2): Let $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < C_{1L} \le Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} < \bar{C}_1$. For this parameter set-up the maximum can be reached either only in (P6) where

(P6): $C_{1L} \leq C_1 \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ is the set of feasible solutions and for $\lambda > \max\{0, \lambda_1^{P6}, \lambda_2^{P6}\}$ is $C_1^{P6,*} = \bar{C}_1^{P6}$, see (61), for $\bar{C}_1 < \bar{C}_1^{T1}$ and $C_1^{P6,*} = Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ for $\bar{C}_1 > \bar{C}_1^{T1}$. This follows from (P6)-(iv)-(2) and (P6)-(iv)-(3).[6]

Note that for $C_{1L} > Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ is the set of feasible solutions empty for case (i).

As $C_1 = Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ is part of the set of feasible solutions for both (P5) and (P6), values of utility functions coincide at this point and it is the point where (P5) reaches its maximum (not (P6)) and thus the objective function of (P6) at its maximum exceeds the objective function of (P5) at its maximum.

Case (ii): Let $C_{1L} \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < \bar{C}_1$, $\bar{C}_1^T < \bar{C}_1 < \bar{C}_1^{T3}$ and $\lambda > \max\{0, \lambda_U^{P5}, \lambda_1^{P6}, \lambda_3^{P6}, \lambda_1^{P8}\}$. Then the following two sub-cases can be considered. Sub-case (ii)-(1): Let $C_{1L} \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < \bar{C}_1$. For this parameter set-up the maximum can be reached either in (P5), (P6) or (P8) where

(P5): $C_{1L} \leq C_1 \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ is the set of feasible solutions and $C_1^{P5,*} = Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ for $\lambda > \lambda_U^{P5}$, see (P5)-(i) and (P5)-(iii)

(P6): $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} \leq C_1 \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ is the set of feasible solutions and $C_1^{P6,*} = Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ for $\lambda > \max\{0, \lambda_U^{P5}, \lambda_3^{P6}\}$, see (P6)-(v)

(P8): $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq C_1 \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ is the set of feasible solutions and $C_1^{P8,*} = \bar{C}_1^{P8}$ for $\lambda > \max\{0, \lambda_3^{P6}, \lambda_1^{P8}\}$, see (P8)-(ii)-(2)

Sub-case (ii)-(2): Let $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < C_{1L} \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < \bar{C}_1$. For this parameter set-up the maximum can be reached either in (P6) or (P8) where

(P6): $C_{1L} \leq C_1 \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ is the set of feasible solutions and $C_1^{P6,*} = Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ for $\lambda > \max\{0, \lambda_1^{P6}, \lambda_3^{P6}\}$, see (P6)-(vi)-(3)

(P8): $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq C_1 \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ is the set of feasible solutions and $C_1^{P8,*} = \bar{C}_1^{P8}$ for $\lambda > \max\{0, \lambda_3^{P6}, \lambda_1^{P8}\}$, see (P8)-(ii)-(2)

The objective function of (P6) at its maximum exceeds the objective function of (P5) at its maximum, in sub-case (ii)-(1) as argued in the proof of case (i). And it can be

[6] Note that under stated conditions $\bar{C}_1$ can not be below $\bar{C}_1^{T2}$ and thus case (P6)-(iv)-(1) will not apply.

argued in the similar lines that objective function of (P8) exceeds at its maximum the objective function of (P6) at its maximum, in both sub-cases (ii)-(1) and (ii)-(2).

Statement of case (ii) for $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq C_{1L} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < \bar{C}_1$, $\bar{C}_1^{T4} < \bar{C}_1 < \bar{C}_1^{T3}$ and $\lambda > \max\{0, \lambda_1^{P8}, \lambda_2^{P8}\}$ follows from (P8)-(iii)-(2).

Case (iii): Let $C_{1L} \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < \bar{C}_1 < \bar{C}_1^T$ and $\lambda > \max\{0, \lambda_U^{P5}, \lambda_1^{P6}, \lambda_3^{P6}\}$. Then the following two sub-cases can be considered.

Sub-case (iii)-(1): Let $C_{1L} \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < \bar{C}_1 < \bar{C}_1^T$. For this parameter set-up the maximum can be reached either in (P6) or (P8) where

(P5): $C_{1L} \leq C_1 \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ is the set of feasible solutions and
$C_1^{P5,*} = Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ for $\lambda > \lambda_U^{P5}$, see (P5)-(i) and (P5)-(iii)

(P6): $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} \leq C_1 \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ is the set of feasible solutions and
$C_1^{P6,*} = \bar{C}_1^{P6}$ for $\lambda > \max\{0, \lambda_U^{P5}, \lambda_3^{P6}\}$, see (P6)-(v)

(P8): $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq C_1 \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ is the set of feasible solutions and
$C_1^{P8,*} = Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ for $\lambda > \max\{0, \lambda_3^{P6}, \lambda_1^{P8}\}$, see (P8)-(ii)-(1)

(P5)<(P6) as argued in the proof of case (i).

Sub-case (iii)-(2): Let $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < C_{1L} \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < \bar{C}_1 < \bar{C}_1^T$. For this parameter set-up the maximum can be reached either in (P6) or (P8) where

(P6): $C_{1L} \leq C_1 \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ is the set of feasible solutions and
$C_1^{P6,*} = \bar{C}_1^{P6}$ for $\lambda > \max\{0, \lambda_1^{P6}, \lambda_3^{P6}\}$, see (P6)-(vi)-(2)

(P8): $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq C_1 \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ is the set of feasible solutions and
$C_1^{P8,*} = Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ for $\lambda > \max\{0, \lambda_3^{P6}, \lambda_1^{P8}\}$, see (P8)-(ii)-(1)

As $C_1 = Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ is part of the set of feasible solutions for both (P6) and (P8), values of utility functions coincide at this point and it is the point where (P8) reaches its maximum (not (P6)) and thus the objective function of (P6) at its maximum exceeds the objective function of (P8) at its maximum. This applies for both sub-cases (iii)-(1) and (iii)-(2).

Statement for case (iii) when $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq C_{1L} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < \bar{C}_1 < \bar{C}_1^T$ and $\lambda > \max\{0, \lambda_1^{P8}, \lambda_2^{P8}\}$ follows from (P8)-(iii)-(3).

Case (iv): Let $C_{1L} \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < \bar{C}_1$, $\bar{C}_1 > \bar{C}_1^{T3}$ and $\lambda > \max\{0, \lambda_U^{P5}, \lambda_1^{P6}, \lambda_3^{P6}, \lambda_1^{P8}\}$. Then the following two sub-cases can be considered.

Sub-case (iv)-(1): Let $C_{1L} \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < \bar{C}_1 > \bar{C}_1^{T3}$. For this parameter set-up the maximum can be reached either in (P5), (P6) or (P8) where

(P5): $C_{1L} \leq C_1 \leq Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ is the set of feasible solutions and
$C_1^{P5,*} = Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f}$ for $\lambda > \lambda_U^{P5}$, see (P5)-(i) and (P5)-(iii)

(P6): $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} \leq C_1 \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ is the set of feasible solutions and
$C_1^{P6,*} = Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ for $\lambda > \max\{0, \lambda_U^{P5}, \lambda_3^{P6}\}$, see (P6)-(v)

(P8): $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq C_1 \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ is the set of feasible solutions and
$C_1^{P8,*} = Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ for $\lambda > \max\{0, \lambda_3^{P6}, \lambda_1^{P8}\}$, see (P8)-(ii)-(3)

Based on the similar lines of arguments we can see that (P5)<(P6)<(P8).

Sub-case (iv)-(2): Let $Y_1 + \frac{Y_{2b}-\bar{C}_2}{1+r_f} < C_{1L} \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f} < \bar{C}_1 > \bar{C}_1^{T3}$. For this parameter set-up the maximum can be reached either in (P6) or (P8) where

(P6): $C_{1L} \leq C_1 \leq Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ is the set of feasible solutions and
$C_1^{P6,*} = Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f}$ for $\lambda > \max\{0, \lambda_1^{P6}, \lambda_3^{P6}\}$, see (P6)-(vi)-(3)

(P8): $Y_1 + \frac{Y_{2g}-\bar{C}_2}{1+r_f} \leq C_1 \leq Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ is the set of feasible solutions and
$C_1^{P8,*} = Y_1 + \frac{Y_{2b}-C_{2L}}{1+r_f}$ for $\lambda > \max\{0, \lambda_3^{P6}, \lambda_1^{P8}\}$, see (P8)-(ii)-(3)

Based on the similar lines of arguments we can see that (P6)<(P8).

Finally, statement of case (v) follows from (P8)-(iii)-(1). ■