COMENIUS UNIVERSITY, BRATISLAVA
FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS

ADVANCED BITCOIN DYNAMICS AND RISK MANAGEMENT

MASTER THESIS

2017 Bc. Veronika STARINSKÁ

COMENIUS UNIVERSITY, BRATISLAVA
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MASTER THESIS

Study Programme:	Mathematical Economics, Finance and Modelling
Field of Study:	1114 Applied Mathematics
Department:	Department of Applied Mathematics and Statistics
Supervisor:	Mgr. Pedro R.C.F. Pólvora

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Aim: The aim of this MSc Thesis is to contribute to the growing research of bitcoin markets by focusing on the application of advanced time-series modelling methods to bitcoin dynamics and its impact on various risk metrics.

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ZADANIE ZÁVEREČNEJ PRÁCE

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Pokročilé modely dynamiky bitcoinu a riadenie rizík

Cieľ: Cieľom diplomovej práce je prispieť k zväčšujúcemu sa akademickému výskumu bitcoinového trhu, aplikovať pokročilé metódy z oblasti časových radov na vývoj ceny bitcoinu a skúmať vplyv na rôzne miery rizika.

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Abstract

STARINSKÁ, Veronika: Advanced bitcoin dynamics and risk management [Master thesis], Comenius University in Bratislava, Faculty of Mathematics, Physics and Informatics, Department of Applied Mathematics and Statistics, Supervisor: Mgr. Pedro R.C.F. Pólvora, Bratislava, 2017, 58 p.

The aim of this Master Thesis is to study bitcoin dynamics by focusing on the application of advanced time series modelling methods and particular risk measures. In the first part of this thesis, a robust risk measure called expected shortfall (ES or cVaR) is applied to bitcoin daily returns. Besides, we pay attention to the risk linked with a portfolio which includes bitcoin and show that bitcoin is becoming more stable. As for the second part of this thesis, the important theory of time series modelling useful to study bitcoin returns is provided. Finally, in the last part, a broad range of time series methods is considered and applied to bitcoin historical price, concluding in obtaining the model which fit the data best.

Keywords: Bitcoin, Time Series, Expected Shortfall, Risk Measure, Price Dynamics

Abstrakt v štátnom jazyku

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Cieľom tejto diplomovej práce je skúmať vývoj ceny bitcoinu zameraním sa na aplikáciu pokročilých metód z oblasti časových radov a výberom vhodnej miery rizika. V prvej časti práce predstavíme robustnú mieru rizika zvanú expected shortfall (ES alebo cVaR) a následne ju aplikujeme na denné výnosy bitcoinu. Naviac sa pozrieme na riziko spojené s portfóliom zahŕňajúce bitcoin a ukážeme, že bitcoin v posledných rokoch ukazuje stabilnejší priebeh. Druhá časť práce sa venuje teórii časových radov vhodných na modelovanie dynamiky bitcoinových výnosov. Nakoniec aplikujeme vybrané modely časových radov na historické ceny bitcoinu a nájdeme model, ktorý popisuje dáta najlepšie.

Kľúčové slová: bitcoin, časové rady, Expected Shortfall, miera rizika, vývoj ceny

Contents

Introduction

Bitcoin is a digital and decentralized currency introduced by Satoshi Nakamoto. It provides users with many advantages such as more anonymity, independence from third party in form of a financial institution and low fees of transactions. In spite of mentioned benefits, bitcoin was considered as highly volatile since its deployment.

However, approximately from year 2015 something remarkable can be seen on the graphs of bitcoin price. One might notice that the fluctuations in price became less volatile. The stabilization of bitcoin price might make more people to start perceive bitcoin as a transaction tool or a business opportunity. The aforementioned facts lead us to study the price dynamics and try to find a suitable model for data.

Therefore, this thesis firstly introduce a reader to the definition of volatility and its basic characteristics. We will present possible ways of approaching the volatility and finish the chapter with short induction into bitcoin volatility. We will mention some of the research which have been done in aforementioned topic by now.

The second chapter is dedicated to the fundamental theory crucial to understand the advanced time series modelling. Since the volatility of bitcoin is clearly changing over the time, we focus our attention to the conditional heteroskedastic models from ARCH/GARCH family.

A robust risk metric, the expected shortfall or also known as the conditional Value-at-Risk is introduced in the third chapter. Here we make a use of the bitcoin daily returns and calculate the expected shortfall value by several approaches. Finally, we will build a portfolio including bitcoin, Japanese yen and Argentine peso. We will study the impact of bitcoin to the overall risk which the portfolio is facing.

Naturally, we expect that the negative news linked with bitcoin will have a higher influence on its volatility than the positive news of the same magnitude. Thus, a measure called news impact curve is presented together with the test for asymmetric effects in the fourth chapter.

And finally, we will apply the time series theory introduced in chapter 2 on the daily returns of bitcoin with the intention to find an appropriate model describing the data. We will provide the results of fitting the models and closing comparison of used approaches.

1 Volatility

The value of a portfolio depends on the market's components, including the interest rates, the exchange rates, the equity prices and many others. Thus, the tracking of their volatilities plays an important role for the financial institutions [15].

In the following chapter we discuss the definition of volatility and its basic characteristics, then we introduce exponentially weighted moving average (EWMA) model and afterwards we move on to the volatility of bitcoin dynamics.

1.1 Definition of volatility

Firstly, let us remind the well-known basic knowledge of the volatility - σ, which is defined as the standard deviation of the variable's return per unit of time. The period of time differs depending on the particular problem. In case of an option pricing, the unit of time is usually one year. Using volatility for risk management, the unit of time usually stands for one day period.

Now, assume the daily returns are independent with the same variance equals to σ^2. The variance of return over T days is T times larger than the variance of the return over one day. When calculating volatilities, usually the assumption is that there are 252 days per year, because the volatility on business days is much higher than the one on non-business days. Applying the previous statement, we obtain the relation between annual and daily standard deviations of return:

$$\sigma_{year} = \sigma_{day}\sqrt{252},$$

showing that the daily volatility is approximately equal to 6% of the annual volatility.

What Causes Volatility?

The financial market consists of possible buyers and sellers of a certain good or service and the transactions between them. Meaning the market is full of various events happening mostly during the business days. Every new information on the market makes people to re-evaluate their portfolios. The change in their opinion about the value of an asset implies changing the asset's price, resulting in volatility change.

Considering a short-run (like in terms of seconds) then this is based on quantity of transactions. The changes of volatility are usually very small. Complicated algorithms are designed to detect these quick changes. This is where high-frequency trading comes into play.

1.2 Characteristics of Volatility

In the following subsection we discuss the attributes of volatility, based on [15] and [25]. An important feature to mention is that volatility per se can not be forecasted. We cannot obtain the daily volatility from the daily returns, since there is only one observation per day. But one can estimate the daily volatility from, for example, intraday returns. Despite this, the accuracy of such an estimate require better study.

Among commonly observed characteristics of volatility we can put the existence of *volatility clustering*. In other words, large changes tend to be followed by large changes, and small changes tend to be followed by small changes. Figure 1 shows an example of the volatility clustering shown on the returns of bitcoin.

Further, the volatility jumps are rarely to be seen, so one can assume that volatility changes continuously. Its value does not diverge to infinity - statistically speaking, volatility is often stationary, and it reacts differently to a big price increase or a big price drop.

Even though risk managers usually compute a volatility from historical data, they should not forget to pay attention to **implied volatility**. The implied volatility is what we obtain when we use a pricing formula that takes volatility as one parameter and we solve that formula with respect to the volatility. So, given an asset's price we can obtain a volatility, which we denote implied volatility.

The example of such formula is the Black-Scholes formula, which is often used in option market. But it is necessary to mentioned that usually the implied volatility is larger than the one obtained by using a GARCH type of volatility model.

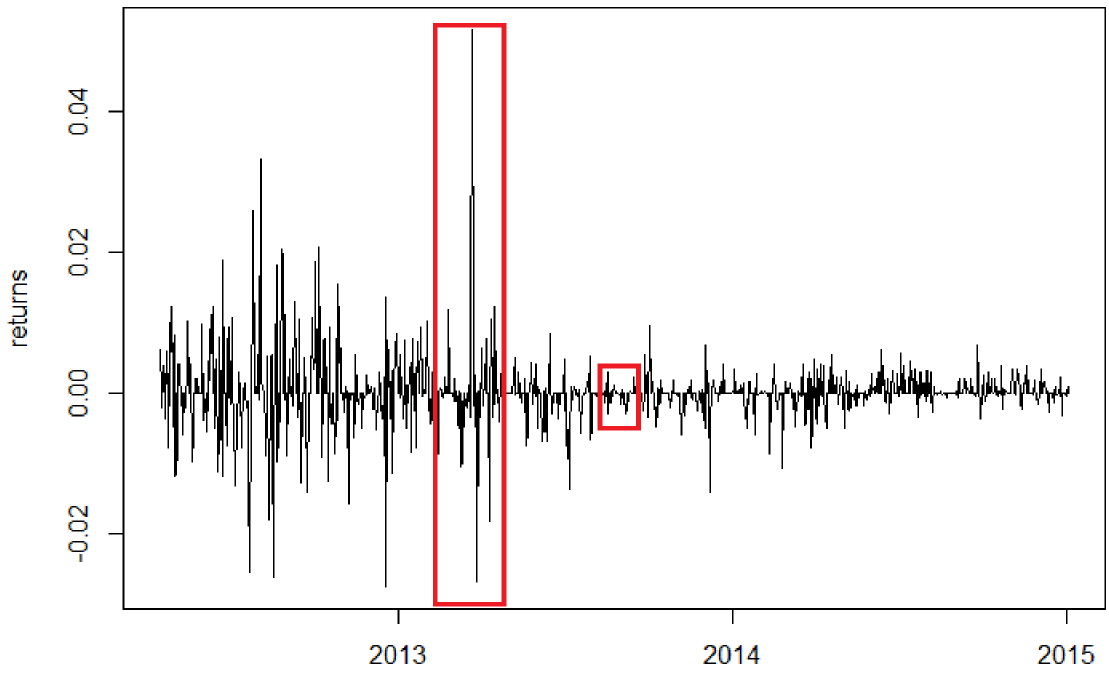


Figure 1: Volatility clustering

The VIX Index

The Chicago Board Options Exchange (CBOE) publishes indices of implied volatility. The most popular one is the VIX index, representing an implied volatility of 30-day option on the S&P 500 calculated from a wide range of calls and puts.

1.3 Approaching Daily Volatility

Since, volatility evolves over time it is necessary to study its behaviour and try to predict it. In this section we discuss some of the basic approaches to calculate the daily volatility of a variable.

Let σ_t be the estimated daily volatility of return on day t, the square of the volatility, σ_t^2, be the *variance rate*, and P_t be the price of the variable at time t. Define the (simple) daily return - R_t

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}}.$$

Using the natural logarithm, we obtain the *log returns*, which we use in the following chapters.

$$r_t = \log \frac{P_t}{P_{t-1}} = \log P_t - \log P_{t-1}.$$

Now, the first estimate of σ_t is equal to the standard deviation of the returns. Applying the well-known definition of the standard deviation with the most recent m observations, we obtain

$$\hat{\sigma}_t^2 = \frac{1}{m-1} \sum_{i=1}^{m} (r_{t-i} - \overline{r})^2 \tag{1.1}$$

where $\overline{r}$ is the mean of the r_t

$$\overline{r} = \frac{1}{m} \sum_{i=1}^{m} r_{t-i}$$

The expected change of variable in one day is very small comparing to the standard deviation of changes, so one can assume $\overline{r}$ to be zero. By replacing $m-1$ with m we move from an unbiased estimate of the volatility to a maximum likelihood estimate. All together, we obtain the formula for the variance rate

$$\hat{\sigma}_t^2 = \frac{1}{m} \sum_{i=1}^{m} r_{t-i}^2 \tag{1.2}$$

Weighting Schemes

In previous equation 1.2 we assign same weights to all of the returns. But it is more reasonable to attribute greater importance to the recent data. A following model ensures such requirements

$$\sigma_t^2 = \sum_{i=1}^{m} \alpha_i r_{t-i}^2, \tag{1.3}$$

where α_i are the weights given to the observations, which must satisfy conditions such as $\alpha_i > 0$ for all i and $\sum_{i=1}^{m} \alpha_i = 1$. By choosing $\alpha_i < \alpha_j$ for $i > j$ we assign less weight to the older observations.

Including a long-run average variance rate (V_L) in the upper equation 1.3 changes the model as follows

$$\sigma_t^2 = \gamma V_L + \sum_{i=1}^{m} \alpha_i r_{t-i}^2, \tag{1.4}$$

with condition on the weights $\gamma + \sum_{i=1}^{m} \alpha_i = 1$. This approach is equivalent to modelling the variance as an ARCH(p) stochastic process that we define in greater detail in chapter 2. But before we do so, we introduce the EWMA model.

The Exponentially Weighted Moving Average Model

The EWMA model is a special case of the equation 1.3 with exponentially decreasing weights α_i. Let λ be a constant between zero and one, then $\alpha_{i+1} = \lambda\alpha_i$ for all i.

The daily volatility using the EWMA model is sum of the daily volatility estimate from previous day and the most recent daily return with associated weights.

$$\sigma_t^2 = \lambda\sigma_{t-1}^2 + (1-\lambda) r_{t-1}^2 \tag{1.5}$$

Now, we take a closer look at equation 1.5 to show exponential decreasing of weights. Firstly, we substitute for σ_{t-1}^2

$$\sigma_t^2 = \lambda[\lambda\sigma_{t-2}^2 + (1-\lambda) r_{t-2}^2] + (1-\lambda) r_{t-1}^2,$$
$$\sigma_t^2 = (1-\lambda)(r_{t-1}^2 + \lambda r_{t-2}^2) + \lambda^2\sigma_{t-2}^2.$$

Repeating previous substitution for σ_{t-2}^2 give us

$$\sigma_t^2 = (1-\lambda)(r_{t-1}^2 + \lambda r_{t-2}^2 + \lambda^2 r_{t-3}^2) + \lambda^3\sigma_{t-3}^2$$

By continuing with the iteration process we obtain

$$\sigma_t^2 = (1-\lambda)\sum_{i=1}^{m} \lambda^{i-1} r_{t-i}^2 + \lambda^m \sigma_{t-m}^2.$$

The limit of $\lambda^m\sigma_{t-m}^2$ goes to zero for large m and now the equation above with $\alpha_i = (1-\lambda)\lambda^{i-1}$ corresponds with equation 1.3. As one can see, moving back through time, the weights decrease at rate λ.

Model's storage requirements are undemanding because at any point in time, we need only the most recent return and the last estimate of volatility to calculate new volatility estimate. The EWMA model tracks changes in the volatility. A high value of λ produce volatility which responds relatively slowly to new information about the return and vice versa.

1.4 Volatility of Bitcoin

Until now we introduced the basic theory of volatility, including its definition, basic properties and some of the possible ways how to calculate the daily volatility of a variable. The following section deals with bitcoin volatility.

Since its deployment, bitcoin is enjoying much more popularity by media than the other cryptocurrencies. It is mostly because bitcoin provide many advantages such as its complete decentralization from all financial authorities, provision of more anonymity for users and fast processing of payments due to a peer-to-peer network called Blockchain introduced by Satoshi Nakamoto, see [22]. Despite mentioned benefits, bitcoin is highly volatile to be used as currency.

In spite of the period of high volatility in 2014, bitcoin started to evince a relatively stable progress since early 2015. The article [4] noticed the stabilization of the volatility too. It compares the periods [December 2010 - June 2015] and [January 2015 - June 2015]. Although the second interval shows less volatility persistence, the degree of asymmetry remains strong. In other words, it is likely driven by negative rather than positive shocks.

The volume of bitcoin transactions is considered in research published in article [21]. The raw annualised volatility of Bitcoin and the adjusted returns are compared to major exchange rates.

$$\text{Adjusted Return} = \frac{\Delta\text{Exchange rate}}{\text{Volume of trades}}$$

This study shows that when comparing only the raw changes on the bitcoin exchange rate, the volatility is much higher than the other major currencies, such as Euro, Sterling Pound, Ruble, Franc and others. Taking into account the volume of trades, the volatility of the bitcoin exchange rate reduces significantly.

2 Conditional Heteroskedastic Models

In the following chapter, we discuss various univariate volatility models. Firstly, we focus on the ARCH model and its basic properties, then we introduce the generalized autoregressive conditional heteroskedastic (GARCH) model. We pay attention to the benefits and drawbacks of mentioned volatility models. Eventually, we go through some of the volatility models designed specifically to correct the weaknesses of the existing ones, such as EGARCH model. This chapter is based on the information from [25].

We can divide the conditional heteroskedastic models into two categories:

- **1st category** - use an exact function to determine the dynamics of σ_t^2 (e.g. GARCH model),
- **2nd category** - use a stochastic equation to describe σ_t^2 (e.g. the stochastic volatility model).

Now, we discuss volatility models from the first category in more detail. Let's start with the basic definition and properties of the ARCH model.

2.1 The ARCH Model

In the ARCH model the volatility at time t is completely pre-determined (deterministic) given previous values. It is one of the earliest time series models for heteroskedasticity, which is used in modelling financial time series that show time-varying volatility clustering. Throughout the text, $\mathbf{a_t}$ is referred to as **the return residual at time t**.

Definition 2.1 (ARCH(p) model)**.** *We define the autoregressive conditional heteroskedasticity model of order p, or simply ARCH(p) model as*

$$a_t = \sqrt{\sigma_t^2}\epsilon_t; \qquad \sigma_t^2 = \omega + \alpha_1 a_{t-1}^2 + ... + \alpha_p a_{t-p}^2 \tag{2.1}$$

where $\{\epsilon_t\}$ is a sequence of independent and identically distributed (iid) random variables with mean zero and variance 1 (white noise with unit variance), i.e., $a_t \sim N(0, \sigma_t^2)$.

One could expect that the older information, the less impact it has on current volatility, hence $\alpha_i < \alpha_j$ for $i > j$. In an ARCH(p) model, only the information recorded at the market less than p periods ago has an effect on volatility, meaning that $\alpha_{p+j} = 0$ for $j = 1, 2, ...$

The coefficients α_i must satisfy some conditions to ensure:

- **the positive variance** $\omega > 0, \quad \alpha_1, ..., \alpha_{p-1} \geq 0, \quad \alpha_p > 0,$
- **stationarity** $\alpha_1 + ... + \alpha_p < 1.$

2.1.1 Properties of ARCH Models

For better understanding the basic properties of the ARCH model, we start with conditions regarding to the particular case of the ARCH(1) model.

Definition 2.2 (ARCH(1) model)**.** *We define the autoregressive conditional heteroskedasticity model of order 1, or simply ARCH(1) model as*

$$a_t = \sqrt{\sigma_t^2}\epsilon_t; \qquad \sigma_t^2 = \omega + \alpha_1 a_{t-1}^2 \tag{2.2}$$

where $\omega > 0$, *and* $\alpha_1 \geq 0$.

First of all, the unconditional mean of a_t remains zero because

$$E(a_t) = E[E(a_t|\mathcal{F}_{t-1})] = E[\sigma_t E(\epsilon_t)] = 0.$$

Secondly, the unconditional variance of a_t is equal to:

$$\begin{aligned} Var(a_t) &= E(a_t^2) = E[E(a_t^2|\mathcal{F}_{t-1})] \\ &= E(\omega + \alpha_1 a_{t-1}^2) = \omega + \alpha_1 E(a_{t-1}^2) \\ &= \omega + \alpha_1 Var(a_t) \end{aligned}$$

We obtained the last equation using the fact that a_t is a stationary process with $E(a_t) = 0, Var(a_t) = Var(a_{t-1}) = E(a_{t-1}^2)$. Now, we rearrange mentioned equation resulting in $Var(a_t) = \frac{\omega}{(1-\alpha_1)}$. Due to the positive variance, we require $0 \leq \alpha_1 < 1$. Furthermore, in some cases we need higher order moments of a_t to exist, implying α_1 must satisfy some additional constraints, see [25].

For a general case of a process ARCH(p) we can determine its order by using the partial autocorrelation function (PACF) of a_t^2. To check the adequacy of fitted model, one can use the Ljung-Box statistics of $\tilde{a}_t = \frac{a_t}{\sigma_t}$ and the forecast of the ARCH model can be acquire recursively.

2.1.2 Drawbacks of ARCH Models

It is necessary to mention some disadvantages of these models. ARCH models respond slowly to large isolated shocks causing overpredicting the volatility. This is a drawback because such shocks are likely to happen in case of bitcoin dynamics.

A small number of terms a_{t-1}^2 is often not sufficient, causing that the squares of residuals are still often correlated. A larger number of terms does not always guarantee successfulness of the model, since the terms are often not significant or the constraints on the parameters are not satisfied.

Moreover, the model responds equally to positive and negative shocks. Thus, it depends on the square of the previous shocks. From the real life, we know that the impact of positive and negative shock on financial asset is different.

2.2 The GARCH Model

The ARCH model often needs many parameters to appropriately describe the volatility dynamics. By a simple generalization of the ARCH model we introduce the GARCH model.

Definition 2.3 (GARCH(p,q) model)**.** *The generalized autoregressive conditional heteroskedasticity model, simply GARCH(p,q) model is given by*

$$a_t = \sqrt{\sigma_t^2}\epsilon_t; \qquad \sigma_t^2 = \omega + \alpha_1 a_{t-1}^2 + ... + \alpha_p a_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + ... + \beta_q \sigma_{t-q}^2 \tag{2.3}$$

where $\{\epsilon_t\}$ is a sequence of iid random variables with mean zero and variance 1.

As in case of the ARCH model, there are some constraints on parameters:

- **variance has to be positive** $\omega > 0, \quad \alpha_1, ..., \alpha_{p-1} \geq 0, \quad \alpha_p > 0,$ $\beta_1, ..., \beta_{q-1} \geq 0, \quad \beta_q > 0,$
- **stationarity** $(\alpha_1 + ... + \alpha_p) + (\beta_1 + ... + \beta_q) < 1.$

The α_i and β_i are known as ARCH and GARCH parameters, respectively.

2.2.1 Properties of GARCH models

Consider frequently used simple GARCH(1,1) model.

$$a_t = \sqrt{\sigma_t^2}\epsilon_t; \qquad \sigma_t^2 = \omega + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$
$$0 \leq \alpha_1, \quad \beta_1 \leq 1, \quad (\alpha_1 + \beta_1) < 1 \tag{2.4}$$

Note that a large a_{t-1}^2 or σ_{t-1}^2 causes a large σ_t^2 and therefore a large a_t^2. Stated implication is described in chapter 1 as volatility clustering. The aforementioned model also gives a simple parametric function for calculating the volatility.

Moreover, it can be shown that after fulfilling a condition $1 - 2\alpha_1^2 - (\alpha_1 + \beta_1)^2 > 0$, the value of a kurtosis (the fourth standardized moment) is more than three. This leads us to conclusion that the GARCH(1,1) model has heavier tails than a normal distribution.

Regarding the weaknesses of GARCH models, they are the same as those of the ARCH models. For example, a GARCH model assumes the same effect on volatility by both positive and negative shocks.

2.3 Other Models for Volatility

In the theory of time series one might find special models designed for enhancing the weaknesses of above mentioned GARCH model. Now, we briefly introduce some of them, focusing on the asymmetric effect (also known as the leverage effect).

2.3.1 TGARCH(1,1) model

Regarding to financial time series, it is usual to observe that "bad news" seem to have higher impact on volatility than "good news". For instance, **the threshold GARCH (TGARCH) model** is designed to cope with this leverage effect. Therefore, the model uses zero as a *threshold* to distinguish between the "good news" - $a_t > 0$ and the "bad news" - $a_t < 0$.

The notation of a TGARCH(1,1) model is

$$a_t = \sqrt{\sigma_t^2}\epsilon_t, \qquad \sigma_t^2 = \omega + \alpha_1 a_{t-1}^2 + \lambda_1 d_{t-1} a_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \tag{2.5}$$

where d_{t-1} is a dummy variable. In this case, when threshold is set to zero, the value of d_{t-1} is equal to one or zero based on the sign of a_{t-1}

$$d_{t-1} = \begin{cases} 1 & \text{if } a_{t-1} < 0, \\ 0 & \text{if } a_{t-1} > 0, \end{cases}$$

In other words, the effect of a_{t-1} on volatility caused by negative return residual is $(\alpha_1 + \lambda_1)a_{t-1}^2$ and of the positive one is only equal to $\alpha_1 a_{t-1}^2$. So the leverage effect of shocks is ensured.

2.3.2 EGARCH(1,1) model

Another approach used to handle the asymmetric effects of positive and negative shocks is **the exponential GARCH (EGARCH) model**. Unlike the GARCH model, EGARCH approach models the logarithm of volatility to detect the aforementioned asymmetry in observed data.

One of the advantages of an EGARCH model is that it does not require the estimated coefficients to be positive. On the other hand, It is difficult to compute unbiased forecasts of volatility over multiperiod intervals.

$$a_t = \sqrt{\sigma_t^2}\epsilon_t; \quad \log \sigma_t^2 = \omega + \alpha_1 \left[\frac{|a_{t-1}|}{\sigma_{t-1}} - E\frac{|a_{t-1}|}{\sigma_{t-1}} \right] + \gamma_1 \left(\frac{a_{t-1}}{\sigma_{t-1}} \right) + \beta_1 \log \sigma_{t-1}^2 \quad (2.6)$$

Since γ_1 is typically negative, positive shocks affect volatility less than an unexpected decrease in price (bad news) of similar magnitude.

2.3.3 NGARCH(1,1) Model

Another enhanced model is **nonlinear GARCH model - NGARCH**, sometimes known as **the power GARCH model**. It covers the leverage effect, indicating that negative news increase future volatility by a larger amount than positive news of the same magnitude.

The approach is modelling the conditional standard deviation to the power δ as function of the lagged conditional standard deviations and the lagged absolute innovations raised to the same power.

$$a_t = \sqrt{\sigma_t^2}\epsilon_t; \qquad \sigma_t^\delta = \omega + \alpha_1 |a_{t-1}|^\delta + \beta_1 \sigma_{t-1}^\delta \quad (2.7)$$

In case $\delta = 2$, this formulation reduces to a standard GARCH(1,1) model, see 2.4

And similarly we could continue naming more models of GARCH. For more detailed information see, for example [9], [25] or [27].

3 Risk metrics

In terms of the financial world, a risk represents the chance that an investment's actual return will be different than expected. So the basic idea of the risk is strongly associated with the uncertainty. High levels of the uncertainty are usually related with high potential returns. Measuring and controlling risk has become inseparable part of decision making. The financial markets are no exception.

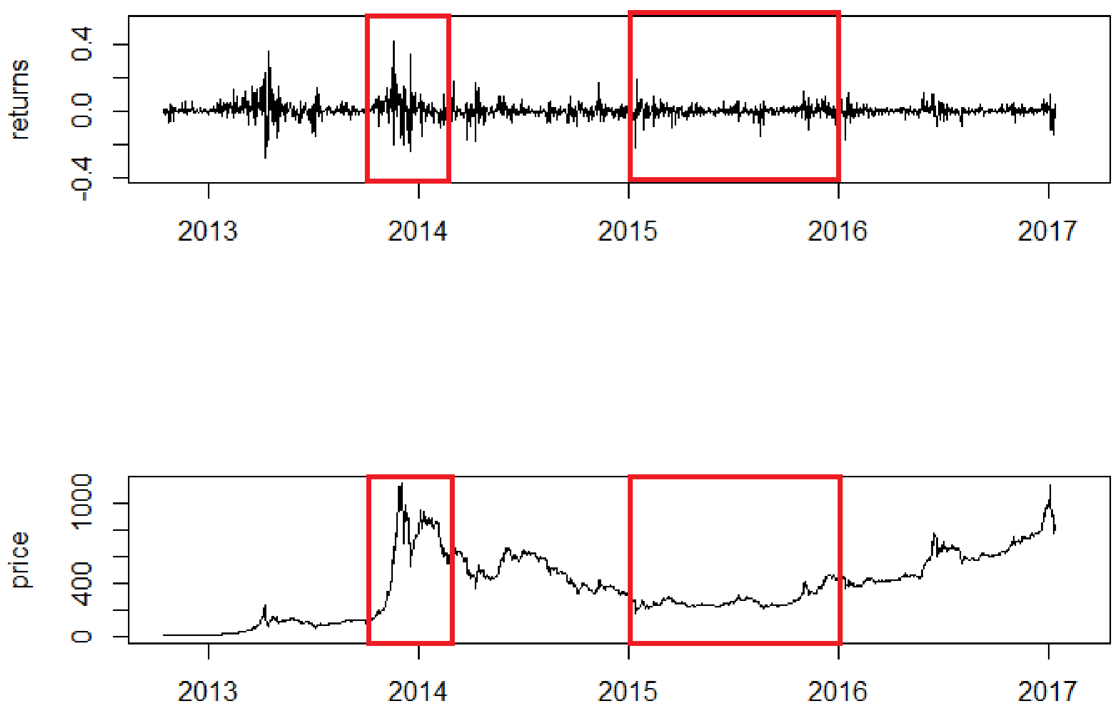


Figure 2: Price and corresponding returns of bitcoin

3.1 Previous analysis

In [24] we discussed the measure of risk that considers the extreme events as the source of a risk - *the value at risk (VaR).* After introducing two approaches of modelling VaR (modelling VaR from the parametric distribution, and modelling an historical VaR), we applied aforementioned methods on the bitcoin data.

The concept of Value at risk includes some inconsistencies, so in this chapter we focus on the robust risk measures such as *a conditional value at risk* known as an expected shortfall.

3.2 Expected shortfall

The conditional value at risk (**cVaR** or **ES**) is the average loss conditioned to the fact that certain threshold is exceeded. It is an alternative to VaR which gives us more accurate estimate of the amplitude of large losses and is usually used to reduce the probability that a portfolio will generate large losses. Therefore sometimes the expected shortfall is called as tail VaR (TVaR) or expected tail loss (ETL).

Now we present the definition of mentioned risk measure associated with VaR - expected shortfall.

Definition 3.1 (Expected Shortfall (ES)). *For a given random variable X, the expected shortfall is defined as the expected size of a loss that exceeds* VaR_α

$$ES_\alpha = E\left[X|X > VaR_\alpha\right] \tag{3.1}$$

$$= VaR_\alpha + E\left[X - VaR_\alpha|X > VaR_\alpha\right] \tag{3.2}$$

Unlike the Value at Risk, Expected shortfall is considered as coherent risk measure. We state the definition of a coherent risk measure. A more detailed information regarding the coherent risk measures can be found for example in [8].

Definition 3.2 (Coherent Risk Measure). *A mapping* $\rho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ *is called a coherent risk measure if the following properties hold*

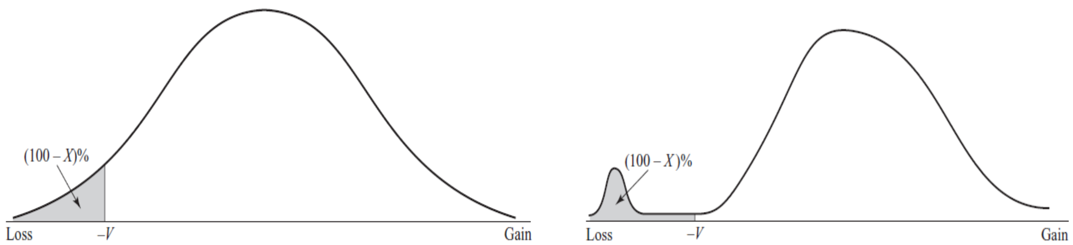
(i) *If* $X \geq 0$ *then* $\rho(X) \leq 0$.

(ii) *Subadditivity:* $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$.

(iii) *Positive homogeneity: for* $\lambda \geq 0$ *we have* $\rho(\lambda X) = \lambda\rho(X)$.

(iv) *For every constant function c we have that* $\rho(c + X) = \rho(X) - c$.

The expected shortfall is a weighted average of the VaR and losses exceeding the VaR. In other words, as it is said in [15], the VaR asks question: "How bad can the situation get?", and the expected shortfall asks: "If the situation does get bad, what is our expected loss?". The figure 3 from [15] shows us the difference between VaR and ES. Unlike the VaR, the ES focuses on the events when the VaR is exceeded, which is represented by a bump in figure.



(a) VaR **(b)** ES

Figure 3: Example of probability distribution of the gain - VaR and ES

3.3 Estimation Methods for Expected Shortfall

There are several possible methods how to estimate the expected shortfall. We divide them into the following categories

- non-parametric methods,
- parametric methods,
- semi-parametric methods.

Now, we take shortly mentioned some of them, more information regarding the estimation methods might be found for example in [18].

3.3.1 Non-parametric Methods

The non-parametric methods use empirical distribution to estimate risk measure. One of them is the historical method.

Historical method

This approach of estimating expected shortfall is given by

$$\widehat{ES}_{\alpha}(X) = \frac{\left(\sum_{i=[n\alpha]}^{n} X_{(i)}\right)}{(n - [n\alpha])} \tag{3.3}$$

where $[x]$ denotes the largest integer not greater than x and $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ are the variables ordered in the ascending order.

3.3.2 Parametric Methods

The basic idea of parametric methods is to specify parametric distribution for returns, estimate parameters of distribution and estimate risk measures as functions of estimated parameters.

In this category belong methods which consider the distribution of observations (for example the Gaussian distribution, Student-t distribution, Generalized Pareto distribution etc.) or assume that the returns follow GARCH(1,1) process or many other models. To improve the idea behind mentioned approaches we shortly present the methods which involve normal and generalized pareto distributions.

Normal Distribution

In general, for a $N(\mu, \sigma^2)$ the ES is given by

$$ES_{\alpha} = \mu + \frac{f(VaR_{\alpha})}{1-\alpha}\sigma,$$

where $f(x)$ is the probability density function.

If the random variable is following the Student-t distribution with degrees of freedom $\nu > 2$, then the ES is given by

$$ES_{\alpha} = \mu + \frac{g_{\nu}(t_{\nu}^{-1}(\alpha))}{1-\alpha} \cdot \frac{\nu + (t_{\nu}^{-1}(\alpha))^2}{\nu - 1}\sigma$$

where g_{ν} is the density of the standard Student-t distribution. In similar way one can derive the evaluation of ES for any random variable with a probability distribution parametrized by a location parameter and a non-negative scale parameter.

Generalized Pareto Distribution

The GPD is used to model the tails of a distribution and it is specified by three parameters: location μ, scale σ, and shape ξ.

Definition 3.3 (Generalized Pareto Distribution (GPD)). *The cumulative distribution function of the GPD is defined by*

$$F_{(\xi,\mu,\sigma)} = \begin{cases} 1 - \left(1 + \frac{\xi(x-\mu)}{\sigma}\right)^{\frac{1}{\xi}} & \text{for } \xi \neq 0, \\ 1 - \exp\left(-\frac{x-\mu}{\sigma}\right) & \text{for } \xi = 0, \end{cases} \tag{3.4}$$

for $x \geq \mu$ when $\xi \geq 0$, and $\mu \leq x \leq \mu - \sigma/\xi$ when $\mu \in \mathbb{R}, \sigma > 0$ and $\xi \in \mathbb{R}$.

Then the estimation of expected shortfall when VaR is exceeded for GPD has the following form

$$\widehat{ES_\alpha} = \frac{\widehat{VaR_\alpha}}{1-\hat{\xi}} + \frac{\hat{\sigma} - \hat{\xi}\mu}{1-\hat{\xi}}.$$

3.3.3 Semi-parametric Methods

The third branch of estimation methods is a mixture of non-parametric and parametric estimation. Some parts of the probability distribution of returns are treated non-parametrically and some parts are treated parametrically.

Example of such method is the Cornish-Fisher expansion which provides a relatively easy way of dealing with non-normality in return distribution. The calculation of VaR is accurate when returns are close to the Gaussian distribution since it takes into account the higher moments such as skewness and kurtosis of data. More detailed information regarding the semi-parametric methods might be find for example in [23].

3.4 Bitcoin Expected Shortfall

In this section we apply the methods described above to bitcoin prices.

3.4.1 Historical method

To start with the estimation of ES for bitcoin, we look at the historical value of the ES of daily returns, setting the level of significance equal to 5%. By changing the amount of considered returns we obtained the historical ES for each day from returns of the last 60, 120, and 180 days, see figure 4.

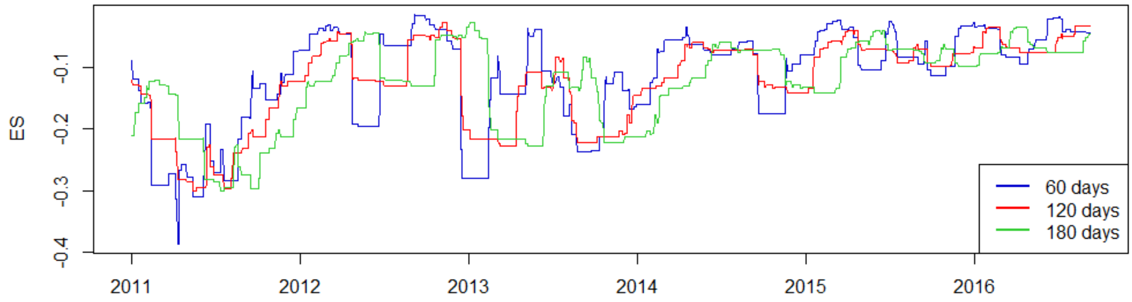


Figure 4: Historical estimate of ES at 95% confidence level- different time intervals

From the figure with drawn historical estimations we can see that the highest variability is shown by the ES estimated from 60 previous returns. On the contrary, estimates from 180 prior returns indicate more stable development.

3.4.2 Parametric method

We start with the comparison of historical and gaussian estimate of the expected shortfall for bitcoin daily returns. In table 1 are the ES estimates at 95% and 99% level. The first column represents the computation of ES from historical sample. In the second one, the Gaussian ES is recorded.

	Historical VaR	Historical ES	Gaussian VaR	Gaussian ES
95%	0.0706	0.1394	0.0889	0.1124
99%	0.1835	0.2656	0.1272	0.1463

Table 1: 95% and 99% ES and VaR - historical and Gaussian approach

Thus, if you are on one of those days when the VaR is exceeded, in this case when you expect to lose more than 7%, you can actually end up with loss approximately 14%, which is quite risky.

The Gaussian estimate of ES is smaller than the historical expected shortfall. This means that an investment in bitcoin is linked with higher risk than we would expect from a normal distribution, which is consistent with previous studies that the bitcoin price does not follow a geometric Brownian motion.

3.5 Portfolio with Bitcoin

For the next analysis we create a portfolio consisting of bitcoin **(BTC)**, Japanese yen **(JPY)** and Argentine peso **(ARS)**. After the United States dollar and Euro, the Japanese yen is the third most traded currency in the world and over the decades, yen is considered a safe-haven currency. On the other hand, Argentina's underground economy is likely to take a hit thanks to developed black market for pesos. Figure 5 shows the dynamics of yen and Argentine peso since 2011. Historical exchange rates for JPY/USD and ARS/USD are from website [13].

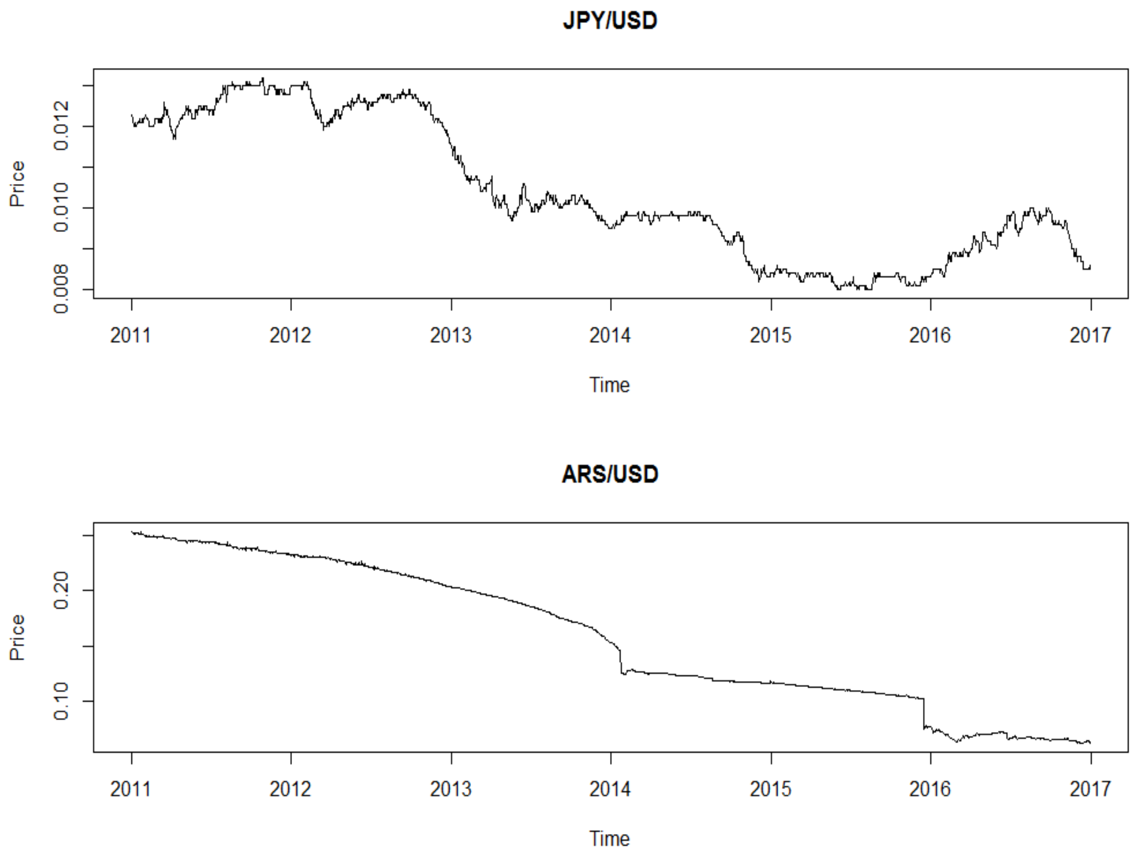


Figure 5: Exchange rate for Japanese yen and Argentine peso

Before we start with the estimation of an expected shortfall for the portfolio, let us go back to the definition of a coherent risk measure. As it is stated in [15], the properties mentioned in the definition 3.2 can be easily demonstrated in the following way

- *monotonicity* - if a portfolio has lower returns than another portfolio for every state of the world, its risk measure is greater. In other words, if one portfolio produces a worse result than another one, it should be considered as risky, therefore its risk measure should be higher,
- *subadditivity* - the risk measure for two merged portfolios in no greater than the sum of their risk measures before they were merged,
- *positive homogeneity* - change of the portfolio size by a factor λ while keeping the relative amounts of different items in the portfolio the same, results in the risk measure being multiplied by λ,
- *translation invariance* - if we add an amount of cash c to a portfolio, its risk measure goes down by c.

Unlike the expected shortfall, a VaR does not always satisfy the property of a subadditivity.

Now, we can continue with estimation of expected shortfall for portfolio made from JPY, ARS and BTC. The portfolio return is given by

$$R_p = \sum_{i=1}^{n} w_i R_i \tag{3.5}$$

where the portfolio weight satisfy $\sum_{i=1}^{n} w_i$ is the percentage composition of a particular holding in a portfolio. Then the ES of a portfolio has the following form

$$\begin{aligned} ES_\alpha &= E\left[R_p | R_p \leq VaR_\alpha\right] \\ &= \sum_{i=1}^{n} w_i E\left[R_i | R_p \leq VaR_\alpha\right] \end{aligned} \tag{3.6}$$

We make several portfolios which consist of daily returns of JPY/USD exchange rate, ARS/USD exchange rate and BTC/USD exchange rate. Our data sample contains daily returns for overlapping trading days for all three currencies, starting on 2nd January 2011 until 30th December 2016. The mentioned portfolios has the following form

$$P_1 = 1/3JPY_1 + 1/3ARS_1 + 1/3BTC_1,$$
$$P_2 = 1/2JPY_1 + 1/2ARS_1,$$
$$P_3 = 1/3JPY_2 + 1/3ARS_2 + 1/3BTC_2,$$

where the index 1 represents time period from 2011 until 2016 and index 2 stands for period 2014-2016.

To start with, we assign equal weights for all three holdings in portfolio. Table 2 show historical, gaussian and modified (Cornish-Fisher) value of the expected shortfall at 95% and 99% confidence level.

	Historical ES	Gaussian ES	Modified ES
95%	0.0531	0.0505	-0.0004
99%	0.0915	0.0659	0.1406

Table 2: Historical, Gaussian and modified ES of portfolio at 95% and 99% confidence level (2011-2016)

For 95% confidence level, the ES is significantly higher than the ES values obtained for the portfolio consisting only from Japanese yen and Argentine peso, see table 3 Bitcoin has the highest contribution on the values of expected shortfall, in all calculations it is more than 90%. On the other hand, the modified expected shortfall at 99% confidence level is almost the same for both portfolios. This is caused by Argentine peso which at December 2015 tumbled as much as 30 percent when the government let currency float free.

	Historical ES	Gaussian ES	Modified ES
95%	0.0115	0.0115	-0.0094
99%	0.0250	0.0147	0.1448

Table 3: Historical, Gaussian and modified ES of equally weighted portfolio of Japanese yen and Argentine peso at 95% and 99% confidence level

What we would like to show is that the dynamic of bitcoin is becoming more stable so we repeat aforementioned estimation of the expected shortfall for shorter period of time. Now, we consider returns for Japanese yen, Argentine peso, and bitcoin for three years since January 2014 until the end of year 2016. Results are recorded in table 4.

	Historical ES	Gaussian ES	Modified ES
95%	0.0357	0.0291	0.0351
99%	0.0628	0.0376	0.0583

Table 4: Historical, Gaussian and modified ES of portfolio at 95% and 99% confidence level (2014-2016)

It is clearly see that the risk linked with mentioned equally weighted portfolio decreased. The bitcoin contribution percentage declined as well. In case of modified method it is at almost the same level with the Argentine peso.

4 Asymmetric Impact of News

As we showed before, throughout the years, many approaches of modelling and forecasting the volatility were developed. Amongst the most commonly used, definitely belong ARCH and GARCH models and also those which take into consideration a leverage affect of news on volatility (e.g. EGARCH, NGARCH, TGARCH etc.).

4.1 News Impact Curve

In the following section, base on an article [11] we define a standard measure of how the new information is incorporated into volatility estimates, called **a news impact curve (NIC)**. This measure was suggested in mentioned article published by Engle and Ng in 1993. The NIC reflects a relationship between the new information and future volatility.

It shows a volatility asymmetry or leverage effect (a negative shock yields higher volatility than a positive shock) for asymmetric GARCH models such as the EGARCH. Moreover, the news impact curve may be used to compare the properties of different volatility models in the ARCH class.

4.1.1 Shape of News Impact Curve

The shape of the news impact curve differs regarding the applied model. A GARCH model's NIC is a quadratic function centred on $a_{t-1} = 0$. The news impact curve for EGARCH model has minimum at $a_{t-1} = 0$ and it is exponentially increasing in both directions but with different parameters. In the case of nonlinear asymmetric GARCH model is the news impact curve symmetric and centred at $a_{t-1} = -\gamma\sqrt{\sigma^2_{t-1}}$.

From figure 6 (source: [11]) a reader can see how volatility responds to good (right side of the graph) and bad news (left side of the graph). Moreover, we can observe two main differences between a GARCH and an EGARCH model. Firstly, previously mentioned, the EGARCH allowance of different impact on volatility by news. Secondly, the EGARCH model in comparison to GARCH model, allow big news to have a greater effect on volatility.

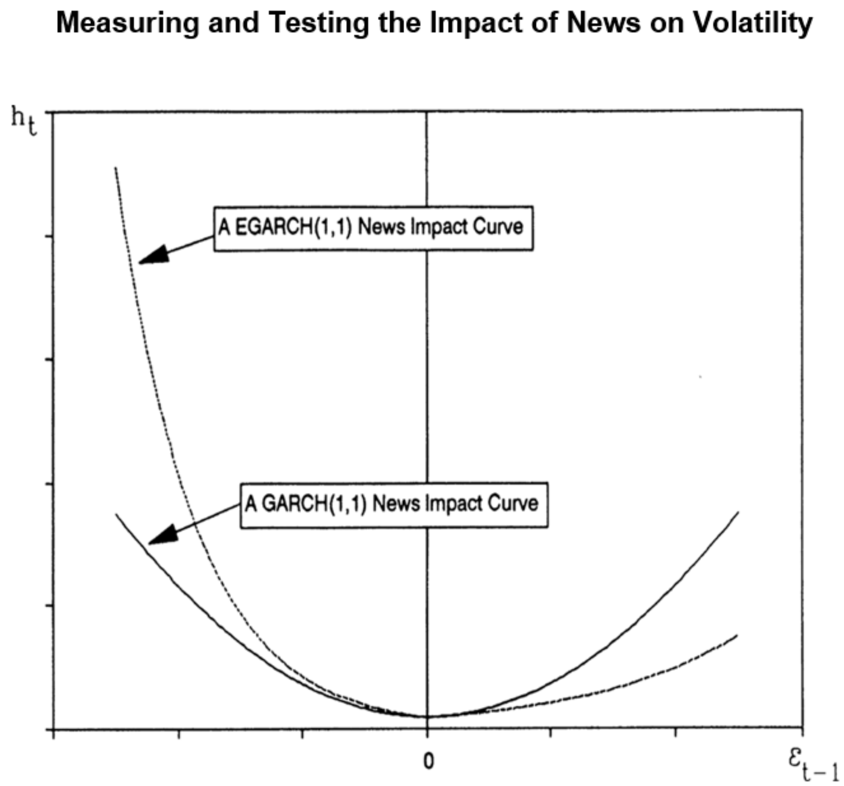


Figure 6: News impact curves of GARCH(1,1) and EGARCH(1,1) models, where ϵ_t represents a_t and h_t stands for volatility

To sum up, the leverage effect of volatility models is covered by the news impact curve by letting the slope to be different for positive and negative or allowing the curve to be centred at $a_{t-1} > 0$.

4.2 Test for Asymmetric Effects

Consider that bad news causes more volatility than good news of the same magnitude. In this particular case, the GARCH model underpredicts the volatility following negative news and overpredicts the volatility following positive news.

A part of an article [11] by Engle and Ng was dedicated to three diagnostic tests, namely **Sign Bias Test, Negative Size Bias Test** and **Positive Size Bias Test**.

Having a volatility model, mentioned test examine whether the squared normalized residuals can be predicted by some not included variable observed in the past. In case that these variables can predict the squared normalized residual, the variance model is misspecified.

Sign bias test is used to examine the significance of positive and negative news on volatility not predicted by the model under consideration. There is a dummy variable which is equal to one when a_{t-1} is negative, otherwise it takes a value of zero.

The different effects of large and small negative return shocks on volatility which is not predicted by the volatility model, are the subject of interest in **the negative size bias test**.

Positive size bias test is working in a similar way, but in this case it examines the different impacts that large and small positive return shocks may have on volatility.

5 Application of Time Series Models to Bitcoin

After introducing the theory needed to understand the basic concept of the advanced time series models, such as ARCH(p) and GARCH(p,q) model, we focus on modelling the dynamics of bitcoin returns. The ARCH/GARCH approach is commonly used by many institutions and companies to gain better idea of the risk their business is facing.

Mean	St. dev.	Skewness	Kurtosis
0.0037	0.0563	-0.3423	13.1556

Table 5: Standard parameters of bitcoin returns

Firstly, we compute the logarithmic returns for the years 2011-2016 from the bitcoin daily prices we obtained from website [12]. Since the bitcoin market is working 24/7 we include the weekends in our data set. Table 5 shows the evidence of fat tails since the kurtosis is equal to more than 3 (the Normal value).

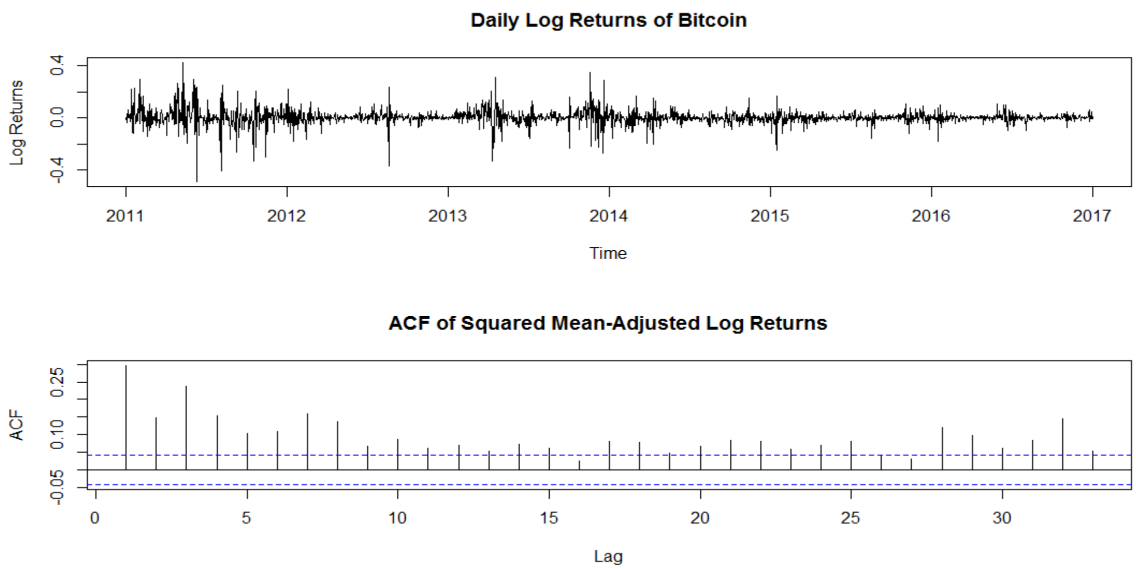


Figure 7: Bitcoin returns and correlogram of squared mean-adjusted log-returns

From figure 7, one might notice the difference in amplitudes of bitcoin returns over the years. In other words, the conditional volatility varies over time (clustering). The correlogram of squared mean-adjusted returns of bitcoin shows evidence of serial correlation implying a conditional heteroskedastic behaviour.

5.1 Fitting ARIMA Model and Testing for ARCH Effect

In this section we focus on fitting ARIMA and GARCH models to obtained bitcoin returns. Firstly, we fit a suitable ARIMA(p,d,q) model. Since we've already applied differencing on bitcoin price so now, we are working with returns, one should expect the degree of differencing - **d** - to be equal zero. The resulting order of the fitted ARIMA model is (p,d,q) = (3,0,1).

Afterwards, we continue with testing whether the residuals of fitted model show evidence of conditional heteroskedasticity. The ACF of residuals of the ARIMA(3,0,1) fit are shown in figure 8. The plot of autocorrelations looks like a realisation of a white noise which might confirm the goodness of the fit.

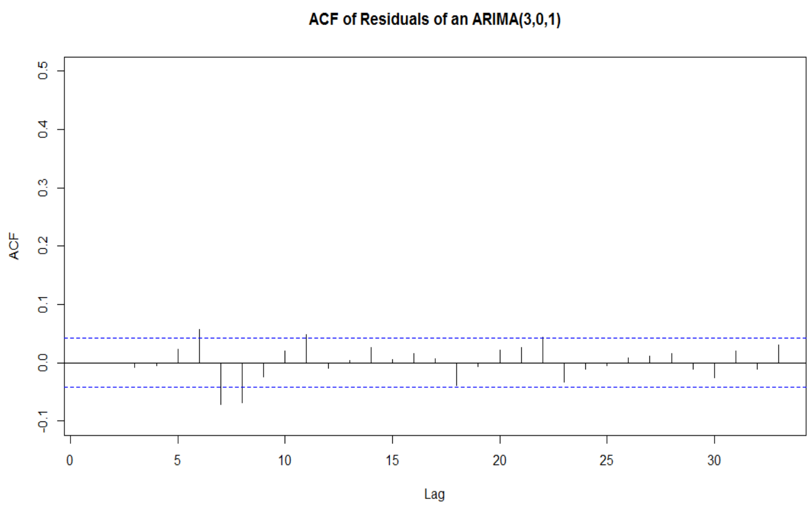


Figure 8: ACF of residuals of fitted ARIMA(3,0,1) model

Let's look at the ACF of squared residuals of fitted ARIMA(3,0,1) model. Figure 10 indicates evidence of serial correlation. We test the autocorrelation of ARIMA(3,0,1) squared residuals via the Ljung-Box test to address the hypothesis of "no ARCH effect" (no conditional heteroskedasticity). Since the p-value is less then 5%, we reject the null hypothesis of no autocorrelation, see results of the Ljung-Box test in figure 9.

```
        Box-Ljung test

data:  resid(lrfinal.arima)^2
X-squared = 212.86, df = 1, p-value < 2.2e-16
```

Figure 9: Ljung-Box test of squared residuals of an ARIMA(3,0,1)

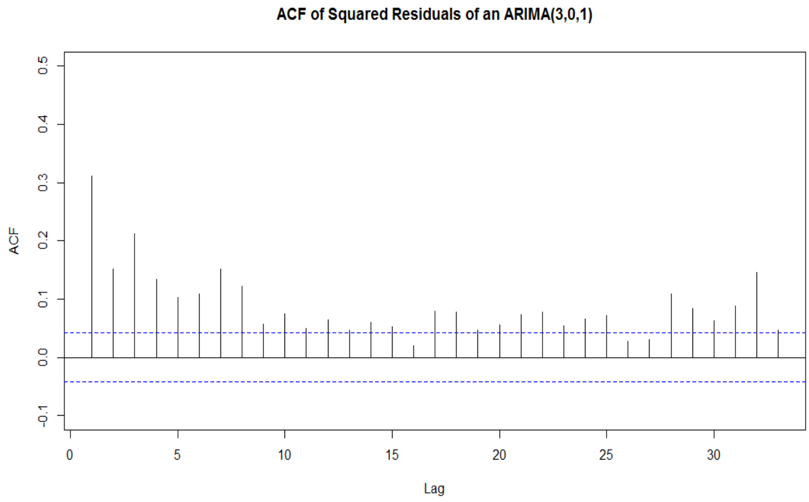


Figure 10: ACF of squared residuals of fitted ARIMA(3,0,1) model

5.2 Estimation of Volatility

We have already shown the presence of the ARCH effect in daily bitcoin returns. Now, we focus on trying to find the best fitting heteroskedastic model on the data. Firstly, we do the estimation of bitcoin volatility via GARCH(1,1) model, afterwards we involve asymmetric models such as EGARCH(1,1) and TGARCH(1,1).

5.2.1 GARCH Approach

Regarding the observation of volatility, one might only try to estimate its value. Figure 11 shows an example of a GARCH process. The GARCH approach is based on an upward spike of volatility which is decaying away until there is another spike. The estimation obtained by a GARCH model is, broadly speaking, an estimation of how fast the decay is.

The aforementioned feature of GARCH-based models can be visible also on the application of GARCH, EGARCH and TGARCH models on daily returns of bitcoin. We chose them to compare the goodness of fit and test the asymetric impact of positive and negative news. In most cases GARCH(1,1) model does good job. In practice, up to GARCH(2,2) model is used. Therefore, we start with a GARCH(1,1) model applied on returns obtained from the beginning of January 2011 until the end of December 2016.

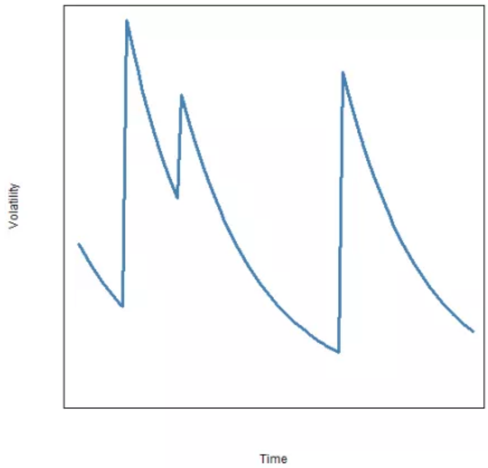


Figure 11: Example of a GARCH process (figure from [20])

5.3 The Whole Data Set

In the following modelling process, we take into consideration bitcoin daily returns from 1st January 2011 until 31 December 2016. As we mentioned before, the bitcoin markets are opened 24/7 therefore we involved the weekends as well. For the whole modelling of bitcoin dynamics we used software **R**.

5.3.1 GARCH(1,1) Model

As we check before the daily bitcoin returns shows an evidence of autocorrelation, in other words ARCH effect is present. Firstly, we specify the model and afterwards we fit the specification to logarithm returns.

Figure 12 shows the estimated coefficients of GARCH(1,1) model and the Ljung-Box statistics of standardized residuals and standardized squared residuals. One can see that the stationarity condition $\alpha_1 + \beta_1 = 0.200371 + 0.798629 = 0.999 < 1$ is fulfilled.

On other hand, based on the *p-values* of the Ljung-Box statistics, there is a clear evidence of serial correlation in residuals of model.

Among the results we obtained was included the Sign Bias Test indicating the evidence of asymmetric effect, see figure 13. Thus our next chosen models are those which consider a leverage effect, namely an EGARCH(1,1) and a TGARCH(1,1).

```
Optimal Parameters
------------------------------------
        Estimate  Std. Error  t value Pr(>|t|)
mu      0.001843    0.000689   2.6759 0.007452
omega   0.000070    0.000012   5.8095 0.000000
alpha1  0.200371    0.022965   8.7251 0.000000
beta1   0.798629    0.019925  40.0813 0.000000

Robust Standard Errors:
        Estimate  Std. Error  t value Pr(>|t|)
mu      0.001843    0.000792   2.3282 0.019900
omega   0.000070    0.000057   1.2334 0.217413
alpha1  0.200371    0.075860   2.6413 0.008258
beta1   0.798629    0.079557  10.0384 0.000000

LogLikelihood : 3870.55
```

```
Weighted Ljung-Box Test on Standardized Residuals
------------------------------------
                        statistic   p-value
Lag[1]                      16.33 5.314e-05
Lag[2*(p+q)+(p+q)-1][2]     17.15 2.493e-05
Lag[4*(p+q)+(p+q)-1][5]     21.87 5.928e-06
d.o.f=0
H0 : No serial correlation

Weighted Ljung-Box Test on Standardized Squared Residuals
------------------------------------
                        statistic p-value
Lag[1]                     0.4776  0.4895
Lag[2*(p+q)+(p+q)-1][5]    0.7240  0.9183
Lag[4*(p+q)+(p+q)-1][9]    1.2787  0.9717
d.o.f=2
```

Figure 12: Coefficients and Ljung-Box statistics for GARCH(1,1) model

```
Sign Bias Test
------------------------------------
                   t-value   prob sig
Sign Bias           0.2159 0.8291
Negative Sign Bias  0.6237 0.5329
Positive Sign Bias  0.1078 0.9142
Joint Effect        0.6319 0.8891
```

Figure 13: The evidence of leverage effect

5.3.2 Asymmetric Models

Exponential GARCH(1,1) Model

First asymmetric model we use is an exponential GARCH(1,1). In this case more result values suggest that the model is not describing the returns in the best way.

```
Optimal Parameters
------------------------------------
        Estimate  Std. Error  t value Pr(>|t|)
mu      0.002286    0.000672   3.4017  0.00067
omega  -0.306124    0.051680  -5.9235  0.00000
alpha1  0.001980    0.014724   0.1345  0.89301
beta1   0.941912    0.008667 108.6790  0.00000
gamma1  0.398956    0.036407  10.9583  0.00000
```

```
Weighted Ljung-Box Test on Standardized Residuals
------------------------------------
                        statistic   p-value
Lag[1]                      16.13 5.909e-05
Lag[2*(p+q)+(p+q)-1][2]     17.14 2.511e-05
Lag[4*(p+q)+(p+q)-1][5]     21.63 6.886e-06
d.o.f=0
H0 : No serial correlation
```

Figure 14: Coefficients and Ljung-Box statistics for EGARCH(1,1) model

For example, we expected the coefficient $\gamma_1 < 0$ due to the leverage effect, which is not meet, see figure 14 showing the estimation of model's coefficients. Moreover, the Ljung-Box test confirms that the serial correlation of residuals is still present.

Treshold GARCH(1,1) Model

The following figure 15 shows a results of a treshold GARCH(1,1) model. The estimate of coefficient γ_1 is equal to $0.040328 > 0$ (in figure represented by parameter eta11), which corresponds with the leverage effect but the autocorrelation of residuals remains unsolved.

```
Optimal Parameters
------------------------------------
        Estimate  Std. Error  t value Pr(>|t|)
mu      0.002913    0.000611   4.7710 0.000002
omega   0.002532    0.000333   7.6112 0.000000
alpha1  0.268260    0.022540  11.9014 0.000000
beta1   0.774560    0.018187  42.5875 0.000000
eta11   0.040328    0.039218   1.0283 0.303805
```

```
Weighted Ljung-Box Test on Standardized Residuals
------------------------------------
                        statistic   p-value
Lag[1]                      13.11 2.936e-04
Lag[2*(p+q)+(p+q)-1][2]     14.17 1.515e-04
Lag[4*(p+q)+(p+q)-1][5]     18.81 4.107e-05
d.o.f=0
H0 : No serial correlation
```

Figure 15: Coefficients and Ljung-Box statistics for TGARCH(1,1) model

5.3.3 Summary of Applied Models

Previous results of applied model shows that they cannot capture the data properly. There is a significant serial correlation of residuals. Moreover, the Pearson Goodness-of-Fit test shows that the normality assumption included in modelling the process is strongly rejected in all three cases.

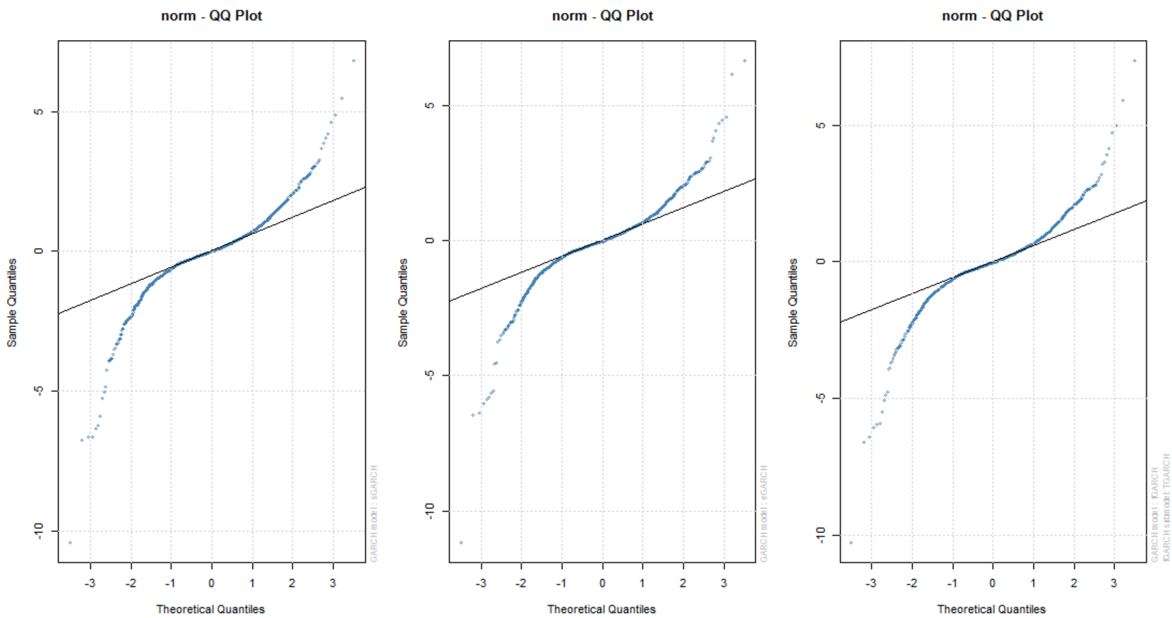


Figure 16: Q-Q plots of GARCH(1,1), EGARCH(1,1) and TGARCH(1,1) respectively

We provide the comparison of aforementioned models using different information criteria. All of the criteria (Akaike, Bayes, Shibata and Hannan-Quinn) prefer the EGARCH(1,1) to the rest of models, see table 6.

	GARCH(1,1)	EGARCH(1,1)	TGARCH(1,1)
Akaike (AIC)	-3.52949	**-3.54166**	-3.54151
Bayes (BIC)	-3.51909	**-3.52867**	-3.52852
Shibata	-3.52949	**-3.54167**	-3.54152
Hannan-Quinn	-3.52569	**-3.53691**	-3.53676

Table 6: Information criteria for used models

5.4 Shorter Time Period

Looking at the dynamics of bitcoin price at figure 2, one may spot more calm development in the last few years than in the period around the end of year 2013, when bitcoin was becoming well-known and its price exceeded USD 1,100.

We decided to look more at the volume traded during the time period starting in January 2011 until December 2016. In two-years long period during 2011 and 2012 only approximately 0.5% of all the transaction volume was traded. Thus, we decided to reduce the data sample we are going to working with. The amount of transactions traded in years 2014-2016 is around 83% of transaction volume.

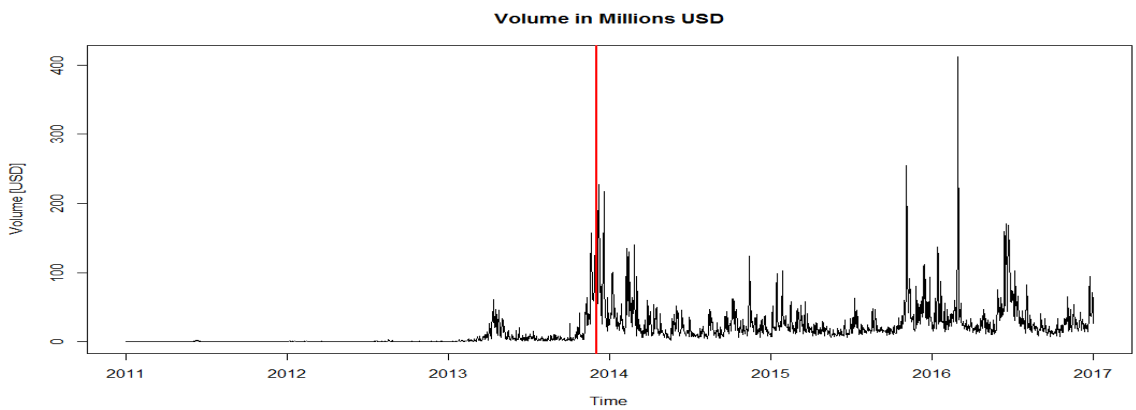


Figure 17: Traded volume of bitcoin in millions USD. The 90% of traded volume during years 2011-2016 is to the right of the red line.

Therefore we use data from 28th November 2013 until 31st December 2016 because for the mentioned time interval the amount of transaction made is slightly over 90% of the whole volume. We also take into consideration a fact that during the chosen period, bitcoin was already known among the people in comparison to its induction weeks. Also, the volume traded during the day influences bitcoin dynamics too. Table 7 states the standard parameters of our new data set.

Mean	St. dev.	Skewness	Kurtosis
-0.0001	0.0390	-0.8229	11.3918

Table 7: Standard parameters of bitcoin returns referring to the truncated dataset

As in previous section, we firstly checked the serial correlation of squared residuals and an evidence of ARCH effect in new set of data. In both cases, the null hypotheses regarding the "no correlation" and "no ARCH effect" we rejected. So we continue and apply the GARCH model and afterwards the leverage models, especially EGARCH and TGARCH with the assumption of normal distribution of the errors.

5.4.1 GARCH(1,1) Model

The specification for an GARCH(1,1) model was made and fitted to data. Now, unlike the previous fitting to the whole data, the autocorrelation of both, residuals and squared residuals is no longer present. On the top of that, the estimates of coefficient satisfy the stationarity condition: $\alpha_1 + \beta_1 = 0.122652 + 0.876348 = 0.999 < 1$, see figure 18.

```
           mu          omega         alpha1          beta1
0.00095139903 0.00002034035 0.12265210999 0.87634786800

Weighted Ljung-Box Test on Standardized Residuals
------------------------------------
                        statistic p-value
Lag[1]                      3.001 0.08323
Lag[2*(p+q)+(p+q)-1][2]     3.187 0.12488
Lag[4*(p+q)+(p+q)-1][5]     4.520 0.19595
d.o.f=0
H0 : No serial correlation

Weighted Ljung-Box Test on Standardized Squared Residuals
------------------------------------
                        statistic p-value
Lag[1]                     0.5655  0.4520
Lag[2*(p+q)+(p+q)-1][5]    1.1615  0.8222
Lag[4*(p+q)+(p+q)-1][9]    1.6177  0.9454
d.o.f=2
```

Figure 18: Coefficients and Ljung-Box statistics for GARCH(1,1) model

Even the relatively positive results we obtained, the Sign bias test shows some evidence of the leverage effect and as well as in prior models the normal distribution assumption is rejected so we move to the asymmetric models.

5.4.2 Asymmetric Models

Again, we apply the exponential GARCH(1,1) and threshold GARCH(1,1) models to bitcoin daily returns. For both models the Ljung-Box test rejected hypothesis for the serial correlation in data.

As we mentioned in chapter 2, the EGARCH approach does not require the estimated coefficients to be positive. The leverage effect implies the coefficient $\alpha_1 < 0$ and as one can see the figure 19 shows that the estimate of coefficient α_1 is negative.

EGARCH(1,1)

```
Optimal Parameters
------------------------------------
        Estimate  Std. Error  t value Pr(>|t|)
mu      0.001278    0.000394   3.2421 0.001187
omega  -0.240205    0.053490  -4.4906 0.000007
alpha1 -0.020578    0.017017  -1.2092 0.226576
beta1   0.959605    0.007840 122.3996 0.000000
gamma1  0.280603    0.030036   9.3421 0.000000

Weighted Ljung-Box Test on Standardized Residuals
------------------------------------
                        statistic p-value
Lag[1]                      2.493  0.1143
Lag[2*(p+q)+(p+q)-1][2]     2.657  0.1738
Lag[4*(p+q)+(p+q)-1][5]     3.941  0.2615
d.o.f=0
H0 : No serial correlation

Weighted Ljung-Box Test on Standardized Squared Residuals
------------------------------------
                        statistic p-value
Lag[1]                     0.4081  0.5229
Lag[2*(p+q)+(p+q)-1][5]    0.9508  0.8707
Lag[4*(p+q)+(p+q)-1][9]    1.4218  0.9617
d.o.f=2
```

TGARCH(1,1)

```
Optimal Parameters
------------------------------------
        Estimate  Std. Error  t value Pr(>|t|)
mu      0.001669    0.000529   3.1555 0.001602
omega   0.001313    0.000337   3.8988 0.000097
alpha1  0.169713    0.021173   8.0156 0.000000
beta1   0.850815    0.020294  41.9250 0.000000
eta11   0.079238    0.065025   1.2186 0.222999

Weighted Ljung-Box Test on Standardized Residuals
------------------------------------
                        statistic p-value
Lag[1]                      1.721  0.1895
Lag[2*(p+q)+(p+q)-1][2]     2.090  0.2482
Lag[4*(p+q)+(p+q)-1][5]     3.577  0.3117
d.o.f=0
H0 : No serial correlation

Weighted Ljung-Box Test on Standardized Squared Residuals
------------------------------------
                        statistic p-value
Lag[1]                     0.3959  0.5292
Lag[2*(p+q)+(p+q)-1][5]    1.0656  0.8447
Lag[4*(p+q)+(p+q)-1][9]    1.7258  0.9351
d.o.f=2
```

Figure 19: Coefficients and Ljung-Box statistics for EGARCH and TGARCH models

5.4.3 Summary of Applied Models

Having the shorter time period of daily bitcoin returns, we provided fitting of GARCH(1,1), EGARCH(1,1) and TGARCH(1,1) models. The table 8 presents the results of four different information criteria where all prefer the EGARCH(1,1) model.

After fitting aforementioned models we draw the news impact curves and a reader can clearly see the differences we mentioned in section 4.1.1 dedicated to the shape of news impact curve. The figure 20 clearly shows us that in case of EGARCH and TGARCH approach, bad news have higher impact on volatility than good news of the

	GARCH(1,1)	EGARCH(1,1)	TGARCH(1,1)
Akaike (AIC)	-4.0655	**-4.0727**	-4.0432
Bayes (BIC)	-4.0477	**-4.0504**	-4.0210
Shibata	-4.0655	**-4.0727**	-4.0433
Hannan-Quinn	-4.0588	**-4.0642**	-4.0348

Table 8: Information criteria for used models

same magnitude. Besides this leverage effect, one can spot the symmetry of GARCH news impact curve.

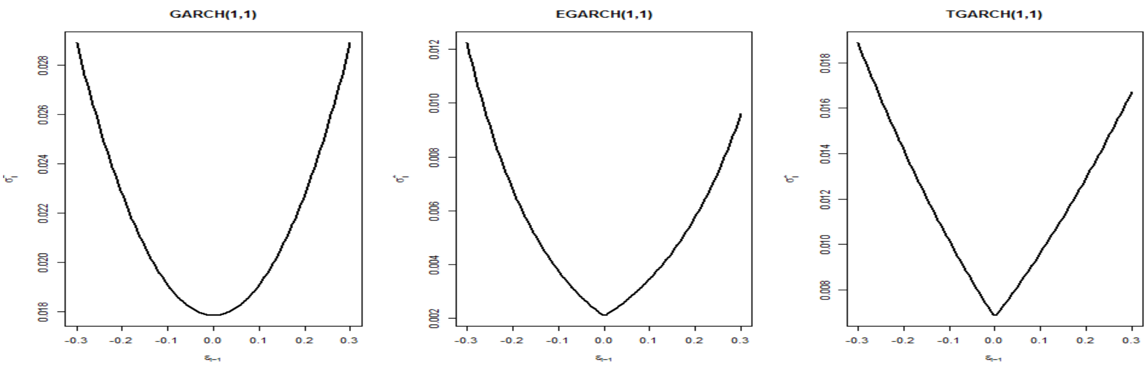


Figure 20: News impact curves for GARCH, EGARCH and TGARCH models

Concerning the goodness of fit, we plotted the Q-Q plots again, see figure 21. Even in these terms, Q-Q plots show we should reject the normal distribution assumption. We confirm this by providing the Pearson Goodness-of-Fit test as well. The normality assumption was rejected in all three applied models.

5.5 GARCH Models with Non-normal Errors

To remind, so far we used the GARCH models with normal errors:

$$a_t = \sqrt{\sigma_t^2}\epsilon_t,$$

where $\epsilon_t \overset{iid}{\sim} N(0,1)$.

As it is stated in [26], we often come across the estimated standard residuals $\hat{\epsilon_t} = \frac{\hat{a_t}}{\hat{\sigma_t}}$ from a GARCH model with normal errors, which still have fat and/or asymmetric tails.

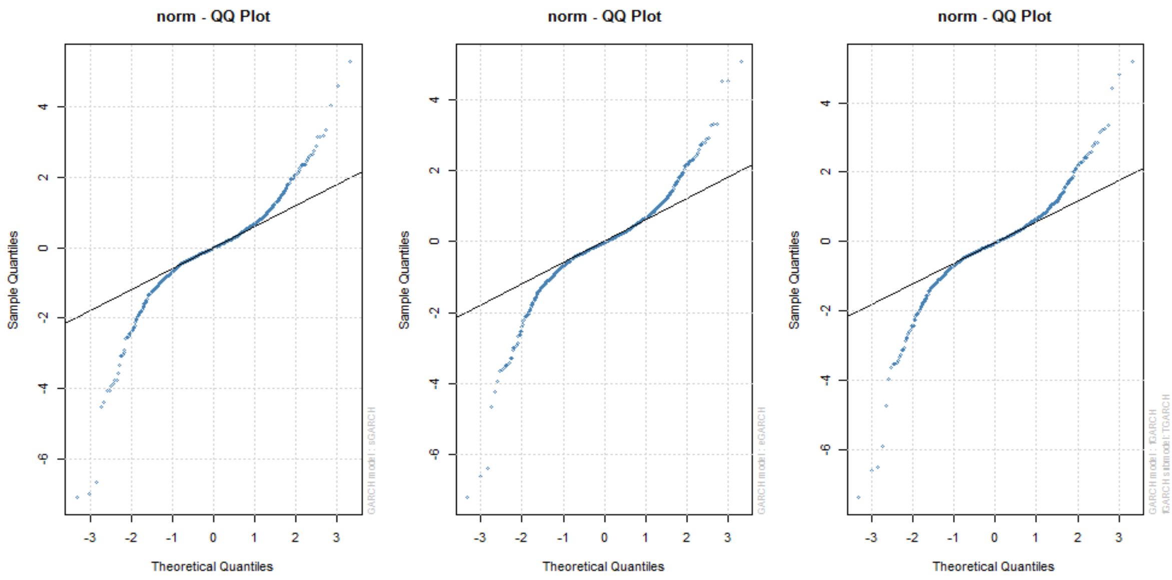


Figure 21: Q-Q plots of GARCH(1,1), EGARCH(1,1) and TGARCH(1,1) respectively

In such cases, instead of using the N(0,1) distribution, one might use a standardized fat-tailed and/or asymmetric error distribution for ϵ_t.

Usually, the following fat-tailed error distributions are mostly considered: the Student-t distribution, the double exponential distribution, the generalized error distribution, and the generalized hyperbolic distribution.

The most common fat-tailed and asymmetric distribution is skewed-t. One of another possible asymmetric distributions with fat tails is the generalized hyperbolic skew Student distribution.

5.5.1 GARCH with Student-t Distributed Errors

Now, to gain better idea about the change in the GARCH model we are about to do, let us briefly introduce the non-normal GARCH model assuming the errors to have Student-t distribution.

Definition 5.1 (GARCH model with Student-t errors). *Let u_t be Student-t random variable, $\nu > 0$ represent the degrees of freedom parameter and s_t be scale parameter.*

Then

$$f(u_t) = \frac{\Gamma\left[\frac{\nu+1}{2}\right]}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu s_t}}\left(1+\frac{u_t^2}{s_t\nu}\right)^{-\frac{\nu+1}{2}} \tag{5.1}$$

$$var(u_t) = \frac{s_t\nu}{\nu-2}, \quad \nu > 2 \tag{5.2}$$

If the distribution of a_t in the GARCH model is Student-t with $E[\alpha_t^2|\mathcal{F}_{t-1}] = \sigma_t^2$, where $\mathcal{F}_{t-1}$ is the information at time $(t-1)$, then set

$$s_t = \frac{\sigma_t^2(\nu-2)}{\nu} \tag{5.3}$$

to obtain a standardized Student-t distribution for ϵ_t.

Application of GARCH Model with Non-normal Errors

From Q-Q plots in figure 21 is clear that the errors of the model are not normally distributed. Moreover, figure 22 shows the errors have fat tails. Therefore we decided to take into consideration a non-normal GARCH model to fit bitcoin returns.

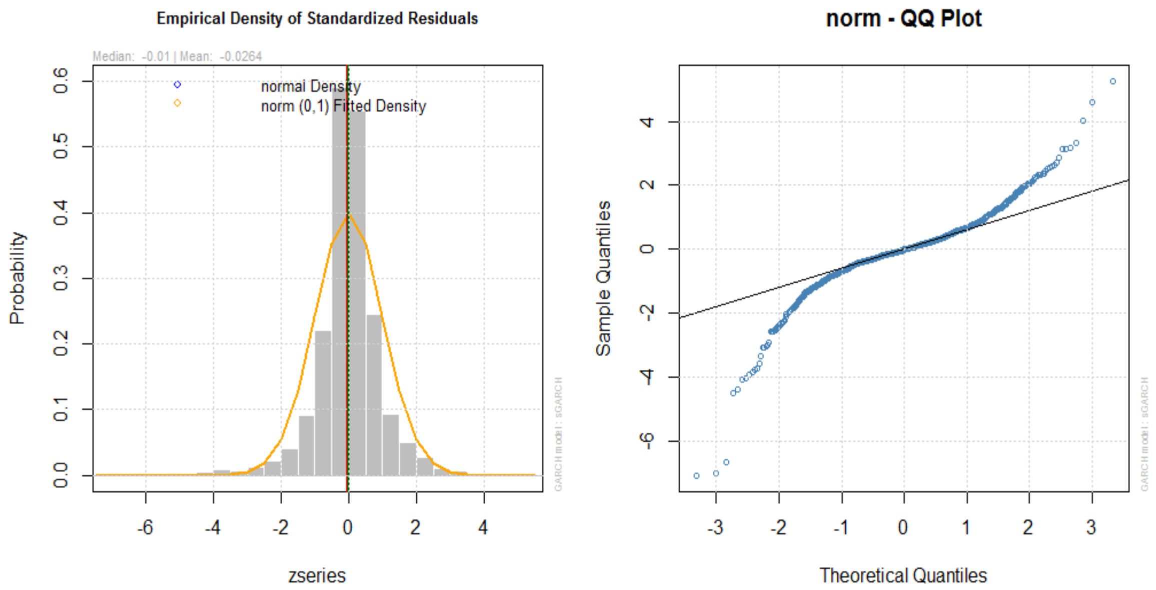


Figure 22: Empirical density of standard residuals of GARCH(1,1) and Q-Q plot

Adjusting our previous model by setting the error distribution to be Student-t we obtain below mentioned results, see 23. The estimation of the degree of freedom is equal to $df = 2.95$. The stationarity condition is still fulfilled hence $\alpha_1 + \beta_1 = 0.1909 + 0.8081 = 0.999 < 1$.

Regarding the autocorrelation of returns, both Ljung-Box tests reject hypothesis for standardized returns and their squared values to be serially correlated.

But more interesting is the result of adjusted Pearson Goodness-of-Fit test. Using the GARCH model with Student-t distributed errors we finally obtain that the tests for distribution goodness-of-fit indicates not to reject the hypothesis.

```
Optimal Parameters
------------------------------------
        Estimate  Std. Error  t value Pr(>|t|)
mu      0.000808    0.000499   1.6207 0.105082
omega   0.000033    0.000013   2.5933 0.009507
alpha1  0.190852    0.032625   5.8499 0.000000
beta1   0.808148    0.030126  26.8254 0.000000
shape   2.948745    0.189138  15.5904 0.000000
```

```
Weighted Ljung-Box Test on Standardized Residuals
------------------------------------
                        statistic p-value
Lag[1]                      3.423 0.06431
Lag[2*(p+q)+(p+q)-1][2]     3.453 0.10587
Lag[4*(p+q)+(p+q)-1][5]     4.481 0.19984
d.o.f=0
H0 : No serial correlation
```

```
Adjusted Pearson Goodness-of-Fit Test:
------------------------------------
  group statistic p-value(g-1)
1    20     23.93      0.19878
2    30     36.16      0.16884
3    40     56.23      0.03646
4    50     44.74      0.64647
```

```
Weighted Ljung-Box Test on Standardized Squared Residuals
------------------------------------
                        statistic p-value
Lag[1]                    0.01361  0.9071
Lag[2*(p+q)+(p+q)-1][5]   1.03977  0.8506
Lag[4*(p+q)+(p+q)-1][9]   1.69222  0.9384
d.o.f=2
```

Figure 23: Results of fitted GARCH(1,1) with Student-t distribution of errors

Although we obtained much better model for daily bitcoin returns, the GARCH(1,1) model is not sufficient enough. A reader may notice, the Q-Q plot of Student-t errors is straighter than when using the normality assumption but still there are many of those which does not correspond with the Student-t distribution.

The previous attempt showed us that the EGARCH(1,1) model was the best from the applied once. Therefore, we carry on with the modification of an EGARCH(1,1) model by assuming the Student-t error distribution.

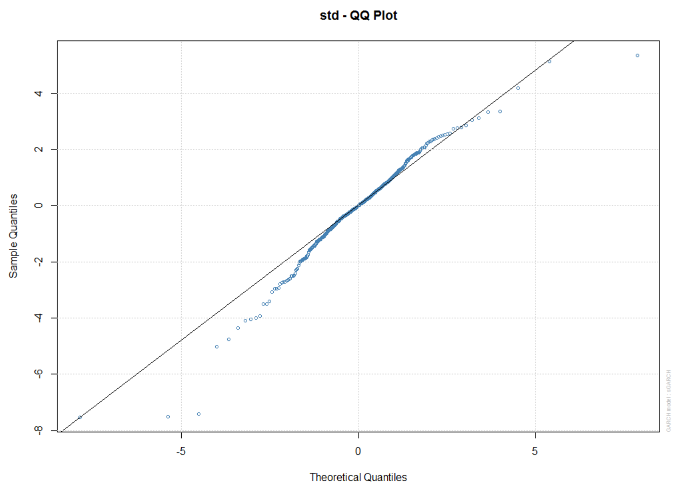


Figure 24: Q-Q plot of GARCH(1,1) with Student-t distribution of errors

5.6 EGARCH Models with Non-normal Errors

5.6.1 Student-t Distributed Errors

Again we repeat whole process of fitting an EGARCH(1,1) model to dataset but this time with the Student-t error distribution assumption. Now the estimation of degree of freedom is 2.36 and the serial correlation is rejected too, see 25.

```
Optimal Parameters
------------------------------------
        Estimate  Std. Error   t value Pr(>|t|)
mu      0.000842    0.000492  1.711267 0.087032
omega  -0.254455    0.091026 -2.795409 0.005183
alpha1  0.001116    0.042573  0.026205 0.979094
beta1   0.959074    0.013945 68.774924 0.000000
gamma1  0.558363    0.132943  4.200018 0.000027
shape   2.360383    0.186973 12.624175 0.000000

Adjusted Pearson Goodness-of-Fit Test:
------------------------------------
  group statistic p-value(g-1)
1    20     14.65       0.7446
2    30     26.54       0.5963
3    40     26.89       0.9290
4    50     43.94       0.6779
```

```
Weighted Ljung-Box Test on Standardized Residuals
------------------------------------
                        statistic p-value
Lag[1]                      3.691 0.05471
Lag[2*(p+q)+(p+q)-1][2]     3.699 0.09087
Lag[4*(p+q)+(p+q)-1][5]     4.843 0.16608
d.o.f=0
H0 : No serial correlation

Weighted Ljung-Box Test on Standardized Squared Residuals
------------------------------------
                        statistic p-value
Lag[1]                    0.06748  0.7950
Lag[2*(p+q)+(p+q)-1][5]   0.94735  0.8715
Lag[4*(p+q)+(p+q)-1][9]   1.65816  0.9416
d.o.f=2
```

Figure 25: Results of fitted EGARCH(1,1) with Student-t distribution of errors

In addition to that, from the Q-Q plot 26 we can see that there are less outliers than in case of GARCH(1,1) model with Studet-t distributed errors in figure 24, but still some of them remain so we move to another type of innovations distribution to provide better fit.

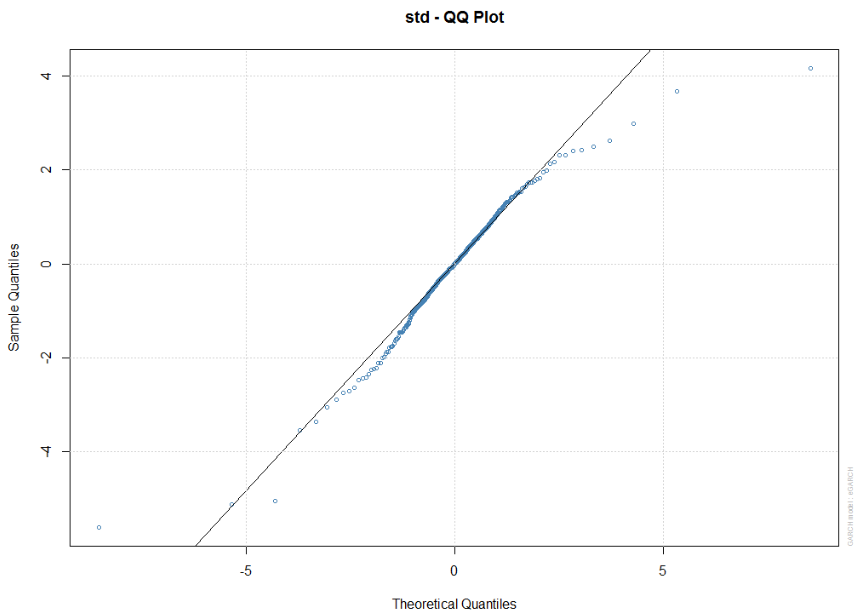


Figure 26: Q-Q plot of EGARCH(1,1) with Student-t distribution of errors

5.6.2 Generalized Hyperbolic Distributed Errors

For the next fitting we choose the EGARCH(1,1) model where the innovations have generalized hyperbolic (GH) distribution. Firstly we introduce the definition.

The generalized hyperbolic distribution describes the exponentially decreasing tails which are often observed in asset returns. It can be parametrized in many ways, we define the distribution by the following density function.

Definition 5.2 (Generalized Hyperbolic Distribution)**.** *For* $(\lambda, \alpha, \beta, \delta, \mu) \in \mathbb{R}^5$ *with* $\delta > 0$ *and* $\alpha > |\beta| > 0$, *the generalized hyperbolic density function has the following notation*

$$f(x) = \frac{\left(\frac{\sqrt{\alpha^2-\beta^2}}{\delta}\right)^{\lambda}}{\sqrt{2\pi}K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)} \frac{K_{\lambda-1/2}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{\left(\frac{\sqrt{\delta^2+(x-\mu)^2}}{\alpha}\right)^{1/2-\lambda}} \exp(\beta(x-\mu)) \tag{5.4}$$

where K_λ *is the modified Bessel function of the third kind.*

For more detailed information about the generalized hyperbolic distribution and its properties see for example [6] or [14].

Let us continue now with applying the EGARCH(1,1) model with GH distributed errors. The results stated in figure 27 shows the model assures no evidence of serial correlation of residuals and the quantiles are almost a straight line on Q-Q plot, see 29.

```
Optimal Parameters
------------------------------------
          Estimate  Std. Error   t value Pr(>|t|)
mu        0.000433    0.000672   0.64472 0.519112
omega    -0.296880    0.105232  -2.82120 0.004784
alpha1   -0.006785    0.032125  -0.21121 0.832727
beta1     0.957333    0.015105  63.37707 0.000000
gamma1    0.404722    0.057910   6.98884 0.000000
skew     -0.060007    0.077769  -0.77160 0.440353
shape     0.281964    0.067786   4.15963 0.000032
ghlambda -0.600297    0.264928  -2.26589 0.023458

Adjusted Pearson Goodness-of-Fit Test:
------------------------------------
  group statistic p-value(g-1)
1    20     13.69       0.8013
2    30     21.76       0.8299
3    40     28.09       0.9025
4    50     40.22       0.8099
```

```
Weighted Ljung-Box Test on Standardized Residuals
------------------------------------
                        statistic p-value
Lag[1]                      3.801 0.05124
Lag[2*(p+q)+(p+q)-1][2]     3.801 0.08531
Lag[4*(p+q)+(p+q)-1][5]     4.984 0.15436
d.o.f=0
H0 : No serial correlation

Weighted Ljung-Box Test on Standardized Squared Residuals
------------------------------------
                        statistic p-value
Lag[1]                    0.04958  0.8238
Lag[2*(p+q)+(p+q)-1][5]   1.00634  0.8582
Lag[4*(p+q)+(p+q)-1][9]   1.73448  0.9342
d.o.f=2
```

Figure 27: Results of fitted EGARCH(1,1) with generalized hyperbolic distribution of errors

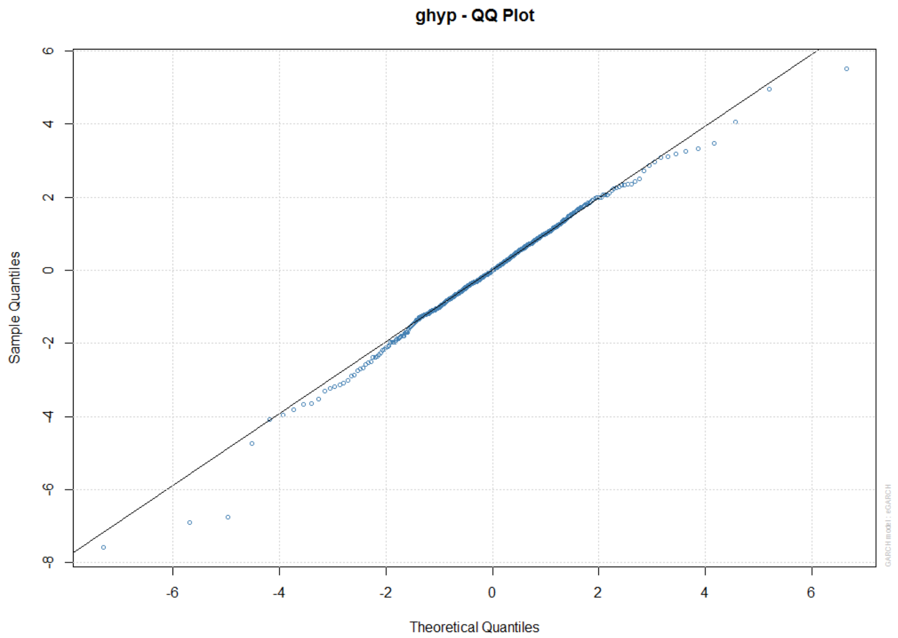


Figure 28: Q-Q plot of EGARCH(1,1) with generalized hyperbolic distribution of errors

5.7 Summary and Comparison of Applied Models

Having the daily bitcoin returns we were trying to find the best appropriate model from time series approaches to describe bitcoin dynamics. We started with the daily returns from its almost the very beginning - January 2011 until the end of year 2016.

Taking into account that the bitcoin dynamics went through its very volatile and unstable times until it became well-known among the masses, we decided to focus on shorter time interval, precisely on daily returns since the end of November 2013. For this modified data set we used models from GARCH family.

	GARCH Normal	EGARCH Normal	GARCH Student-t	EGARCH Student-t	EGARCH GenHyp
Akaike (AIC)	-4.0655	-4.0727	-4.4127	-4.4216	**-4.4263**
Bayes (BIC)	-4.0477	-4.0504	-4.3905	**-4.3949**	-4.3907
Shibata	-4.0655	-4.0727	-4.4128	-4.4216	**-4.4264**
Hannan-Quinn	-4.0588	-4.0642	-4.4043	-4.4115	**-4.4128**

Table 9: Information criteria for used models

Firstly, the basic GARCH and EGARCH models which were not sufficient enough to describe the data because of the normality assumption of errors. Therefore we concentrated on more advanced GARCH and EGARCH models with non-normal distribution of innovations, such as approaches considering the distribution of errors to be Student-t or generalized hyperbolic.

The EGARCH(1,1) model assuming Student-t distributed errors gave us much better results than the previous models. Using the generalized hyperbolic distribution enhanced fitting even a bit more. The table 9 shows the information criteria for used models. Majority of them prefer the last one applied model, namely the EGARCH(1,1) with generalized hyperbolic distributed errors.

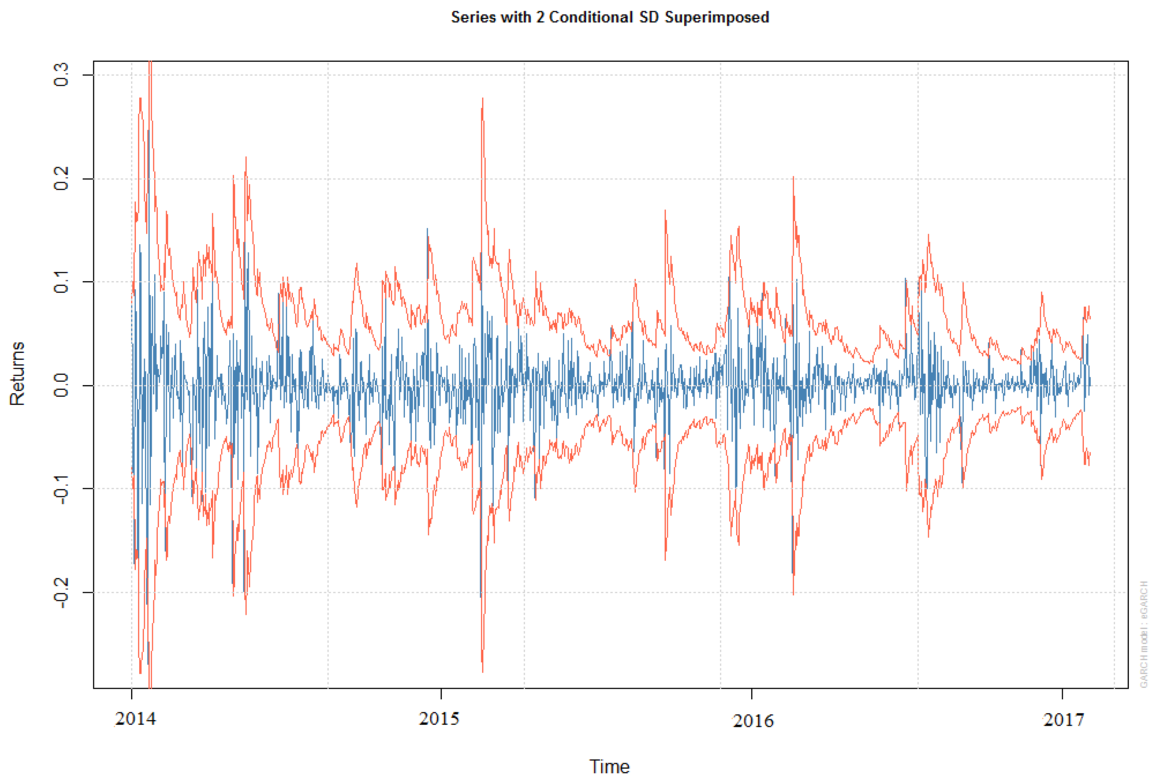
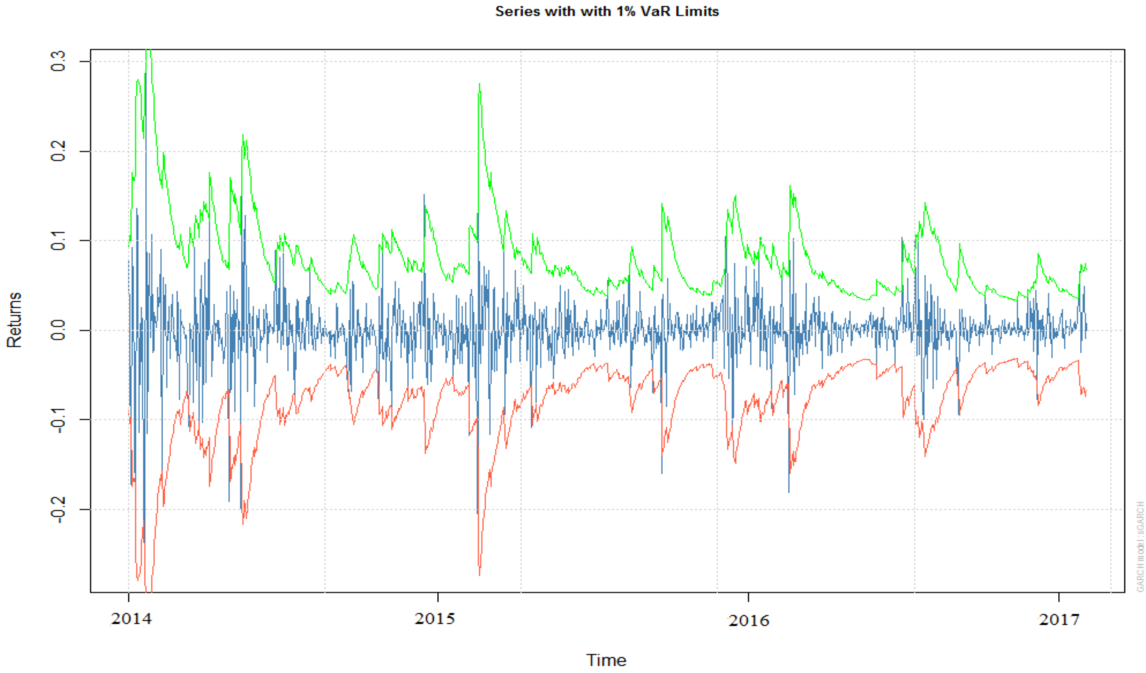
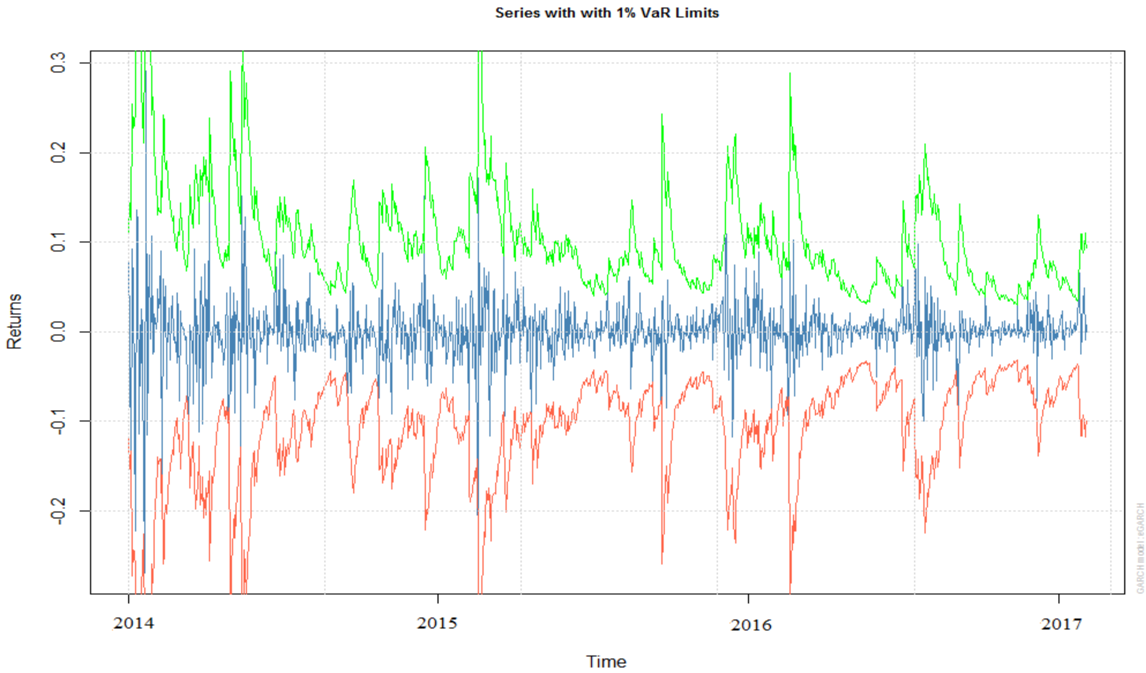


Figure 29: Returns and 2 conditional SD of EGARCH(1,1) with GH distribution

Finally, the figure 30 shows 1% VaR limits for GARCH(1,1) model with normal distributed errors and for EGARCH(1,1) model with generalized hyperbolic error distribution. A reader can see that the EGARCH model capture more of the returns' peaks.



(a) 1% VaR - GARCH(1,1) - normal distribution



(b) 1% VaR - EGARCH(1,1) - gen. hyperbolic distribution

Figure 30: Comparison of 1% VaR limits for GARCH(1,1) model with assumption of normal distribution and EGARCH(1,1) model with generalized hyperbolic distribution of errors

Conclusion

In this master thesis we focused on digital cryptocurrency called bitcoin, started with presenting the bitcoin volatility and the basic characteristics of volatility itself together with the alternative ways of approaching it.

In the following chapter we introduced the theory of time series. Firstly, we defined an ARCH(p) model and its basic properties then we presented the generalization of model called GARCH(p,q) and we finished the second chapter with the enhanced models which focus on the asymmetric effect such as EGARCH and TGARCH.

The third chapter presented a robust risk measure - an expected shortfall. After shortly describing various methods of estimating its value, we applied it to the daily returns of bitcoin. We showed the fluctuation of historical ES and compared Gaussian ES estimation to Value-at-risk estimate. It was shown that during the years 2011-2016 bitcoin was quite risky investment.

Moreover, we created two equally weighted portfolios consisting of Japanese yen, Argentine peso and bitcoin. One considering the whole time period since 2011 until 2016, and the second with a truncated data set made of daily returns from 2014 to the end of 2016. The risk linked with the second portfolio was significantly lower than for the first one, indicating bitcoin dynamics is becoming more stable with lesser unexpected shocks.

The chapter four provided information regarding a standard measure of the leverage effect of the news called news impact curve with three diagnostic tests for asymmetric effect.

Finally, we applied gathered theoretical knowledge from previous chapters to bitcoin returns. Firstly, we showed the presence of the heteroskedasticity in our data set and then we tried to find a model which provided the best fit. After applying a basic GARCH model we found out that there was a serial correlation in residuals of model and the evidence of asymmetric effect, therefore we tried EGARCH and TGARCH model. The results remained unsatisfactory since we assumed a normal distribution of innovations.

Based on the former results we decided to look at the traded volume during considered period. This led us to conclusion that 90% of trades were made since the end of year 2013. Thus, for the next analysis we use only this shorter time interval. Moreover, we changed the assumption of the error distribution from normal to Student-t and/or generalized hyperbolic distribution. Our analysis ended up with pretty accurate fit of data by EGARCH(1,1) model with generalized hyperbolic distribution of innovations, which solved a serial correlation of errors and the Q-Q plot was almost a straight line as one can see in figure 29.

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